A FINITE APPROXIMATE LIVŠIC THEOREM FOR ANOSOV DIFFEOMORPHISMS

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Abstract. In this paper, we prove a finite approximate version of the Livšic theorem for Anosov diffeomorphisms. Let f be a transitive Anosov diffeomorphism and $\varphi \in C^{\alpha}(M)$. We show that there exist $0 < \beta \leq \alpha, C > 0$ and $\tau > 0$ such that for any $\varepsilon > 0$, if $\left|\sum_{i=0}^{n} \varphi\left(f^{i}(p)\right)\right| \leq \varepsilon$ for all periodic point $p = f^{n}(p)$ with $n \leq \varepsilon^{-\frac{1}{2}}$, then there exist $u \in C^{\beta}(M)$ and $h \in C^{\beta}(M)$ such that $\varphi = u \circ f - u + h$. Moreover, $\|u\|_{C^{\beta}} \leq C$ and $\|h\|_{C^{\beta}} \leq C\varepsilon^{\tau}$.

1. Introduction

Given a dynamical system $f : M \to M$ and a function $\varphi : M \to \mathbb{R}$, φ is called a *coboundary*, if there exists a function $u : M \to \mathbb{R}$ such that

$$\varphi = u \circ f - u. \tag{1.1}$$

Equation (1.1) is usually called the *cohomological equation* and u is called a solution to this equation. The study of the cohomological equation has applications in many problems including conjugacy of dynamical systems; ergodic optimization; rigidity of group actions.

If f is a hyperbolic system, the fundamental work on the existence of solutions to cohomological equations were first studied by Livšic [15, 14]. He proved that if f is a hyperbolic system and φ is Hölder continuous, then φ is a coboundary if and only if

$$\sum_{i=0}^{n-1} \varphi(f^{i}p) = 0, \quad \forall p = f^{n}(p), n \geq 1.$$

In recent decades, cohomological equations was widely studied in different directions. For instance, the smooth regularity of solutions [4]; the function φ could be replaced by a sequence of non-additive functions [8], or a general map $\varphi : M \to G$, where G is a metric group [10, 1]; cohomology of two cocycles [2, 3, 16]; the " $=$ " in (1.1) could be replaced by " $\geq$ " [17]; the system f could be nonuniformly hyperbolic [11, 20], partially hyperbolic [19], random Anosov [9], integrable systems [5], or Axiom A flows [13]and so on.

In this article, we study a finite approximate version Livšic theorem over Anosov diffeomorphisms.

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Theorem 1.1. *Let f be a transitive Anosov diffeomorphism of a compact manifold M and let $0 < \alpha \leq 1$. There exist $0 < \beta \leq \alpha, C > 0$ and $\tau > 0$ such that for any $\varepsilon > 0$ and any $\varphi \in C^\alpha(M)$ with $\|\varphi\|_{C^\alpha} \leq 1$ and*

$$\left|\sum_{i=0}^{n} \varphi\left(f^i(p)\right)\right| \leq \varepsilon, \quad \forall p = f^n(p) \text{ with } n \leq \varepsilon^{-\frac{1}{2}}.$$

Then there exist $u \in C^\beta(M)$ and $h \in C^\beta(M)$ such that

$$\varphi(x) = u \circ f(x) - u(x) + h(x), \quad \forall x \in M.$$

Moreover, $\|u\|_{C^\beta} \leq C$ and $\|h\|_{C^\beta} \leq C\varepsilon^\tau$.

We note that a similar finite Livšic theorem was first studied by S. Katok [12] in the case of contact Anosov flows on 3-dimensional manifolds. In [7], S. Gouëzel and T. Lefeuvre generalized the result to general Anosov flows.

The proof of Theorem 1.1 relies on the following proposition. We say a subset S is ε-separated, if $S \cap B(x, \varepsilon) = \{x\}$ for every $x \in S$.

Proposition 1.2. *Let f be a transitive Anosov diffeomorphism of a compact manifold M. Fix any $\varepsilon_1 > 0$. There exist $\beta_s, \beta_d > 0$ such that for any $\varepsilon > 0$ small enough, there exists a periodic point $p = f^n(p)$ with $n \leq \varepsilon^{-\frac{1}{2}}$, such that the orbit $\mathcal{O}(p)$ is ε^{β_s}-separated and $\{p, \dots, f^{n-2}(p)\}$ is ε^{β_d}-dense in M. Moreover, there exists $N_1 = N(\varepsilon_1) \in \mathbb{N}$ such that $\{p, \dots, f^{N_1-1}(p)\}$ is ε_1-dense in M.*

We note that the orbit segment $\{p, \dots, f^{n-2}(p)\}$ considered in Proposition 1.2 is utilized in equation (3.5) and (3.6), to make sure the function $h = \varphi - (u \circ f - u) \equiv 0$ on $\{p, \dots, f^{n-2}(p)\}$. We also mention the main difference between our setting and that in [7]. In the Anosov flow setting, S. Gouëzel and T. Lefeuvre obtained a periodic orbit that is separated only transversally to the flow direction, while the periodic orbit in Proposition 1.2 is required separated in M, which creates difficulties in the construction of the periodic orbit.

Given a periodic orbit $\mathcal{O}$, denote by $T(\mathcal{O})$ the period of $\mathcal{O}$ and

$$d(\mathcal{O}) = \min\{d(x, y) : x, y \in \mathcal{O}, x \neq y\}.$$

As a corollary of Proposition 1.2, we obtain the following result.

Corollary 1.3. *Let f be a transitive Anosov diffeomorphism of a compact manifold M. Then there exist $C > 0, 0 < \beta < 1$ and a sequence of periodic orbits $\{\mathcal{O}_k\}$ such that*

$$d(\mathcal{O}_k)^{\beta \dim(M)} T(\mathcal{O}_k) \geq C, \text{ and } \lim_{k \to +\infty} T(\mathcal{O}_k) = +\infty.$$

Proof. By Proposition 1.2, there exist $\beta_s, \beta_d > 0$ such that for any $\varepsilon = 2^{-k}$ small enough, there exists a periodic point p_k such that the orbit $\mathcal{O}_k := \mathcal{O}(p_k)$ is $2^{-k\beta_s}$-separated and $2^{-k\beta_d}$-dense in M. Note that there exists $\widetilde{C} > 0$ such that for any $x \in \mathcal{O}_k$, $\operatorname{vol}\left(B\left(x, 2^{-k\beta_d}\right)\right) \leq \widetilde{C} \cdot 2^{-k\beta_d \cdot \dim(M)}$. As $\mathcal{O}_k$ is $2^{-k\beta_d}$-dense in M, it follows that

$$\operatorname{vol}(M) \leq T(\mathcal{O}_k) \cdot \widetilde{C} \cdot 2^{-k\beta_d \cdot \dim(M)}.$$

Therefore, $T(\mathcal{O}_k) \to +\infty$ as $k \to +\infty$, and

$$d(\mathcal{O}_k)^{\frac{\beta_d}{\beta_s} \dim(M)} T(\mathcal{O}_k) \geq (2^{-k\beta_s})^{\frac{\beta_d}{\beta_s} \dim(M)} T(\mathcal{O}_k) \geq \operatorname{vol}(M) \cdot \widetilde{C}^{-1}.$$

This proves the corollary. □

We note that this result is related to a result of S. Gan and D. Yu in [6]. They proved that if f is an endomorphism on a torus M which is ergodic with respect to the Lebesgue measure, then there exists a sequence of of periodic orbits $\{\mathcal{O}_k\}$ such that

$$d(\mathcal{O}_k)^{\dim(M)}T(\mathcal{O}_k) \geq C, \text{ and } \lim_{k\to+\infty} T(\mathcal{O}_k) = +\infty.$$

We mention that in Corollary 1.3, f is a general Anosov diffeomorphism of a general compact manifold M, and without the ergodic assumption. However, the result is also weaker. We do not know whether β in Corollary 1.3 could be chosen equal to 1 or not if f is ergodic with respect to the Lebesgue measure.

2. Preliminaries

For any $0 < \alpha \leq 1$, we denote by $C^\alpha(M)$ the set of α-Hölder continuous functions on M. The α-Hölder norm of φ is defined by

$$\|\varphi\|_{C^\alpha} := \sup_{x\in M} |\varphi(x)| + \sup_{x,y\in M, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{d(x,y)^\alpha} = \|\varphi\|_{C^0} + \|\varphi\|_\alpha.$$

Let f be a transitive Anosov diffeomorphism of a compact manifold M. Given $x \in M$ and $n \in \mathbb{N}$, denote $(x,n) = \{x, f(x), \dots, f^{n-1}(x)\}$. A sequence of orbit segments $\{(x_i,n_i)\}_{i\in\mathbb{Z}}$ is called an *ε-pseudo orbit*, if

$$d(f^{n_i}(x_i), x_{i+1}) < \varepsilon, \ \forall i \in \mathbb{Z}.$$

A pseudo orbit$\{(x_i,n_i)\}_{i\in\mathbb{Z}}$ is called *periodic*, if there exists some $m > 0$ such that $(x_i,n_i) = (x_{i+m}, n_{i+m})$ for all $i \in \mathbb{Z}$.

The following shadowing lemma is the main tool we shall use to construct periodic points. The exponential shadowing property in (2.1) is due to the local product structure. Part of the proof can be found in [18, Proposition 2.7].

Lemma 2.1 (Bowen)**.** *Let f be a transitive Anosov diffeomorphism of a compact manifold M. There exist $C_0 > 0, \lambda > 0$ and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any ε-pseudo orbit $\{(x_i,n_i)\}_{i\in\mathbb{Z}}$, there exists an orbit $\mathcal{O}(x)$ which shadows $\{(x_i,n_i)\}_{i\in\mathbb{Z}}$ up to $C_0\varepsilon$. More precisely, for any $0 \leq j \leq n_i$,*

$$d\left(f^{t_i+j}(x), f^j(x_i)\right) < C_0\varepsilon \cdot e^{-\lambda\cdot\min\{j,n_i-j\}}, \tag{2.1}$$

where

$$t_i = \begin{cases} \sum_{k=0}^{i-1} n_k, & \text{if } i > 0,\\ 0, & \text{if } i = 0,\\ -\sum_{k=i}^{-1} n_k, & \text{if } i < 0.\end{cases}$$

Finally, if the pseudo orbit $\{(x_i,n_i)\}_{i\in\mathbb{Z}}$ is periodic, then $\mathcal{O}(x)$ is also periodic.

Remark. *As a special case, if the ε-pseudo orbit is*

$$\{\dots, f^{-1}(x_0), x_0\}, (x_1,n_1), \dots, (x_{k-1}, n_{k-1}), \{x_k, f(x_k), \dots\},$$

then (2.1) *could be replaced by*

$$d\left(f^{-j}(x), f^{-j}(x_0)\right) < C_0\varepsilon\cdot e^{-\lambda j}, \text{ and } d\left(f^{t_k+j}(x), f^j(x_k)\right) < C_0\varepsilon\cdot e^{-\lambda j}. \tag{2.2}$$

3. Proof of the approximate Livšic theorem

We begin by proving Proposition 1.2.

Proof of Proposition 1.2. For the sake of convenience, one can refer to Figure 1 for a visual representation of the proof. Let us fix two periodic points $p_1 = f^{l_1}(p_1)$ and $p_2 = f^{l_2}(p_2)$ such that $W^u_{\text{loc}}(p_1) \cap W^s_{\text{loc}}(p_2) \neq \varnothing$ and $W^s_{\text{loc}}(p_1) \cap W^u_{\text{loc}}(p_2) \neq \varnothing$. Denote $\{z_1\} = W^u_{\text{loc}}(p_1) \cap W^s_{\text{loc}}(p_2)$ and $\{z_2\} = W^s_{\text{loc}}(p_1) \cap W^u_{\text{loc}}(p_2)$. Since the orbit of z_1 is asymptotic to $\mathcal{O}(p_1)$ in negative time and to $\mathcal{O}(p_2)$ in positive time, one can choose $0 < \rho_0 < \varepsilon_0$ small enough (where ε_0 is given by Lemma 2.1) such that

$$B(z_1, 3\rho_0) \cap \mathcal{O}(z_1) = \{z_1\}, \text{ and } B(p_1, 3\rho_0) \cap \mathcal{O}_+(z_1) = \{p_1\}. \tag{3.1}$$

As $z_1 \in W^u_{\text{loc}}(p_1)$, for any $0 < \varepsilon < \varepsilon_0$, one can choose $C_1 > 0$ large enough (C_1 is independent of ε) such that

$$d(f^{-i}z_1, f^{-i}p_1) < \varepsilon, \ \forall i \geq C_1|\log \varepsilon|.$$

It follows that there exists $n_1 = [C_1|\log \varepsilon|] + k$ with $1 \leq k \leq l_1$ such that

$$d(f^{-n_1}(z_1), p_1) = d(f^{-n_1}(z_1), f^{-n_1}(p_1)) < \varepsilon.$$

We denote $n_1 = [C_1|\log \varepsilon|] + O(1)$ for simplicity. Similarly, if $C_1 > 0$ is large enough, there exists $n_2 = [C_1|\log \varepsilon|] + O(1)$ such that

$$d(f^{n_2}(z_2), p_1) < \varepsilon.$$

We claim that there exists $T_1 \in \mathbb{N}_+$ such that for any $x, y \in M$, there exists an orbit segment $\{w, fw, \dots, f^k w\}$ with $1 \leq k \leq T_1$ such that $d(x, w) < \frac{\rho_0}{C_0}$ and $d(y, f^k w) < \frac{\rho_0}{C_0}$, where C_0 is given by Lemma 2.1. Indeed, as f is transitive, there exists a dense orbit $\mathcal{O}(x_0)$. Choose $T_0 \in \mathbb{N}_+$ large enough such that $\{x_0, fx_0, \dots, f^{T_0}x_0\}$ is $\frac{\rho_0}{C_0}$-dense in M. Then take $T_1 > T_0$ large enough such that $\{f^{T_0+1}x_0, \dots, f^{T_1}x_0\}$ is $\frac{\rho_0}{C_0}$-dense in M. Now for any $x, y \in M$, there exist $n_x \in [0, T_0]$ and $n_y \in [T_0+1, T_1]$ such that $d(x, f^{n_x}x_0) < \frac{\rho_0}{C_0}$ and $d(y, f^{n_y}x_0) < \frac{\rho_0}{C_0}$. This proves the claim. We denote by $\gamma_{x,y}$ the orbit segment $\{w, fw, \dots, f^{k-1}w\}$.

For any $x \in M$, we construct a periodic pseudo orbit γ_x as follows. Consider the orbit segments

$$\left\{\dots, f^{4[C_1|\log\varepsilon|]-1}(z_1), f^{4[C_1|\log\varepsilon|]}(z_1)\right\}, \ \gamma_{f^{4[C_1|\log\varepsilon|]+1}(z_1), f^{-[C_1|\log\varepsilon|]}(x)},$$
$$\left\{f^{-[C_1|\log\varepsilon|]}(x), \dots, f^{[C_1|\log\varepsilon|]}(x)\right\}, \ \gamma_{f^{[C_1|\log\varepsilon|]+1}(x), z_2}, \ \left\{z_2, f(z_2), \dots\right\}.$$

Note that these 5 segments form a $\frac{\rho_0}{C_0}$-pseudo orbit. Then Lemma 2.1 yields an orbit $\mathcal{O}(y)$ ρ_0-shadows the above pieces of orbits. Suppose that y shadows $f^{-n_1}(z_1)$ and $f^N(y)$ shadows $f^{n_2}(z_2)$ for some $N \geq 1$. If C_1 is large enough, (2.2) gives

$$d\left(y, f^{-n_1}(z_1)\right) < \varepsilon, \text{ and } d\left(f^N(y), f^{n_2}(z_2)\right) < \varepsilon.$$

Which implies

$$d\left(y, p_1\right) < 2\varepsilon, \text{ and } d\left(f^N(y), p_1\right) < 2\varepsilon. \tag{3.2}$$

As y and N are determined by x, we denote $\gamma_x := \left\{y, \dots, f^{N-1}(y)\right\}$. Then (2.1) implies x is within distance ε of γ_x if C_1 is large enough. Moreover,

$$\text{Card}(\gamma_x) = 8[C_1|\log\varepsilon|] + O(1) < 9[C_1|\log\varepsilon|], \tag{3.3}$$

if $\varepsilon > 0$ is small enough.

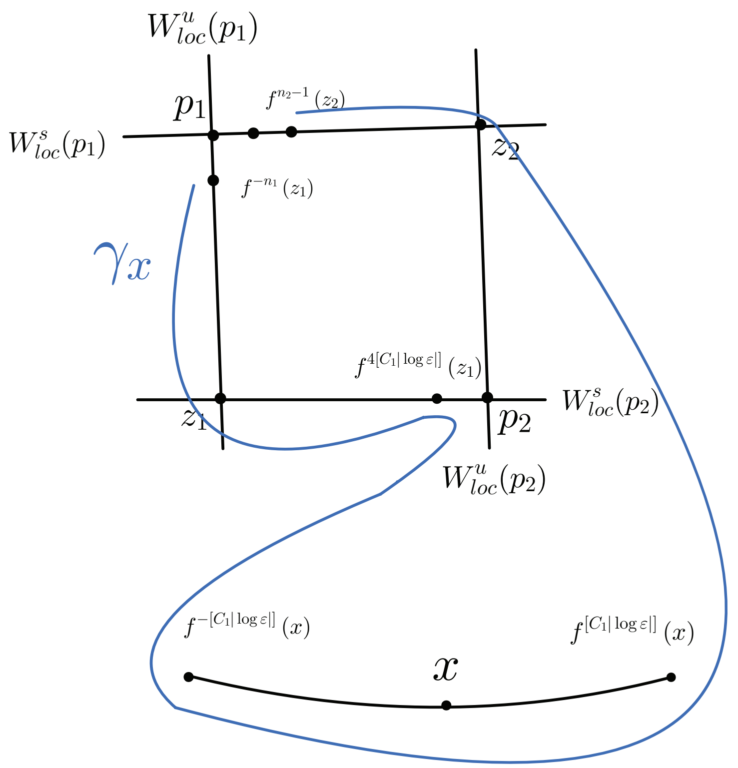

FIGURE 1. Construction of γ_x.

Let us fix a point $x_1 \in M$ such that $\mathcal{O}(x_1)$ is dense in M and then choose $N_1 \geq 1$ such that $\{x_1, \ldots, f^{N_1-1}(x_1)\}$ is $\frac{\varepsilon_1}{2}$-dense in M, to guarantee the last conclusion of the proposition holds. Let $\beta_d = \frac{1}{3\dim(M)}$. For any $x \in M$, note that $d(\gamma_x, x) < \varepsilon < \frac{1}{2}\varepsilon^{\beta_d}$ if ε is small enough, that is, $B(\gamma_x, \frac{1}{2}\varepsilon^{\beta_d})$ is a neighborhood of x. If γ_{x_1} is not $\frac{1}{2}\varepsilon^{\beta_d}$-dense in M, choose $x_2 \in M \setminus B(\gamma_{x_1}, \frac{1}{2}\varepsilon^{\beta_d})$. If $\gamma_{x_1} \cup \gamma_{x_1}$ is still not $\frac{1}{2}\varepsilon^{\beta_d}$-dense, choose $x_3 \in M \setminus B(\gamma_{x_1} \cup \gamma_{x_2}, \frac{1}{2}\varepsilon^{\beta_d})$ and so on. By the compactness of M, there exist finite points $x_2, \ldots, x_K$ such that $\gamma_{x_1} \cup \cdots \cup \gamma_{x_K}$ is $\frac{1}{2}\varepsilon^{\beta_d}$-dense in M. By (3.2),

$$\ldots, \gamma_{x_K}, \gamma_{x_1}, \ldots, \gamma_{x_K}, \gamma_{x_1}, \ldots$$

is a periodic 4ε-pseudo orbit. Then by Lemma 2.1, there exists a periodic orbit $\{p, \ldots, f^{n-1}(p)\}$ which $4C_0\varepsilon$-shadows them. We prove $p = f^n(p)$ is the required periodic point.

We check the period first. For any $1 \leq i < j \leq K$,

$$d(x_j, x_i) \geq d(x_j, \gamma_{x_i}) - d(\gamma_{x_i}, x_i) \geq \frac{1}{2}\varepsilon^{\beta_d} - \varepsilon > \frac{1}{3}\varepsilon^{\beta_d},$$

if ε is small enough. Thus the balls $\{B(x_i, \frac{1}{6}\varepsilon^{\beta_d})\}_{1\leq i\leq K}$ are disjoint, with volume

$$\mathrm{vol}\left(B\Big(x_i, \frac{1}{6}\varepsilon^{\beta_d}\Big)\right) \geq C_2^{-1} \cdot \varepsilon^{\beta_d \cdot \dim(M)} = C_2^{-1} \cdot \varepsilon^{\frac{1}{3}},$$

where $C_2 > 0$ is a constant. This proves $K \le \mathrm{vol}(M)C_2 \cdot \varepsilon^{-\frac{1}{3}}$. By (3.3), the period n is bounded by $n \le K \cdot 9[C_1|\log\varepsilon|] \le \varepsilon^{-\frac{1}{2}}$ if ε is small enough.

Let us estimate the density. As $\gamma_{x_1} \cup \cdots \cup \gamma_{x_K}$ is $\frac{1}{2}\varepsilon^{\beta_d}$-dense in M and $\mathcal{O}(p)$ $4C_0\varepsilon$-shadows them, it follows that $\mathcal{O}(p)$ is $\frac{1}{2}\varepsilon^{\beta_d} + 4C_0\varepsilon \le \varepsilon^{\beta_d}$ dense in M if ε is small enough. However, in the statement of the proposition, we require a little stronger that $\{p, \dots, f^{n-2}(p)\}$ is ε^{β_d}-dense. Note that for every $1 \le i \le K$, the last element in γ_{x_i} is within $e^{\lambda}\varepsilon$ of $f^{n_2-1}(z_2)$. Denote by y_K the last element in γ_{x_K}. Then y_K is within $2e^{\lambda}\varepsilon$ of the last element in γ_{x_1}, which implies $\cup_{i=1}^{K}\gamma_{x_i} \setminus \{y_K\}$ is $\frac{1}{2}\varepsilon^{\beta_d} + 2e^{\lambda}\varepsilon$ dense in M. Therefore, $\{p, \dots, f^{n-2}(p)\}$ is $\frac{1}{2}\varepsilon^{\beta_d} + 2e^{\lambda}\varepsilon + 4C_0\varepsilon \le \varepsilon^{\beta_d}$ dense in M if ε is small enough. Since $\{x_1, \dots, f^{N_1-1}(x_1)\}$ is $\frac{\varepsilon_1}{2}$-dense in M, if ε is small enough, one has $\{p, \dots, f^{N_1-1}(p)\}$ is $\frac{\varepsilon_1}{2} + 4C_0\varepsilon \le \varepsilon_1$ dense in M,

At last, we check the separation. Let $\beta_s = 30C_1 \log\|Df\| + 1$, where $\|Df\| = \sup_{x\in M}\|D_xf\|$. Then for any $x, y \in M$ with $d(x,y) < \varepsilon^{\beta_s}$, and any $1 \le k \le 30[C_1|\log\varepsilon|]$, one has

$$d(f^k(x), f^k(y)) \le \|Df\|^k d(x,y) \le \|Df\|^{30C_1|\log\varepsilon|} \cdot \varepsilon^{\beta_s} = \varepsilon. \tag{3.4}$$

Suppose that there exist $x, y \in \mathcal{O}(p)$ with $d(x,y) < \varepsilon^{\beta_s}$. As every γ_{x_k} has at most $9[C_1|\log\varepsilon|]$ elements, there exists $0 \le t_1 \le 9[C_1|\log\varepsilon|]$ such that $f^{t_1}(x)$ shadows the first element in γ_{x_i} for some $1 \le i \le K$. Denote by $\gamma_{x_i}(k)$ the $(k+1)$-th element in γ_{x_i}. Then

$$d\left(f^{t_1}(x), \gamma_{x_i}(0)\right) \le 4C_0\varepsilon.$$

Denote $\widetilde{x} = f^{t_1+n_1}(x)$ and $\widetilde{y} = f^{t_1+n_1}(y)$. Then

$$\begin{aligned} d(\widetilde{x}, z_1) &\le d(\widetilde{x}, \gamma_{x_i}(n_1)) + d(\gamma_{x_i}(n_1), z_1) \\ &\le 4C_0\varepsilon + \rho_0. \end{aligned}$$

Claim. *$f^{t_1}(y)$ could not be within $4C_0\varepsilon$ of $\gamma_{x_j}(k)$ for any $k \ne 0$ and any $1 \le j \le K$.*

Proof of the Claim. We prove by contradiction. If $d\left(f^{t_1}(y), \gamma_{x_j}(k)\right) \le 4C_0\varepsilon$ for some $k \ge 1$, we split it into two cases.

Case 1: $1 \le k \le 4[C_1|\log\varepsilon|]$, that is, $\gamma_{x_j}(k+n_1)$ shadows $f^k(z_1)$ up to ρ_0. Then

$$\begin{aligned} d(\widetilde{y}, f^k(z_1)) &\le d(\widetilde{y}, \gamma_{x_j}(k+n_1)) + d(\gamma_{x_i}(k+n_1), f^k(z_1)) \\ &\le 4C_0\varepsilon + \rho_0. \end{aligned}$$

By (3.4), $d(\widetilde{x}, \widetilde{y}) \le \varepsilon$. It follows that

$$\begin{aligned} d(z_1, f^k(z_1)) &\le d(z_1, \widetilde{x}) + d(\widetilde{x}, \widetilde{y}) + d(\widetilde{y}, f^k(z_1)) \\ &\le (8C_0+1)\varepsilon + 2\rho_0 \\ &< 3\rho_0, \end{aligned}$$

if ε is small enough. This is a contradiction with (3.1).

Case 2: $4[C_1|\log\varepsilon|] < k < \mathrm{Card}(\gamma_{x_j})$, that is, $\gamma_{x_j}(k+n_1)$ shadows a point in $\gamma_{f^{4[C_1|\log\varepsilon|]+1}(z_1), f^{-[C_1|\log\varepsilon|]}(x)}$, $\left\{f^{-[C_1|\log\varepsilon|]}(x), \dots, f^{[C_1|\log\varepsilon|]}(x)\right\}$, $\gamma_{f^{[C_1|\log\varepsilon|]+1}(x), z_2}$, or $\left\{z_2, \dots, f^{n_2-1}(z_2)\right\}$. Then there exists $1 \le m \le 3[C_1|\log\varepsilon|] + O(1)$ such that $f^m(\widetilde{y})$ shadows $f^{n_2}(z_2)$ up to $4C_0\varepsilon + \varepsilon$, and hence is within $(4C_0+2)\varepsilon$ of p_1. However, as $3[C_1|\log\varepsilon|] + O(1) < 4[C_1|\log\varepsilon|]$, $f^m(\widetilde{x})$ shadows $f^m(z_1)$ up to $4C_0\varepsilon + \rho_0$. It implies that

$$\begin{aligned} d(p_1, f^m(z_1)) &\le d(p_1, f^m(\widetilde{y})) + d(f^m(\widetilde{y}), f^m(\widetilde{x})) + d(f^m(\widetilde{x}), f^m(z_1)) \\ &\le (8C_0+3)\varepsilon + \rho_0 < 3\rho_0, \end{aligned}$$

if ε is small enough. This contradicts with (3.1). The proof of the claim completes. $\square$

The above claim shows that there exists $1 \le j \le K$ such that $f^{t_1}(y)$ shadows $\gamma_{x_j}(0)$ up to $4C_0\varepsilon$. To end the proof, it's left to prove $i = j$. Otherwise, suppose that $i < j$. As x_j is within distance ε of γ_{x_j}, there exists $k = 6[C_1|\log \varepsilon|] + O(1)$ such that $d(x_j, \gamma_{x_j}(k)) < \varepsilon$. Note that

$$\begin{aligned} d(\gamma_{x_j}(k), \gamma_{x_i}(k)) &\le d(\gamma_{x_j}(k), f^{t_1+k}(y)) + d(f^{t_1+k}(y), f^{t_1+k}(x)) + d(f^{t_1+k}(x), \gamma_{x_i}(k)) \\ &\le 4C_0\varepsilon + \varepsilon + 4C_0\varepsilon = (8C_0+1)\varepsilon. \end{aligned}$$

It follows that $d(x_j, \gamma_{x_i}(k)) \le (8C_0+2)\varepsilon$. This is a contradiction with $d(x_j, \gamma_{x_i}(k)) > \frac{1}{2}\varepsilon^{\beta_d}$ and $\frac{1}{2}\varepsilon^{\beta_d} > (8C_0+2)\varepsilon$ if ε is small enough. $\square$

We now prove the main result Theorem 1.1.The proof mainly relies on Proposition1.2.

Proof of Theorem 1.1. Let $\varepsilon_2 > 0$ be small enough such that Proposition 1.2 applies for any $0 < \varepsilon < \varepsilon_2$. If $\varepsilon \ge \varepsilon_2$. Take $u = 0$ and $h = \varphi$. Then it's clear that u and φ satisfy the requirement of Theorem 1.1. Thus we may assume $0 < \varepsilon < \varepsilon_2$. Let $p = f^n(p)$ be a periodic point given by Proposition 1.2, with $\varepsilon_1 = \varepsilon_0$. Denote $\mathcal{O}_{n-2}(p) := \{p, f(p), \ldots, f^{n-2}(p)\}$ and define $\widetilde{u} : \mathcal{O}_{n-2}(p) \to \mathbb{R}$ by $\widetilde{u}(p) = 0$ and

$$\widetilde{u}(f^k(p)) = \sum_{i=0}^{k-1} \varphi\left(f^i(p)\right), \quad \forall 1 \le k \le n-2. \tag{3.5}$$

We first study the Hölder regularity of $\widetilde{u}$. Suppose that $x = f^k(p), y = f^{k+m}(p) \in \mathcal{O}_{n-2}(p)$ with $d(x,y) < \varepsilon_0$, where ε_0 is given by Lemma 2.1. Then there exists a periodic point $q = f^m(q)$ such that $d\left(f^i(x), f^i(q)\right) < C_0 d(x,y) \cdot e^{-\lambda \cdot \min\{i, m-i\}}$ for every $0 \le i \le m$. By Proposition 1.2, $d(x,y) > \varepsilon^{\beta_s}$ and $m < n \le \varepsilon^{-\frac{1}{2}}$. Hence by assumption,

$$\left|\sum_{i=0}^{m-1} \varphi(f^i(q))\right| \le m\varepsilon \le \varepsilon^{\frac{1}{2}} < d(x,y)^{\frac{1}{2\beta_s}}.$$

As $\|\varphi\|_{C^\alpha} \le 1$, it follows that

$$\begin{aligned} |\widetilde{u}(x) - \widetilde{u}(y)| &\le \left|\sum_{i=0}^{m-1} \varphi(f^i(x)) - \sum_{i=0}^{m-1} \varphi(f^i(q))\right| + \left|\sum_{i=0}^{m-1} \varphi(f^i(q))\right| \\ &\le \sum_{i=0}^{m-1} \left(C_0 \cdot e^{-\lambda \cdot \min\{i, m-i\}} d(x,y)\right)^\alpha + d(x,y)^{\frac{1}{2\beta_s}} \\ &\le C_1 d(x,y)^{\beta_1}, \end{aligned}$$

where C_1 is a constant independent of ε and $\beta_1 = \min\{\alpha, \frac{1}{2\beta_s}\}$.

We now estimate the C^0 norm of $\widetilde{u}$ on $\mathcal{O}_{n-2}(p)$. By Proposition 1.2, there exists $N_1 = N_1(\varepsilon_0)$ such that $\{p, f(p), \ldots, f^{N_1-1}(p)\}$ is ε_0-dense in M. Since $\|\varphi\|_{C^\alpha} \le 1$, for any $0 \le k < N_1$,

$$|\widetilde{u}(f^k(p))| \le \left|\sum_{i=0}^{k-1} \varphi(f^i(q))\right| \le N_1.$$

For any $x \in \mathcal{O}_{n-2}(p)$, choose $0 \le k < N_1$ such that $d(x, f^k(p)) < \varepsilon_0$. Then we conclude that

$$|\widetilde{u}(x)| \le |\widetilde{u}(x) - \widetilde{u}(f^k p)| + |\widetilde{u}(f^k p)| \le C_1 \varepsilon_0^{\beta_1} + N_1.$$

Therefore, there exists $C_2 > 0$ independent of ε such that

$$\|\widetilde{u}\|_{C^{\beta_1}(\mathcal{O}_{n-2}(p))} \le C_2.$$

We extend the function $\widetilde{u}$ to M as follows:

$$u(x) = \sup_{z \in \mathcal{O}_{n-2}(p)} \left(\widetilde{u}(z) - C_2 d(x, z)^{\beta_1} \right), \quad \forall x \in M.$$

Then by definition, $u = \widetilde{u}$ on $\mathcal{O}_{n-2}(p)$. Let's estimate the C^{β_1} norm of u. For any $x_1, x_2 \in M$, suppose that $u(x_i) = \widetilde{u}(z_i) - C_2 d(x_i, z_i)^{\beta_1}$, where $i = 1, 2$ and $z_i \in \mathcal{O}_{n-2}(p)$. Then

$$\widetilde{u}(z_1) - C_2 d(x_2, z_1)^{\beta_1} \le \sup_{z \in \mathcal{O}_{n-2}(p)} \left(\widetilde{u}(z) - C_2 d(x_2, z)^{\beta_1} \right) = \widetilde{u}(z_2) - C_2 d(x_2, z_2)^{\beta_1}.$$

It follows that

$$\begin{aligned} u(x_1) - u(x_2) &= \widetilde{u}(z_1) - C_2 d(x_1, z_1)^{\beta_1} - \left(\widetilde{u}(z_2) - C_2 d(x_2, z_2)^{\beta_1} \right) \\ &\le \widetilde{u}(z_1) - C_2 d(x_1, z_1)^{\beta_1} - \left(\widetilde{u}(z_1) - C_2 d(x_2, z_1)^{\beta_1} \right) \\ &\le C_2 d(x_1, x_2)^{\beta_1}. \end{aligned}$$

Swapping x_1 and x_2, we obtain $|u(x_1) - u(x_2)| \le C_2 d(x_1, x_2)^{\beta_1}$. Since $\|u\|_{C^0} \le \|\widetilde{u}\|_{C^0} + C_2 \mathrm{diam}(M)^{\beta_1} \le C_2 + C_2 \mathrm{diam}(M)^{\beta_1} =: C_3$, we conclude that

$$\|u\|_{C^{\beta_1}} \le C_3.$$

We now set

$$h = \varphi - (u \circ f - u). \tag{3.6}$$

Then h is C^{β_1} Hölder continuous with $\|h\|_{C^{\beta_1}} \le C_4$, where C_4 is a constant independent of ε. By definition, $h|_{\mathcal{O}_{n-2}(p)} \equiv 0$. As $\mathcal{O}_{n-2}(p)$ is ε^{β_d}-dense, for any $x \in M$, there exists $p_x \in \mathcal{O}_{n-2}(p)$ such that $d(x, p_x) \le \varepsilon^{\beta_d}$. Thus

$$|h(x)| = |h(x) - h(p_x)| \le C_4 d(x, p_x)^{\beta_1} \le C_4 \varepsilon^{\beta_1 \beta_d},$$

and hence $\|h\|_{C^0} \le C_4 \varepsilon^{\beta_1 \beta_d}$. To finish the proof, we estimate the $C^{\frac{1}{2}\beta_1}$ norm of h. For any $x \ne y \in M$, if $d(x, y) \le \varepsilon^{\beta_d}$, by $\|h\|_{C^{\beta_1}} \le C_4$,

$$\frac{|h(x) - h(y)|}{d(x, y)^{\frac{1}{2}\beta_1}} \le C_4 d(x, y)^{\frac{1}{2}\beta_1} \le C_4 \varepsilon^{\frac{1}{2}\beta_1 \beta_d}.$$

If $d(x, y) > \varepsilon^{\beta_d}$, by $\|h\|_{C^0} \le C_4 \varepsilon^{\beta_1 \beta_d}$,

$$\frac{|h(x) - h(y)|}{d(x, y)^{\frac{1}{2}\beta_1}} \le \frac{2C_4 \varepsilon^{\beta_1 \beta_d}}{\varepsilon^{\frac{1}{2}\beta_1 \beta_d}} = 2C_4 \varepsilon^{\frac{1}{2}\beta_1 \beta_d}.$$

Therefore, $\|h\|_{C^{\frac{1}{2}\beta_1}} = \|h\|_{C^0} + \|h\|_{\frac{1}{2}\beta_1} \le 3C_4 \varepsilon^{\frac{1}{2}\beta_1 \beta_d}$. Let $C = 3C_4, \beta = \frac{1}{2}\beta_1$ and $\tau = \frac{1}{2}\beta_1 \beta_d$. Then the proof of Theorem 1.1 completes. □

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