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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR AN INTEGRAL BOUNDARY VALUE PROBLEM INVOLVING THE CAPUTO-FABRIZIO-TYPE FRACTIONAL DERIVATIVE*

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Abstract In this work we study the existence of nontrivial solutions for a Caputo-Fabrizio-type fractional integral boundary value problem. We first construct a new linear operator, which can include the integral boundary condition, and then under some conditions involving the spectral radius of the linear operator, we use topological degree methods to obtain some existence theorems for our considered problem.

Keywords Caputo-Fabrizio-type fractional-order differential equations, integral boundary value problems, nontrivial solutions, topological degree.

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1. Introduction

In this paper we study the Caputo-Fabrizio-type fractional integral boundary value problem

$$\begin{cases} {}^{\mathrm{CFR}}_{\ 0}D^{\alpha} z(t) + f(t, z(t)) = 0, \ 0 < t < 1, \\ z(0) = 0, \ z(1) = \int_0^1 g(t, z(t)) d\gamma(t), \end{cases} \tag{1.1}$$

where ${}^{\mathrm{CFR}}_{\ 0}D^{\alpha}$ is the Caputo-Fabrizio-type fractional derivative with $\alpha \in (1, 2)$ and the functions f, g, γ satisfy the conditions:

(H0) $f, g \in C([0, 1] \times \mathbb{R}, \mathbb{R})$,

(H1) $\gamma(t)$ is a nondecreasing and nonconstant function on $t \in [0, 1]$.

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Fractional calculus is a major research field, and many physical problems have been expressed using fractional calculus. The main reasons for using fractional derivative models are that many systems show memory, history, or nonlocal effects, which can be difficult to model using integer order derivatives. Caputo and Fabrizio presented a new definition of fractional operator without a singular kernel based on the decay exponential law, the Caputo-Fabrizio fractional-order operator, and for recent results we refer the reader to [1–21] and the references therein. For example, in [1] the authors studied the following system of nonlinear fractional derivative equations of the COVID-19 mathematical model involving the Caputo-Fabrizio fractional derivative

$$\begin{cases} {}^{\mathrm{CFR}}_{0}D^{\alpha}\mathcal{S}(t) = -(\omega+\nu)\mathcal{S} - \mu\mathcal{I}\mathcal{S}, \\ {}^{\mathrm{CFR}}_{0}D^{\alpha}\mathcal{V}(t) = \nu\mathcal{S} - (1-\varepsilon)\mu\mathcal{V}, \\ {}^{\mathrm{CFR}}_{0}D^{\alpha}\mathcal{I}(t) = \mu\mathcal{S}\mathcal{I} - (\eta+\omega)\mathcal{I}, \\ {}^{\mathrm{CFR}}_{0}D^{\alpha}\mathcal{R}(t) = \nu\mathcal{S} + \eta\mathcal{I}, \\ {}^{\mathrm{CFR}}_{0}D^{\alpha}\mathcal{D}(t) = \varpi\mathcal{I}, \end{cases}$$

with the initial conditions

$$\mathcal{S}(0) = \mathcal{S}_0, \quad \mathcal{V}(0) = \mathcal{V}_0, \quad \mathcal{I}(0) = \mathcal{I}_0, \quad \mathcal{R}(0) = \mathcal{R}_0, \quad \mathcal{D}(0) = \mathcal{D}_0.$$

They investigated the existence, uniqueness, and stability of the solution for the above system by means of fixed point theorems.

In [2] the authors studied the nonlinear coupled system of fractional q-integro-differential equations involving the derivation and integration of fractional Caputo-Fabrizio

$$\begin{cases} {}^{\mathrm{CFR}}_{0}D^{\beta_1}u(t) = \varphi_1\left(t, u(t), {}^{\mathrm{CFR}}_{0}D^{\alpha_1}v(t), {}^{\mathrm{CFR}}_{0}I^{\gamma_1}u(t), I^{\psi_1}_{q_1}v(t)\right), t \in (0,1], \\ {}^{\mathrm{CFR}}_{0}D^{\beta_2}v(t) = \varphi_2\left(t, v(t), {}^{\mathrm{CFR}}_{0}D^{\alpha_2}u(t), {}^{\mathrm{CFR}}_{0}I^{\gamma_2}v(t), I^{\psi_2}_{q_2}u(t)\right), t \in (0,1], \\ (1-q_1)\,\lambda_1 \sum_{l=0}^{\kappa_1} q_1^l u\left(q_1^l\lambda_1\right) = c_1, \quad \lambda_1 \in (0,1], \\ (1-q_2)\,\lambda_2 \sum_{l=0}^{\kappa_2} q_2^l v\left(q_2^l\lambda_2\right) = c_2, \quad \lambda_2 \in (0,1], \end{cases}$$

and they used some fixed-point methods to obtain the existence, uniqueness, and continuous dependence of solutions for their problem.

In [3] the authors used the consecutive interval division and the midpoint approach to study a problem involving the Caputo-Fabrizio derivatives

$$\begin{cases} {}^{\mathrm{CFR}}_{0}D^{\alpha}y(t) = f(t, y(t)), \text{ if } t \in (0,1], \\ y(0) = y_0, \text{ if } t = 0, \end{cases}$$

where $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function. In [4] the author discussed positive solutions for (1.1) with nonnegative nonlinearity f and $\gamma(t) \equiv 0, t \in [0,1]$, and using the Guo-Krasnosel'skii fixed-point theorem, the author obtained existence theorems under the following growth conditions:

($\mathrm{H_{Wang1}}$) $\lim_{z\to 0}\min_{t\in[0,1]} \frac{f(t,z)}{z} = +\infty$, $\lim_{z\to +\infty}\max_{t\in[0,1]} \frac{f(t,z)}{z} = 0$,

($\mathrm{H_{Wang2}}$) $\lim_{z\to 0}\max_{t\in[0,1]} \frac{f(t,z)}{z} = 0$, $\lim_{z\to +\infty}\min_{t\in[0,1]} \frac{f(t,z)}{z} = +\infty$.

Motivated by the aforementioned works, in this paper we use topological degree methods to study the existence of nontrivial solutions for the Caputo-Fabrizio-type fractional integral boundary value problem (1.1). We first study a new linear operator, which can include the integral boundary condition, and then using some conditions concerning the spectral radius of the linear operator, some existence theorems are derived. Our nonlinearities f, g need to satisfy some growth conditions (see (H2)-(H5) in Section 3), but they can be sign-changing, and in addition they are more general than $(\mathrm{H}_{\mathrm{Wang1}})$-$(\mathrm{H}_{\mathrm{Wang2}})$.

2. Preliminaries

In this section, we only present the definitions of the left Caputo-Fabrizio-type fractional derivatives in the left Riemann-Liouville sense. For more details, we refer the reader to [5,6].

Definition 2.1. Let $f \in H^1(a,b)$ with $a < b$, and $\alpha \in [0,1]$. Then the α-order left Caputo-Fabrizio-type fractional derivative in the left Riemann-Liouville sense is defined by:

$$\left({}^{\mathrm{CFR}}_{\ a}D^{\alpha} f\right)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)e^{\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]}dx,$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

Definition 2.2. Let $n < \alpha \leq n+1$ and f be such that $f^{(n)} \in H^1(a,b)$, $\beta = \alpha - n$. Then $\beta \in (0,1]$ and the α-order left Caputo-Fabrizio-type fractional derivative in the left Riemann-Liouville sense is defined by:

$$\left({}^{\mathrm{CFR}}_{\ a}D^{\alpha} f\right)(t) = \left({}^{\mathrm{CFR}}_{\ a}D^{\beta} f^{(n)}\right)(t).$$

Lemma 2.3(see [4]). Let $h, V \in C[0,1]$, and $\alpha \in (1,2)$. Then the boundary value problem

$$\begin{cases} {}^{\mathrm{CFR}}_{\ 0}D^{\alpha} z(t) + h(t) = 0,\ 0 < t < 1, \\ z(0) = 0,\ z(1) = \int_0^1 V(t)d\gamma(t) \end{cases}$$

has a solution in the form:

$$z(t) = \int_0^1 G(t,s)h(s)ds + t\int_0^1 V(t)d\gamma(t),$$

where

$$G(t,s) = \frac{1}{2B(\beta)}\begin{cases} 2(1-\beta)s(1-t) + \beta t(1-s)^2 - \beta(t-s)^2, 0 \leq s \leq t \leq 1, \\ 2(1-\beta)t(1-s) + \beta t(1-s)^2, 0 \leq t \leq s \leq 1,\ \beta = \alpha - 1. \end{cases}$$

Proof. From Lemma 1 in [4] we have

$$z(t) = c_1 + c_2 t - \frac{1-\beta}{B(\beta)}\int_0^t (t-s)h(s)ds - \frac{\beta}{2B(\beta)}\int_0^t (t-s)^2 h(s)ds,$$

where $c_i \in \mathbb{R}, i = 1,2$. Since $z(0) = 0$, we have $c_1 = 0$. Consequently,

$$z(1) = c_2 - \frac{1-\beta}{B(\beta)}\int_0^1 (1-s)h(s)ds - \frac{\beta}{2B(\beta)}\int_0^1 (1-s)^2 h(s)ds = \int_0^1 V(t)d\gamma(t),$$

and hence we obtain

$$\begin{aligned} z(t) &= t\int_0^1 V(t)d\gamma(t) + \frac{1-\beta}{B(\beta)}\int_0^1 t(1-s)h(s)ds + \frac{\beta}{2B(\beta)}\int_0^1 t(1-s)^2h(s)ds \\ &\quad - \frac{1-\beta}{B(\beta)}\int_0^t (t-s)h(s)ds - \frac{\beta}{2B(\beta)}\int_0^t (t-s)^2h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds + t\int_0^1 V(t)d\gamma(t). \end{aligned}$$

This completes the proof. □

Lemma 2.4(see [4]). Let $\sigma=\min\{\alpha-1,2-\alpha\}$ and $\psi(s)=B^{-1}(\beta)(1-s),\ \theta(t)=\frac{\sigma}{2}t(1-t), t,s\in[0,1]$. Then G has the following properties:

(i) $G(t,s)\geq 0, t,s\in[0,1]$;

(ii) $\theta(t)\psi(s)\leq G(t,s)\leq\psi(s),\ t,s\in[0,1]$.

Let $E:=C[0,1], \|z\|:=\max_{t\in[0,1]}|z(t)|$ and $P:=\{z\in E: z(t)\geq 0,\forall t\in[0,1]\}$. Then $(E,\|\cdot\|)$ is a real Banach space and P is a cone on E. Moreover, the conjugate space of E, denoted by E^*, is $\{\gamma:\gamma$ has bounded variation on $[0,1]\}$. From [22] we obtain the dual cone of P and the bounded linear functional on E can be expressed by

$$P^*:=\{\gamma\in E^*:\gamma \text{ is non-decreasing on } [0,1]\} \text{ and } \langle\gamma,z\rangle=\int_0^1 z(t)d\gamma(t), z\in E,\gamma\in E^*.$$

In view of Lemma 2.3, we define an operator $\Psi: E\to E$ as follows

$$(\Psi z)(t)=\int_0^1 G(t,s)f(s,z(s))ds+t\int_0^1 g(t,z(t))d\gamma(t), z\in E, t\in[0,1]. \tag{2.1}$$

It is easy to see that if Ψ has a fixed point z^* in $E\backslash\{0\}$, i.e., $\Psi z^*=z^*$, then this z^* is a nontrivial solution for (1.1). To study our problem, we define a linear operator as follows:

$$(L_{\eta_1,\eta_2}z)(t)=\eta_1\int_0^1 G(t,s)z(s)ds+\eta_2 t\int_0^1 z(t)d\gamma(t),\ z\in E,\ \eta_i>0, i=1,2. \tag{2.2}$$

Lemma 2.5. Let $r(L_{\eta_1,\eta_2})$ be the spectral radius of L_{η_1,η_2}. Then it satisfies

$$\eta_1\int_0^1\theta(t)\psi(t)dt+\eta_2\int_0^1 td\gamma(t)\leq r(L_{\eta_1,\eta_2})\leq\eta_1\int_0^1\psi(t)dt+\eta_2\int_0^1 d\gamma(t). \tag{2.3}$$

Proof. Define two linear operators as follows:

$$(L_1z)(t)=\int_0^1 G(t,s)z(s)ds,\ (L_2z)(t)=t\int_0^1 z(t)d\gamma(t), z\in E.$$

Then $L_i: P\to P (i=1,2)$ and for all $n\in\mathbb{N}^+$ we have

$$(L_1^nz)(t)=\underbrace{\int_0^1\cdots\int_0^1}_{n} G(t,s_1)\cdots G(s_{n-1},s_n)z(s_n)ds_1\cdots ds_n,\ (L_2^nz)(t)=t\left(\int_0^1 td\gamma(t)\right)^{n-1}\int_0^1 z(t)d\gamma(t).$$

Therefore, the Gelfand theorem implies that

$$\begin{aligned} r(L_1) &= \liminf_{n\to\infty} \sqrt[n]{\|L_1^n\|} \geq \liminf_{n\to\infty} \sqrt[n]{\max_{t\in[0,1]} (L_1^n \mathbf{1})(t)} \\ &\geq \liminf_{n\to\infty} \sqrt[n]{\max_{t\in[0,1]} \theta(t) \left(\int_0^1 \theta(t)\psi(t)dt\right)^{n-1} \int_0^1 \psi(t)dt} \\ &= \int_0^1 \theta(t)\psi(t)dt, \end{aligned}$$

and

$$r(L_2) = \liminf_{n\to\infty} \sqrt[n]{\|L_2^n\|} \geq \liminf_{n\to\infty} \sqrt[n]{\max_{t\in[0,1]} (L_2^n \mathbf{1})(t)} = \int_0^1 t d\gamma(t),$$

where $\mathbf{1}(t) \equiv 1, t \in [0,1]$.

On the other hand, we know that

$$r(L_1) \leq \max_{t\in[0,1]} \int_0^1 G(t,s)\mathbf{1}(s)ds \leq \int_0^1 \psi(t)dt, \quad r(L_2) \leq \max_{t\in[0,1]} \left(t\int_0^1 \mathbf{1}(t)d\gamma(t)\right) = \int_0^1 d\gamma(t).$$

From the relations of L_{η_1,η_2} and $L_i (i = 1, 2)$ we have that (2.3) holds, as required. This completes the proof. □

Note that (2.3), $r(L_{\eta_1,\eta_2}) > 0$. Then the Krein-Rutman theorem [23] enables us to obtain that there exist $\zeta_{\eta_1,\eta_2} \in P\backslash\{0\}$ and $\varrho_{\eta_1,\eta_2} \in P^*\backslash\{0\}$ such that

$$L_{\eta_1,\eta_2}\zeta_{\eta_1,\eta_2} = r(L_{\eta_1,\eta_2})\zeta_{\eta_1,\eta_2}, \ L^*_{\eta_1,\eta_2}\varrho_{\eta_1,\eta_2} = r\left(L_{\eta_1,\eta_2}\right)\varrho_{\eta_1,\eta_2}, \tag{2.4}$$

where $L^*_{\eta_1,\eta_2} : E^* \to E^*$ is the conjugate operator of L_{η_1,η_2}, denoted by

$$\left(L^*_{\eta_1,\eta_2}\vartheta\right)(t) := \eta_1 \int_0^t ds \int_0^1 G(\tau,s)d\vartheta(\tau) + \eta_2\gamma(t) \int_0^1 t d\vartheta(t), \vartheta \in E^*.$$

Lemma 2.6. Let $P_0 = \{z \in P : \int_0^1 z(t)d\varrho_{\eta_1,\eta_2}(t) \geq \omega_{\eta_1,\eta_2}\|z\|\}$. Then $L_{\eta_1,\eta_2}(P) \subset P_0$, where $\omega_{\eta_1,\eta_2} = \int_0^1 \theta(t)d\varrho_{\eta_1,\eta_2}(t)$.

Proof. Note that if $z \in P$, we have

$$(L_{\eta_1,\eta_2}z)(t) \leq \eta_1 \int_0^1 \psi(s)z(s)ds + \eta_2 \int_0^1 z(t)d\gamma(t).$$

From Lemma 2.4(ii) we obtain

$$\begin{aligned} (L_{\eta_1,\eta_2}z)(t) &\geq \eta_1 \int_0^1 \theta(t)\psi(s)z(s)ds + \eta_2\theta(t) \int_0^1 z(t)d\gamma(t) \\ &\geq \theta(t)\|L_{\eta_1,\eta_2}z\|. \end{aligned}$$

Hence, we find that

$$\int_0^1 (L_{\eta_1,\eta_2}z)(t)d\varrho_{\eta_1,\eta_2}(t) \geq \int_0^1 \theta(t)d\varrho_{\eta_1,\eta_2}(t)\|L_{\eta_1,\eta_2}z\|.$$

This completes the proof. □

Lemma 2.7(see [24]). Let E be a Banach space, $\Omega \subset E$ a bounded open set, and $T : \Omega \to E$ a continuous compact operator. If that there exists $z_0 \in E\backslash\{0\}$ such that

$$z - Tz \neq \mu z_0, \forall z \in \partial\Omega, \mu \geq 0, \tag{2.5}$$

then the topological degree $\deg(I - T, \Omega, 0) = 0$.

Lemma 2.8(see [24]). Let E be a Banach space, $\Omega \subset E$ a bounded open set with $0 \in \Omega$, and $T : \Omega \to E$ a continuous compact operator. If

$$Tz \neq \mu z, \forall z \in \partial\Omega, \mu \geq 1, \tag{2.6}$$

then the topological degree $\deg(I - T, \Omega, 0) = 1$.

3. Main Results

We list our hypotheses on f, g:

(H2) There exist $\xi_i > 0 (i = 1, 2)$ with $r(L_{\xi_1,\xi_2}) = 1$ such that

$$\liminf_{z\to+\infty} \frac{f(t,z)}{z} > \xi_1, \text{ and } \limsup_{z\to-\infty} \frac{f(t,z)}{z} < \xi_1 \text{ uniformly in } t \in [0,1],$$

and

$$\liminf_{z\to+\infty} \frac{g(t,z)}{z} > \xi_2, \text{ and } \limsup_{z\to-\infty} \frac{g(t,z)}{z} < \xi_2 \text{ uniformly in } t \in [0,1].$$

(H3) There exist $\xi_i > 0 (i = 3, 4)$ with $r(L_{\xi_3,\xi_4}) < 1$ such that

$$\limsup_{|z|\to0^+} \frac{|f(t,z)|}{|z|} < \xi_3 \text{ uniformly in } t \in [0,1],$$

and

$$\limsup_{|z|\to0^+} \frac{|g(t,z)|}{|z|} < \xi_4 \text{ uniformly in } t \in [0,1].$$

(H4) There exist $\xi_i > 0 (i = 5, 6)$ with $r(L_{\xi_5,\xi_6}) < 1$ such that

$$\limsup_{|z|\to+\infty} \frac{|f(t,z)|}{|z|} < \xi_5 \text{ uniformly in } t \in [0,1],$$

and

$$\limsup_{|z|\to+\infty} \frac{|g(t,z)|}{|z|} < \xi_6 \text{ uniformly in } t \in [0,1].$$

(H5) There exist $\xi_i > 0 (i = 7, 8)$ with $r(L_{\xi_7,\xi_8}) = 1$ such that

$$\liminf_{z\to0^+} \frac{f(t,z)}{z} > \xi_7, \text{ and } \limsup_{z\to0^-} \frac{f(t,z)}{z} < \xi_7 \text{ uniformly in } t \in [0,1],$$

and

$$\liminf_{z\to0^+} \frac{g(t,z)}{z} > \xi_8, \text{ and } \limsup_{z\to0^-} \frac{g(t,z)}{z} < \xi_8 \text{ uniformly in } t \in [0,1].$$

In this section, we use Lemma 2.7-2.8 to obtain our main results. The basic idea in the proof of Theorem 3.1 is as follows: by means of (H2) we show that (2.5) holds in an open ball with a large enough radius, and using (H3) we show that (2.6) is satisfied in an open ball with a small enough radius. Then we obtain the existence of fixed points from properties of the degree.

Theorem 3.1. Suppose that (H0)-(H3) hold. Then (1.1) has at least one nontrivial solution.

Proof. Note that (H2) implies that there are $\varepsilon \in (0,1)$ and $C_1 > 0$ such that

$$f, g(t,z) \geq \xi_i\,(1+\varepsilon)\,z - C_1,\ \ z \geq 0, t \in [0,1],\ \ i = 1,2,$$

and

$$f, g(t,z) \geq \xi_i\,(1-\varepsilon)\,z - C_1,\ \ z \leq 0, t \in [0,1],\ \ i = 1,2.$$

This shows that

$$f, g(t,z) \geq \xi_i\,(1+\varepsilon)\,z - C_1 \geq \xi_i\,(1-\varepsilon)\,z - C_1$$

if $(t,z) \in [0,1] \times \mathbb{R}^+$, and

$$f, g(t,z) \geq \xi_i\,(1-\varepsilon)\,z - C_1 \geq \xi_i\,(1+\varepsilon)\,z - C_1$$

if $(t,z) \in [0,1] \times \mathbb{R}^-$. Consequently, we have

$$f, g(t,z) \geq \xi_i\,(1+\varepsilon)\,z - C_1,\ \ z \in \mathbb{R}, t \in [0,1],\ \ i = 1,2, \tag{3.1}$$

and

$$f, g(t,z) \geq \xi_i\,(1-\varepsilon)\,z - C_1,\ \ z \in \mathbb{R}, t \in [0,1],\ \ i = 1,2. \tag{3.2}$$

Note that from (2.4), there exist $\zeta_{\xi_1,\xi_2} \in P\backslash\{0\}$ and $\varrho_{\xi_1,\xi_2} \in P^*\backslash\{0\}$ such that

$$L_{\xi_1,\xi_2}\zeta_{\xi_1,\xi_2} = r(L_{\xi_1,\xi_2})\zeta_{\xi_1,\xi_2},\ \ L^*_{\xi_1,\xi_2}\varrho_{\xi_1,\xi_2} = r\,(L_{\xi_1,\xi_2})\,\varrho_{\xi_1,\xi_2}. \tag{3.3}$$

From Lemma 2.6 we have

$$\zeta_{\xi_1,\xi_2} \in P_0. \tag{3.4}$$

Let $M_1 := \{z \in E : z = \Psi z + \lambda\zeta_{\xi_1,\xi_2}, \lambda \geq 0\}$. Then we claim that M_1 is bounded in E. Indeed, if $z_0 \in M_1$, there is a $\lambda_0 \geq 0$ such that

$$z_0(t) = (\Psi z_0)\,(t) + \lambda_0\zeta_{\xi_1,\xi_2}(t) = \int_0^1 G(t,s) f\,(s, z_0(s))\,ds + t\int_0^1 g(t, z_0(t))d\gamma(t) + \lambda_0\zeta_{\xi_1,\xi_2}(t). \tag{3.5}$$

From (3.1) we have

$$\begin{aligned} z_0(t) &\geq \int_0^1 G(t,s)[\xi_1\,(1+\varepsilon)\,z_0(s) - C_1]ds + t\int_0^1 [\xi_2\,(1+\varepsilon)\,z_0(t) - C_1]d\gamma(t) \\ &\geq \xi_1\,(1+\varepsilon)\int_0^1 G(t,s)z_0(s)ds + \xi_2\,(1+\varepsilon)\,t\int_0^1 z_0(t)d\gamma(t) - C_2, \end{aligned} \tag{3.6}$$

where $C_2 = C_1[\int_0^1 \psi(s)ds + \int_0^1 d\gamma(t)]$. Multiplying by $d\varrho_{\xi_1,\xi_2}(t)$ on both sides of (3.6) and integrating over $[0,1]$, from (3.3) we have

$$\begin{aligned}\int_0^1 z_0(t)d\varrho_{\xi_1,\xi_2}(t) &\geq (1+\varepsilon)\int_0^1 d\varrho_{\xi_1,\xi_2}(t)\left[\xi_1\int_0^1 G(t,s)z_0(s)ds + \xi_2 t\int_0^1 z_0(t)d\gamma(t)\right] - C_2\int_0^1 d\varrho_{\xi_1,\xi_2}(t)\\ &= (1+\varepsilon)\int_0^1 z_0(s)d\left[\xi_1\int_0^s d\tau\int_0^1 G(t,\tau)d\varrho_{\xi_1,\xi_2}(t) + \xi_2\gamma(s)\int_0^1 td\varrho_{\xi_1,\xi_2}(t)\right] - C_2\int_0^1 d\varrho_{\xi_1,\xi_2}(t)\\ &= (1+\varepsilon)\left\langle L^*_{\xi_1,\xi_2}\varrho_{\xi_1,\xi_2}, z_0\right\rangle - C_2\int_0^1 d\varrho_{\xi_1,\xi_2}(t)\\ &= (1+\varepsilon)\left\langle r\left(L_{\xi_1,\xi_2}\right)\varrho_{\xi_1,\xi_2}, z_0\right\rangle - C_2\int_0^1 d\varrho_{\xi_1,\xi_2}(t)\\ &= (1+\varepsilon)\int_0^1 z_0(t)d\varrho_{\xi_1,\xi_2}(t) - C_2\int_0^1 d\varrho_{\xi_1,\xi_2}(t).\end{aligned}$$

This implies that

$$\int_0^1 z_0(t)d\varrho_{\xi_1,\xi_2}(t) \leq \frac{C_2}{\varepsilon}\int_0^1 d\varrho_{\xi_1,\xi_2}(t). \tag{3.7}$$

Note that (3.5) is equivalent to

$$\begin{aligned}&z_0(t) - (1-\varepsilon)\left[\xi_1\int_0^1 G(t,s)z_0(s)ds + \xi_2 t\int_0^1 z_0(t)d\gamma(t)\right]\\ &\quad + C_1\int_0^1 G(t,s)ds + C_1 t\int_0^1 d\gamma(t)\\ &= z_0(t) - (1-\varepsilon)\left(L_{\xi_1,\xi_2}z_0\right)(t) + (L_{C_1,C_1}\mathbf{1})(t)\\ &= \int_0^1 G(t,s)[f\left(s,z_0(s)\right) - \xi_1\left(1-\varepsilon\right)z_0(s) + C_1]ds\\ &\quad + t\int_0^1 [g(t,z_0(t)) - \xi_2\left(1-\varepsilon\right)z_0(t) + C_1]d\gamma(t) + \lambda_0\zeta_{\xi_1,\xi_2}(t).\end{aligned}$$

Note that (3.2) implies that $f\left(s,z_0(s)\right) - \xi_1\left(1-\varepsilon\right)z_0(s) + C_1 \in P$ and $g(t,z_0(t)) - \xi_2\left(1-\varepsilon\right)z_0(t) + C_1 \in P$. Then we can use a similar method as in Lemma 2.6 to prove that

$$\int_0^1 G(t,s)[f\left(s,z_0(s)\right) - \xi_1\left(1-\varepsilon\right)z_0(s) + C_1]ds + t\int_0^1 [g(t,z_0(t)) - \xi_2\left(1-\varepsilon\right)z_0(t) + C_1]d\gamma(t) \in P_0.$$

Combining this with (3.4), we have

$$z_0 - (1-\varepsilon)L_{\xi_1,\xi_2}z_0 + L_{C_1,C_1}\mathbf{1} \in P_0.$$

Therefore, we find

$$\begin{aligned}
\|z_0-(1-\varepsilon)L_{\xi_1,\xi_2}z_0+L_{C_1,C_1}\mathbf{1}\| &\le \frac{1}{\omega_{\xi_1,\xi_2}}\int_0^1[z_0(t)-(1-\varepsilon)(L_{\xi_1,\xi_2}z_0)(t)+(L_{C_1,C_1}\mathbf{1})(t)]d\varrho_{\xi_1,\xi_2}(t)\\
&=\frac{1}{\omega_{\xi_1,\xi_2}}\int_0^1[z_0(t)+(L_{C_1,C_1}\mathbf{1})(t)]d\varrho_{\xi_1,\xi_2}(t)\\
&\quad-\frac{1-\varepsilon}{\omega_{\xi_1,\xi_2}}\int_0^1 z_0(s)d\left[\xi_1\int_0^s d\tau\int_0^1 G(t,\tau)d\varrho_{\xi_1,\xi_2}(t)+\xi_2\gamma(s)\int_0^1 td\varrho_{\xi_1,\xi_2}(t)\right]\\
&=\frac{1}{\omega_{\xi_1,\xi_2}}\int_0^1[z_0(t)+(L_{C_1,C_1}\mathbf{1})(t)]d\varrho_{\xi_1,\xi_2}(t)-\frac{1-\varepsilon}{\omega_{\xi_1,\xi_2}}\langle L^*_{\xi_1,\xi_2}\varrho_{\xi_1,\xi_2},z_0\rangle\\
&=\frac{1}{\omega_{\xi_1,\xi_2}}\int_0^1[z_0(t)+(L_{C_1,C_1}\mathbf{1})(t)]d\varrho_{\xi_1,\xi_2}(t)-\frac{1-\varepsilon}{\omega_{\xi_1,\xi_2}}\langle r(L_{\xi_1,\xi_2})\varrho_{\xi_1,\xi_2},z_0\rangle\\
&=\frac{\varepsilon}{\omega_{\xi_1,\xi_2}}\int_0^1 z_0(t)d\varrho_{\xi_1,\xi_2}(t)+\frac{1}{\omega_{\xi_1,\xi_2}}\int_0^1(L_{C_1,C_1}\mathbf{1})(t)d\varrho_{\xi_1,\xi_2}(t),
\end{aligned}$$

where $\omega_{\xi_1,\xi_2}=\int_0^1\theta(t)d\varrho_{\xi_1,\xi_2}(t)$. Note that (3.7), we have

$$\|z_0-(1-\varepsilon)L_{\xi_1,\xi_2}z_0+L_{C_1,C_1}\mathbf{1}\|\le\frac{C_2}{\omega_{\xi_1,\xi_2}}\int_0^1 d\varrho_{\xi_1,\xi_2}(t)+\frac{1}{\omega_{\xi_1,\xi_2}}\int_0^1(L_{C_1,C_1}\mathbf{1})(t)d\varrho_{\xi_1,\xi_2}(t).$$

Since $(1-\varepsilon)r(L_{\xi_1,\xi_2})<1$, $I-(1-\varepsilon)L_{\xi_1,\xi_2}$ has the bounded inverse operator $(I-(1-\varepsilon)L_{\xi_1,\xi_2})^{-1}$. Hence, there exists a $\overline{Q}>0$ such that $\|z_0\|\le\overline{Q}$. This proves the boundedness of M_1. Take $R_1>\sup M_1$, and then we have

$$z\ne\Psi z+\lambda\zeta_{\xi_1,\xi_2},z\in\partial B_{R_1},\lambda\ge0,$$

where $B_{R_1}:=\{z\in E:\|z\|<R_1\}$. Lemma 2.7 implies that

$$\deg(I-\Psi,B_{R_1},0)=0. \tag{3.8}$$

From (H3) there exists a sufficiently small $r_1\in(0,R_1)$ such that

$$|f(t,z)|\le\xi_3|z|,\ |g(t,z)|\le\xi_4|z|,\ |z|\le r_1,\ t\in[0,1].$$

Consequently, we have

$$\begin{aligned}
|(\Psi z)(t)|&=\left|\int_0^1 G(t,s)f(s,z(s))ds+t\int_0^1 g(t,z(t))d\gamma(t)\right|\\
&\le\int_0^1 G(t,s)|f(s,z(s))|ds+t\int_0^1|g(t,z(t))|d\gamma(t)\\
&\le\xi_3\int_0^1 G(t,s)|z(s)|ds+\xi_4 t\int_0^1|z(t)|d\gamma(t)\\
&=(L_{\xi_3,\xi_4}|z|)(t),\ z\in\overline{B}_{r_1},\ t\in[0,1],
\end{aligned}$$

where $B_{r_1}:=\{z\in E:\|z\|<r_1\}$. In what follows, we claim that

$$z\ne\lambda\Psi z,\ z\in\partial B_{r_1},\ \lambda\in[0,1]. \tag{3.9}$$

Suppose that the contrary. Then there exist $z_1 \in \partial B_{r_1}$ and $\lambda_1 \in [0,1]$ such that

$$z_1(t) = \lambda_1(\Psi z_1)(t), t \in [0,1].$$

Let $w_1(t) = |z_1(t)|, t \in [0,1]$. Then we have

$$w_1(t) = |z_1(t)| = \lambda_1|(\Psi z_1)(t)| \leq (L_{\xi_3,\xi_4} w_1)(t),\ t \in [0,1].$$

The nth iteration of this inequality shows that

$$w_1 \leq L^n_{\xi_3,\xi_4} w_1, n = 1, 2, \cdots .$$

This implies that

$$\|w_1\| \leq \|L^n_{\xi_3,\xi_4}\|\|w_1\| \text{ and thus } 1 \leq \|L^n_{\xi_3,\xi_4}\|.$$

This means that

$$1 \leq \liminf_{n\to\infty} \sqrt[n]{\|L^n_{\xi_3,\xi_4}\|} = r\left(L_{\xi_3,\xi_4}\right).$$

This contradicts $r\left(L_{\xi_3,\xi_4}\right) < 1$ in (H3). As a result of this, (3.9) holds, as required. Lemma 2.8 gives

$$\deg\left(I - \Psi, B_{r_1}, 0\right) = 1. \tag{3.10}$$

From (3.8) with (3.10) we have

$$\deg\left(I - \Psi, B_{R_1}\backslash\overline{B}_{r_1}, 0\right) = \deg\left(I - \Psi, B_{R_1}, 0\right) - \deg\left(I - \Psi, B_{r_1}, 0\right) = -1.$$

Therefore the operator Ψ has at least one fixed point in $B_{R_1}\backslash\overline{B}_{r_1}$, and thus (1.1) has at least one nontrivial solution. This completes the proof. □

Theorem 3.2. Suppose that (H0)-(H1) and (H4)-(H5) hold. Then (1.1) has at least one nontrivial solution.

Proof. Note that (H5) implies that there are $\varepsilon \in (0,1)$ and $r_2 > 0$ such that

$$f, g(t,z) \geq \xi_i\left(1+\varepsilon\right)z,\ z \in [0, r_2], t \in [0,1], i = 7, 8,$$

and

$$f, g(t,z) \geq \xi_i\left(1-\varepsilon\right)z,\ z \in [-r, 0], t \in [0,1], i = 7, 8.$$

Note that

$$f, g(t,z) \geq \xi_i\left(1+\varepsilon\right)z \geq \xi_i\left(1-\varepsilon\right)z$$

if $(t,z) \in [0,1] \times [0, r_2]$, and

$$f, g(t,z) \geq \xi_i\left(1-\varepsilon\right)z \geq \xi_i\left(1+\varepsilon\right)z$$

if $(t,z) \in [0,1] \times [-r_2, 0]$. Hence, we have

$$f, g(t,z) \geq \xi_i\left(1+\varepsilon\right)z,\ z \in [-r_2, r_2], t \in [0,1], i = 7, 8, \tag{3.11}$$

and

$$f, g(t,z) \geq \xi_i\left(1-\varepsilon\right)z,\ z \in [-r_2, r_2], t \in [0,1], i = 7, 8. \tag{3.12}$$

Note that from (2.4), there exist $\zeta_{\xi_7,\xi_8} \in P\backslash\{0\}$ and $\varrho_{\xi_7,\xi_8} \in P^*\backslash\{0\}$ such that

$$L_{\xi_7,\xi_8}\zeta_{\xi_7,\xi_8} = r(L_{\xi_7,\xi_8})\zeta_{\xi_7,\xi_8},\ L^*_{\xi_7,\xi_8}\varrho_{\xi_7,\xi_8} = r\left(L_{\xi_7,\xi_8}\right)\varrho_{\xi_7,\xi_8}. \tag{3.13}$$

From Lemma 2.6 we have

$$\zeta_{\xi_7,\xi_8} \in P_0. \tag{3.14}$$

Now, we claim that

$$z \neq \Psi z + \lambda\zeta_{\xi_7,\xi_8},\ z \in \partial B_{r_2}, \lambda \geq 0. \tag{3.15}$$

Indeed, if the claim is false, then there are $z_2 \in \partial B_{r_2}$ and $\lambda_2 \geq 0$ such that

$$z_2 = \Psi z_2 + \lambda_2\zeta_{\xi_7,\xi_8}.$$

Using (3.11) we find

$$z_2(t) \geq \int_0^1 G(t,s)\xi_7\left(1+\varepsilon\right)z_2(s)ds + t\int_0^1 \xi_8\left(1+\varepsilon\right)z_2(t)d\gamma(t). \tag{3.16}$$

Multiplying by $d\varrho_{\xi_7,\xi_8}(t)$ on both sides of (3.16) and integrating over $[0,1]$, from (3.13) we have

$$\begin{aligned}\int_0^1 z_2(t)d\varrho_{\xi_7,\xi_8}(t) &\geq (1+\varepsilon)\int_0^1 d\varrho_{\xi_7,\xi_8}(t)\left[\xi_7\int_0^1 G(t,s)z_2(s)ds + \xi_8 t\int_0^1 z_2(t)d\gamma(t)\right]\\ &= (1+\varepsilon)\int_0^1 z_2(s)d\left[\xi_7\int_0^s d\tau\int_0^1 G(t,\tau)d\varrho_{\xi_7,\xi_8}(t) + \xi_8\gamma(s)\int_0^1 td\varrho_{\xi_7,\xi_8}(t)\right]\\ &= (1+\varepsilon)\left\langle L^*_{\xi_7,\xi_8}\varrho_{\xi_7,\xi_8}, z_2\right\rangle\\ &= (1+\varepsilon)\left\langle r\left(L_{\xi_7,\xi_8}\right)\varrho_{\xi_7,\xi_8}, z_2\right\rangle\\ &= (1+\varepsilon)\int_0^1 z_2(t)d\varrho_{\xi_7,\xi_8}(t).\end{aligned}$$

This implies that

$$\int_0^1 z_2(t)d\varrho_{\xi_7,\xi_8}(t) \leq 0. \tag{3.17}$$

On the other hand, we note that

$$\begin{aligned}&z_2(t) - (1-\varepsilon)(L_{\xi_7,\xi_8}z_2)(t) = (\Psi z_2)(t) - (1-\varepsilon)(L_{\xi_7,\xi_8}z_2)(t) + \lambda_2\zeta_{\xi_7,\xi_8}(t)\\ &\quad= \int_0^1 G(t,s)[f(s,z_2(s)) - \xi_7(1-\varepsilon)z_2(s)]ds + t\int_0^1 [g(t,z_2(t)) - \xi_8(1-\varepsilon)z_2(t)]d\gamma(t) + \lambda_2\zeta_{\xi_7,\xi_8}(t).\end{aligned}$$

From (3.12), $f(s,z_2(s)) - \xi_7(1-\varepsilon)z_2(s) \in P$ and $g(t,z_2(t)) - \xi_8(1-\varepsilon)z_2(t) \in P$. By a similar method as in Lemma 2.6 and (3.14), it can be proven that

$$z_2 - (1-\varepsilon)L_{\xi_7,\xi_8}z_2 \in P_0.$$

Hence, from (3.17) we have

$$
\begin{aligned}
\|z_2-(1-\varepsilon)L_{\xi_7,\xi_8}z_2\| &\le \frac{1}{\omega_{\xi_7,\xi_8}}\int_0^1[z_2(t)-(1-\varepsilon)(L_{\xi_7,\xi_8}z_2)(t)]d\varrho_{\xi_7,\xi_8}(t)\\
&=\frac{1}{\omega_{\xi_7,\xi_8}}\int_0^1 z_2(t)d\varrho_{\xi_7,\xi_8}(t)-\frac{1-\varepsilon}{\omega_{\xi_7,\xi_8}}\int_0^1 z_2(s)d\Bigg[\xi_7\int_0^s d\tau\int_0^1 G(t,\tau)d\varrho_{\xi_7,\xi_8}(t)\\
&\quad+\xi_8\gamma(s)\int_0^1 td\varrho_{\xi_7,\xi_8}(t)\Bigg]\\
&=\frac{1}{\omega_{\xi_7,\xi_8}}\int_0^1 z_2(t)d\varrho_{\xi_7,\xi_8}(t)-\frac{1-\varepsilon}{\omega_{\xi_7,\xi_8}}\langle L^*_{\xi_7,\xi_8}\varrho_{\xi_7,\xi_8},z_2\rangle\\
&=\frac{1}{\omega_{\xi_7,\xi_8}}\int_0^1 z_2(t)d\varrho_{\xi_7,\xi_8}(t)-\frac{1-\varepsilon}{\omega_{\xi_7,\xi_8}}\langle r\left(L_{\xi_7,\xi_8}\right)\varrho_{\xi_7,\xi_8},z_2\rangle\\
&=\frac{\varepsilon}{\omega_{\xi_7,\xi_8}}\int_0^1 z_2(t)d\varrho_{\xi_7,\xi_8}(t)\\
&\le 0,
\end{aligned}
$$

where $\omega_{\xi_7,\xi_8}=\int_0^1\theta(t)d\varrho_{\xi_7,\xi_8}(t)$. Note that $(1-\varepsilon)r\left(L_{\xi_7,\xi_8}\right)<1$, i.e., $I-(1-\varepsilon)L_{\xi_7,\xi_8}$ has the bounded inverse operator $(I-(1-\varepsilon)L_{\xi_7,\xi_8})^{-1}$. As a result, we have $z_2(t)\equiv 0, t\in[0,1]$, contradicting $z_2\in\partial B_{r_2}, r_2>0$. Thus (3.15) holds, as required. Lemma 2.7 implies that

$$\deg\left(I-\Psi,B_{r_2},0\right)=0. \tag{3.18}$$

From (H4) there exists $C_3>0$ such that

$$|f(t,z)|\le\xi_5|z|+C_3,\ |g(t,z)|\le\xi_6|z|+C_3,\ z\in\mathbb{R},\ t\in[0,1]. \tag{3.19}$$

Let $M_2:=\{z\in E: z=\lambda\Psi z, 0\le\lambda\le 1\}$. Now we shall show that M_2 is bounded in E. Indeed, if there exists $z_3\in M_2$, then $z_3=\lambda_3\Psi z_3$ for some $\lambda_3\in[0,1]$. In view of (3.19), we have

$$
\begin{aligned}
|z_3(t)| &\le |(\Psi z_3)(t)|\\
&\le \int_0^1 G(t,s)[\xi_5|z_3(s)|+C_3]ds+t\int_0^1[\xi_6|z_3(t)|+C_3]d\gamma(t)\\
&=(L_{\xi_5,\xi_6}|z_3|)(t)+(L_{C_3,C_3}\mathbf{1})(t).
\end{aligned}
$$

Let $w_3(t)=|z_3(t)|$ and $\widetilde{w}(t)=(L_{C_3,C_3}\mathbf{1})(t), t\in[0,1]$. Then $w_3,\widetilde{w}\in P$, and

$$w_3\le L_{\xi_5,\xi_6}w_3+\widetilde{w}.$$

On iterating this inequality, we find

$$w_3\le L^{n+1}_{\xi_5,\xi_6}w_3+L^n_{\xi_5,\xi_6}\widetilde{w}+\cdots+L_{\xi_5,\xi_6}\widetilde{w}+\widetilde{w}.$$

Note that $r\left(L_{\xi_5,\xi_6}\right)<1$, we have

$$\lim_{n\to\infty}L^{n+1}_{\xi_5,\xi_6}w_3=0,\ \lim_{n\to\infty}\sum_{i=0}^n L^i_{\xi_5,\xi_6}\widetilde{w}=(I-L_{\xi_5,\xi_6})^{-1}\widetilde{w}.$$

Thus, we have

$$w_3 \leq (I - L_{\xi_5,\xi_6})^{-1}\widetilde{w}.$$

This proves the boundedness of M_2. Choose R_2 sufficiently large such that $R_2 > \max\{\sup M_2, r_2\}$ and

$$z \neq \lambda \Psi z, z \in \partial B_{R_2}, 0 \leq \lambda \leq 1.$$

Now Lemma 2.8 implies

$$\deg (I - \Psi, B_{R_2}, 0) = 1. \tag{3.20}$$

From (3.18) with (3.20) we have

$$\deg \left(I - \Psi, B_{R_2} \backslash \overline{B}_{r_2}, 0\right) = \deg (I - \Psi, B_{R_2}, 0) - \deg (I - \Psi, B_{r_2}, 0) = 1.$$

Therefore the operator Ψ has at least one fixed point in $B_{R_2} \backslash \overline{B}_{r_2}$, and thus (1.1) has at least one nontrivial solution. This completes the proof. □

In what follows, we provide some examples to verify our main results. Choose $a_i \geq 0 (i = 1, \cdots, m-2, m \geq 3)$ and let

$$\gamma(t) = \begin{cases} 0, t \in [0, \nu_1), \\ a_1, t \in [\nu_1, \nu_2), \\ a_1 + a_2, t \in [\nu_2, \nu_3), \\ \vdots \\ \sum\limits_{i=1}^{m-3} a_i, t \in [\nu_{m-3}, \nu_{m-2}), \\ \sum\limits_{i=1}^{m-2} a_i, t \in [\nu_{m-2}, 1], \end{cases}$$

where $\{\nu_i\}_{i=1}^{m-2}$ with $0 < \nu_1 < \nu_2 < \cdots < \nu_{m-2} < 1$. Then this function γ satisfies (H1). Moreover, by Lemma 2.5 we know that it is appropriate to choose some positive constants $\xi_1, \xi_2, \cdots, \xi_8$ such that the spectral radius $r(L_{\xi_1,\xi_2})$, $r(L_{\xi_3,\xi_4})$, $r(L_{\xi_5,\xi_6})$ and $r(L_{\xi_7,\xi_8})$ satisfy the appropriate conditions in (H2)-(H5), respectively.

Example 3.3. (i) Let

$$f(t,z) = \begin{cases} 1 + \sum\limits_{i=1}^{n} (-1)^i \sigma_i - |z|^{1/2}, & z \in (-\infty, -1], t \in [0,1], \\ \sum\limits_{i=1}^{n} \sigma_i z^i, & z \in [-1, +\infty), t \in [0,1], \end{cases}$$

and

$$g(t,z) = \begin{cases} 1 + \sum\limits_{j=1}^{n} (-1)^j \delta_j - |z|^{1/2}, & z \in (-\infty, -1], t \in [0,1], \\ \sum\limits_{j=1}^{n} \delta_j z^j, & z \in [-1, +\infty), t \in [0,1], \end{cases}$$

where $\sigma_i, \delta_j \in \mathbb{R}$, $i, j = 1, \cdots, n, n \in \mathbb{N}$ with $n \geq 1$, $\sigma_1 \in (0, \xi_3), \delta_1 \in (0, \xi_4), \sigma_n, \delta_n > 0$. Then

$$\liminf_{z \to +\infty} \frac{f(t,z)}{z} = \liminf_{z \to +\infty} \frac{\sum\limits_{i=1}^{n} \sigma_i z^i}{z} = +\infty, \quad \liminf_{z \to +\infty} \frac{g(t,z)}{z} = \liminf_{z \to +\infty} \frac{\sum\limits_{j=1}^{n} \delta_j z^j}{z} = +\infty,$$

$$\limsup_{z\to-\infty}\frac{f(t,z)}{z}=\limsup_{z\to-\infty}\frac{1+\sum\limits_{i=1}^{n}(-1)^i\sigma_i-|z|^{1/2}}{z}=0,\quad \limsup_{z\to-\infty}\frac{g(t,z)}{z}=\limsup_{z\to-\infty}\frac{1+\sum\limits_{j=1}^{n}(-1)^j\delta_j-|z|^{1/2}}{z}=0,$$

and

$$\limsup_{|z|\to0^+}\frac{|f(t,z)|}{|z|}=\limsup_{|z|\to0^+}\frac{\left|\sum\limits_{i=1}^{n}\sigma_i z^i\right|}{|z|}=\sigma_1<\xi_3\quad \limsup_{|z|\to0^+}\frac{|g(t,z)|}{|z|}=\limsup_{|z|\to0^+}\frac{\left|\sum\limits_{j=1}^{n}\delta_j z^j\right|}{|z|}=\delta_1<\xi_4,$$

uniformly in $t\in[0,1]$. Therefore, (H0) and (H2)-(H3) hold, as required, and by Theorem 3.1, (1.1) has at least one nontrivial solution.

(ii) Let

$$f(t,z)=\begin{cases}\sum\limits_{i=2}^{n}\overline{\sigma}_i|z|^{1/i}, & z\in(-\infty,-1],t\in[0,1],\\ \sum\limits_{i=2}^{n}\overline{\sigma}_i|z|^{i}, & z\in[-1,0],t\in[0,1],\\ \sum\limits_{i=2}^{n}\overline{\kappa}_i z^{1/i}, & z\in[0,+\infty),t\in[0,1],\end{cases}$$

and

$$g(t,z)=\begin{cases}\sum\limits_{i=2}^{n}\widetilde{\sigma}_i|z|^{1/i}, & z\in(-\infty,-1],t\in[0,1],\\ \sum\limits_{i=2}^{n}\widetilde{\sigma}_i|z|^{i}, & z\in[-1,0],t\in[0,1],\\ \sum\limits_{i=2}^{n}\widetilde{\kappa}_i z^{1/i}, & z\in[0,+\infty),t\in[0,1],\end{cases}$$

where $\overline{\sigma}_i,\widetilde{\sigma}_i\in\mathbb{R},\overline{\kappa}_i,\widetilde{\kappa}_i\in\mathbb{R}^+,i=2,\cdots,n\in\mathbb{N}$ with $n\geq2$. Then we have

$$\limsup_{z\to+\infty}\frac{|f(t,z)|}{|z|}\leq\limsup_{z\to+\infty}\frac{\sum\limits_{i=2}^{n}\overline{\kappa}_i z^{1/i}}{z}=0<\xi_5$$

and

$$\limsup_{z\to-\infty}\frac{|f(t,z)|}{|z|}\leq\limsup_{z\to-\infty}\frac{\sum\limits_{i=2}^{n}|\overline{\sigma}_i||z|^{1/i}}{|z|}=0<\xi_5,$$

uniformly in $t\in[0,1]$. Moreover,

$$\liminf_{z\to0^+}\frac{f(t,z)}{z}=\liminf_{z\to0^+}\frac{\sum\limits_{i=2}^{n}\overline{\kappa}_i z^{1/i}}{z}=+\infty>\xi_7$$

and

$$\limsup_{z\to0^-}\frac{f(t,z)}{z}=\limsup_{z\to0^-}\frac{\sum\limits_{i=2}^{n}\overline{\sigma}_i|z|^{i}}{z}=0<\xi_7$$

uniformly in $t\in[0,1]$. Therefore, f satisfies the condition (H0), (H4)-(H5). Similarly, g also satisfies these conditions. As a result, from Theorem 3.2 we see that (1.1) has at least one nontrivial solution.

4. Conclusions

In this paper, we studied the existence of nontrivial solutions for the Caputo-Fabrizio-type fractional integral boundary value problem (1.1). We first use the Gelfand theorem and the Krein-Rutman theorem to investigate a related positive linear operator, which can include the integral boundary condition. Then under some conditions concerning the spectral radius of the linear operator, we obtain our main existence theorems.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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