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SOME STUDIES ON EULER-LAGRANGE QUARTIC FUNCTIONAL EQUATIONS IN β-NORMED SPACES

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Abstract In this paper, we study two types of Euler-Lagrange functional equations that map $\jmath$ from normed spaces to β-normed spaces using quartic functional equations. We will investigate the inequality under various transformations and perturbations using both fixed point and direct methods, to establish the Hyers-Ulam stability of quartic Euler-Lagrange functional equations.

Keywords Quartic functional equations, Hyers-Ulam stability, β-normed spaces, Euler-Lagrange functional equations.

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1. Introduction

The original stability problem was first articulated by S.M. Ulam [35] in a mathematical discussion conducted at the University of Wisconsin. The discussion addressed several unresolved issues, including the stability of group homomorphisms. One of the key questions posed was:"Given an approximately linear mapping, when does a linear mapping estimate exist?" In 1941, D.H. Hyers [7] provided a favorable response to S.M. Ulam's question and offered a partial answer to the inquiry. Hyers introduced the Banach spaces $\mathcal{V}$ and $\mathcal{W}$ along with $\varepsilon > 0$. When the mapping $\jmath$ satisfies

$$\|\jmath(s+t) - \jmath(s) - \jmath(t)\| < \varepsilon, \tag{1.1}$$

there exists a unique linear transformation $\mathcal{K}(s)$, such that $\mathcal{K}(s) = \lim_{n\to\infty} \frac{\jmath(2^n s)}{2^n}$ and it holds that $\|\jmath(s) - \mathcal{K}(s)\| < \varepsilon$, where $\mathcal{K}(s)$ is the unique linear transformation that satisfies this inequality.

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The approximate linear transformations originally defined by D.H. Hyers were based on a fixed error bound (1.1). T. Aoki [36] generalized the stability results of D.H. Hyers concerning additive mappings, particularly in dealing with unbounded Cauchy differences. T. Aoki introduced parameters $\mathfrak{p}$ and $\mathcal{N}$, allowing the error bound to vary with the $\mathfrak{p}$-th power of the norm of the input vector as

$$\|\jmath(s+t)-\jmath(s)-\jmath(t)\| \leq \mathcal{N}\left(\|s\|^{\mathfrak{p}}+\|t\|^{\mathfrak{p}}\right),$$

where $0 \leq \mathfrak{p} < 1$ and $\mathcal{N} \geq 0$. Subsequently, T. M. Rassias [37] introduced a weaker inequality

$$\|\jmath(s+t)-\jmath(s)-\jmath(t)\| \leq \theta\|s\|^{\mathfrak{p}}\|t\|^{\mathfrak{q}},$$

where $\theta \geq 0$ and $\mathfrak{p}+\mathfrak{q} \neq 1$, and proved a generalization of D.H. Hyer's result by means of unbounded Cauchy differentials. In particular, J. M. Rassias [18, 19] introduced Euler-Lagrange quadratic functions

$$\Im\left(\sum_{i=1}^{n}\sigma_i s_i\right)+\sum_{1\leq i<j\leq n}\Im\left(\sigma_j s_i-\sigma_i s_j\right)=m\sum_{i=1}^{n}\Im\left(s_i\right), \quad s_i\in\mathcal{V}, \sigma_i\neq 0, i\in\mathbb{N}$$

and solved the Hyers-Ulam stability problem for quadratic multi-dimensional mappings. Based on these studies, J.M. Rassias [6, 9, 17–20, 29] discussed different types of Hyers-Ulam stability and responded to S.M. Ulam's question from different angles. Many scholars have further generalized the Hyers-Ulam stability problem and made important contributions to the development of stability theory (for example, [1–5, 8, 11, 12, 24, 30–33, 39])

J.M. Rassias [16] was the first to propose and solve the earliest quartic functional equations

$$\jmath(s+2t)+\jmath(s-2t)=24\jmath(t)+4[\jmath(s+t)+\jmath(s-t)]-6\jmath(s). \tag{1.2}$$

Next, J.K. Chung [15] succeeded in deriving a generalized solution of (1.2) through a method that does not depend on arbitrary regularity assumptions for the unknown function. Despite the fundamental nature of this method of solving the quartic functional equation, it cleverly draws on an important result of M. Hosszú [28], which played a key role in determining the general solution of (1.2). Subsequently, many experts in the field of functional equations and inequalities have proposed and solved various quartic and equations (see [16, 22, 26, 27, 38]).

This paper introduces two classes of Euler-Lagrange quartic functional equations that map from a normed space to a β-normed space, namely the General Quartic Functional Equation ($GQFE_\sigma$) and the Differential Analogue of the Quartic Functional Equation ($DAQFE_\varsigma$). For simplification of the expression we define the mapping $\jmath:\mathcal{V}\to\mathcal{W}$, where $\jmath$ is defined by the functions

$$\begin{aligned}\Phi_\sigma\jmath(s,t)=&2[\jmath(\sigma s+t)+\jmath(s+\sigma t)]+\sigma(\sigma-1)^2\jmath(s-t)\\&-2\left(\sigma^2-1\right)^2[\jmath(s)+\jmath(t)]-\sigma(\sigma+1)^2\jmath(s+t)\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\Omega_\varsigma\jmath(s,t)=&2[\jmath(s+\varsigma t)-\jmath(\varsigma s-t)]\\&-\varsigma\left(\varsigma^2+1\right)[\jmath(s+t)-\jmath(s-t)]-2\left(1-\varsigma^4\right)[\jmath(s)-\jmath(t)]\end{aligned} \tag{1.4}$$

to describe the Euler-Lagrange quartic functional equation. We study the Hyers-Ulam stability of the quartic functions (1.3) and (1.4) under various β conditions and explore the general solutions of related quartic functional inequalities using the fixed point and direct methods.

2. Basic Notations

This paper examines two types of Euler-Lagrange quartic functional equations: the $GQFE$ and the $DAQFE$. These equations map from a real linear space $\mathcal{V}$ to a completely β-normed space $\mathcal{W}$. Consequently, we have carried out a review of certain definitions related to abstract spaces by relying on [13, 14, 23].

Definition 2.1. [23] Let $\mathcal{V}$ be a nonempty set and let $r, s, t \in \mathcal{V}$. A function $\varrho : \mathcal{V}^2 \to [0, \infty]$ is designated as a generalized metric on $\mathcal{V}$ provided that ϱ fulfills conditions

(i) $\varrho(r, s) = 0$ if and only if $r = s$;

(ii) $\varrho(r, s) = \varrho(s, r)$;

(iii) $\varrho(r, t) \leqslant \varrho(r, s) + \varrho(s, t)$.

Then, $(\mathcal{V}, \varrho)$ is designated as a generalized complete metric space provided that for every Cauchy sequence $\{s_n\}_{n=1}^{\infty} \subseteq \mathcal{V}$, it converges to an element $s \in \mathcal{V}$.

It is important to note that the only difference between generalized metric and traditional metric is that generalized metric cover the concept of infinity.

Definition 2.2. [13, 14] Let $\mathcal{V}$ is a real vector space. Let $s, t \in \mathcal{V}$ and $\beta \in \mathbb{R}$ be given. Suppose that a function $\|\cdot\| : \mathcal{V} \to [0, +\infty)$ satisfies condition

(i) $\|s\| = 0$ if and only if $s = 0$;

(ii) $\|\mu s\| = |\mu|^{\beta}\|s\|$, where $\mu \in \mathbb{R}$;

(iii) $\|s + t\| \leq \|s\| + \|t\|$,

then $\|\cdot\|$ is referred to as a β-norm. In this case, $\mathcal{V}$ is a β-norm space of the function $\|\cdot\|$. $(\mathcal{V}, \|\cdot\|)$ is complete if every Cauchy sequence $\{s_n\}_{n=1}^{\infty} \subseteq \mathcal{V}$ converges to $s \in \mathcal{V}$.

Definition 2.3. The $GQFE_\sigma$ for $\sigma \in \mathbb{R}$ with the conditions $0 < |\sigma| < 1$ and $|\sigma| > 1$ is defined by

$$\begin{aligned} &2[\jmath(\sigma s + t) + \jmath(s + \sigma t)] + \sigma(\sigma - 1)^2\jmath(s - t) \\ =&2\left(\sigma^2 - 1\right)^2[\jmath(s) + \jmath(t)] + \sigma(\sigma + 1)^2\jmath(s + t), \end{aligned} \tag{2.1}$$

that is, $\Phi_\sigma \jmath(s, t) = 0$.

Definition 2.4. Given $0 < |\varsigma| < 1$ and $|\varsigma| > 1$, and $\varsigma^3 + \varsigma - 2 \neq 0$ and $\varsigma \in \mathbb{R}$. Assuming $\jmath(0) = 0$, the $DAQFE_\varsigma$ is defined by

$$\begin{aligned} &2[\jmath(s + \varsigma t) - \jmath(\varsigma s - t)] \\ =&\varsigma\left(\varsigma^2 + 1\right)[\jmath(s + t) - \jmath(s - t)] + 2\left(1 - \varsigma^4\right)[\jmath(s) - \jmath(t)], \end{aligned} \tag{2.2}$$

that is, $\Omega_\varsigma \jmath(s, t) = 0$

Clearly, $\jmath(s) = s^4$ serves as a real-valued solution to $GQFE_\sigma$ and $DAQFE_\varsigma$.

3. Stability of Euler-Lagrange quartic functional equations:Fixed point theory

In this section, we utilize the theoretical proposed by L. Cădariu and V. Radu [25] and apply fixed point theory to investigate the Hyers-Ulam stability of two distinct classes of Euler-Lagrange quartic functional equations under the specified constraints of $0 < \beta < 1$.

Lemma 3.1. *[10, 13] Let $(\mathcal{V}, \varrho)$ be a generalized complete metric space, $r, s \in \mathcal{V}$. Let Lipschitz constant $0 < \wp < 1$ be given. Suppose that $\Lambda : \mathcal{V} \to \mathcal{V}$ be a strictly contractive function, that is, $\varrho(\Lambda r, \Lambda s) \leq \wp\varrho(r, s)$, then a positive constant i exists such that $\varrho\left(\Lambda^{i+1}s, \Lambda^i s\right) < \infty$. Therefore*

- (i) *the sequence $\{\Lambda^n s\}_{n=1}^{\infty}$ converges to a fixed point $s' \in \mathcal{V}$ of Λ;*
- (ii) *s' is the unique fixed point of $\Lambda \in \mathcal{M}$, where $\mathcal{M} = \left\{t \in \mathcal{V} : \varrho\left(\Lambda^i s, t\right) < \infty\right\}$;*
- (iii) *$\varrho(t, s') \leq \frac{1}{1-\wp}\varrho(t, \Lambda t)$ for all $t \in \mathcal{M}$.*

We first study the Hyers-Ulam stability of $GQFE_\sigma$ by using the fixed point method.

Theorem 3.1. *Suppose $\psi : \mathcal{V}^2 \to [0, +\infty)$ is a given function and satisfies*

$$\lim_{n\to\infty} \frac{\psi\left(\sigma^n s, \sigma^n t\right)}{\sigma^{4n\beta}} = 0, \quad \psi(\sigma s, 0) \leq \sigma^{4\beta}\wp\psi(s, 0), \quad s, t \in \mathcal{V}, \tag{3.1}$$

where $0 < \wp < 1$. Additionally, let $\jmath : \mathcal{V} \to \mathcal{W}$ be a function satisfies $\jmath(0) = 0$ with

$$\|\Phi_\sigma \jmath(s, t)\| \leq \psi(s, t), \quad s, t \in \mathcal{V}. \tag{3.2}$$

Then, there exists a unique quartic function $\Im : \mathcal{V} \to \mathcal{W}$ satisfies

$$\|\jmath(s) - \Im(s)\| \leq \frac{1}{(2\sigma^4)^\beta(1-\wp)}\psi(s, 0), \quad s \in \mathcal{V}. \tag{3.3}$$

Proof. Let $\Upsilon = \{g : \mathcal{V} \to \mathcal{W}\}$. We define $\varrho : \Upsilon^2 \to [0, +\infty]$ by

$$\varrho(g, \ell) = \inf\left\{\hbar \in [0, +\infty] : \|g(s) - \ell(s)\| \leq \hbar\psi(s, 0), s \in \mathcal{V}\right\}. \tag{3.4}$$

Consider a Cauchy sequence $\{\jmath_n\} \subseteq (\Upsilon, \varrho)$. Given that $\varepsilon > 0$ is arbitrary, and there exists a non-negative integer N_ε associated with ε, then by applying the Cauchy sequence and (3.4), we obtain

$$\|\jmath_m(s) - \jmath_n(s)\| \leq \varepsilon\psi(s, 0), \quad m, n \geq N_\varepsilon \quad s \in \mathcal{V}. \tag{3.5}$$

If s is an arbitrary point in $\mathcal{V}$, then according to (3.5), the sequence $\{\jmath_n(s)\}$ forms a Cauchy sequence in $\mathcal{W}$. Given that $\mathcal{W}$ is complete, $\{\jmath_n(s)\} \subseteq \mathcal{W}$ is converges for each $s \in \mathcal{V}$. Consequently, we define $\jmath : \mathcal{V} \to \mathcal{W}$ by

$$\jmath(s) = \lim_{n\to\infty} \jmath_n(s), \quad s \in \mathcal{V}.$$

If let $m \to \infty$, then by applying (3.5), we get

$$\|\jmath(s) - \jmath_n(s)\| \leq \varepsilon\psi(s, 0), \quad s \in \mathcal{V}, \quad n \geq N_\varepsilon,$$

that is, $\varrho\left(\jmath_n, \jmath\right) \leq \varepsilon$. Hence, the sequence $\{\jmath_n\} \subseteq \mathcal{W}$ is converges. Consequently, $(\mathcal{W}, \varrho)$ is a complete space. We define $\Lambda : \Upsilon \to \Upsilon$ by

$$(\Lambda g)(s) = \frac{g(\sigma s)}{\sigma^4}, \quad s \in \mathcal{V}. \tag{3.6}$$

Subsequently, we prove the following inequality $\varrho(\Lambda g, \Lambda \ell) \leq \wp\varrho(g, \ell)$ for all $g, \ell \in \Upsilon$. Given $g, \ell \in \Upsilon$ with $\varrho(g, \ell) < \infty$ and an arbitrary $\varepsilon > 0$, we conclude that

$$\|g(s) - \ell(s)\| \leq \hbar\psi(s, 0), \quad s \in \mathcal{V}, \tag{3.7}$$

where $\hbar = \varrho(g, \ell) + \varepsilon$. It follows from (3.1),(3.6) and (3.7), we get

$$\|\Lambda g(s) - \Lambda\ell(s)\| = \left\|\frac{g(\sigma s)}{\sigma^4} - \frac{\ell(\sigma s)}{\sigma^4}\right\| \leq \frac{\hbar\psi(\sigma s, 0)}{\sigma^{4\beta}} \leq \hbar\wp\psi(s, 0), \quad s \in \mathcal{V}.$$

By applying (3.4), we have

$$\varrho(\Lambda g, \Lambda\ell) \leq \wp\hbar = \wp(\varrho(g, \ell) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, this leads us to

$$\varrho(\Lambda g, \Lambda\ell) \leq \wp\varrho(g, \ell).$$

Thus, Λ is a strictly contractive map on Υ with Lipschitz constant $\wp$.

By substituting $t = 0$ into in (3.2) and then dividing both sides by $(2\sigma^4)^\beta$, we obtain

$$\|\Lambda\jmath(s) - \jmath(s)\| \leq \frac{\psi(s, 0)}{(2\sigma^4)^\beta}, \quad s \in \mathcal{V}.$$

By applying (3.4), we get

$$\varrho(\Lambda\jmath, \jmath) \leq \frac{1}{(2\sigma^4)^\beta} < \infty.$$

Based on the conclusion of Theorem (3.1), it follows that the sequence $\{\Lambda^n\jmath\}_{n=1}^{\infty}$ converges to a fixed point $\Im \in \Upsilon$ of Λ. Consequently, we define $\Im : \mathcal{V} \to \mathcal{W}$ by

$$\Im(s) = \lim_{n\to\infty} \frac{\jmath(\sigma^n s)}{\sigma^{4n}}, \quad s \in \mathcal{V}. \tag{3.8}$$

Therefore, by Theorem (3.1), $\Im$ is the unique fixed point of the set $\mathcal{M} := \{g \in \Upsilon : \varrho(g, \jmath) < \infty\}$. This leads us to the following conclusion

$$\varrho(\jmath, \Im) \leq \frac{1}{1 - \wp}\varrho(\Lambda\jmath, \jmath) \leq \frac{1}{(2\sigma^4)^\beta(1 - \wp)}.$$

It is obvious from (3.4) that we can verify the validity of inequality (3.3). By combining applications (3.1), (3.2), and (3.8), we conclude

$$\Big\| 2[\Im(\sigma s + t) + \Im(s + \sigma t)] + \sigma(\sigma - 1)^2\Im(s - t)$$
$$- 2\left(\sigma^2 - 1\right)^2[\Im(s) + \Im(t)] - \sigma(\sigma + 1)^2\Im(s + t)\Big\|$$

$$= \lim_{n\to\infty} \left\| 2\left[\frac{\jmath(\sigma^n(\sigma s+t))}{\sigma^{4n}} + \frac{\jmath(\sigma^n(s+\sigma t))}{\sigma^{4n}}\right] + \sigma(\sigma-1)^2\frac{\jmath(\sigma^n(s-t))}{\sigma^{4n}} \right.$$
$$\left. - 2\left(\sigma^2-1\right)^2\left[\frac{\jmath(\sigma^n s)}{\sigma^{4n}} + \frac{\jmath(\sigma^n t)}{\sigma^{4n}}\right] - \sigma(\sigma+1)^2\frac{\jmath(\sigma^n s+\sigma^n t)}{\sigma^{4n}} \right\|$$
$$= \lim_{n\to\infty} \|\Phi_\sigma \jmath(\sigma^n s, \sigma^n t)\|$$
$$\leq \lim_{n\to\infty} \frac{\psi(\sigma^n s, \sigma^n t)}{\sigma^{4n\beta}} = 0, \quad s,t \in \mathcal{V}.$$

Thus, $\Im$ is a solution of (2.1). This proves the existence of $\Im$.

Let us consider the following scenario: $\mathcal{H}_f : \mathcal{V} \to \mathcal{W}$ is another solution of (2.1) that also satisfies (3.3). In this case, we can assert that $\mathcal{H}_f$ represents the unique fixed point of the set $\mathcal{M} := \{g \in \Upsilon : \varrho(g,\jmath) < \infty\}$. This conclusion aligns with the one presented in Theorem 3.1, which states that $\Im = \mathcal{H}_f$. As such, the uniqueness of $\Im$ is proven.

□

Corollary 3.1. *Suppose $\psi : \mathcal{V}^2 \to [0,+\infty)$ is a given function and satisfies*

$$\lim_{n\to\infty} \sigma^{4n\beta}\psi\left(\frac{s}{\sigma^n}, \frac{t}{\sigma^n}\right) = 0, \quad \sigma^{4\beta}\psi(s,0) \leq \wp\psi\left(\frac{s}{\sigma}, 0\right), s,t \in \mathcal{V},$$

where $0 < \wp < 1$. Furthermore, let $\jmath : \mathcal{V} \to \mathcal{W}$ be a function that satisfies $\jmath(0) = 0$ with

$$\|\Phi_\sigma \jmath(s,t)\| \leq \psi(s,t), \quad s,t \in \mathcal{V}.$$

Then, there exists a unique quartic function $\Im : \mathcal{V} \to \mathcal{W}$ satisfies

$$\|\jmath(s) - \Im(s)\| \leq \frac{\wp}{(2\sigma^4)^\beta(1-\wp)}\psi(s,0), \quad s \in \mathcal{V}.$$

Next, we study the Hyers-Ulam stability of $DAQFE_\varsigma$ using fixed point theory.

Theorem 3.2. *Suppose $\psi : \mathcal{V}^2 \to [0,+\infty)$ is a given function and satisfies*

$$\lim_{n\to\infty} \varsigma^{4n\beta}\psi\left(\frac{s}{\varsigma^n}, \frac{t}{\varsigma^n}\right) = 0, \quad \psi\left(\frac{s}{\varsigma}, 0\right) \leq \frac{\wp\psi(s,0)}{\varsigma^{4\beta}}, \quad s,t \in \mathcal{V}, \tag{3.9}$$

where $0 < \wp < 1$. Furthermore, let $\jmath : \mathcal{V} \to \mathcal{W}$ be a function that satisfies $f(0) = 0$ and

$$\|\Omega_\varsigma \jmath(s,t)\| \leq \psi(s,t), \quad s,t \in \mathcal{V}. \tag{3.10}$$

Then, there exists a unique quartic function $\Im : \mathcal{V} \to \mathcal{W}$ satisfies

$$\|\jmath(s) - \Im(s)\| \leq \frac{\wp}{(2\varsigma)^{4\beta}(1-\wp)}\psi(s,0), \quad s \in \mathcal{V}. \tag{3.11}$$

Proof. We apply the proof method related to (Υ, ϱ) in Theorem 3.1 proof. Therefore, we conclude that (Υ, ϱ) is a complete generalized metric space. Then, we define $\Lambda : \Upsilon \to \Upsilon$ by

$$\Lambda g(s) = \varsigma^4 g\left(\frac{s}{\varsigma}\right), \quad s \in \mathcal{V}. \tag{3.12}$$

It follows from (3.9) and (3.12) that

$$\|\Lambda g(s)-\Lambda\ell(s)\| = \left\|\varsigma^4 g\left(\frac{s}{\varsigma}\right)-\varsigma^4\ell\left(\frac{s}{\varsigma}\right)\right\| \le \varsigma^{4\beta}\hbar\psi\left(\frac{s}{\varsigma},0\right) \le \wp\hbar\psi(s,0), \quad s\in\mathcal{V},$$

where $\hbar=\varrho(g,\ell)+\varepsilon$, that is, $\varrho(\Lambda g,\Lambda\ell)\le\wp\hbar$. Thus, we obtain

$$\varrho(\Lambda g,\Lambda\ell)\le\wp\varrho(g,\ell), \quad g,\ell\in\Upsilon.$$

Therefore, Λ is strictly contractive with Lipschitz constant $\wp$ on Υ.

Thereafter, if we set $s=\frac{s}{\varsigma}, t=0$ in (3.10), and then divide by 2^β, we get

$$\left\|\jmath(s)-\varsigma^4\jmath\left(\frac{s}{\varsigma}\right)\right\| \le \frac{\psi\left(\frac{s}{\varsigma},0\right)}{2^\beta} \le \frac{\wp\psi(s,0)}{2^\beta\varsigma^{4\beta}}$$

for all $s\in\mathcal{V}$, which implies

$$\varrho(\jmath,\Lambda\jmath)\le\frac{\wp}{2^\beta\varsigma^{4\beta}}<\infty.$$

By Theorem 3.1 we conclude that the sequence $\{\Lambda^n\jmath\}_{n=1}^\infty$ converges to a fixed point $\Im\in\Upsilon$ of Λ. Therefore

$$\Im:\mathcal{V}\to\mathcal{W}, \quad \Im(s)=\lim_{n\to\infty}\varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right), \quad s\in\mathcal{V}. \tag{3.13}$$

Accordingly, Theorem 3.1 indicates that $\Im$ is the unique fixed point of $\mathcal{M}:=\{g\in\Upsilon:\varrho(g,\jmath)<\infty\}$. Furthermore,

$$\varrho(\jmath,\Im)\le\frac{1}{1-\wp}\varrho(\Lambda\jmath,\jmath)\le\frac{\wp}{(2\varsigma^4)^\beta(1-\wp)},$$

gives rise to the inequality (3.11). Using (3.9), (3.10), and (3.13), we get

$$\begin{aligned}
&\Big\|2[\Im(\varsigma s+t)+\Im(s+\varsigma t)]+\varsigma(\varsigma-1)^2\Im(s-t)\\
&-2\left(\varsigma^2-1\right)^2[\Im(s)+\Im(t)]-\varsigma(\varsigma+1)^2\Im(s+t)\Big\|\\
&=\lim_{n\to\infty}\Big\|2\left[\varsigma^{4n}\jmath\left(\frac{\varsigma s+t}{\varsigma^n}\right)+\varsigma^{4n}\jmath\left(\frac{s+\varsigma t}{\varsigma^n}\right)\right]+\varsigma(\varsigma-1)^2\varsigma^{4n}\jmath\left(\frac{s-t}{\varsigma^n}\right)\\
&-2\left(\varsigma^2-1\right)^2\left[\varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right)+\varsigma^{4n}\jmath\left(\frac{t}{\varsigma^n}\right)\right]-\varsigma(\varsigma+1)^2\varsigma^{4n}\jmath\left(\frac{s+t}{\varsigma^n}\right)\Big\|\\
&\le\lim_{n\to\infty}\varsigma^{4n\beta}\psi\left(\frac{s}{\varsigma^n},\frac{t}{\varsigma^n}\right)=0, \quad s,t\in\mathcal{V}.
\end{aligned}$$

This indicates that $\Im$ is a solution to (2.2). As with the proof of uniqueness in Theorem 3.1, we can similarly demonstrate the uniqueness of $\Im$. Therefore, the uniqueness of $\Im$ is proven. □

Corollary 3.2. *Suppose $\psi:\mathcal{V}^2\to[0,+\infty)$ is a given function and satisfies*

$$\lim_{n\to\infty}\frac{\psi\left(\varsigma^n s,\varsigma^n t\right)}{\varsigma^{4n\beta}}=0, \quad \psi(\varsigma s,0)\le\varsigma^{4\beta}\wp\psi(s,0), \quad s,t\in\mathcal{V}, \tag{3.14}$$

where $0 < \wp < 1$. *Furthermore, let* $\jmath : \mathcal{V} \to \mathcal{W}$ *be a function that satisfies* $\jmath(0) = 0$ *and*

$$\|\Omega_\varsigma \jmath(s,t)\| \leq \psi(s,t), \quad s,t \in \mathcal{V}. \tag{3.15}$$

Then, there exists a unique quartic function $\Im : \mathcal{V} \to \mathcal{W}$ *such that*

$$\|\jmath(s) - \Im(s)\| \leq \frac{1}{(2\varsigma)^{4\beta}(1-\wp)}\psi(s,0), \quad s \in \mathcal{V}. \tag{3.16}$$

Remark 3.1. If $\jmath$ is an even function, the above theorem and inference can be used to consider the case of $\psi(0,t)$ and still get a similar result.

4. Stability of Euler-Lagrange quartic functional equations:Direct method

In this section, we study the Hyers-Ulam stability of two classes of Euler-Lagrange quartic functional equations under parity of β by direct method.

Theorem 4.1. *Let* $\varepsilon > 0$ *be given. Suppose* $\jmath : \mathcal{V} \to \mathcal{W}$ *is a given function and satisfies*

$$\|\Phi_\sigma \jmath(s,t)\| \leq \varepsilon, \quad s,t \in \mathcal{V} \tag{4.1}$$

where $0 < |\sigma| < 1$ *and* $|\sigma| > 1$. *Then there exists a unique mapping* $\Im : \mathcal{V} \to \mathcal{W}$ *such that*

$$\begin{aligned} &2[\Im(\sigma s + t) + \Im(s + \sigma t)] + \sigma(\sigma-1)^2\Im(s-t) \\ =&2\left(\sigma^2-1\right)^2[\Im(s) + \Im(t)] + \sigma(\sigma+1)^2\Im(s+t) \end{aligned} \tag{4.2}$$

and

$$\|\jmath(s) - \Im(s)\| \leq \begin{cases} \frac{(2\sigma^2)^\beta + (\sigma^2-1)^\beta}{(\sigma^{4\beta}-1)(4\sigma^2)^\beta}\varepsilon, & \beta \text{ is odd (even)}, |\sigma| > 1 ; \\ \frac{(2\sigma^2)^\beta - (\sigma^2-1)^\beta}{(1-\sigma^{4\beta})(4\sigma^2)^\beta}\varepsilon, & \beta \text{ is odd}, 0 < |\sigma| < 1 ; \\ \frac{(2\sigma^2)^\beta + (\sigma^2-1)^\beta}{(1-\sigma^{4\beta})(4\sigma^2)^\beta}\varepsilon, & \beta \text{ is even}, 0 < |\sigma| < 1 , \end{cases}$$

with

$$\Im(s) = \lim_{n\to\infty} \Im_n(s) = \lim_{n\to\infty} \begin{cases} \frac{\jmath(\sigma^n s)}{\sigma^{4n}}, & |\sigma| > 1, \\ \sigma^{4n}\jmath\left(\frac{s}{\sigma^n}\right), & 0 < |\sigma| < 1 , \end{cases} \quad s \in \mathcal{V}, \quad n \in \mathbb{N}.$$

Proof. Letting $s = t = 0$ in (4.1), we get

$$\|\jmath(0)\| \leq \frac{\varepsilon}{(4\sigma^2\,|\sigma^2-1|)^\beta}. \tag{4.3}$$

Letting $t = 0$ in (4.1), we get

$$\|\jmath(\sigma s) - \sigma^4\jmath(s) - (\sigma^2-1)^2\jmath(0)\| \leq \frac{\varepsilon}{2^\beta}, \quad s \in \mathcal{V}. \tag{4.4}$$

By employing the triangle inequality in (4.3) and (4.4), we obtain

$$\begin{aligned}\left\|\jmath(\sigma s)-\sigma^4\jmath(s)\right\| &\le \|\jmath(\sigma s)-\sigma^4\jmath(s)-(\sigma^2-1)^2\jmath(0)\|+\|(\sigma^2-1)^2\jmath(0)\| \\ &\le \frac{(2\sigma^2)^\beta+\left|\sigma^2-1\right|^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s\in\mathcal{V}.\end{aligned} \tag{4.5}$$

Case 1. When β is odd (or even) and $|\sigma|>1$, we obtain

$$\left\|\jmath(\sigma s)-\sigma^4\jmath(s)\right\| \le \frac{(2\sigma^2)^\beta+(\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s\in\mathcal{V}.$$

Therefore

$$\left\|\jmath(s)-\frac{\jmath(\sigma s)}{\sigma^4}\right\| \le \frac{(2\sigma^2)^\beta+(\sigma^2-1)^\beta}{(4\sigma^6)^\beta}\varepsilon, \quad s\in\mathcal{V}. \tag{4.6}$$

Substituting $\sigma^n s$ for s in (4.6) yields

$$\left\|\jmath(\sigma^n s)-\frac{\jmath(\sigma^{n+1}s)}{\sigma^4}\right\| \le \frac{(2\sigma^2)^\beta+(\sigma^2-1)^\beta}{(4\sigma^6)^\beta}\varepsilon, \quad s\in\mathcal{V}, \quad n\in\mathbb{N}. \tag{4.7}$$

Applying the inequality (4.7), we obtain

$$\begin{aligned}\left\|\jmath(s)-\frac{\jmath(\sigma^n s)}{\sigma^{4n}}\right\| &\le \sum_{i=1}^{n}\frac{1}{\sigma^{4(i-1)\beta}}\left\|\jmath(\sigma^{i-1}s)-\frac{\jmath(\sigma^i s)}{\sigma^4}\right\| \\ &\le \frac{1-\frac{1}{\sigma^{4n\beta}}}{1-\frac{1}{\sigma^{4\beta}}}\cdot\frac{(2\sigma^2)^\beta+(\sigma^2-1)^\beta}{(4\sigma^6)^\beta}\varepsilon, \quad s\in\mathcal{V}.\end{aligned} \tag{4.8}$$

Thus, the sequence $\left\{\frac{\jmath(\sigma^n s)}{\sigma^{4n}}\right\}_n$ is a Cauchy sequence, and by $\mathcal{W}$ being complete, we get that the sequence $\left\{\frac{\jmath(\sigma^n s)}{\sigma^{4n}}\right\}_n$ is convergent. Then, we define the function $\Im:\mathcal{V}\to\mathcal{W}$ by

$$\Im(s) := \lim_{n\to\infty}\frac{\jmath(\sigma^n s)}{\sigma^{4n}}, \quad s\in\mathcal{V}. \tag{4.9}$$

Therefore

$$\|\jmath(s)-\Im(s)\| \le \frac{(2\sigma^2)^\beta+(\sigma^2-1)^\beta}{(\sigma^{4\beta}-1)(4\sigma^2)^\beta}\varepsilon, \quad s\in\mathcal{V}. \tag{4.10}$$

Replacing s with $\sigma^n s, t$ with $\sigma^n t$ in (4.1), dividing the resultant inequality by $\sigma^{4n\beta}$, we obtain

$$\|\Phi_\sigma\jmath(\sigma^n s,\sigma^n t)\| \le \frac{\varepsilon}{\sigma^{4n\beta}}, \quad s,t\in\mathcal{V}, \quad n\in\mathbb{N}. \tag{4.11}$$

By applying (1.3) and (4.9)and by setting $n\to\infty$ in (4.11), we obtain

$$\begin{aligned}&2[\Im(\sigma s+t)+\Im(s+\sigma t)]+\sigma(\sigma-1)^2\Im(s-t) \\ =&2\left(\sigma^2-1\right)^2[\Im(s)+\Im(t)]+\sigma(\sigma+1)^2\Im(s+t)\end{aligned} \tag{4.12}$$

Therefore, the existence of $\Im$ is completed.

Next, let us prove the uniqueness of $\Im$. Let $\mathcal{H}_{d1} : \mathcal{V} \to \mathcal{W}$ be another mapping satisfies (4.10) and (4.12), we get

$$\begin{aligned}\|\Im(s) - \mathcal{H}_{d1}(s)\| &\leq \|\Im(s) - \jmath(s)\| + \|\jmath(s) - \mathcal{H}_{d1}(s)\| \\ &\leq \frac{2\left((2\sigma^2)^\beta + (\sigma^2-1)^\beta\right)}{(\sigma^{4\beta}-1)(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}.\end{aligned}$$

Therefore

$$\begin{aligned}\|\Im(s) - \mathcal{H}_{d1}(s)\| &\leq \frac{1}{n^{4\beta}}\left\{\|\Im(ns) - \jmath(ns)\| + \|\jmath(ns) - \mathcal{H}_{d1}(ns)\|\right\} \\ &\leq \frac{2\left\{(2\sigma^2)^\beta + (\sigma^2-1)^\beta\right\}}{n^{4\beta}(\sigma^{4\beta}-1)(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}, \quad n \in \mathbb{N}.\end{aligned}$$

Letting $n \to \infty$, the proof of the uniqueness is completed.

Case 2. When β is odd and $0 < |\sigma| < 1$. Substituting $\frac{s}{\sigma}$ for s in inequality (4.5) yields

$$\left\|\jmath(s) - \sigma^4 \jmath\left(\frac{s}{\sigma}\right)\right\| \leq \frac{(2\sigma^2)^\beta - (\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}. \tag{4.13}$$

By replacing s with $\frac{s}{\sigma^n}$ in (4.13), we get

$$\left\|\jmath\left(\frac{s}{\sigma^n}\right) - \sigma^4 \jmath\left(\frac{s}{\sigma^{n+1}}\right)\right\| \leq \frac{(2\sigma^2)^\beta - (\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}, \quad n \in \mathbb{N}. \tag{4.14}$$

By using a similar method as (4.8), we obtain the following conclusion

$$\begin{aligned}\left\|\jmath(s) - \sigma^{4n} \jmath\left(\frac{s}{\sigma^n}\right)\right\| &\leq \sum_{i=1}^{n} \sigma^{4(i-1)\beta} \left\|\jmath\left(\frac{s}{\sigma^i}\right) - \sigma^4 \jmath\left(\frac{s}{\sigma^{i+1}}\right)\right\| \\ &\leq \frac{1-\sigma^{4\beta}}{1-\sigma^{4\beta}} \frac{(2\sigma^2)^\beta - (\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}.\end{aligned} \tag{4.15}$$

By **Case 1**, it is clear that the sequence $\left\{\sigma^{4n}\jmath\left(\frac{s}{\sigma^n}\right)\right\}_n$ is converges, then the function $\Im : \mathcal{V} \to \mathcal{W}$ is defined by

$$\Im(s) := \lim_{n\to\infty} \sigma^{4n} \jmath\left(\frac{s}{\sigma^n}\right), \quad s \in \mathcal{V}. \tag{4.16}$$

Therefore

$$\|\jmath(s) - \Im(s)\| \leq \frac{(2\sigma^2)^\beta - (\sigma^2-1)^\beta}{(1-\sigma^{4\beta})(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}. \tag{4.17}$$

In addition, substituting $\frac{s}{\sigma^n}$ and $\frac{t}{\sigma^n}$ for s and t in (4.1) and multiply the resulting inequality by $\sigma^{4n\beta}$, we obtain

$$\|\Phi_\sigma \jmath(\sigma^n s, \sigma^n t)\| \leq \sigma^{4n\beta}\varepsilon, \quad s, t \in \mathcal{V}, \quad n \in \mathbb{N}. \tag{4.18}$$

By applying (1.3) and (4.16) and by setting $n \to \infty$ in (4.18), which gives a equation (4.12). Therefore, the existence of $\Im$ is completed.

Next, we prove the uniqueness of $\Im$. Let $\mathcal{H}_{d2} : \mathcal{V} \to \mathcal{W}$ be another mapping satisfies (4.12) and (4.17), we conclude

$$\begin{aligned}\|\Im(s) - \mathcal{H}_{d2}(s)\| \le& \|\Im(s) - \jmath(s)\| + \|\jmath(s) - \mathcal{H}_{d2}(s)\| \\ \le& \frac{1}{n^{4\beta}} \{\|\Im(ns) - \jmath(ns)\| + \|\jmath(ns) - \mathcal{H}_{d2}(ns)\|\} \\ =& \frac{2\left((2\sigma^2)^\beta - (\sigma^2-1)^\beta\right)}{(1-\sigma^{4\beta})(4\sigma^2 n^4)^\beta}\varepsilon, \quad s \in \mathcal{V}.\end{aligned} \tag{4.19}$$

Letting $n \to \infty$, the proof of the uniqueness is completed.

Case 3. When β is even and $0 < |\sigma| < 1$,

$$\left\|\jmath(\sigma s) - \sigma^4 \jmath(s)\right\| \le \frac{(2\sigma^2)^\beta + (\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}. \tag{4.20}$$

By replacing s with $\frac{s}{\sigma}$ in inequality (4.20), follows

$$\left\|\jmath(s) - \sigma^4 \jmath\left(\frac{s}{\sigma}\right)\right\| \le \frac{(2\sigma^2)^\beta + (\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}. \tag{4.21}$$

Substituting $\frac{s}{\sigma^n}$ for s in inequality (4.21) yields

$$\left\|\jmath\left(\frac{s}{\sigma^n}\right) - \sigma^4 \jmath\left(\frac{s}{\sigma^{n+1}}\right)\right\| \le \frac{(2\sigma^2)^\beta + (\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}, \quad n \in \mathbb{N}. \tag{4.22}$$

Using the same method as (4.8), we obtain the followings

$$\begin{aligned}\left\|\jmath(s) - \sigma^{4n} \jmath\left(\frac{s}{\sigma^n}\right)\right\| \le& \sum_{i=1}^{n} \sigma^{4(i-1)\beta} \left\|\jmath\left(\frac{s}{\sigma^i}\right) - \sigma^4 \jmath\left(\frac{s}{\sigma^{i+1}}\right)\right\| \\ \le& \frac{1-\sigma^{4n\beta}}{1-\sigma^{4\beta}} \frac{(2\sigma^2)^\beta + (\sigma^2-1)^\beta}{(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}.\end{aligned} \tag{4.23}$$

Similarly, by (4.16), we obtain

$$\|\jmath(s) - \Im(s)\| \le \frac{(2\sigma^2)^\beta + (\sigma^2-1)^\beta}{(1-\sigma^{4\beta})(4\sigma^2)^\beta}\varepsilon, \quad s \in \mathcal{V}. \tag{4.24}$$

By **Case 2** to prove in the same argument, we obtained the equation (4.12), and the uniqueness of the $\Im$. We can easily get the thesis. □

Theorem 4.2. *Let $\varepsilon > 0$ be given. Suppose $\jmath : \mathcal{V} \to \mathcal{W}$, is a given function and satisfies $\|\jmath(0)\| \le \varepsilon$ and*

$$\|\Omega_\varsigma \jmath(s,t)\| \le \varepsilon, \quad s, t \in \mathcal{V}, \tag{4.25}$$

where $0 < |\varsigma| < 1$ and $|\varsigma| > 1$. Then, there exists a unique mapping $\Im : \mathcal{V} \to \mathcal{W}$ satisfies

$$2[\Im(s+\varsigma t) - \Im(\varsigma s - t)] = \varsigma\left(\varsigma^2+1\right)[\Im(s+t) - \Im(s-t)] + 2\left(1-\varsigma^4\right)[\Im(s) - \Im(t)]$$

and

$$\|\jmath(s)-\Im(s)\| \leq \begin{cases} \frac{\varsigma^{4n\beta}-1}{\varsigma^{4n\beta}(\varsigma^4-1)}\left(\frac{1}{2^\beta}+\left(\varsigma^4-1\right)^\beta\right)\varepsilon, & \beta \text{ is odd (even)}, |\varsigma|>1 \ ; \\ \frac{1-\varsigma^{4n\beta}}{1-\varsigma^{4\beta}}\left(\frac{1}{2^\beta}-\left(\varsigma^4-1\right)^\beta\right)\varepsilon, & \beta \text{ is odd}, 0<|\varsigma|<1 \ ; \\ \frac{1-\varsigma^{4n\beta}}{1-\varsigma^{4\beta}}\left(\frac{1}{2^\beta}+\left(\varsigma^4-1\right)^\beta\right)\varepsilon, & \beta \text{ is even}, 0<|\varsigma|<1 \ , \end{cases}$$

with

$$\Im(s)=\lim_{n\to\infty}\Im_n(s)=\lim_{n\to\infty}\begin{cases}\frac{\jmath(\varsigma^n s)}{\varsigma^{4n}}, & |\varsigma|>1 \\ \varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right), & 0<|\varsigma|<1.\end{cases} \quad s\in\mathcal{V}.$$

Proof. Letting $t=0$ in (4.25), we obtain

$$\left\|\jmath(\varsigma s)-\varsigma^4\jmath(s)+\left(\varsigma^4-1\right)\jmath(0)\right\| \leq \frac{\varepsilon}{2^\beta}, \quad s\in\mathcal{V}. \tag{4.26}$$

By employing the triangle inequality in (4.26) and $\|\jmath(0)\|\leq\varepsilon$, we obtain

$$\left\|\jmath(\varsigma s)-\varsigma^4\jmath(s)\right\| \leq \left(\frac{1}{2^\beta}+\left|\varsigma^4-1\right|^\beta\right)\varepsilon, \quad s\in\mathcal{V}. \tag{4.27}$$

Case 1. When β is odd (or even) and $|\varsigma|>1$. By dividing both sides of (4.27) by $\varsigma^{4\beta}$, we obtain

$$\left\|\jmath(s)-\frac{\jmath(\varsigma s)}{\varsigma^4}\right\| \leq \left(\frac{1}{2^\beta\varsigma^{4\beta}}+\left(1-\frac{1}{\varsigma^4}\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}. \tag{4.28}$$

Replacing s by $\varsigma^n s$ in (4.28) gives

$$\left\|\jmath(\varsigma^n s)-\frac{\jmath(\varsigma^{n+1}s)}{\varsigma^4}\right\| \leq \left(\frac{1}{2^\beta\varsigma^{4\beta}}+\left(1-\frac{1}{\varsigma^4}\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}, \quad n\in\mathbb{N}. \tag{4.29}$$

Similar to (4.8), we obtain the followings

$$\begin{aligned}\left\|\jmath(s)-\frac{\jmath(\varsigma^n s)}{\varsigma^{4n}}\right\| &\leq \sum_{\varsigma=1}^{n}\frac{1}{\varsigma^{4(i-1)\beta}}\left\|\jmath\left(k^i s\right)-\frac{\jmath\left(\varsigma^{i+1}s\right)}{\varsigma^4}\right\| \\ &\leq \frac{\varsigma^{4n\beta}-1}{\varsigma^{4(n-1)\beta}(\varsigma^4-1)}\left(\frac{1}{2^\beta\varsigma^{4\beta}}+\left(1-\frac{1}{\varsigma^4}\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}.\end{aligned} \tag{4.30}$$

Thus, the sequence $\left\{\frac{\jmath(\varsigma^n s)}{\varsigma^{4n}}\right\}_n$ is a Cauchy sequence, and by $\mathcal{W}$ being complete, the sequence $\left\{\frac{\jmath(\varsigma^n s)}{\varsigma^{4n}}\right\}_n$ is convergent. Then, we define the function $\Im:\mathcal{V}\to\mathcal{W}$ by

$$\Im(s):=\lim_{n\to\infty}\frac{\jmath\left(\varsigma^n s\right)}{\varsigma^{4n}}, \quad s\in\mathcal{V}. \tag{4.31}$$

Then, we have

$$\|\jmath(s)-\Im(s)\| \leq \frac{\varsigma^{4n\beta}-1}{\varsigma^{4n\beta}(\varsigma^4-1)}\left(\frac{1}{2^\beta}+\left(\varsigma^4-1\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}.$$

Moreover, replacing s by $\varsigma^n s, t$ by $\varsigma^n t$, and dividing the resultant inequality by $\varsigma^{4n\beta}$, we obtain

$$\left\|2\left[\frac{\jmath(\varsigma^n(s+\varsigma t))}{\varsigma^{4n}}-\frac{\jmath(\varsigma^n(\varsigma s-t))}{\varsigma^{4n}}\right]-\varsigma(\varsigma^2+1)\left[\frac{\jmath(\varsigma^n(s+t))}{\varsigma^{4n}}-\frac{\jmath(\varsigma^n(s-t))}{\varsigma^{4n}}\right]\right.$$
$$\left.-2\left(1-\varsigma^4\right)\left[\frac{\jmath(\varsigma^n s)}{\varsigma^{4n}}-\frac{\jmath(\varsigma^n t)}{\varsigma^{4n}}\right]\right\| \leq \frac{\varepsilon}{\varsigma^{4n\beta}}$$

for all $s,t \in \mathcal{V}$, where $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain

$$\begin{aligned} &2[\Im(s+\varsigma t)-\Im(\varsigma s-t)] \\ =&\varsigma\left(\varsigma^2+1\right)[\Im(s+t)-\Im(s-t)]+2\left(1-\varsigma^4\right)[\Im(s)-\Im(t)]. \end{aligned} \tag{4.32}$$

Therefore, the existence of $\Im$ is completed.
Case 2. As β is odd and $0<|\varsigma|<1$, by replacing s with $\frac{s}{\varsigma}$ in (4.27), we obtain

$$\left\|\jmath(s)-k^4\jmath\left(\frac{s}{\varsigma}\right)\right\| \leq \left(\frac{1}{2^\beta}-\left(\varsigma^4-1\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}. \tag{4.33}$$

Substituting $\frac{s}{\varsigma^n}$ for s in inequality (4.33) yields

$$\left\|\jmath\left(\frac{s}{\varsigma^n}\right)-\varsigma^4\jmath\left(\frac{s}{\varsigma^{n+1}}\right)\right\| \leq \left(\frac{1}{2^\beta}-\left(\varsigma^4-1\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}, \quad n\in\mathbb{N}. \tag{4.34}$$

Similar to (4.8), we obtain the followings

$$\begin{aligned} \left\|\jmath(s)-\varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right)\right\| &\leq \sum_{i=1}^{n}\varsigma^{4(i-1)\beta}\left\|\jmath\left(\frac{s}{\varsigma^n}\right)-\varsigma^4\jmath\left(\frac{s}{\varsigma^{n+1}}\right)\right\| \\ &\leq \frac{1-\varsigma^{4n\beta}}{1-\varsigma^{4\beta}}\left(\frac{1}{2^\beta}-\left(\varsigma^4-1\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}. \end{aligned} \tag{4.35}$$

By **Case 1**, it is obvious that the sequence $\left\{\varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right)\right\}_n$ is complete, so $\Im:\mathcal{V}\to W$ is defined by

$$\Im(s) := \lim_{n\to\infty}\varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right), \quad s\in\mathcal{V}. \tag{4.36}$$

Then, we obtain

$$\|\jmath(s)-\Im(s)\| \leq \frac{1-\varsigma^{4n\beta}}{1-\varsigma^{4\beta}}\left(\frac{1}{2^\beta}-\left(\varsigma^4-1\right)^\beta\right)\varepsilon, \quad s\in\mathcal{V}.$$

Moreover, replacing s by $\frac{s}{\varsigma^n}, t$ by $\frac{t}{\varsigma^n}$, and multiply the resulting inequality by $\varsigma^{4n\beta}$, we obtain

$$\left\|2\left[\varsigma^{4n}\jmath\left(\frac{s+\varsigma t}{\varsigma^n}\right)-\varsigma^{4n}\jmath\left(\frac{\varsigma s-t}{\varsigma^n}\right)\right]-\varsigma(\varsigma^2+1)\left[\varsigma^{4n}\jmath\left(\frac{s-t}{\varsigma^n}\right)\right]\right.$$
$$\left.-2\left(1-\varsigma^4\right)\left[\varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right)-\varsigma^{4n}\jmath\left(\frac{t}{\varsigma^n}\right)\right]\right\| \leq \varsigma^{4n\beta}\varepsilon$$

for all $s,t\in\mathcal{V}$, where $n\in\mathbb{N}$. Similarly, letting $n\to\infty$, which gives you equation (4.32). Therefore, the existence of $\Im$ is completed.
Case 3. When β is even and $0<|a|<1$, replacing s by $\frac{s}{\varsigma}$ in (4.27), we obtain

$$\left\|\jmath(s)-\varsigma^4\jmath\left(\frac{s}{\varsigma}\right)\right\|\le\left(\frac{1}{2^\beta}+\left(\varsigma^4-1\right)^\beta\right)\varepsilon,\quad s\in\mathcal{V}. \tag{4.37}$$

By replacing s with $\frac{s}{\varsigma^n}$ in (4.37), then

$$\left\|\jmath\left(\frac{s}{\varsigma^n}\right)-\varsigma^4\jmath\left(\frac{s}{\varsigma^{n+1}}\right)\right\|\le\left(\frac{1}{2^\beta}+\left(\varsigma^4-1\right)^\beta\right)\varepsilon,\quad s\in\mathcal{V}. \tag{4.38}$$

Applying a similar method to (4.8), we get the following conclusions

$$\begin{aligned}\left\|\jmath(s)-\varsigma^{4n}\jmath\left(\frac{s}{\varsigma^n}\right)\right\|\le&\sum_{i=1}^{n}\varsigma^{4(i-1)\beta}\left\|\jmath\left(\frac{s}{\varsigma^n}\right)-\varsigma^4\jmath\left(\frac{s}{\varsigma^{n+1}}\right)\right\|\\ \le&\frac{1-\varsigma^{4n\beta}}{1-\varsigma^{4\beta}}\left(\frac{1}{2^\beta}+\left(\varsigma^4-1\right)^\beta\right)\varepsilon,\quad s\in\mathcal{V}.\end{aligned} \tag{4.39}$$

Using a similar method to (4.8), we come to the conclusion

$$\|\jmath(s)-\Im(s)\|\le\frac{1-\varsigma^{4n\beta}}{1-\varsigma^{4\beta}}\left(\frac{1}{2^\beta}+\left(\varsigma^4-1\right)^\beta\right)\varepsilon.$$

By **Case 2** to prove in the same argument, we obtained the equation (4.32). This proves the existence of $\Im$, and the proof of the uniqueness of $\Im$ is omitted because it is similar to the proof of Theorem 4.1. □

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