CORRECTIONS TO ERRORS IN THE PAPER "HADAMARD-TYPE INEQUALITIES FOR s-CONVEX FUNCTIONS I" AND NEW INTEGRAL INEQUALITIES OF s-CONVEX FUNCTIONS IN THE SECOND SENSE

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Abstract. In the work, the authors correct some errors appeared in the paper "S. Hussain, M. I. Bhatti, and M. Iqbal, *Hadamard-type inequalities for s-convex functions I*, Punjab Univ. J. Math. (Lahore) **41** (2009), 51–60" and establish some new integral inequalities of s-convex functions in the second sense.

1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. A real-valued function $f : I \to \mathbb{R}$ is said to be convex (or concave, respectivey) on I if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on an interval I such that $a, b \in I$ and $a < b$. Then the well-known Hermite–Hadamard integral inequality reads that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

In [1, 4], the concept of s-convex functions was innovated below.

Definition 1 ([1, 4])**.** Let $s \in (0,1]$ be a real number. A function $f : \mathbb{R} \to \mathbb{R}_0 = [0,\infty)$ is said to be s-convex in the second sense if the inequality

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

It is easy to see that for $s = 1$ the s-convexity reduces to the classical and ordinary convexity of functions defined on $[0,\infty)$.

The Hermite–Hadamard type integral inequalities for s-convex functions in the second sense are a very active research topic. We now recall some of them as follows.

Theorem 1 ([9])**.** *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, the numbers $a, b \in I$ with $a < b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then*

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$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^q+|f'(b)|^q\right]^{1/q}.$$

Theorem 2 ([11])**.** *Let $f: I\subseteq\mathbb{R}_0\to\mathbb{R}$ be differentiable on I°, let $a,b\in I$ with $a<b$, and let $f'\in L_1([a,b])$. If $|f'|$ is s-convex on $[a,b]$ for some $s\in(0,1]$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ \le \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\bigl(|f'(a)|+|f'(b)|\bigr).$$

For some other related papers on Hermite–Hadamard type inequalities for convex functions and s-convex functions, please refer to [3, 7, 13, 16]

In [5], Hussain and his two coauthors studied the Hermite–Hadamard type inequality of s-convex functions in the second sense, established several Hermite–Hadamard type inequalities for differentiable and twice differentiable functions based on concavity and s-convexity, and applied to construct some special means.

2. Hermite–Hadamard type inequalities by Hussain and his coauthors

Hussain and his two coauthors introduced in [5] the following lemma.

Lemma 1 ([5, Lemma 3])**.** *Let $I\subseteq\mathbb{R}$ denote an interval, $f: I\to\mathbb{R}$ be a differentiable function on I° (the interior of I), and $a,b\in I^\circ$ with $a<b$. If $f'\in L_1([a,b])$, then*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \\ =\frac{(b-a)^2}{4}\int_0^1(1-t)\left[f'\left(ta+(1-t)\frac{a+b}{2}\right)+f'\left(tb+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t. \quad (1)$$

Using Lemma 1, Hussain and his coauthors established the following Theorems 3 to 6 in the paper [5].

Theorem 3 ([5, Theorem 4])**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $q\ge 1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\left(\frac{1}{2}\right)^{1/p}\left\{\frac{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right. \\ \left.+\frac{\left[(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right\}.$$

Remark 1. If $q\ge 1$, the factor $\left(\frac{1}{2}\right)^{1/p}$ in [5, Theorem 4] should be modified to $\left(\frac{1}{2}\right)^{1-1/q}$. Otherwise, if $q=1$, the number $p=\frac{q}{q-1}$ is meaningless.

Theorem 4 ([5, Theorem 5]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is a concave function on $[a,b]$ for $q > 1$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{4(p+1)^{1/p}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right],$$

where $\frac{1}{q} + \frac{1}{q} = 1$.

Theorem 5 ([5, Theorem 6]). *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{(b-a)^2}{4(p+1)^{1/p}}\left(\frac{1}{s+1}\right)^{1/q} \\ &\times \left[\left(|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q} + \left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right)^{1/q}\right],\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q} = 1$.

Theorem 6 ([5, Theorem 7]). *Let $f : I \subseteq [0,\infty) \to \mathbb{R}_0$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an s-concave function on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\qquad\le \frac{(b-a)^2}{4(p+1)^{1/p}} 2^{(s-1)/q}\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right],\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q} = 1$.

We note that many typos in the above lemma and theorems quoted from the paper [5] have been corrected.

In this article, we will modify and correct the conditions and errors in Theorems 3 to 6 about s-convex functions in the second sense.

3. Errors and two lemmas

We first give a counterexample of [5, Lemma 3], that is, Lemma 1 mentioned above in this paper.

Example 1. *Letting $f(x) = x^2$ for $x \in [a,b]$ and taking $a = 0$ and $b = 1$ in Lemma 1, then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{(b-a)^2}{4}\int_0^1 (1-t)\biggl[f'\left(ta + (1-t)\frac{a+b}{2}\right) \\ + f'\left(tb + (1-t)\frac{a+b}{2}\right)\biggr]\,\mathrm{d}t = -\frac{1}{3}.\end{aligned}$$

Therefore, we can be sure that Lemma 1, that is, [5, Lemma 3]*, is not valid.*

In [14, Remark 1]*, among other things, the invalidness of the integral identity in* [5, Lemma 3] *has been pointed out and alternatively corrected.*

Making use of [8, Lemma 2.1], we correct [5, Lemma 3] as follows.

Lemma 2. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I$ with $a < b$. If $f' \in L_1([a,b])$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \\ = \frac{b-a}{4}\int_0^1 (1-t)\left[f'\left(ta+(1-t)\frac{a+b}{2}\right) - f'\left(tb+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t. \end{aligned} \tag{2}$$

Example 2. *Let $f(x) = x^2$ for $x \in [a,b]$. Then $|f'(x)|^q$ is an s-convex function in the second sense on $[a,b]$ for $s = 1$ and $q = 1$.*

If setting $a = 0$, $b = 12.12$, and $s = q = 1$ in Theorem 3, then

$$\begin{aligned} \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| = \frac{1}{12}(b-a)^2 = 12.2412 \\ > 12.12 = \frac{|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right|}{6} + \frac{2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|}{6}. \end{aligned}$$

If letting $a = 0$, $b = 6$, and $s = q = 1$ in Theorem 3, then

$$\begin{aligned} \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| = \frac{1}{12}(b-a)^2 = 3 \\ < 6 = \frac{|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right|}{6} + \frac{2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|}{6}. \end{aligned}$$

These numerical computations reveal that Theorems 3 to 6 are not necessarily true.

Remark 2. Comparing the factors $\frac{(b-a)^2}{4}$ and $\frac{b-a}{4}$ on the right hand sides of the integral equalities (1) and (2), we can illustrate that Theorems 5 to 6 are not necessarily true.

Now we establish the Jensen type integral inequalities for s-concave functions.

Lemma 3. *Let $\varphi: [a,b] \to \mathbb{R}_0$ be continuous and $g, p: [a,b] \to \mathbb{R}$ be integrable functions with $g(x) \in [a,b]$, $p(x) \ge 0$ for $x \in [a,b]$, and $\int_a^b p(x)\,\mathrm{d}x > 0$. If φ is an s-concave function in the second sense for some $s \in (0,1]$, then the Jensen type integral inequality*

$$\varphi\left(\frac{\int_a^b p(x)g(x)\,\mathrm{d}x}{\int_a^b p(x)\,\mathrm{d}x}\right) \ge \frac{\int_a^b [p(x)]^s \varphi(g(x))\,\mathrm{d}x}{\left[\int_a^b p(x)\,\mathrm{d}x\right]^s} \tag{3}$$

is sound.

Proof. Let $x_0 < x_1 < \cdots < x_n$ be a partition of $[a,b]$ and denote $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \dots, n$ such that $\max_{1\le i\le n}\{\Delta x_i\} \le 1$. In this way, we see that $(\Delta x_i)^s \ge \Delta x_i$ for $i = 1, 2, \dots, n$. By the s-concavity in the second sense of φ on $[a,b]$, see [15, Corollary 4], we obtain

$$\begin{aligned} \varphi\left(\frac{\sum_{i=1}^n p(x_i)g(x_i)\Delta x_i}{\sum_{i=1}^n p(x_i)\Delta x_i}\right) &\ge \frac{\sum_{i=1}^n [p(x_i)\Delta x_i]^s \varphi(g(x_i))}{\left[\sum_{i=1}^n p(x_i)\Delta x_i\right]^s} \\ &\ge \frac{\sum_{i=1}^n [p(x_i)]^s \varphi(g(x_i))\Delta x_i}{\left[\sum_{i=1}^n p(x_i)\Delta x_i\right]^s}. \end{aligned}$$

Further taking the limit of $n \to \infty$ on both sides of the above inequality leads to the inequality (3). The proof of Lemma 3 is completed. ∎

4. Modifications and corrections of integral inequalities of s-convex functions in the second sense

In this section, we modify and correct the conditions and errors in Theorems 3 to 6 about s-convex functions in the second sense.

Theorem 7 (Modifications and corrections of Theorem 3)**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $q\ge 1$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\\ &\times\left\{\frac{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}+\frac{\left[(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right\}. \quad (4)\end{aligned}$$

Proof. Since $|f'|^q$ is s-convex in the second sense on $[a,b]$, using Lemma 2 and by the Hölder integral inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le \frac{b-a}{4}\int_0^1(1-t)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\\ &\le \frac{b-a}{4}\left[\int_0^1(1-t)\,\mathrm{d}t\right]^{1-1/q}\left\{\left[\int_0^1(1-t)\left(t^s|f'(a)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\ &\quad\left.+\left[\int_0^1(1-t)\left(t^s|f'(b)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}\\ &=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right.\\ &\quad\left.+\frac{\left[(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right\}.\end{aligned}$$

The proof of Theorem 7 is completed. □

Theorem 8 (Generalization of Theorem 5)**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and for $q>1$ and $q\ge \ell\ge 0$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le\frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\left\{\left[\frac{sB(s,\ell+1)|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+\ell+1}\right]^{1/q}\right.\\ &\quad\left.+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+sB(s,\ell+1)|f'(b)|^q}{s+\ell+1}\right]^{1/q}\right\},\end{aligned}$$

where $B(u,v)$ denotes the classical beta function defined by

$$B(u,v)=\int_0^1 z^{u-1}(1-z)^{v-1}\,\mathrm{d}z,\quad \Re(u)>0,\Re(v)>0.$$

Proof. Similar to the proof of the inequality (4) in Theorem 7, using Lemma 2, employing the Hölder integral inequality, and utilizing the s-convexity in the second sense on $[a,b]$ of $|f'|^q$, we derive

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\left[\int_0^1(1-t)^{(q-\ell)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\\
&\times\left\{\left[\int_0^1(1-t)^{\ell}\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right.\\
&\left.+\left[\int_0^1(1-t)^{\ell}\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}\\
&\le\frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\\
&\times\left\{\left[\int_0^1(1-t)^{\ell}\left(t^s|f'(a)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\
&\left.+\left[\int_0^1(1-t)^{\ell}\left(t^s|f'(b)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}\\
&=\frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\left\{\left[\frac{sB(s,\ell+1)|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+\ell+1}\right]^{1/q}\right.\\
&\left.+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+sB(s,\ell+1)|f'(b)|^q}{s+\ell+1}\right]^{1/q}\right\}.
\end{aligned}$$

The proof of Theorem 8 is completed. □

If $q>1$ and $\frac{1}{p}=1-\frac{1}{q}$, then $\frac{1}{(p+1)^{1/p}}=\left(\frac{q-1}{2q-1}\right)^{1-1/q}$. Therefore, putting $\ell=0$ in Theorem 8 yields

Corollary 1 (Modifications and corrections of Theorem 5)**.** *Under conditions of Theorem 8 applied to $\ell=0$, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}\\
&\times\left[\left(s|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+s|f'(b)|^q\right)^{1/q}\right].
\end{aligned}$$

Next, we will study the Hermite–Hadamard type integral inequalities of s-concave functions. We first establish an Hermite–Hadamard type integral inequality of s-concave functions in the case of $q\ge1$.

Theorem 9. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-concave function in the second sense on $[a,b]$ for $q\ge1$ and some fixed $s\in(0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{2^{3-1/q}}\left(\frac{s}{s+1}\right)^{s/q}$$

$$\times\left[\left|f'\left(\frac{(3s+1)a+(s+1)b}{2(2s+1)}\right)\right|+\left|f'\left(\frac{(s+1)a+(3s+1)b}{2(2s+1)}\right)\right|\right]. \quad (5)$$

Proof. Using Lemma 2 and employing the Hölder integral inequality, we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\int_0^1(1-t)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t+\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t \\
&\le \frac{b-a}{4}\left(\int_0^1(1-t)\,\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1(1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right. \\
&\quad\left.+\left[\int_0^1(1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}.
\end{aligned}$$

Taking $p(t)=(1-t)^{1/s}$ for $[0,1]$ in Lemma 3, utilizing Lemma 3, and using the s-convexity of $|f'|^q$ in the second sense of $[a,b]$, we derive

$$\begin{aligned}
&\int_0^1(1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t \\
&\le\left(\int_0^1(1-t)^{1/s}\,\mathrm{d}t\right)^s\left|f'\left(\frac{\int_0^1(1-t)^{1/s}\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}t}{\int_0^1(1-t)^{1/s}\,\mathrm{d}t}\right)\right|^q \\
&=\left(\frac{s}{s+1}\right)^s\left|f'\left(\frac{(3s+1)a+(s+1)b}{2(2s+1)}\right)\right|^q
\end{aligned}$$

and

$$\int_0^1(1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\le\left(\frac{s}{s+1}\right)^s\left|f'\left(\frac{(s+1)a+(3s+1)b}{2(2s+1)}\right)\right|^q.$$

Substituting these two inequalities into the first inequality in this proof yields the inequality (5). The proof of Theorem 9 is completed. □

If taking $s=1$ in Theorem 9, we have

Corollary 2. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is a concave function in the second sense on $[a,b]$ for $q\ge1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{8}\left[\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right].$$

Theorem 10 (Generalization of Theorems 4 and 6)**.** *Suppose $q>1$ and $q\ge\ell\ge0$. Let $f: I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-concave function on $[a,b]$ for some fixed $s\in(0,1]$, then*

$$\begin{aligned}
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|&\le\frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\left(\frac{s}{s+\ell}\right)^{s/q} \\
&\times\left[\left|f'\left(\frac{(3s+\ell)a+(s+\ell)b}{2(2s+\ell)}\right)\right|+\left|f'\left(\frac{(s+\ell)a+(3s+\ell)b}{2(2s+\ell)}\right)\right|\right]. \quad (6)
\end{aligned}$$

Proof. Using Lemma 2 and utilizing the Hölder integral inequality, we obtain

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{4}\left(\int_0^1 (1-t)^{(q-\ell)/(q-1)}\right)^{1-1/q}\\ &\times\left\{\left[\int_0^1 (1-t)^\ell\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right]^{1/q}\right.\\ &+\left.\left[\int_0^1 (1-t)^\ell\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}.\end{aligned}\tag{7}$$

Taking $p(t)=(1-t)^{\ell/s}$ for $t\in[0,1]$ in Lemma 3 and using the s-convexity of $|f'|^q$ in the second sense on $[a,b]$, we have

$$\begin{aligned}&\int_0^1 (1-t)^\ell\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\\ &\le\left[\int_0^1 (1-t)^{\ell/s}\,\mathrm{d}t\right]^s\left|f'\left(\frac{\int_0^1 (1-t)^{\ell/s}\big(ta+(1-t)\frac{a+b}{2}\big)\,\mathrm{d}t}{\int_0^1 (1-t)^{\ell/s}\,\mathrm{d}t}\right)\right|^q\\ &=\left(\frac{s}{s+\ell}\right)^s\left|f'\left(\frac{(3s+\ell)a+(s+\ell)b}{2(2s+\ell)}\right)\right|^q\end{aligned}$$

and

$$\int_0^1 (1-t)^\ell\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\le\left(\frac{s}{s+\ell}\right)^s\left|f'\left(\frac{(s+\ell)a+(3s+\ell)b}{2(2s+\ell)}\right)\right|^q.$$

Substituting these two inequalities into the inequality (7) yields the inequality (6). The proof of Theorem 10 is completed. □

If putting $s=1$ and $\ell=0$ in Theorem 10, we acquire

Corollary 3. (Modifications and corrections of Theorem 4) *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is a concave function on $[a,b]$ for $q>1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\qquad\le\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right].\end{aligned}$$

If letting $\ell=0$ in Theorem 10, we obtain

Corollary 4. (Modifications and corrections of Theorem 6) *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-concave function on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\qquad\le\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right].\end{aligned}$$

5. Conclusions

In this paper, we pointed out many errors appeared in the article [5], corrected these errors, and established several new integral inequalities of s-convex functions in the second sense.

For more information on recent developments of this topic, please refer to the papers [2, 6, 10, 12, 14, 17] and closely-related references therien.

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