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Asymptotic properties of a stochastic eco-epidemiological model with fear effect and hunting cooperation

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Abstract: In this paper, we put forward and analyze a stochastic eco-epidemiological model with disease in the prey population, which incorporates fear effect of predators on prey and hunting cooperation among predators. We find out sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the system by using the stochastic Lyapunov function methods. Moreover, we also derive sufficient criteria for extinction of the infected prey population and the predator population. Additionally, we give the specific expression of the probability density function of the stochastic model near the unique endemic quasi-equilibrium by solving the Fokker-Planck equation. In the end, the supporting theoretical results are verified by numerical simulation.

Keywords: Stochastic eco-epidemiological model; Hunting cooperation functional response; Fear effect; Stationary distribution and ergodicity; Extinction; Density function.

1 Introduction

Recent studies have shown that the dynamic relationship between predators and their prey may play an important role in both ecology and mathematical ecology due to its universal existence and importance in population dynamics. For most of the predator-prey models, the impact of predators on the prey population usually reflects through direct killing. However, some researchers have showed that the fear effect of predator on the prey may play a vital role in the predator-prey system[1–3] . Wang et al. incorporate the fear effect to the predator-prey model and show that strong fear can stabilize the predator-prey system by excluding the existence of periodic solutions and relatively weak fear can induce multiple limit cycles via subcritical Hopf bifurcations. More details can be seen in [1]. As universally recognized, eco-epidemiology is one of the most interesting issues in the investigation of mathematical biology, which combines epidemiology with ecology [4–6]. Chattopadhyay et al. [7] proposed a three species eco-epidemiological model and found the conditions for local stability, extinction and Hopf-bifurcation. Liu et al. [8] proposed an eco-epidemiological model with disease in the prey population, incorporates fear effect of predators on prey and hunting cooperation among predators. They divide the prey population into two classes, one is the susceptible prey, the other is the infected prey. They also assumed that the predator eats only the infected prey. The eco-epidemiological model can be written as

$$\begin{cases} \dfrac{dS}{dt} = \dfrac{rS}{1+K_1 y} - \mu S - aS^2 - \dfrac{\beta SI}{1+K_2 y}, \\ \dfrac{dI}{dt} = \dfrac{\beta SI}{1+K_2 y} - \delta I - (p+by)Iy, \\ \dfrac{dy}{dt} = c(p+by)Iy - my, \end{cases} \tag{1.1}$$

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where $S(t), I(t), y(t)$ represent the population density of the susceptible prey, the infected prey and the predator at time t respectively. The parameters r, μ, a, K_1, K_2, β, δ, p, c and m are positive constants and b is a nonnegative constant, r, μ denote the intrinsic growth rate and natural death rate of the susceptible prey, respectively. a stands for the mortality rate of prey population due to intra-specific competition among the individuals of the susceptible prey population, β represents the disease transmission rate, p stands for the attack rate of the predator on the prey and b describes the predator cooperation in hunting, $c \in (0, 1)$ represents the conversion efficiency from prey biomass to predator biomass, δ and m represent the death rates of the infected prey and predator populations, respectively. The term $\frac{1}{1+K_1 y}$ denotes the fear function which represents the cost of anti-predator defence of prey due to fear induced by predator and K_1 reflects the level of fear that reduces the growth of the prey. The term $\frac{\beta}{1+K_2 y}$ represents the fear of predators reduces the foraging activity of the prey population and K_2 reflects the level of fear which reduces the disease transmission.

According to the theory of Liu et al. [8], system (1.1) may have the following four nonnegative equilibria.

(i) The trivial equilibrium $E_0 = (0, 0, 0)$ which always exists and it is a unstable saddle point.

(ii) The disease-free and predator-free prey equilibrium $E_1 = (\frac{r-\mu}{a}, 0, 0)$ which always exists under the condition $r > \mu$ and it is locally asymptotically stable if $R_p < 0$ and unstable provided that $R_p > 0$, where $R_p = \frac{cp(\beta(r-\mu)-a\delta)}{m\beta^2}$.

(iii) The predator-free prey equilibrium $E_2 = (\bar{S}, \bar{I}, 0)$ exists if and only if $R_p > 0$ and it is locally asymptotically stable if $0 < R_p < 1$ and unstable provided that $R_p > 1$, where

$$\bar{S} = \frac{\delta}{\beta},\ \bar{I} = \frac{\beta(r-\mu) - a\delta}{\beta^2}.$$

(iv) The coexistence equilibrium $E^* = (S^*, I^*, y^*)$ exists if and only if $R_p > 1$.

Model (1.1) is a deterministic model which assumes that the parameters are deterministic irrespective environmental fluctuations. However, the ecology and epidemiology systems are always affected by the environmental noise. Therefore, the deterministic systems have some limitations to predict the future dynamics accurately[9–18]. Motivated by those previous works, in this paper, we consider fluctuations in the environment, which are assumed to manifest themselves as fluctuations in parameters μ, δ and m involved in the previous deterministic model (1.1), that is

$$\mu \to \mu - \sigma_1 \dot{B}_1(t),\ \delta \to \delta - \sigma_2 \dot{B}_2(t) \text{ and } m \to m - \sigma_3 \dot{B}_3(t)$$

respectively, where $\{B_1(t)\}_{t\geq 0}$, $\{B_2(t)\}_{t\geq 0}$ and $\{B_3(t)\}_{t\geq 0}$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) [19]; σ_i^2 $(i = 1, 2, 3)$ denotes the intensities of white noise. Incorporating the above perturbations in the deterministic system (1.1), we get the following stochastic model:

$$\begin{cases} dS = \left[\frac{rS}{1+K_1 y} - \mu S - aS^2 - \frac{\beta SI}{1+K_2 y}\right]dt + \sigma_1 S dB_1(t), \\ dI = \left[\frac{\beta SI}{1+K_2 y} - \delta I - (p + by)Iy\right]dt + \sigma_2 I dB_2(t), \\ dy = [c(p + by)Iy - my]dt + \sigma_3 y dB_3(t). \end{cases} \tag{1.2}$$

Throughout this paper, let $\mathbb{R}^d$ be a d-dimensional Euclidean space and

$$\mathbb{R}_+^d = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d\}\ ,\ \mathbb{R}_+^{d,\circ} = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}.$$

If G is a matrix, its transpose is denoted by G^T and $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

The rest of this paper is structured as follows. In Section 3, we find out sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the stochastic system (1.2). In Section 3, we derive sufficient criteria for extinction of the infected prey population and the predator population. In Section 4, we discuss the probability density function of the stochastic model near the unique endemic quasi-equilibrium. In Section 5, the theoretical results are showed by numerical simulation. Finally, a brief conclusion scope of the investigation is provided to end this paper in Section 6.

2 Existence of ergodic stationary distribution

In this section, we will find out sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the stochastic system (1.2).

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^d$ described by the stochastic differential equation

$$dX(t) = f(X(t))dt + \sum_{r=1}^{k} g_r(X(t))dB_r(t).$$

The diffusion matrix of the process $X(t)$ is defined as follows

$$A(x) = (a_{ij}(x)),\ a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Lemma 2.1. *[20]. The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded open domain $D \subset \mathbb{R}^d$ with regular boundary Γ, having the following properties:*

($\mathcal{A}_1$) In the domain D and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

($\mathcal{A}_2$) If $x \in \mathbb{R}^d \setminus D$, the mean time τ at which a path issuing from x reaches the set D is finite, and $\sup_{x\in U} \mathbb{E}^x\tau < \infty$ for every compact subset $U \subset \mathbb{R}^d$.

Lemma 2.2. *Assume that $\min\{\delta, m\} > \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)$, then for any initial value $(S(0), I(0), y(0)) \in \mathbb{R}_+^{3,\circ}$, there is a unique solution $(S(t), I(t), y(t))$ to system (1.2) on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^{3,\circ}$ with probability one, namely, the solution $(S(t), I(t), y(t)) \in \mathbb{R}_+^{3,\circ}$ for all $t \geq 0$ almost surely (a.s).*

Proof. The proof is similar to the statement of Theorem 3.1 in Liu and Jiang [21], hence we only construct a C^2-function $U : \mathbb{R}_+^{3,\circ} \to \mathbb{R}_+$ as follows

$$U(S, I, y) = \left(S + I + \frac{y}{c}\right)^{\theta+2} - \ln S - \ln I - \ln y,$$

where $\theta \in (0, \frac{2\min\{\delta,m\}}{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2} - 1)$ is a sufficiently small number. Now we are in the position to give our main results of this section.

Theorem 2.1. *Assume that $R_p^S := \frac{cp(\beta(r-\mu-\frac{\sigma_1^2}{2})-a(\delta+\frac{\sigma_2^2}{2}))}{(m+\frac{\sigma_3^2}{2})\beta^2} > 1$, $m > \frac{\sigma_3^2}{2}$ and $\min\{\delta, m\} > 2(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)$, then the process $(S(t), I(t), y(t))$ has an invariant probability measure π^* on $\mathbb{R}_+^{3,\circ}$.*

Proof. To prove Theorem 2.1, we should verify conditions ($\mathcal{A}_1$) and ($\mathcal{A}_2$) in Lemma 2.1. We first need to show the condition ($\mathcal{A}_1$). The diffusion matrix of system (1.2) is given by

$$A_0 = \begin{pmatrix} \sigma_1^2S^2 & 0 & 0 \\ 0 & \sigma_2^2I^2 & 0 \\ 0 & 0 & \sigma_3^2y^2 \end{pmatrix}.$$

Apparently, the matrix A_0 is positive definite for any compact subset of $\mathbb{R}_+^{3,\circ}$, ($\mathcal{A}_1$) is obvious. Now we show the condition $\mathcal{A}_2$. For any adequately small number $\epsilon_0 \in (0, \frac{a}{K_2(r-\mu)})$, define

$$R_p^S(\epsilon_0) = \frac{cp(\beta(1-\frac{K_2(r-\mu)\epsilon_0}{a})(r-\mu-\frac{\sigma_1^2}{2}) - a(\delta+\frac{\sigma_2^2}{2}))}{(m+\frac{\sigma_3^2}{2})\beta^2(1-\frac{K_2(r-\mu)\epsilon_0}{a})}.$$

Evidently, $\lim_{\epsilon_0\to0^+} R_p^S(\epsilon_0) = R_p^S$. Since the function $R_p^S(\epsilon_0)$ is continuous with respect to ϵ_0 and $R_p^S > 1$, we can select ϵ_0 small enough such that $R_p^S(\epsilon_0) > 1$. From system (1.2) it follows that

$$\begin{aligned} L(-\ln S) &= -\frac{r}{1+K_1y} + \mu + aS + \frac{\beta I}{1+K_2y} + \frac{\sigma_1^2}{2} \\ &= -r + \mu + \frac{\sigma_1^2}{2} + aS + \beta I + \frac{rK_1y}{1+K_1y} \\ &\leq -r + \mu + \frac{\sigma_1^2}{2} + aS + \beta I + rK_1y, \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
L(S) =& \frac{rS}{1+K_1 y} - \mu S - aS^2 - \frac{\beta SI}{1+K_2 y} \\
\leq& (r-\mu)S - aS^2,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
L(-\ln y) =& -c(p+by)I + m + \frac{\sigma_3^2}{2} \\
\leq& -cpI + m + \frac{\sigma_3^2}{2},
\end{aligned}
\tag{2.3}
$$

$$
L(y) = c(p+by)Iy - my,
\tag{2.4}
$$

$$
\begin{aligned}
L(y^2) =& 2y[c(p+by)Iy - my] + \sigma_3^2 y^2 \\
=& 2c(p+by)Iy^2 - (2m - \sigma_3^2)y^2
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
L(-\ln I) =& -\frac{\beta S}{1+K_2 y} + \delta + \frac{\sigma_2^2}{2} + (p+by)y \\
=& -\beta S + \frac{\beta K_2 S y}{1+K_2 y} + \delta + \frac{\sigma_2^2}{2} + py + by^2 \\
\leq& -\beta S + \beta K_2 S y + \delta + \frac{\sigma_2^2}{2} + py + by^2 \\
\leq& -\beta S + \beta K_2 \epsilon_0 S^2 + \delta + \frac{\sigma_2^2}{2} + py + \left(\frac{\beta K_2}{4\epsilon_0} + b\right) y^2,
\end{aligned}
\tag{2.6}
$$

where in the inequality of (2.6), we have used the Young inequality

$$
Sy \leq \epsilon_0 S^2 + \frac{y^2}{4\epsilon_0}
$$

and $\epsilon_0 \in (0, \frac{a}{K_2(r-\mu)})$ is a sufficiently small number. Define

$$
\begin{aligned}
V_1(S,I,y) =& -\ln I + \frac{\beta K_2 \epsilon_0}{a} S - \frac{\beta(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a} \ln S + \frac{1}{m}\left(\frac{\beta r K_1 (1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a} + p\right) y \\
& - \frac{\beta^2(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{acp} \ln y + \frac{\frac{\beta K_2}{4\epsilon_0} + b}{2m - \sigma_3^2} y^2,
\end{aligned}
$$

then from (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) it follows that

$$
\begin{aligned}
LV_1 \leq& -\frac{\beta(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a}\left(r - \mu - \frac{\sigma_1^2}{2}\right) + \delta + \frac{\sigma_2^2}{2} + \frac{\beta^2(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a} I \\
& + \frac{c}{m}\left(\frac{\beta r K_1(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a} + p\right)(p+by)Iy - \frac{\beta^2(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a} I \\
& + \frac{\beta^2(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{acp}\left(m + \frac{\sigma_3^2}{2}\right) + \frac{2c(\frac{\beta K_2}{4\epsilon_0} + b)}{2m - \sigma_3^2}(p+by)Iy^2 \\
=& -\frac{\beta^2(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{acp}\left(m + \frac{\sigma_3^2}{2}\right)(R_p^S(\epsilon_0) - 1) + \frac{c}{m}\left(\frac{\beta r K_1(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a} + p\right)(p+by)Iy \\
& + \frac{2c(\frac{\beta K_2}{4\epsilon_0} + b)}{2m - \sigma_3^2}(p+by)Iy^2 \\
=& -\lambda(\epsilon_0) + \frac{c}{m}\left(\frac{\beta r K_1(1 - \frac{K_2(r-\mu)\epsilon_0}{a})}{a} + p\right)(p+by)Iy + \frac{2c(\frac{\beta K_2}{4\epsilon_0} + b)}{2m - \sigma_3^2}(p+by)Iy^2,
\end{aligned}
\tag{2.7}
$$

where

$$
R_p^S(\epsilon_0) = \frac{cp(\beta(1 - \frac{K_2(r-\mu)\epsilon_0}{a})(r - \mu - \frac{\sigma_1^2}{2}) - a(\delta + \frac{\sigma_2^2}{2}))}{(m + \frac{\sigma_3^2}{2})\beta^2(1 - \frac{K_2(r-\mu)\epsilon_0}{a})},
$$

$$\lambda(\epsilon_0) = \frac{\beta^2(1-\frac{K_2(r-\mu)\epsilon_0}{a})}{acp}\left(m+\frac{\sigma_3^2}{2}\right)(R_p^S(\epsilon_0)-1) > 0.$$

Next, define

$$V_2(S,I,y) = \frac{1}{\theta+5}\left(S+I+\frac{y}{c}\right)^{\theta+5},$$

where θ is a sufficiently small number satisfying the following condition

$$\min\{\delta,m\} > \frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2).$$

Then

$$\begin{aligned}
LV_2 =&\left(S+I+\frac{y}{c}\right)^{\theta+4}\left[\frac{rS}{1+K_1y}-\mu S-aS^2-\delta I-\frac{m}{c}y\right]+\frac{\theta+4}{2}\left(S+I+\frac{y}{c}\right)^{\theta+3}\left(\sigma_1^2S^2+\sigma_2^2I^2+\frac{\sigma_3^2}{c^2}y^2\right)\\
\leq&\left(S+I+\frac{y}{c}\right)^{\theta+4}\left[(r+\min\{\delta,m\})S-aS^2-\min\{\delta,m\}\left(S+I+\frac{y}{c}\right)\right]\\
&+\frac{\theta+4}{2}\left(S+I+\frac{y}{c}\right)^{\theta+5}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\\
\leq&\frac{(r+\min\{\delta,m\})^2}{4a}\left(S+I+\frac{y}{c}\right)^{\theta+4}-\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)\left(S+I+\frac{y}{c}\right)^{\theta+5}\\
\leq&-\frac{1}{2}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)\left(S+I+\frac{y}{c}\right)^{\theta+5}+C\\
\leq&-\frac{1}{2}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)\left(S^{\theta+5}+I^{\theta+5}+\frac{y^{\theta+5}}{c^{\theta+5}}\right)+C,
\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}
C:=&\sup_{(S,I,y)\in\mathbb{R}_+^{3,\circ}}\left\{-\frac{1}{2}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)\left(S+I+\frac{y}{c}\right)^{\theta+5}\right.\\
&\left.+\frac{(r+\min\{\delta,m\})^2}{4a}\left(S+I+\frac{y}{c}\right)^{\theta+4}\right\}\\
<&\infty.
\end{aligned}$$

Define a Lyapunov function $\tilde{V}:\mathbb{R}_+^{3,\circ}\to\mathbb{R}$ as follows

$$\tilde{V}(S,I,y) = MV_1(S,I,y)+V_2(S,I,y),$$

where $M>0$ is a sufficiently large constant satisfying

$$-M\lambda(\epsilon_0)+g_1^u+g_2^u\leq -2$$

and functions g_i $(i=1,2)$ will be determined later. Furthermore, note that

$$\liminf_{k\to\infty,(S,I,y)\in\mathbb{R}_+^{3,\circ}\setminus D_k}\tilde{V}(S,I,y)=\infty,$$

where $D_k=(\frac{1}{k},k)\times(\frac{1}{k},k)\times(\frac{1}{k},k)$. Thus, $\tilde{V}$ has a minimal value $\tilde{V}(S_0,I_0,y_0)$ in $\mathbb{R}_+^{3,\circ}$ and we can define a C^2-function $V:\mathbb{R}_+^{3,\circ}\to\mathbb{R}_+$ as follows

$$\begin{aligned}
V(S,I,y)=&\tilde{V}(S,I,y)-\tilde{V}(S_0,I_0,y_0)\\
=&MV_1(S,I,y)+V_2(S,I,y)-\tilde{V}(S_0,I_0,y_0).
\end{aligned}$$

Thus, from (2.7) and (2.8) it follows that

$$
\begin{aligned}
LV \leq & -M\lambda(\epsilon_0)+\frac{Mc}{m}\left(\frac{\beta rK_1(1-\frac{K_2(r-\mu)\epsilon_0}{a})}{a}+p\right)(p+by)Iy+\frac{2Mc(\frac{\beta K_2}{4\epsilon_0}+b)}{2m-\sigma_3^2}(p+by)Iy^2\\
&-\frac{1}{2}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)\left(S^{\theta+5}+I^{\theta+5}+\frac{y^{\theta+5}}{c^{\theta+5}}\right)+C\\
\leq & -M\lambda(\epsilon_0)+\frac{Mc}{m}\left(\frac{\beta rK_1(1-\frac{K_2(r-\mu)\epsilon_0}{a})}{a}+p\right)(p+by)Iy+\frac{2Mc(\frac{\beta K_2}{4\epsilon_0}+b)}{2m-\sigma_3^2}(p+by)Iy^2\\
&-\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)S^{\theta+5}-\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)I^{\theta+5}\\
&-\frac{1}{4c^{\theta+5}}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)y^{\theta+5}-\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)I^{\theta+5}\\
&-\frac{1}{4c^{\theta+5}}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)y^{\theta+5}+C\\
:= & H(S,I,y).
\end{aligned}
$$

In view of the expression of $H(S,I,y)$, we can obtain:

Case 1. If $S\to\infty$ or $I\to\infty$ or $y\to\infty$, then

$$
\begin{aligned}
H(S,I,y)\leq & -\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)S^{\theta+5}-\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)I^{\theta+5}\\
&-\frac{1}{4c^{\theta+5}}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)y^{\theta+5}-\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)I^{\theta+5}\\
&-\frac{1}{4c^{\theta+5}}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)y^{\theta+5}\\
&+\frac{Mc}{m}\left(\frac{\beta rK_1(1-\frac{K_2(r-\mu)\epsilon_0}{a})}{a}+p\right)(p+by)Iy+\frac{2Mc(\frac{\beta K_2}{4\epsilon_0}+b)}{2m-\sigma_3^2}(p+by)Iy^2+C\\
\leq & -\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)S^{\theta+5}-\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)I^{\theta+5}\\
&-\frac{1}{4c^{\theta+5}}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)y^{\theta+5}+J\\
\to & -\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
J:=\sup_{(I,y)\in\mathbb{R}_+^{2,\circ}}&\left\{-\frac{1}{4}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)I^{\theta+5}\right.\\
&-\frac{1}{4c^{\theta+5}}\left(\min\{\delta,m\}-\frac{\theta+4}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right)y^{\theta+5}\\
&\left.+\frac{Mc}{m}\left(\frac{\beta rK_1(1-\frac{K_2(r-\mu)\epsilon_0}{a})}{a}+p\right)(p+by)Iy+\frac{2Mc(\frac{\beta K_2}{4\epsilon_0}+b)}{2m-\sigma_3^2}(p+by)Iy^2+C\right\}\\
<&\infty.
\end{aligned}
$$

Case 2. If $I \to 0^+$ or $y \to 0^+$, then

$$
\begin{aligned}
H(S,I,y) \leq & -M\lambda(\epsilon_0) + \frac{Mc}{m}\bigg(\frac{\beta r K_1(1-\frac{K_2(r-\mu)\epsilon_0}{a})}{a} + p\bigg)(p+by)Iy + \frac{2Mc(\frac{\beta K_2}{4\epsilon_0}+b)}{2m-\sigma_3^2}(p+by)Iy^2 \\
& - \frac{1}{4}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)I^{\theta+5} - \frac{1}{4c^{\theta+5}}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)y^{\theta+5} \\
& - \frac{1}{4}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)I^{\theta+5} - \frac{1}{4c^{\theta+5}}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)y^{\theta+5} \\
& + C \\
\leq & -M\lambda(\epsilon_0) + g_1^u + g_2^u + \frac{Mc}{m}\bigg(\frac{\beta r K_1(1-\frac{K_2(r-\mu)\epsilon_0}{a})}{a} + p\bigg)(p+by)Iy + \frac{2Mc(\frac{\beta K_2}{4\epsilon_0}+b)}{2m-\sigma_3^2}(p+by)Iy^2 \\
& - \frac{1}{4}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)I^{\theta+5} - \frac{1}{4c^{\theta+5}}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)y^{\theta+5} \\
\to & -\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(I) &:= -\frac{1}{4}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)I^{\theta+5} + C, \\
g_2(y) &:= -\frac{1}{4c^{\theta+5}}\bigg(\min\{\delta,m\} - \frac{\theta+4}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\bigg)y^{\theta+5}.
\end{aligned}
$$

As a result, we can select a sufficiently small number $0 < \epsilon < 1$ such that

$$
H(S,I,y) \leq -1 \text{ for any } (S,I,y) \in \mathbb{R}_+^{3,\circ} \setminus D_\epsilon, \tag{2.9}
$$

where

$$
D_\epsilon = \bigg\{(S,I,y) \in \mathbb{R}_+^{3,\circ} : 0 < S \leq \frac{1}{\epsilon}, \epsilon \leq I \leq \frac{1}{\epsilon}, \epsilon \leq y \leq \frac{1}{\epsilon}\bigg\}.
$$

Moreover, there is also a positive constant P such that

$$
H(S,I,y) \leq P \text{ for any } (S,I,y) \in \mathbb{R}_+^{3,\circ}. \tag{2.10}
$$

Therefore, we derive

$$
\begin{aligned}
-\mathbb{E}(V(S(0),I(0),y(0))) \leq & \mathbb{E}(V(S(t),I(t),y(t))) - \mathbb{E}(V(S(0),I(0),y(0))) \leq \int_0^t \mathbb{E}(LV(S(s),I(s),y(s)))ds \\
\leq & \int_0^t \mathbb{E}(H(S(s),I(s),y(s)))ds.
\end{aligned}
$$

By means of (2.9) and (2.10), we derive

$$
\begin{aligned}
0 \leq & \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}(H(S(s),I(s),y(s)))ds \\
= & \liminf_{t\to\infty} \frac{1}{t}\int_0^t (\mathbb{E}(H(S(s),I(s),y(s)))1_{\{(S(s),I(s),y(s))\in D_\epsilon^c\}}) + \mathbb{E}(H(S(s),I(s),y(s)))1_{\{(S(s),I(s),y(s))\in D_\epsilon\}}))ds \\
\leq & \liminf_{t\to\infty} \frac{1}{t}\int_0^t (-\mathbb{P}((S(s),I(s),y(s)) \in D_\epsilon^c) + P\mathbb{P}((S(s),I(s),y(s)) \in D_\epsilon)ds \\
= & -1 + (1+P)\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}((S(s),I(s),y(s)) \in D_\epsilon)ds,
\end{aligned}
$$

this indicates that

$$
\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}((S(s),I(s),y(s)) \in D_\epsilon)ds \geq \frac{1}{1+P} \text{ a.s.}
$$

Consequently

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}(s,(S,I,y),D_\epsilon)ds \geq \frac{1}{1+P} \text{ a.s. for any } (S,I,y)\in\mathbb{R}_+^{3,\circ}, \tag{2.11}$$

where $\mathbb{P}(t,(S,I,y),\cdot)$ is the transition probability of $(S(t),I(t),y(t))$. According to the invariance of $\mathcal{M}=\{S\geq 0, I>0, y>0\}$ under system (1.2), we can consider the Markov process $(S(t),I(t),y(t))$ on the state space $\mathcal{M}$. It is easy to show that $(S(t),I(t),y(t))$ has the Feller property. Therefore, inequality (2.11) indicates that there exists an invariant probability measure π^* on $\mathcal{M}$; see [22]. Since $I(t)\to 0$ and $y(t)\to 0$ provided that $S(0)=0$, $\lim_{t\to\infty}\mathbb{P}(t,(0,I,y),Q)=0$ for all compact set $Q\subset\mathcal{M}$. Hence, we must get $\pi^*(\{S=0,I>0,y>0\})=0$ (equivalently $\pi^*(\mathbb{R}_+^{3,\circ})=1$). Moreover, in view of the invariance of $\mathbb{R}_+^{3,\circ}$, π^* is an invariant probability measure of $(S(t),I(t),y(t))$ on $\mathbb{R}_+^{3,\circ}$. This completes the proof.

3 Extinction

In the research of eco-epidemiological systems, extinction is one of the most important issues. In this section, we verify sufficient conditions for the extinction of the infected prey population and the predator population in stochastic system.

3.1 Extinction of the infected prey population

In this subsection, we shall investigate that under what conditions the infected prey population will go to extinction exponentially with probability one.

Theorem 3.1. *Let $(S(t),I(t),y(t))$ be a solution to system (1.2) with any initial value $(S(0),I(0),y(0))\in\mathbb{R}_+^{3,\circ}$. If $r-\mu>\frac{\sigma_1^2}{2}$ and $R_p^S<0$, then the infected prey population will go to extinction exponentially with probability one, i.e., $\lim_{t\to\infty}I(t)=0$ a.s.*

Proof. Consider the following one-dimensional stochastic differential equation

$$dX = X[(r-\mu)-aX]dt+\sigma_1 X dB_1(t). \tag{3.1}$$

Let $X(t)$ be the solution to Eq. (3.1) with any initial value $X(0)=S(0)>0$. According to Lemma A.1 of Appendix A in Ji et al. [23], we can derive

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)ds = \frac{r-\mu-\frac{\sigma_1^2}{2}}{a} \text{ a.s.} \tag{3.2}$$

Applying Itô's formula [19] to $\ln I$ leads to that

$$\begin{aligned} d(\ln I) =& \left[\frac{\beta S}{1+K_2 y}-\delta-\frac{\sigma_2^2}{2}-(p+by)y\right]dt+\sigma_2 dB_2(t) \\ \leq& \left(\beta S-\delta-\frac{\sigma_2^2}{2}\right)dt+\sigma_2 dB_2(t). \end{aligned} \tag{3.3}$$

Integrating from 0 to t and then dividing by t on both sides of (3.3), we get

$$\begin{aligned} \frac{\ln I(t)-\ln I(0)}{t} \leq& \frac{\beta}{t}\int_0^t S(s)ds-\delta-\frac{\sigma_2^2}{2}+\frac{\sigma_2 B_2(t)}{t} \\ \leq& \frac{\beta}{t}\int_0^t X(s)ds-\delta-\frac{\sigma_2^2}{2}+\frac{\sigma_2 B_2(t)}{t}. \end{aligned} \tag{3.4}$$

Taking the superior limit on both sides of (3.4) and combining with (3.2) and noting that $\lim_{t\to\infty}\frac{B_2(t)}{t}=0$ a.s., we obtain

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq \frac{\beta(r-\mu-\frac{\sigma_1^2}{2})}{a}-\delta-\frac{\sigma_2^2}{2}<0 \text{ a.s.},$$

which indicates that $\lim_{t\to\infty}I(t)=0$ a.s. and so the infected prey population dies out exponentially with probability one. This completes the proof.

Remark 3.1. *In Theorem 3.1, we only show the susceptible prey population survives and the infected prey population dies out. We don't consider the case $r-\mu<\frac{\sigma_1^2}{2}$. In fact, when parameters satisfy $r-\mu<\frac{\sigma_1^2}{2}$, then from comparison theorem for one-dimensional stochastic differential equation and*

$$dS \leq S[(r-\mu)-aS]dt+\sigma_1 SdB_1(t),$$

we can see that $\lim_{t\to\infty} S(t)=0$ a.s., applying the theory of asymptotically autonomous systems to the second and third equations in system (1.2), we also obtain $\lim_{t\to\infty} I(t)=0$ and $\lim_{t\to\infty} y(t)=0$ a.s. In other words, when $r-\mu<\frac{\sigma_1^2}{2}$, all the prey population and predator population will go to extinction a.s..

3.2 Extinction of the predator population

In this subsection, we shall investigate that under what conditions the predator population will go to extinction exponentially with probability one.

Theorem 3.2. *Let $(S(t),I(t),y(t))$ be a solution to system (1.2) with any initial value $(S(0),I(0),y(0))\in\mathbb{R}_+^{3,\circ}$. If $r-\mu>\frac{\sigma_1^2}{2}$, $K_2=b=0$ and $0<R_p^S<1$, then the predator population will go to extinction exponentially with probability one, i.e., $\lim_{t\to\infty} y(t)=0$ a.s.*

Proof. Consider $K_2=b=0$, then system (1.2) can be transformed as

$$\begin{cases} dS=\left[\frac{rS}{1+K_1y}-\mu S-aS^2-\beta SI\right]dt+\sigma_1 SdB_1(t),\\ dI=\left[\beta SI-\delta I-pIy\right]dt+\sigma_2 IdB_2(t),\\ dy=(pIy-my)dt+\sigma_3 ydB_3(t),\end{cases} \tag{3.5}$$

and it follows that

$$\begin{cases} dS\leq\left(rS-\mu S-aS^2-\beta SI\right)dt+\sigma_1 SdB_1(t),\\ dI\leq\left(\beta SI-\delta I\right)dt+\sigma_2 IdB_2(t).\end{cases}$$

We consider the auxiliary system

$$\begin{cases} d\tilde{S}=\left(r\tilde{S}-\mu\tilde{S}-a\tilde{S}^2-\beta\tilde{S}\tilde{I}\right)dt+\sigma_1\tilde{S}dB_1(t),\\ d\tilde{I}=\left(\beta\tilde{S}\tilde{I}-\delta\tilde{I}\right)dt+\sigma_2\tilde{I}dB_2(t),\end{cases}$$

By the comparison theorem, we obtain that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t I(s)ds\leq\frac{\beta(r-\mu-\frac{\sigma_1^2}{2})-a(\delta-\frac{\sigma_2^2}{2})}{\beta^2}\ \text{a.s..} \tag{3.6}$$

For third of Eq. (3.5), applying Itô's formula to $\ln y$ leads to that

$$d\ln y=(cpI-m-\frac{\sigma_3^2}{2})dt+\sigma_3 dB_3(t). \tag{3.7}$$

Integrating from 0 to t and then dividing by t on both sides of (3.7), we get

$$\frac{\ln y(t)-\ln y(0)}{t}\leq\frac{cp}{t}\int_0^t I(s)ds-(m+\frac{\sigma_3^2}{2})+\frac{\sigma_3 B_2(t)}{t}. \tag{3.8}$$

Taking the superior limit on both sides of (3.8) and combining with (3.6), together with $\lim_{t\to\infty}\frac{B_3(t)}{t}=0$ a.s., one can get

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln y(t)}{t}\leq&\frac{cp\big(\beta(r-\mu-\frac{\sigma_1^2}{2})-a(\delta-\frac{\sigma_2^2}{2})\big)}{\beta^2}-(m+\frac{\sigma_3^2}{2})\\ =&(R_p^S-1)(m+\frac{\sigma_3^2}{2})<0\ \text{a.s.,}\end{aligned}$$

which indicates that $\lim_{t\to\infty} y(t)=0$ a.s.

4 Density function analysis of system (1.2)

In this section, we give the local probability density function of the system (1.2).

First, let $x_1 = lnS, x_2 = lnI, x_3 = lny$, then applying Itô's formula, we can transform system (1.2) into the following form:

$$\begin{cases} dx_1 = [\frac{r}{1+K_1e^{x_3}} - ae^{x_1} - \frac{\beta e^{x_2}}{1+K_2e^{x_3}} - (\mu + \frac{\sigma_1^2}{2})]dt + \sigma_1 dB_1(t), \\ dx_2 = [\frac{\beta e^{x_1}}{1+K_2e^{x_3}} - (p + be^{x_3})e^{x_3} - (\delta + \frac{\sigma_2^2}{2})]dt + \sigma_2 dB_2(t), \\ dx_3 = [c(p + be^{x_3})e^{x_2} - (m + \frac{\sigma_3^2}{2})]dt + \sigma_3 dB_3(t). \end{cases} \tag{4.1}$$

Define $E_3 = (S_1^*, I_1^*, y_1^*) = (e^{x_1^*}, e^{x_2^*}, e^{x_3^*})$, which satisfy

$$\begin{cases} \frac{r}{1+K_1y_1^*} - aS_1^* - \frac{\beta I_1^*}{1+K_2y_1^*} - (\mu + \frac{\sigma_1^2}{2}) = 0, \\ \frac{\beta S_1^*}{1+K_2y_1^*} - (p + by_1^*)y_1^* - (\delta + \frac{\sigma_2^2}{2}) = 0, \\ c(p + by_1^*)I_1^* - (m + \frac{\sigma_3^2}{2}) = 0. \end{cases}$$

Let $(z_1, z_2, z_3) = (x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*)$, then the linearized system of system (4.1) is as follows:

$$\begin{cases} dz_1 = (-a_{11}z_1 - a_{12}z_2 - a_{13}z_3)dt + \sigma_1 dB_1(t), \\ dz_2 = (a_{21}z_1 - a_{23}z_3)dt + \sigma_2 dB_2(t), \\ dz_3 = (a_{32}z_2 + a_{33}z_3)dt + \sigma_3 dB_3(t), \end{cases} \tag{4.2}$$

where

$$\begin{aligned} a_{11} =& ae^{x_1^*}, a_{12} = \frac{\beta e^{x_2^*}}{1+K_2e^{x_3^*}}, a_{13} = \frac{rK_1e^{x_3^*}}{(1+K_1e^{x_3^*})^2} - \frac{\beta K_2 e^{x_2^*+x_3^*}}{(1+K_2e^{x_3^*})^2}, a_{21} = \frac{\beta e^{x_1^*}}{1+K_2e^{x_3^*}}, \\ a_{23} =& \frac{\beta K_2 e^{x_1^*+x_3^*}}{(1+K_2e^{x_3^*})^2} + (p + 2be^{x_3^*})e^{x_3^*}, a_{32} = c(p + be^{x_3^*})e^{x_2^*}, a_{33} = cbe^{x_2^*+x_3^*}. \end{aligned}$$

Lemma 4.1. *[24] For the algebra equation $\Lambda_1^2 + \widetilde{A}_1\overline{\Theta}_1 + \overline{\Theta}_1\widetilde{A}_1^T = 0$, where $\Lambda_1 = diag(1, 0, 0)$, $\overline{\Theta}_1$ is symmetric matrix, and the standard matrix*

$$\widetilde{A}_1 = \begin{pmatrix} -N_1 & -N_2 & -N_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

if $N_1 > 0, N_2 > 0, N_3 > 0$ and $N_1N_2 - N_3 > 0$, then the matrix $\overline{\Theta}_1$ is positive definite.

In the following, we give the explicit local density function near the quasi-stable equilibrium point E_3.

Theorem 4.1. *Let $Z = (z_1, z_2, z_3)^T$ be a solution to (4.2) with any initial value$(z_1(0), z_2(0), z_3(0)) \in \mathbb{R}_+^3$. If $R_p^S > 1$,$a_{11}a_{12} + a_{12}a_{33} \neq a_{13}a_{32}$, then there exists a unique density function $\Phi(Z)$ near the quasi-stable equilibrium point E_3, which can be expressed in the following form:*

$$\Phi(Z) = (2\pi)^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}}e^{-\frac{1}{2}(z_1,z_2,z_3)\Sigma^{-1}(z_1,z_2,z_3)^T},$$

where Σ is positive definite and it will be determined later.

Proof. System (4.2) can be rewritten into the matrix form $dZ = AZdt + \Theta dB(t)$, where $Z = (z_1, z_2, z_3)^T$, $\Theta = diag(\sigma_1, \sigma_2, \sigma_3), B(t) = (B_1(t), B_2(t), B_3(t))^T$ and $b_{ij}(i, j = 1, 2, 3)$ are defined by

$$A = \begin{pmatrix} -a_{11} & -a_{12} & a_{13} \\ a_{21} & 0 & -a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

The characteristic polynomial of matrix A is

$$\varphi_A(\lambda) = \lambda^3 + f_1\lambda^2 + f_2\lambda + f_3.$$

where

$$
\begin{aligned}
f_1 =& a_{11} - a_{33}, \\
f_2 =& a_{12}a_{21} - a_{11}a_{33} + a_{23}a_{32}, \\
f_3 =& a_{11}a_{23}a_{32} + a_{21}(a_{13}a_{32} - a_{12}a_{33}),
\end{aligned}
$$

By [25], the density function $\Phi(Z)$ of the quasi-stationary distribution of system (4.2) near the origin point $Z^* = (0,0,0)$ can be approached by the following three-dimensional Fokker-Planck equation

$$
-\sum_{i=1}^{3} \frac{\sigma_i^2}{2} \frac{\partial^2}{\partial z_i^2}\Phi + \frac{\partial}{\partial z_1}[(-a_{11}z_1 - a_{12}z_2 - a_{13}z_3)\Phi] + \frac{\partial}{\partial z_2}[(a_{21}z_1 - a_{23}z_3)\Phi] + \frac{\partial}{\partial z_3}[(a_{32}z_2 + a_{33}z_3)\Phi] = 0,
$$

which can be approximated by a Gaussian distribution $\Phi(Z) = c_0 e^{-\frac{1}{2}(Z-Z^*)Q(Z-Z^*)^T}$, c_0 is a constant and Q is a real symmetric matrix satisfies $Q\Theta^2 Q + A^T Q + QA = 0$.

If Q is positive definite and $Q^{-1} = \Sigma$, then $\Theta^2 + A\Sigma + \Sigma A^T = 0$. According to the finite independent superposition principle, then $\Theta_i^2 + A\Sigma_i + \Sigma_i A^T = 0, i = 1,2,3$, where

$$
\Theta_1 = diag(\sigma_1, 0, 0), \Theta_2 = diag(0, \sigma_2, 0), \Theta_3 = diag(0,0,\sigma_3),
$$

$$
\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3, \Theta^2 = \Theta_1^2 + \Theta_2^2 + \Theta_3^2.
$$

Case 1. For system (4.2), we consider $\Theta_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0$. Let

$$
J_1 = \begin{pmatrix} a_{21}a_{32} & a_{23}a_{32} & a_{33}^2 - a_{23}a_{32} \\ 0 & a_{32} & a_{33} \\ 0 & 0 & 1 \end{pmatrix},
$$

then we obtain that

$$
B = J_1 A J_1^{-1} = \begin{pmatrix} -f_1 & -f_2 & -f_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

it can be expressed as $\Lambda_1^2 + B\Sigma_0 + \Sigma_0 B^T = 0$, where $\Lambda_1 = diag(1,0,0)$,

$$
\Sigma_0 = \frac{1}{\rho_1^2} J_1 \Sigma_1 J_1^T = \begin{pmatrix} t_{11} & 0 & -t_{22} \\ 0 & t_{22} & 0 \\ -t_{22} & 0 & t_{33} \end{pmatrix}
$$

and $t_{22} = \frac{1}{f_1 f_2 - f_3}$ $t_{11} = f_2 t_{22}$, $t_{33} = \frac{f_1}{f_3} t_{22}$ with $\rho_1 = a_{21}a_{32}\sigma_1$.

Based on [8], we conclude that the positive equilibrium E^* of system (1.2) is local asymptotically stable when $R_p^s > 1$. Therefore, according to Routh-Hurwitz criterion [19], we obtain that $f_1 > 0, f_2 > 0, f_3 > 0, f_1 f_2 - f_3 > 0$. Applying Lemma 4.1, Σ_0 is a positive definite, thus $\Sigma_1 = \rho_1^2 J_1^{-1} \Sigma_0 (J_1^T)^{-1}$ is also positive definite.

Case 2. For system (4.2), we consider $\Theta_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0$. Let

$$
P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
$$

we obtain that

$$
A_1 = P_1 A P_1^{-1} = \begin{pmatrix} 0 & -a_{23} & a_{21} \\ a_{32} & a_{33} & 0 \\ -a_{12} & -a_{13} & -a_{11} \end{pmatrix}.
$$

Let

$$P_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & \frac{a_{12}}{a_{32}} & 1\end{pmatrix},$$

we also obtain that

$$A_2=P_2A_1P_2^{-1}=\begin{pmatrix}0 & -a_{23}-\frac{a_{12}a_{21}}{a_{32}} & a_{21}\\ a_{32} & a_{33} & 0\\ 0 & \omega & -a_{11}\end{pmatrix},$$

where $\omega=\frac{a_{11}a_{12}+a_{12}a_{33}-a_{13}a_{32}}{a_{32}}$.

Let

$$J_2=\begin{pmatrix}\omega a_{32} & \omega(a_{33}-a_{11}) & a_{11}^2\\ 0 & \omega & -a_{11}\\ 0 & 0 & 1\end{pmatrix},$$

then $J_2A_2J_2^{-1}=B$, it can be expressed as $\Lambda_1^2+B\Sigma_0+\Sigma_0B^T=0$, where $\Lambda_1=diag(1,0,0)$ and

$$\Sigma_0=\frac{1}{\rho_2^2}J_2P_2P_1\Sigma_2P_1^TP_2^TJ_2^T=\begin{pmatrix}t_{11} & 0 & t_{13}\\ 0 & t_{22} & 0\\ t_{13} & 0 & t_{33}\end{pmatrix}$$

with $\rho_2=\omega a_{32}\sigma_2$.

The next steps are similar to those in Case 1, we can obtain that

$$\Sigma_2=\rho_2^2P_1^{-1}P_2^{-1}J_2^{-1}\Sigma_0(J_2^T)^{-1}(P_2^T)^{-1}(P_1^T)^{-1}$$

is also a positive definite.

Case 3. For system (4.2), we consider $\Theta_3^2+A\Sigma_3+\Sigma_3A^T=0$. Let

$$P_3=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},$$

then

$$A_3=P_3AP_3^{-1}=\begin{pmatrix}a_{33} & 0 & a_{32}\\ -a_{13} & -a_{11} & -a_{12}\\ -a_{23} & a_{21} & 0\end{pmatrix}.$$

Case 3.1 If $a_{13}\neq 0$, let

$$P_4=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -\frac{a_{23}}{a_{13}} & 1\end{pmatrix},$$

then

$$A_4=P_4A_3P_4^{-1}=\begin{pmatrix}a_{33} & \frac{a_{23}a_{32}}{a_{13}} & a_{32}\\ -a_{13} & -a_{11}-\frac{a_{12}a_{23}}{a_{13}} & -a_{12}\\ 0 & \bar{\omega} & \frac{a_{12}a_{23}}{a_{13}}\end{pmatrix},$$

where $\bar{\omega}=a_{21}+\frac{a_{12}a_{23}^2}{a_{13}^2}+\frac{a_{11}a_{23}}{a_{13}}$.

Let

$$J_3=\begin{pmatrix}-\bar{\omega}a_{13} & -\bar{\omega}a_{11} & -a_{12}a_{21}-\frac{a_{11}a_{12}a_{23}}{a_{13}}\\ 0 & \bar{\omega} & \frac{a_{12}a_{23}}{a_{13}}\\ 0 & 0 & 1\end{pmatrix},$$

then $J_3 A_4 J_3^{-1} = B$, it can be expressed as $\Lambda_1^2 + B\Sigma_0 + \Sigma_0 B^T = 0$, where $\Lambda_1 = diag(1,0,0)$ and

$$\Sigma_0 = \frac{1}{\rho_3^2} J_3 P_4 P_3 \Sigma_3 P_3^T P_4^T J_3^T = \begin{pmatrix} t_{11} & 0 & t_{13} \\ 0 & t_{22} & 0 \\ t_{13} & 0 & t_{33} \end{pmatrix}$$

with $\rho_3 = -\bar{\omega} a_{13} \sigma_3$.

The next steps are similar to those in Case 1, we can obtain that

$$\Sigma_3 = \rho_3^2 P_3^{-1} P_4^{-1} J_3^{-1} \Sigma_0 (J_3^T)^{-1} (P_4^T)^{-1} (P_3^T)^{-1}$$

is also a positive definite.

Case 3.2 If $a_{13} = 0$, let

$$P_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then

$$A_4' = P_5 A_3 P_5^{-1} = \begin{pmatrix} a_{33} & a_{32} & 0 \\ -a_{23} & 0 & a_{21} \\ 0 & -a_{12} & -a_{11} \end{pmatrix}.$$

Let

$$J_4 = \begin{pmatrix} a_{12}a_{23} & a_{11}a_{12} & a_{11}^2 - a_{12}a_{21} \\ 0 & -a_{12} & -a_{11} \\ 0 & 0 & 1 \end{pmatrix},$$

then we obtain that

$$B_0 = J_4 A_4' J_4^{-1} = \begin{pmatrix} -h_1 & -h_2 & -h_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where

$$\begin{aligned} h_1 =& a_{11} - a_{33}, \\ h_2 =& a_{12}a_{21} - a_{11}a_{33} + a_{23}a_{32}, \\ h_3 =& a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33}, \end{aligned}$$

it can be expressed as $\Lambda_1^2 + B_0\Sigma_0 + \Sigma_0 B_0^T = 0$, where $\Lambda_1 = diag(1,0,0)$,

$$\bar{\Sigma}_0 = \frac{1}{\rho_3'^2} J_4 P_5 P_3 \Sigma_3 P_3^T P_5^T J_4^T = \begin{pmatrix} v_{11} & 0 & v_{13} \\ 0 & v_{22} & 0 \\ v_{13} & 0 & v_{33} \end{pmatrix}$$

and $v_{22} = \frac{1}{h_1 h_2 - h_3}$, $v_{13} = -h_{22}$, $v_{11} = h_2 v_{22}$, $v_{33} = \frac{h_1}{h_3} v_{22}$ with $\rho_3' = a_{12}a_{23}\sigma_3$. Similarly, we have that $h_1 > 0, h_2 > 0, h_3 > 0, h_1 h_2 - h_3 > 0$ according to Routh-Hurwitz criterion [19]. Applying Lemma 4.1, $\bar{\Sigma}_0$ is a positive definite, hence we can obtain that

$$\Sigma_3 = \rho_3'^2 P_3^{-1} P_5^{-1} J_4^{-1} \bar{\Sigma}_0 (J_4^T)^{-1} (P_5^T)^{-1} (P_3^T)^{-1}$$

is also a positive definite.

To sum up, we conclude that the symmetric matrix $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$ is positive definite. Thus, there is a local and asymptotic density function $\Phi(Z)$ near the quasi-endemic equilibrium point E_3.

5 Examples and computer simulations

Numerical simulation is performed in this part in order to illustrate above conclusions. We mainly pay attention to verify the following three results:

(1) Theorem 2.1 is satisfied when $R_p^S > 1$, $m > \frac{\sigma_3^2}{2}$ and $\min\{\delta, m\} > 2(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)$, then the system has a stationary distribution;

(2) Theorem 3.1 is satisfied when $r - \mu > \frac{\sigma_1^2}{2}$ and $R_p^S < 0$, then both the infected prey population and the predator population will die out.

(2)Theorem 3.2 is satisfied when $r - \mu > \frac{\sigma_1^2}{2}$, $K_2 = b = 0$ and $0 < R_p^S < 1$, then the predator population will die out.

Therefore, using the Milsteins high-order method [26], the numerical scheme for system (1.2) is given by:

$$\begin{cases} S_{k+1} = S_k + [\frac{rS_k}{1+K_1 y_k} - \mu S_k - aS_k^2 - \frac{\beta S_k I_k}{1+K_2 y_k}]\Delta t + \sigma_1 \zeta_{1,k}\sqrt{\Delta t} S_k + \frac{\sigma_1^2}{2}(\zeta_{1,k}^2 \Delta t - \Delta t) S_k, \\ I_{k+1} = I_k + [\frac{\beta S_k I_k}{1+K_2 y_k} - \delta I_k - (p + b y_k) I_k y_k]\Delta t + \sigma_2 \zeta_{2,k}\sqrt{\Delta t} I_k + \frac{\sigma_2^2}{2}(\zeta_{2,k}^2 \Delta t - \Delta t) I_k, \\ y_{k+1} = y_k + [c(p + b y_k) I_k y_k - m y_k]\Delta t + \sigma_3 \zeta_{3,k}\sqrt{\Delta t} y_k + \frac{\sigma_3^2}{2}(\zeta_{3,k}^2 \Delta t - \Delta t) y_k, \end{cases} \tag{5.1}$$

where the time interval is represented by $\Delta t > 0$, $\zeta_{1,k}, \zeta_{2,k}, \zeta_{3,k}$ are Gaussian random variables and follow the standard normal distribution.

Example 5.1. *(Stationary distribution) We select the parameters and initial values of system (1.2) as follows:* $c = 0.4, p = 0.3, r = 0.7, a = 0.1, \delta = 0.1, b = 0.01, \mu = 0.1, m = 0.1, \beta = 0.5, K_1 = 0.05, K_2 = 0.1;, \sigma_1 = \sigma_2 = \sigma_3 = 0.05, (S(0), I(0), y(0)) = (2, 0.8, 2)$.

After computing, we obtain that $R_p^S = 1.3713 > 1$, *and* $E_3^* = (S_1^*, I_1^*, y_1^*) = (2.0801, 0.7836, 2.3035)$. *Moreover, we know*

$$\Sigma = \begin{pmatrix} 0.0317 & -0.0186 & -0.0139 \\ -0.0186 & 0.0568 & 0.0133 \\ -0.0139 & 0.0133 & 0.0171 \end{pmatrix}.$$

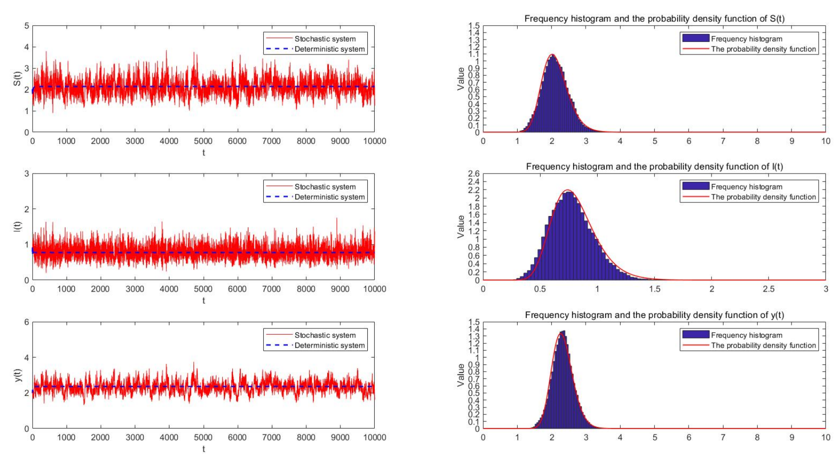

Figure 1: The figure on the left shows the solution of stochastic system and deterministic system when $R_p^S > 1$, the figure on the right shows its histograms and the probability density function of the solution.

According to Theorem 2.1 , the solution $(S(t), I(t), y(t))$ *of system (1.2) has a stationary distribution, which has the ergodic property. The left side of Figure 1 is the solution of stochastic system and deterministic system, the right side are histograms and the probability density function of the solution.*

Example 5.2. *(Extinction of the infected prey population) We select the parameters and initial values of system (1.2) as follows:* $c = 0.4, p = 0.3, r = 0.25, a = 0.35, \delta = 0.25, b = 0.01, \mu = 0.1, m = 0.1, \beta = 0.5, K_1 = 0.05, K_2 = 0.1, \sigma_1 = \sigma_2 = \sigma_3 = 0.05, (S(0), I(0), y(0)) = (2, 0.8, 2)$, *which satisfies* $R_p^S = -0.0643 < 0$. *Obviously, the infected prey population and the infected prey population of system (1.2) will extinct exponentially in a long term, which is supported by Figure 2.*

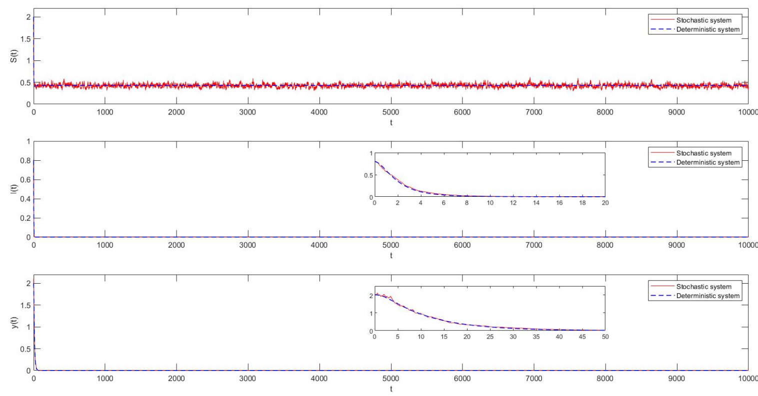

Figure 2: The figure shows the solution of stochastic system and deterministic system when $r - \mu > \frac{\sigma_1^2}{2}$ and $R_p^S < 0$.

Example 5.3. *(Extinction of the predator population) We select the parameters and initial values of system* (1.2) *as follows:* $c = 0.4, p = 0.3, r = 0.5, a = 0.35, \delta = 0.25, b = 0, \mu = 0.1, m = 0.1, \beta = 0.5, K_1 = 0.05, K_2 = 0, \sigma_1 = \sigma_2 = \sigma_3 = 0.05, (S(0), I(0), y(0)) = (2, 0.8, 2)$.

After computing, we obtain that $R_p^S = 0.5283$ *which satisfies* $R_p^S \in (0, 1)$*. Obviously, the infected prey population of system* (1.2) *will extinct exponentially in a long term, which is supported by Figure 3.*

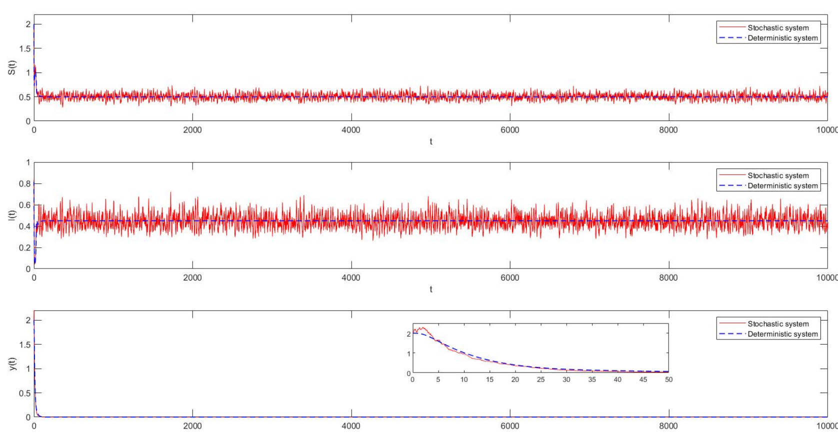

Figure 3: The figure shows the solution of stochastic system and deterministic system when $r - \mu > \frac{\sigma_1^2}{2}$ and $0 < R_p^S < 1$.

6 Conclusion

In the current paper, we have analyzed the stochastic dynamics of a stochastic eco-epidemiological model with disease in the prey population, which incorporates fear effect of predators on prey and hunting cooperation among predators. More precisely, on the one hand, we found out sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solutions to the stochastic system (1.2) by establishing a series of suitable Lyapunov functions. On the other hand, we obtained sufficient criteria for extinction of the infected prey population. In addition, it would be interesting to introduce the white noise into other intrinsic parameters of the associated system incorporating a fear function. Also, it needs to be mentioned that we only consider the influence of fear on the birth rate of prey population, but it can also affect the strength of intra-specific competition among the prey population. These problems are extremely meaningful and more works can be done in this direction in the near future.

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