A NEW SMOOTHING FUNCTION TECHNIQUE FOR SOLVING MINIMAX PROBLEMS

NURULLAH YILMAZ

Abstract. In this study, we consider non-smooth finite minimax problems. A new approach for solving minimax problems is developed, employing indicator functions and smoothing functions. First, the formulation of minimax problems is revised using indicator functions. Then, a new generation smoothing technique is used for the revised formulation. An algorithm is developed to solve the revised and smoothed problems numerically. The efficiency of the algorithm is demonstrated on several test problems, and a comparison is conducted between the numerical results achieved and those of alternative approaches. Finally, the portfolio planning problem is considered as a real-life application, and satisfactory results are obtained.

1. Introduction

We consider the following minimization problem

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1.1}$$

where

$$f(x) = \max_{j\in J} f_j(x) \tag{1.2}$$

and $f_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J = \{1, 2, \ldots, m\}$ are continuously differentiable. The different versions of the problem (1.1) been considered for many papers [25, 16, 3] and appear in many application areas such as engineering design [32], vehicle routing [4], resource-allocation [15], portfolio selection [29], the problem of multi-model regulatory networks under polyhedral uncertainty [30] and etc. [11, 14, 17, 28].

The problem (1.1) is difficult to solve since the objective function defined in (1.2) may be non-differentiable [12]. Many algorithms have been developed in order to solve the problem (1.1) such as sub-gradient based methods [20], bundle-methods [19], homotopy methods [46] and smoothing methods [33, 22, 47].

In particular, we concentrate on smoothing techniques for non-smooth functions. Smoothing techniques provide an opportunity to use the existing gradient-based methods in solving finite minimax problems [10]. The smoothing techniques have been considered for min-max type problems [23, 24] The idea of the smoothing approaches is based on approximating the original, non-smooth functions by using smooth functions [9, 21, 34]. The approximation is controlled by adjustable parameters. There are two important classes of smoothing techniques. The first

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technique is called local smoothing, which is based on smoothing out the original function in a suitable neighborhood of the kink points. The second technique, known as global smoothing, relies on building smooth functions that approximate the original function across the entire domain.

Developing a smoothing function for the mathematical function $f(x)$ described in equation (1.2) is a difficult task since it has many kink points. To address these challenges, alternative formulations have been suggested. Chronologically, we list some of them. In [7], the function $f(x)$ is restated as follows:

(1.3)
$$f(x) = f_1(x) + \max\{f_2(x) - f_1(x) + \max\{ \dots \max\{f_{m-1}(x) - f_{m-2}(x) + \max\{f_m(x) - f_{m-1}(x), 0\}, 0\} \dots, 0\}, 0\},$$

and for the first time, one of the global smoothing approaches is proposed for solving minimax problems. One of the first local smoothing techniques is proposed in [45] for solving minimax problems by considering the form (1.3). However, the above formula is useful, but coding it using computer programs is again complicated when m is large. Alternative penalty form with a smooth approximation is stated in [40] as

$$F(x, \varepsilon) = \beta \ln \sum_{j=1}^{m} \exp\left(\frac{f_j(x)}{\varepsilon}\right), \tag{1.4}$$

where $\varepsilon > 0$ is a smoothing parameter. The formula (1.4) is efficiently used with many gradient-based algorithms [44]. However, when ε is too small ($\varepsilon \to 0$), the numerical stabilization is uncontrolled because of an exponential term. Another interesting formulation of $f(x)$ is given

$$F(x, r) = r + \sum_{j=1}^{m} \max\{f_j(x) - r, 0\} \tag{1.5}$$

by adding a new variable t and the relation

$$f(x) = \min_{r \in \mathbb{R}} F(x, r)$$

is proved by [5, 6, 18]. Moreover, the hyperbolic smoothing technique proposed by [36, 39] is applied to solve minimax problems in [5, 6] by considering formula (1.5). In recent years, there has been considerable attention on smoothing methods, and new generation smoothing techniques are proposed and successfully applied for many non-smooth problems [35, 38, 42, 43]. However, minimax problems have not been studied with these new generation smoothing techniques. In this study, we consider the formula (1.5) and reformulate it in order to make it possible to apply the new generation smoothing techniques to solve problem (1.1). We modify the smoothing technique for minimax problems inspired by the paper [42] and introduce the useful properties of this smoothing technique. We propose a new algorithm to numerically solve the reformulated and smoothed problem. In order to show the efficiency of the algorithm, some numerical examples are considered.

The next section focuses on providing some preliminary knowledge about smoothing approaches. In Section 3, the formulation of the minimax problem is adapted for the new generation smoothing technique, and the convergence properties of the smoothing technique are investigated. In Section 4, we present the minimization algorithm in order to find an approximate solution for the problem (1.1). In Section

5, we apply the algorithms to the important test problems and a portfolio planning problem in order to evaluate the numerical performance of the proposed algorithm. The final section presents concluding remarks.

2. Preliminaries

Throughout the paper, $\|x\| = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$ is used to denote the Euclidean norm in $\mathbb{R}^n$. The $L^1[a,b]-$norm is defined as

$$\|f\|_{L^1[a,b]} = \int_a^b |f(t)|dt,$$

where f is an integrable function. Moreover, x_k^* denotes the $k-$th local minimizer of f and x^* denotes the global minimizer.

The sub-differential of the function f at the point x_0 is defined as $\partial f(x_0) = \text{conv}\{\nabla f_j(x_0) : j \in \{j \in \mathbb{N} : f_j(x_0) = f(x_0)\}\}$ where conv is a convex hull of a set. A point $x_0 \in \mathbb{R}^n$ is called a stationary point of f if $0 \in \partial f(x_0)$.

Definition 1. [10] *Let h be a continuous function defined on $\mathbb{R}^n$ to R. The function $\tilde{h} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $h(x)$, if $\tilde{h}(\cdot, \varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed β, and for any $x \in \mathbb{R}^n$,*

$$\lim_{y\to x, \varepsilon\to 0} \tilde{h}(y, \varepsilon) = h(x).$$

3. A New Formulation of Minimax Problems and Smoothing Approach

In this section we revise the formula of minimax problems give in (1.5) and we apply the smoothing technique to this new formulation.

By considering the technique in [41], let us re-define the function (1.5) as follows:

$$F(x) = r + \sum_{j=1}^{m} (f_j(x) - r)\chi_{A_j}(x), \tag{3.1}$$

where $\chi_{A_j}(x)$ function is the indicator function of the set A_j defined by

$$\chi_{A_j}(x) = \begin{cases} 0, & x \notin A_j, \\ 1, & x \in A_j, \end{cases}$$

where $A_j = \{x \in \mathbb{R}^n : f_j(x) - r \geq 0\}$ for $j = 1, 2, \ldots, m$. It is easy to see that the function $F(x)$ may have non-smooth structure. Indeed, the non-smoothness of $F(x)$ is originated from the existence of the $\chi_{A_j}(x)$ since $f_j(x)$ are continuously differentiable for $j = 1, \ldots, m$. The idea for eliminating this lack is that if the indicator functions $\chi_{B_{ji}}(x)$ is smoothed, then the function $F(x)$ becomes smooth. First, we define the smoothing function for indicator functions.

Definition 2. *Let h be a semi-continuous function (upper or lower) defined on $\mathbb{R}$ to R. The function $\tilde{g} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $g(t)$, if $\tilde{h}(\cdot, \varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed ε, and for any $t \in \mathbb{R}$,*

$$\lim_{z\to t, \beta\to 0} \tilde{g}(z, \varepsilon) = h(t).$$

For $t_j = f_j(x) - r$, we re-define the indicator functions as

$$\chi_{A_j}(t) = \begin{cases} 0, & t_j < 0, \\ 1, & t_j \geq 0. \end{cases}$$

In the following we give the smoothing function of indicator function as

$$\tilde{\chi}_{A_j}(t,\varepsilon)=\begin{cases}0, & t\le -\varepsilon,\\ Q(t,\varepsilon), & -\varepsilon\le t\le\varepsilon,\\ 1, & t\ge\varepsilon,\end{cases} \tag{3.2}$$

where $Q(t_j,\varepsilon)=\frac{3}{16\varepsilon^5}t_j^5-\frac{10}{16\varepsilon^3}t_j^3+\frac{15}{16\varepsilon}t_j+\frac{1}{2}$ and $\varepsilon>0$. The function $Q(t_j,\varepsilon)$ is called smooth transition function. It is designed in order to supply twice continuously differentiability between the pieces of the indicator function. Therefore, $\chi_{A_j}(t_j,\varepsilon)$ is second-order continuously differentiable. We have

$$\tilde{\chi}'_{A_j}(j,\varepsilon)=\begin{cases}0, & t_j\le -\varepsilon,\\ Q'(t_j,\varepsilon), & -\varepsilon\le t_j\le\varepsilon,\\ 0, & t_j\ge\varepsilon,\end{cases} \tag{3.3}$$

where $Q'(t_j,\varepsilon)=\frac{15}{16\varepsilon^5}t_j^4-\frac{30}{16\varepsilon^3}t_j^2+\frac{15}{16\varepsilon}$ and

$$\tilde{\chi}''_{A_j}(t_j,\varepsilon)=\begin{cases}0, & t_j\le -\varepsilon,\\ Q''(t_j,\varepsilon), & -\varepsilon\le t_j\le\varepsilon,\\ 0, & t_j\ge\varepsilon,\end{cases} \tag{3.4}$$

where $Q''(t_j,\varepsilon)=\frac{15}{4\varepsilon^5}t_j^3-\frac{15}{4\varepsilon^3}t_j$.

At the following lemmas, we investigate the relation between $\chi_{A_j}(t)$ and its smoothing function $\tilde{\chi}_{A_j}(t,\varepsilon)$.

Lemma 1. *Assume that $\chi_{A_j}(t_j)$ is an indicator function of the set $A_j\subset\mathbb{R}^n$ and $\tilde{\chi}_{A_j}(t_j,\varepsilon)$ is a smoothing function of $\chi_{A_j}(t_j)$. Then, we have*

$$|\tilde{\chi}_{A_j}(t_j,\varepsilon)-\chi_{A_j}(t_j)|\le\frac{1}{2},$$

for any $\varepsilon>0$.

Proof. Since we have $\tilde{\chi}_{A_j}(t_j,\varepsilon)=\chi_{A_j}(t_j)$ for $t_j\le-\varepsilon$ and $t_j\ge\varepsilon$, we discuss the cases $-\varepsilon\le t_j\le 0$ and $0\le t_j\le\varepsilon$. For $-\varepsilon\le t_j\le 0$, we obtain

$$\left|\tilde{\chi}_{A_j}(t_j,\varepsilon)-\chi_{A_j}(t_j)\right|=|Q(t_j,\varepsilon)|\le\frac{1}{2},$$

and for $0\le t_j\le\varepsilon$

$$\left|\tilde{\chi}_{A_j}(t_j,\varepsilon)-\chi_{A_j}(t_j)\right|=|Q(t_j,\varepsilon)-1|\le\frac{1}{2}.$$

Therefore, the proof is completed. □

Lemma 2. *Assume that $\chi_{A_j}(t_j)$ is an indicator function of the set $A_j\subset\mathbb{R}^n$ and $\tilde{\chi}_{A_j}(t_j,\varepsilon)$ is the smoothing function. Then, we have*

$$\|\tilde{\chi}_{A_j}(t_j,\varepsilon)-\chi_{A_j}(t_j)\|_{L^1(\mathbb{R})}\le\frac{\varepsilon}{2},$$

for any $\varepsilon>0$.

Proof. Since we have $\tilde{\chi}_{A_j}(t_j,\varepsilon)=\chi_{A_j}(t_j)$ for $t_j\le -\varepsilon$ and $t_j\ge\varepsilon$, we deal with the case $-\varepsilon\le t_j\le\varepsilon$. For $-\varepsilon\le t_j\le\varepsilon$,

$$\begin{aligned}\left\|\tilde{\chi}_{A_j}(t_j,\varepsilon)-\chi_A(t_j)\right\|_{L^1(\mathbb{R})} &= \int_{-\varepsilon}^{\varepsilon}\left|\tilde{\chi}_{A_j}(t_j,\varepsilon)-\chi_{A_j}(t_j)\right|dt\\ &= \int_{-\varepsilon}^{0}|Q(t_j,\varepsilon)|\,dt+\int_0^{\varepsilon}|Q(t_j,\varepsilon)-1|\,dt\\ &= \frac{5\varepsilon}{32}+\frac{5\varepsilon}{32}\\ &< \frac{\varepsilon}{2}.\end{aligned}$$

Therefore, the proof is completed. □

Based on the new formulation and smoothing technique we define the smoothing function of the objective function $F(x)$ as

$$\tilde{F}(x,\varepsilon)=r+\sum_{j=1}^{m}t_j\chi_{A_j}(t_j,\varepsilon), \tag{3.5}$$

where $t_j=f_j(x)-r$ and the problem given in (1.1) is re-defined as

$$\min_{x\in\mathbb{R}^n}\tilde{F}\left(x,\varepsilon\right), \tag{3.6}$$

for $\varepsilon>0$. First, we introduce the case $m=2$ and obtain the following results.

Theorem 1. *Let $x\in\mathbb{R}^n$, $\varepsilon>0$*

$$|F(x)-\tilde{F}(x,\varepsilon)|\le\varepsilon.$$

Proof. Since $\tilde{\chi}_{A_1}(t_1,\varepsilon)=\chi_{A_1}(t_1)$ for $t_1\le-\varepsilon$ and $t_1\ge\varepsilon$ and $\tilde{\chi}_{A_2}(t_2,\varepsilon)=\chi_{A_2}(t_2)$ for $t_2\le-\varepsilon$ and $t_2\ge\varepsilon$, we concern with the case $-\varepsilon\le t_1,t_2\le\varepsilon$ for $\varepsilon>0$. Let us consider the case $t_1\in[-\varepsilon,\varepsilon]$ and $t_2\notin[-\varepsilon,\varepsilon]$

$$\begin{aligned}|F(x)-\tilde{F}(x,\varepsilon)| &= |r+t_1\chi_{A_1}(t_1)+t_2\chi_{A_2}(t_2)-(r+t_1\tilde{\chi}_{A_1}(t_1,\varepsilon)+t_2\tilde{\chi}_{A_2}(t_2,\varepsilon))|\\ &= |t_1\chi_{A_1}(t_1)-t_1\tilde{\chi}_{A_1}(t_1,\varepsilon)|\\ &\le \frac{\varepsilon}{2}.\end{aligned}$$

Similar result is obtained for the case $t_1\notin[-\varepsilon,\varepsilon]$ and $t_2\in[-\varepsilon,\varepsilon]$. Now, we consider the case $-\varepsilon\le t_1,t_2\le\varepsilon$. By considering the Lemma 1, we obtain

$$\begin{aligned}|F(x)-\tilde{F}(x,\varepsilon)| &= |r+t_1\chi_{A_1}(t_1)+t_2\chi_{A_2}(t_2)-(r+t_1\tilde{\chi}_{A_1}(t_1,\varepsilon)+t_2\tilde{\chi}_{A_2}(t_2,\varepsilon))|\\ &= |t_1\left(\chi_{A_1}(t_1)-\tilde{\chi}_{A_1}(t_1,\varepsilon)\right)+t_2\left(\chi_{A_2}(t_2)-t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)|\\ &\le |t_1||\chi_{A_1}(t_1)-\tilde{\chi}_{A_1}(t_1,\varepsilon)|+|t_2||\chi_{A_2}(t_2)-\tilde{\chi}_{A_2}(t_2,\varepsilon)|\\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2. *Let $\varepsilon>0$ and $x\in\mathbb{R}^n$*

$$\|\tilde{F}(x,\varepsilon)-F(x)\|_{L^1}\le 2\varepsilon^2.$$

Proof. We start the proof similar to the Lemma 1. Since $\tilde{\chi}_{A_1}(t_1,\varepsilon) = \chi_{A_1}(t_1)$ for $t_1 \leq -\varepsilon$ and $t_1 \geq \varepsilon$ and $\tilde{\chi}_{A_2}(t_2,\varepsilon) = \chi_{A_2}(t_2)$ for $t_2 \leq -\varepsilon$ and $t_2 \geq \varepsilon$, we concern with the case $-\varepsilon \leq t_1, t_2 \leq \varepsilon$ for $\varepsilon > 0$. Let us consider the case $t_1 \in [-\varepsilon, \varepsilon]$ and $t_2 \notin [-\varepsilon, \varepsilon]$

$$
\begin{aligned}
\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} &= \int_{-\varepsilon}^{\varepsilon} |r + t_1\chi_{A_1}(t_1) + t_2\chi_{A_2}(t_2) - \\
&\quad (r + t_1\tilde{\chi}_{A_1}(t_1,\varepsilon) + t_2\tilde{\chi}_{A_2}(t_2,\varepsilon))|\,dt \\
&= \int_{-\varepsilon}^{\varepsilon} |t_1\chi_{A_1}(t_1) - t_1\tilde{\chi}_{A_1}(t_1,\varepsilon)|\,dt \\
&= \int_{-\varepsilon}^{\varepsilon} |t_1|\,|\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)|\,dt.
\end{aligned}
$$

Since $|t_1| \leq \varepsilon$ and from Lemma 2, we have

$$\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \leq \varepsilon^2.$$

Similar result is obtained for the case $t_1 \notin [-\varepsilon, \varepsilon]$ and $t_2 \in [-\varepsilon, \varepsilon]$. Now, we consider the case $-\varepsilon \leq t_1, t_2 \leq \varepsilon$. By considering the Lemma 2, we obtain

$$
\begin{aligned}
\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} &= \int_{-\varepsilon}^{\varepsilon} |r + t_1\chi_{A_1}(t_1) + t_2\chi_{A_2}(t_2) - \\
&\quad (r + t_1\tilde{\chi}_{A_1}(t_1,\varepsilon) + t_2\tilde{\chi}_{A_2}(t_2,\varepsilon))|\,dt \\
&= \int_{-\varepsilon}^{\varepsilon} |t_1\left(\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)\right) + t_2\left(\chi_{A_2}(t_2) - t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)|dt \\
&\leq \int_{-\varepsilon}^{\varepsilon} |t_1||\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)|dt + \int_{-\varepsilon}^{\varepsilon} |t_2||\chi_{A_2}(t_2) - \tilde{\chi}_{A_2}(t_2,\varepsilon)|dt \\
&\leq \varepsilon^2 + \varepsilon^2 = 2\varepsilon^2.
\end{aligned}
$$

Thus, the proof is completed. □

The Theorems 1 and 2 are verify theoretically that the proposed approach is a smoothing approach. In order to visualize the smoothing process we give the following example:

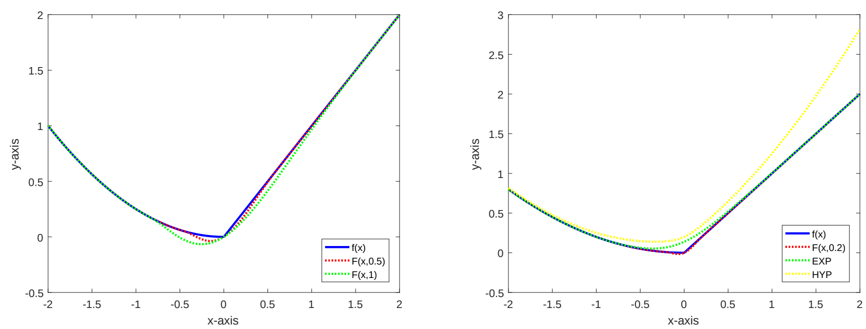
Example 1. *Let the function f is defined as*

$$f(x) = \max\{f_1(x), f_2(x)\},$$

where $f_1(x) = \frac{1}{5}x^2$ and $f_2(x) = x$. It can be observed from the definition of the function f is continuous but non-differentiable and $\partial f(0) = [0,1]$. According to the concept of the sub-differential, the point $x_0 = 0$ is the stationary point. The graph of the function f can be imagined by considering the max *function of f_1 and f_2 at Fig. 1 (blue and solid). By applying the above smoothing technique the smoothing function $\tilde{F}(x,\varepsilon)$ of f is obtained as*

$$\tilde{F}(x,\varepsilon) = r + (f_1(x) - r)\tilde{\chi}_{A_1}(t_1,\varepsilon) + (f_2(x) - r)\tilde{\chi}_{A_2}(t_2,\varepsilon),$$

where $A_1 = \{x \in \mathbb{R} : f_1(x) - r \geq 0\}$, $A_2 = \{x \in \mathbb{R} : f_2(x) - r \geq 0\}$ for $x \in \mathbb{R}$. By choosing $r = 0$, the graph of the function $F(x,\varepsilon)$ is illustrated in Fig. 1 (a) (red and dotted). In fact, we obtain an outer approximation to original function by the help of the above smoothing approach. We can deduce that for any function $f(x) = \max\{f_1(x), f_2(x)\}$, the inequality $f(x) = F(x) \geq \tilde{F}(x,\varepsilon)$ holds.



(a) (b)

FIGURE 1. (a) The blue graph is the graph of $f(x)$, the red and dotted one is the graph of $\tilde{F}(x, 0.5)$ and the green and dotted one is the graph of $\tilde{F}(x, 1)$, and (b) The blue graph is the graph of $f(x)$, the red one is the graph of $\tilde{F}(x, 0.2)$, the green one is the graph of exponential smoothing with $\varepsilon = 0.2$ and the yellow one is the graph of hyperbolic smoothing function with $\varepsilon = 0.2$.

According to Fig. 1 (a), choosing smaller ε values produces better approximations to the original function. In Fig. 1 (b), we illustrate the graph of smoothing functions that we mentioned in the introduction in a single framework in order to compare them visually. When the same value of $\varepsilon = 0.2$ is chosen for all smoothing approaches, the best approximation is achieved by our smoothing approach.

Let us continue giving the results about the degree of approximation of the the smoothing approach. Now, we present the convergence results for any finite value of m.

Theorem 3. *Let $x \in \mathbb{R}^n$, $\varepsilon > 0$*

$$\left| F(x) - \tilde{F}(x, \varepsilon) \right| \leq \frac{m}{2} \varepsilon.$$

Proof. For any $x \in \mathbb{R}^n$, we have

$$\left| F(x) - \tilde{F}(x, \varepsilon) \right| = \left| r + \sum_{j=1}^{m} t_j \chi_{A_j}(t_j) - \left(r + \sum_{j=1}^{m} t_j \tilde{\chi}_{A_j}(t_j, \varepsilon) \right) \right|.$$

By considering the similar way of the proof of Theorem 1, we obtain

$$\begin{aligned} \left| F(x) - \tilde{F}(x, \varepsilon) \right| &\leq \sum_{j=1}^{m} |t_j| \left| \chi_{A_j}(t_j) - \tilde{\chi}_{A_j}(t_j, \varepsilon) \right| \\ &\leq \frac{m\varepsilon}{2}. \end{aligned}$$

Thus, the proof is completed. □

Theorem 4. *Let $x \in \mathbb{R}^n$, $\varepsilon > 0$*

$$\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \leq m\varepsilon^2.$$

Proof. The proof is obtained by following similar ways as Theorems 2 and 3. □

Theorem 5. *Suppose that the point x^* is an optimal solution for the problem* (1.1) *and $\overline{x}$ is an optimal solution for the problem* (3.5)*. Then,*

$$|F(x^*) - \tilde{F}(\overline{x},\varepsilon)| \leq \varepsilon.$$

Proof. Since $F(\bar{x}) \geq F(x^*) \geq \tilde{F}(\bar{x},\varepsilon)$ we have

$$|F(x^*) - \tilde{F}(\bar{x},\varepsilon)| \leq |F(\bar{x}) - \tilde{F}(\bar{x},\varepsilon)|.$$

By the help of Theorem 1 and 3, we obtain

$$|F(\bar{x}) - \tilde{F}(\bar{x},\varepsilon)| \leq \varepsilon.$$

It completes the proof. □

Theorem 6. *Let $\{\varepsilon_j\} \to 0$ and x^k be a solution of* (3.5)*. Assume that $\overline{x}$ is an accumulation point of $\left\{x^k\right\}$. Then $\overline{x}$ is an optimal solution for* (1.1)*.*

Proof. By considering the Theorem 5, the proof is obtained. □

4. Algorithm and Minimization Procedure

In this section the new algorithm inspiring from [6] is given to solve minimax problem defined in (1.1). We propose to use smoothed version of the problem (3.6) instead of the problem given in (1.1).

Algorithm I

- Step 1 Choose a starting point x^0 and set $r_0 = f(x_0)$. Determine $\varepsilon_0 > 0$, $0 < q < 1$ and $\tau = 10^{-4}$ let $k = 0$ and go to Step 2.
- Step 2 Consider x^k as an initial point to solve the problem (3.5) by using smooth optimization solver. Let x^{k+1} be the solution.
- Step 3 If $\|\nabla\tilde{F}(x^k,\varepsilon_k)\| \leq \tau$ then stop and x^{k+1} is the optimal solution otherwise; determine $\varepsilon_{k+1} = q\varepsilon_k$, $r_{k+1} = f(x^{k+1})$ and $k = k+1$, then go to Step 2.

Wee need the following assumption for convergence of the Algorithm I.

Assumption 1. *For a point x^0 consider the level set $\mathcal{L}(x^0) = \left\{x \in \mathbb{R}^n : f(x) \leq f(x^0)\right\}$ is bounded.*

The convergence of Algorithm I is stated by the following theorem:

Theorem 7. *Let Assumption 1 hold. Suppose the set*

$$\underset{x\in\mathbb{R}^n}{argmin}\ \tilde{F}(x,\varepsilon) \neq \emptyset,$$

for $\varepsilon \in (0,\varepsilon_0]$. Let x^k be generated by Algorithm I. If $\{x^k\}$ has an accumulation point, then the accumulation point of $\{x^k\}$ is the solution for (1.1)*.*

Proof. Let us define the set $\mathcal{L}(x^0) = \{x \in \mathbb{R}^n : f(x) \leq f(x^0)\}$ for starting point x_0. Since $\mathcal{L}(x^0)$ is bounded, the sequence $\{x^k\}$ has at least one accumulation point. Let $\overline{x}$ be an accumulation point of $\{x^k\}$. We first show that $\overline{x} \in \mathcal{L}(x_0)$. Since

$$\tilde{F}(x_0, \varepsilon) \geq \tilde{F}(x^k, \varepsilon),$$

and according to Theorem 4, we have $f(x_0) \geq f(x^k)$ and $x^k \in \mathcal{L}(x_0)$. Since $\mathcal{L}(x_0)$ is bounded we obtain $\overline{x} \in \mathcal{L}(x_0)$. By the Theorem 6, $\overline{x}$ is the solution for (1.1). □

5. Numerical Examples

This section is devoted to present the numerical results of *Algorithm I* with the smoothing approach on finite minimax problems. Moreover, the obtained results are compared with *Algorithm I* with exponential smoothing used in [26] and *Algorithm I* with hyperbolic smoothing used in [6, 39]. We consider the BFGS method as a local search for *Algorithm I*. We apply the *Algorithm I* by using MATLAB on PC with configuration of Intel Core i3, 8GB RAM. At this algorithm, the parameters are selected as $\varepsilon = 10^{-1}$ and $q = 10^{-1}$. It is accepted that the problem is solved, if the accuracy 10^{-4} with respect to function value is obtained.

5.1. **Test problems.**

We first consider the well known test problems given in [27, 13] and the obtained results are compared with competing algorithm declared at above. The explicit formula of test problems are presented as follows:

Problem 1. [27] $\min f(x) = \max_{1\leq j\leq 2} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *and*

$$\begin{aligned} f_1(x) &= x_1^2 + (x_2-1)^2 + x_2 - 1, \\ f_2(x) &= -x_1^2 - (x_2-1)^2 + x_2 + 1, \end{aligned}$$

the global minimum value of the objective function f *is* $f^* = 0$.

Problem 2. [27] $\min f(x) = \max_{1\leq j\leq 3} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *and*

$$\begin{aligned} f_1(x) &= x_1^2 + x_2^4, \\ f_2(x) &= (2-x_1)^2 + (2-x_2)^2, \\ f_3(x) &= 2\exp(x_2 - x_1), \end{aligned}$$

the global minimum value of the objective function f *is* $f^* = 1.9522245$.

Problem 3. [27] $\min f(x) = \max_{1\leq j\leq 3} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *and*

$$\begin{aligned} f_1(x) &= 5x_1 + x_2, \\ f_2(x) &= -5x_1 + x_2, \\ f_3(x) &= x_1^2 + x_2^2 + 4x_2, \end{aligned}$$

the global minimum value of the objective function f *is* $f^* = -3$.

Problem 4. [27] $\min f(x) = \max_{1\leq j\leq 6} f_j(x)$ *where* $f : \mathbb{R}^3 \to \mathbb{R}$ *and*

$$\begin{aligned} f_1(x) &= x_1^2 + x_2^2 + x_3^2 - 1, \\ f_2(x) &= x_1^2 + x_2^2 + (x_3 - 2)^2, \\ f_3(x) &= x_1 + x_2 + x_3 - 1, \\ f_4(x) &= x_1 + x_2 - x_3 + 1, \\ f_5(x) &= 2x_1^3 + 6x_2^2 + 2(5x_3 - x_1 + 1)^2, \\ f_6(x) &= x_1^2 - 9x_3, \end{aligned}$$

the global minimum value of the objective function f *is* $f^* = 3.5997$.

Problem 5. [27] $\min f(x) = \max_{1\leq j\leq 4} f_j(x)$ *where* $f : \mathbb{R}^4 \to \mathbb{R}$ *and*

$$\begin{aligned} f_1(x) &= x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4, \\ f_2(x) &= f_1(x) + 10\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8\right), \\ f_3(x) &= f_1(x) + 10\left(x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10\right), \\ f_4(x) &= f_1(x) + 10\left(2x_1^2 + 2x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5\right), \end{aligned}$$

the global minimum value of the objective function f *is* $f^* = -44$.

Problem 6. [27] $\min f(x) = \max_{1\leq j\leq 5} f_j(x)$ *where* $f : \mathbb{R}^7 \to \mathbb{R}$ *and*

$$\begin{aligned} f_1(x) &= (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 \\ &\quad + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 + 8x_7, \\ f_2(x) &= f_1(x) + 10\left(2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 - 127\right), \\ f_3(x) &= f_1(x) + 10\left(7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 - 282\right), \\ f_4(x) &= f_1(x) + 10\left(23x_1 + x_2^2 + 6x_6^2 - 8x_7 - 196\right), \\ f_5(x) &= f_1(x) + 10\left(4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7\right), \end{aligned}$$

the global minimum value of the objective function f *is* $f^* = 680.63006$.

Problem 7. [13] $\min f(x) = \max_{1\leq j<m} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f_j(x) = x_1^2 + 2x_1t_j^2 + \exp(x_1 + x_2) - \exp(t_j),$$

and $t_j = \frac{j}{(q-1)}$, $j = 0, 1, \ldots, m-1$. *The global minimum value of objective function* f *is* $f^* = -1$.

Problem 8. [13] $\min f(x) = \max_{1\leq i,j<m} f_{i,j}(x)$ *where* $f : \mathbb{R}^4 \to \mathbb{R}$,

$$f_{i,j}(x) = \frac{(t_i - x_i)^2}{x_3^2} + \frac{(r_j - x_2)^2}{x_4^2} - 4,$$

and $t_i = \frac{i}{\sqrt{m}-1}$, $t_j = \frac{j}{\sqrt{m}-1}$, $i, j = 0, 1, \ldots, m-1$. *The global minimum value of the objective function* f *is* $f^* = -4$.

The numerical results are reported in Table 1. In the table, the number dimension "n", the number of functions "m" for each of the problems is presented. We illustrate the results on total iteration numbers "iter", total function evaluations "feval", function values "f.val" and the CPU time in seconds "Time" obtained by using Algorithm I with our formulation and smoothing approach (ISA) in the left

TABLE 1. The numerical results comparison with smoothing

			ISA				ESA				HSA			
Problem No.	n	m	*iter*	*feval*	*f.val*	Time	*iter*	*f.eval*	*f.val*	Time	*iter*	*f.eval*	*f.val*	Time
1	2	2	10	93	-0.0000	**0.0131**	65	338	0.0000	0.0947	38	210	0.0000	0.0656
2	2	3	2	21	1.9522	**0.0233**	57	262	1.9582	0.0550	61	252	1.9523	0.0644
3	2	3	17	252	-3.0000	**0.0382**	17	153	-2.9940	0.0601	18	144	-3	0.0423
4	3	6	33	256	3.5997	**0.0563**	99	578	3.6030	0.0685	97	1220	3.5998	0.2136
5	4	4	36	640	-44.0000	**0.0864**	115	705	-43.832	0.1069	114	945	-43.99	0.1270
6	7	5	98	1832	680.3800	**0.1230**	236	2174	678.9000	0.1941	106	2528	693.02	0.3226
7	2	5	47	156	-1.0000	**0.0263**	42	193	-0.9999	0.0354	52	186	-0.9999	0.0362
7	2	10	57	195	-1.0000	**0.0424**	11	99	-1.0000	0.0471	56	222	-0.9999	0.0632
7	2	50	51	186	-1.0000	0.1062	16	105	-1.0000	**0.0672**	79	321	-1	0.2010
7	2	100	43	144	-1.0000	**0.4833**	35	186	-0.9888	0.6020	64	246	-0.9999	0.2951
7	2	500	36	156	-1.0000	**0.8912**	45	201	-0.9876	0.9571	61	216	-0.9999	0.9096
7	2	1000	50	228	-1.0000	7.2473	61	267	-0.9982	**1.1384**	58	231	-0.9999	1.7948
8	4	5	22	115	-4.0000	**0.0342**	24	175	-3.9868	0.0594	33	210	-4	0.0483
8	4	10	70	120	-4.0000	**0.0101**	18	145	-3.9650	0.0495	16	110	-3.9999	0.0427
8	4	50	13	184	-4.0000	0.2096	6	105	-3.9679	**0.0629**	36	255	-4	0.1703
8	4	100	54	432	-4.0000	0.7779	26	195	-3.9731	0.6396	31	210	-3.9999	**0.2575**
8	4	500	69	647	-4.0000	**1.9314**	44	385	-3.9660	1.8309	28	200	-4	1.9911
8	4	1000	96	859	-4.0000	5.7579	21	155	-3.8949	4.7227	17	145	-4	**3.6680**

side of the Table 1. Additionally, we present the results at the same categories as our method for exponential smoothing (in [26]) with Algorithm I (ESA) and hyperbolic smoothing (in [6]) with Algorithm I (HSA) approaches at the rest of the Table 1. The same starting points are considered for both of the algorithms and these points are randomly chosen.

It can be seen from the Table 1 that ISA presents better results than ESA and HSA at the rate of 50% considering all test problems in terms of total number of iterations. In terms of total function evaluations, ISA presents better results than the ESA and HSA at the rate of 56% considering all test problems. Moreover, by using ISA and HSA (except Problem 6) the correct solutions are obtained for all test problems but by using ESA the solutions are not close to desired results except Problem 5. Moreover, if anyone compares ISA with the ESA and HSA in terms of CPU time, it is seen that ISA is faster than than ESA and HSA at the rate of 72% considering all test problems. On the other hand, it is not easy to use the ESA because of the exponential term. When the smoothing parameter $\varepsilon \to 0^+$ again the exponential function $\exp(\frac{f_j(x)}{\varepsilon})$ reaches huge values. Therefore the function "fminunc" gives error and can not continue. The HSA is easy to control and it is possible to obtain the results with desired precision but it is slower than ISA. It can be concluded that ISA is very easy to use and it has no drawbacks as the ESA.

5.2. **Application of *Algorithm I* in portfolio planning problem.** In this section, a minimax portfolio planning model is considered. The problem first defined by [31] and it is reformulated by Cai et al. in [8] and Teo et al. in [37]. The final of this problem given in [29]. The problem is mathematically formulated as follows:

$$\min f(x) = \frac{1}{12}\sum_{t=1}^{12} y_t \tag{5.1}$$

$$\begin{aligned} s.t.\quad & \\ & Ax \leq y, \\ & 0.0207x_1 + 0.0316x_2 + 0.0323x_3 + 0.0337x_4 + 0.0376x_5 \geq 0.03, \\ & \sum_{j=1}^{5} x_j = 1, \\ & 0 \leq x_j \leq 0.75 \quad j = 1, 2, \ldots, 5, \\ & y_i \geq 0 \quad i = 1, 2, \ldots, 12, \end{aligned}$$

where A is a 12×5 matrix given as

$$A = \begin{pmatrix} 0.0333 & 0.0004 & 0.0083 & 0.0043 & 0.0114 \\ 0.0243 & 0.0234 & 0.0203 & 0.0283 & 0.0294 \\ 0.0507 & 0.0676 & 0.0123 & 0.0707 & 0.0766 \\ 0.0387 & 0.0204 & 0.0087 & 0.0163 & 0.0134 \\ 0.0223 & 0.0154 & 0.0173 & 0.0313 & 0.0114 \\ 0.0263 & 0.0024 & 0.0003 & 0.0767 & 0.0006 \\ 0.0334 & 0.0314 & 0.0013 & 0.0283 & 0.0174 \\ 0.0153 & 0.0164 & 0.0113 & 0.0003 & 0.0126 \\ 0.0597 & 0.0066 & 0.0747 & 0.0013 & 0.0144 \\ 0.0637 & 0.0084 & 0.0113 & 0.0223 & 0.0176 \\ 0.0253 & 0.0044 & 0.0043 & 0.0233 & 0.0074 \\ 0.0313 & 0.0486 & 0.0037 & 0.0087 & 0.0024 \end{pmatrix} \tag{5.2}$$

$x = (x_1, x_2, \ldots, x_5)^T$ is decision variable $y = (y_1, y_2, \ldots, y_n)^T$. For more details of the problem, we refer to [29]. We consider the equivalent formulation of the problem (5.2), defined as

$$\min f(x) = \frac{1}{12}\sum_{t=1}^{12} \max_j A(t,j)x_j \tag{5.3}$$

$$\begin{aligned} s.t.\quad & \\ & 0.0207x_1 + 0.0316x_2 + 0.0323x_3 + 0.0337x_4 + 0.0376x_5 \geq 0.03, \\ & \sum_{j=1}^{5} x_j = 1, \\ & 0 \leq x_j \leq 0.75 \quad j = 1, 2, \ldots, 5, \\ & y_i \geq 0 \quad i = 1, 2, \ldots, 12, \end{aligned}$$

By considering *Algorithm I*, the numerical solution of the problem is obtained as $x^* = (0.0000, 0.0000, 0.7500, 0.0000, 0.2500)^T$ with the corresponding function value $f(x^*) = 0.015$ which verifies the solution given in [29].

6. Conclusion

In this study, new generation smoothing techniques are successfully applied to the finite minimax problems. The formulation of minimax problems is revised based on the indicator functions. The error estimates are presented, and the relations between the original and smoothed problems are investigated in detail. This reformulation and suggested smoothing technique not only simplify the formulation of minimax problems but also provide a smooth approximation for such non-smooth problems.

A new algorithm for solving reformulated and smoothed finite minimax problems is presented, and the efficiency of our algorithm on some numerical examples is illustrated. According to the comparison of the results with the other methods, it is shown that our approach is competitive with well-known prestigious approaches.

For future studies, indicator functions can be used to derive effective formulations of minimax problems. The concept of employing new generation smoothing techniques utilizing indicator functions can also be extended to address various non-smooth problems, including complementarity, exact penalty, l_1 signal reconstruction, and so on. On the other hand, the methodology proposed in this article can be considered to solve the minimax part of the problem of the optimization of desirability functions under model uncertainty [1, 2, 30].

Data Availability Statement

Data will be made available on request.

Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

Nurullah Ylmaz: methodology, visualization, software, writing.

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References

[1] Başak Akteke-Öztürk, Gerhard-Wilhelm Weber, and Gülser Köksal. Optimization of generalized desirability functions under model uncertainty. *Optimization*, 66(12):2157–2169, 2017.

[2] Başak Akteke-Öztürk, Gerhard-Wilhelm Weber, and Gülser Köksal. Generalized desirability functions: a structural and topological analysis of desirability functions. *Optimization*, 2020.

[3] Tadeusz Antczak. The minimax exact penalty fuzzy function method for solving convex nonsmooth optimization problems with fuzzy objective functions. *Journal of Industrial and Management Optimization*, 20(1):392–427, 2024.

[4] Esther M. Arkin, Refael Hassin, and Asaf Levin. Approximations for minimum and min-max vehicle routing problems. *Journal of Algorithms*, 59(1):1 – 18, 2006.

[5] A. M. Bagirov, N. Sultanova, A. Al Nuaimat, and S. Taheri. Solving minimax problems: Local smoothing versus global smoothing. In Mehiddin Al-Baali, Lucio Grandinetti, and Anton Purnama, editors, *Numerical Analysis and Optimization*, pages 23–43, Cham, 2018. Springer International Publishing.

[6] A.M. Bagirov, A. Al Nuaimat, and N. Sultanova. Hyperbolic smoothing function method for minimax problems. *Optimization*, 62(6):759–782, 2013.

[7] Dimitri P. Bertsekas. On penalty and multiplier methods for constrained minimization. *SIAM Journal on Control and Optimization*, 14(2):216–235, 1976.

[8] Xiaoqiang Cai, Kok-Lay Teo, Xiaoqi Yang, and Xun Yu Zhou. Portfolio optimization under a minimax rule. *Management Science*, 46(7):957–972, 2000.
[9] Chunhui Chen and O. L. Mangasarian. A class of smoothing functions for nonlinear and mixed complementarity problems. *Computational Optimization and Applications*, 5(2):97–138, Mar 1996.
[10] Xiaojun Chen. Smoothing methods for nonsmooth, nonconvex minimization. *Mathematical Programming*, 134(1):71–99, Aug 2012.
[11] Zeynep Cobandag Guloglu and Gerhard Wilhelm Weber. Risk modeling in optimization problems via value at risk, conditional value at risk, and its robustification. In Alberto A. Pinto and David Zilberman, editors, *Modeling, Dynamics, Optimization and Bioeconomics II*, pages 133–145, Cham, 2017. Springer International Publishing.
[12] V. F. Demyanov and V. N. Malozemov. *Introduction Minimax*. Wiley, New York, 1974.
[13] Li Dong and Bo Yu. A spline smoothing newton method for finite minimax problems. *Journal of Engineering Mathematics*, 93(1):145–158, Aug 2015.
[14] D. Z. Du and P. M. Pardalos. *Minimax and Applications*, volume 4. Kluwer Academic, Dordrecht, 1995.
[15] J. Du, L. Zhao, J. Feng, and X. Chu. Computation offloading and resource allocation in mixed fog/cloud computing systems with min-max fairness guarantee. *IEEE Transactions on Communications*, 66(4):1594–1608, 2018.
[16] Tanil Ergenc, Stefan Wolgang Pickl, Nicole Radde, and Gerhard Wilhelm Weber. Generalized semi-infinite optimization and anticipatory systems. *International Journal of Computing Anticipatory Systems*, 15:3–30, 2004.
[17] W. Fan, J. Liang, H. C. So, and G. Lu. Min-max metric for spectrally compatible waveform design via log-exponential smoothing. *IEEE Transactions on Signal Processing*, 68:1075–1090, 2020.
[18] Ye Feng, Liu Hongwei, Zhou Shuisheng, and Liu Sanyang. A smoothing trust-region newton-cg method for minimax problem. *Applied Mathematics and Computation*, 199(2):581 – 589, 2008.
[19] Antonio Fuduli, Manlio Gaudioso, Giovanni Giallombardo, and Giovanna Miglionico. A partially inexact bundle method for convex semi-infinite minmax problems. *Communications in Nonlinear Science and Numerical Simulation*, 21(1):172 – 180, 2015. Numerical Computations: Theory and Algorithms (NUMTA 2013), International Conference and Summer School.
[20] Manlio Gaudioso, Giovanni Giallombardo, and Giovanna Miglionico. An incremental method for solving convex finite min-max problems. *Mathematics of Operations Research*, 31(1):173–187, 2006.
[21] C. Grossman. *Smoothing techniques for exact penalty function methods*, pages 249–265. American Mathematical Society, Providence, Rhode Island, 2016.
[22] F. Guerra Vazquez, H. Günzel, and H.Th. Jongen. On logarithmic smoothing of the maximum function. *Annals of Operations Research*, 101(1):209–220, Jan 2001.
[23] Hubertus Theodorus Jongen and Oliver Stein. Smoothing by mollifiers. part i: semi-infinite optimization. *Journal of Global Optimization*, 41:319–334, 2008.
[24] Hubertus Theodorus Jongen and Oliver Stein. Smoothing by mollifiers. part ii: nonlinear optimization. *Journal of Global Optimization*, 41:335–350, 2008.
[25] Hubertus Theodorus Jongen, Frank Twilt, and Gerhard Wilhelm Weber. Semi-infinite optimization: structure and stability of the feasible set. *Journal of Optimization Theory and Applications*, 72:529–552, 1992.
[26] J.K. Liu and L. Zheng. A smoothing iterative method for the finite minimax problem. *Journal of Computational and Applied Mathematics*, 374:112741, 2020.
[27] L. Luksan and J. Vlcek. Test problems for non-smooth unconstrained and linearly constrained optimization. *Technical Reports No. 798, Institute of Computer Science, Academy of Sciences of the Czech Republic*, 2000.
[28] Sasan Mahmoudinazlou and Changhyun Kwon. A hybrid genetic algorithm for the minmax multiple traveling salesman problem. *Computers & Operations Research*, 162:106455, 2024.
[29] A. Nezami and M. Mortezaee. A new gradient-based neural dynamic framework for solving constrained min-max optimization problems with an application in portfolio selection model. *Applied Intelligence*, 49(2):369–419, 2019.

[30] Ayşe Özmen, Erik Kropat, and Gerhard-Wilhelm Weber. Robust optimization in spline regression models for multi-model regulatory networks under polyhedral uncertainty. *Optimization*, 66(12):2135–2155, 2017.

[31] Christos Papahristodoulou and Erik Dotzauer. Optimal portfolios using linear programming models. *Journal of the Operational research Society*, 55(11):1169–1177, 2004.

[32] E. Polak. On the mathematical foundations of nondifferentiable optimization in engineering design. *SIAM Review*, 29(1):21–89, 1987.

[33] E. Polak, J. O. Royset, and R. S. Womersley. Algorithms with adaptive smoothing for finite minimax problems. *Journal of Optimization Theory and Applications*, 119(3):459–484, Dec 2003.

[34] Ahmet Sahiner, Gulden Kapusuz, and Nurullah Yilmaz. A new smoothing approach to exact penalty functions for inequality constrained optimization problems. *Numerical Algebra, Control & Optimization*, 6(2):161–173, 2016.

[35] W Sánchez, CA Arias, and R Perez. A jacobian smoothing inexact newton method for solving the nonlinear complementary problem. *Computational and Applied Mathematics*, 43(5):279, 2024.

[36] Michael Souza, Adilson Elias Xavier, Carlile Lavor, and Nelson Maculan. Hyperbolic smoothing and penalty techniques applied to molecular structure determination. *Operations Research Letters*, 39(6):461 – 465, 2011.

[37] Kok Lay Teo and XQ Yang. Portfolio selection problem with minimax type risk function. *Annals of Operations Research*, 101:333–349, 2001.

[38] Hong Wang. A decentralized smoothing quadratic regularization algorithm for composite consensus optimization with non-lipschitz singularities. *Numerical Algorithms*, 96(1):369–396, 2024.

[39] Adilson Elias Xavier. The hyperbolic smoothing clustering method. *Pattern Recognition*, 43(3):731 – 737, 2010.

[40] Song Xu. Smoothing method for minimax problems. *Computational Optimization and Applications*, 20(3):267–279, Dec 2001.

[41] Nurullah Yilmaz and Ayegl Kayacan. A new smoothing algorithm to solve a system of nonlinear inequalities. *Fundamental Journal of Mathematics and Applications*, 6(3):137146, 2023.

[42] Nurullah Yilmaz and Ahmet Sahiner. New smoothing approximations to piecewise smooth functions and applications. *Numerical Functional Analysis and Optimization*, 40(5):513–534, 2019.

[43] Nurullah Yilmaz and Ahmet Sahiner. Smoothing techniques in solving non-lipschitz absolute value equations. *International Journal of Computer Mathematics*, 100(4):867–879, 2023.

[44] Hongxia Yin. An adaptive smoothing method for continuous minimax problems. *Asia-Pacific Journal of Operational Research*, 32(01):1540001, 2015.

[45] Israel Zang. A smoothing-out technique for min—max optimization. *Mathematical Programming*, 19(1):61–77, Dec 1980.

[46] Zhengyong Zhou and Yarui Duan. An aggregate homotopy method for solving unconstrained minimax problems. *Optimization*, 70(8):1791–1808, 2021.

[47] Zhengyong Zhou and Qi Yang. An active set smoothing method for solving unconstrained minimax problems. *Mathematical Problems in Engineering*, 2020:9108150, Jun 2020.

(N. Yilmaz) Department of Mathematics, Suleyman Demirel University, Isparta, Turkey
E-mail address, N. Yilmaz: nurullahyilmaz@sdu.edu.tr
URL: https://orcid.org/0000-0001-6429-7518