# Conformable Exponential Dichotomy and Roughness of Conformable Fractional Differential Equations

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**Abstract**

The solutions of traditional fractional differential equations neither satisfy group property nor generate dynamical systems, so hyperbolicity of these equations is difficult to study. Benefitting from the new proposed conformable fractional derivative, we investigate dichotomy of conformable fractional equations, including conformable exponential dichotomy and stability, roughness and nonuniform dichotomy.

**Keywords:** Conformable fractional differential equations; Conformable exponential dichotomy; Roughness; Nonuniform dichotomy; Stability

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# 1 Introduction

The well-known dichotomy concept on various of hyperbolic systems, e.g. ODEs

$$\dot{x} = A(t)x, \quad t \in J, \tag{1.1}$$

where interval $J \subset \mathbb{R}$, is said that there exist a projection matrix $P$ and a fundamental matrix $X(t)$ of (1.1), and positive constants $K_i$ and $\beta_i$ $(i = 1, 2)$ such that for all $t, s \in J$,

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq K_1 e^{-\beta_1(t-s)}, \quad t \geq s, \\ \|X(t)(I-P)X^{-1}(s)\| &\leq K_2 e^{-\beta_2(s-t)}, \quad s \geq t. \end{aligned} \tag{1.2}$$

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Correspondingly, the roughness of dichotomy is regarded as the persistence of dichotomy undergoing small linear perturbation, i.e., the perturbed system

$$\dot{x} = (A(t) + B(t))x, \quad t \in J$$

still admits dichotomy behaviour in the form of (1.2) along with small variations of $P$, $X(t)$, $K_i$ and $\beta_i$ ($i = 1, 2$). According to the difference of asymptotic rate, there are diverse dichotomies, e.g., the classical exponential dichotomy (1.2) ([16]), $(h, k)$-dichotomy ([30]), polynomial dichotomy ([8]), etc. The dichotomies and their corresponding properties are core issues in the field of dynamical systems, which can be traced back to the papers of Perron ([33]) and Li ([26]) on conditional stability of linear differential and difference equations respectively. And they are gradually formalized, developed and summarized in literatures [28, 29, 16]. Recently decades, many research results were devoted to exploring the existence criteria of exponential dichotomy (see Hale ([22]), Chow and Leiva ([13]), Sasu ([38]), Barreira and Valls ([7]), Battelli and Palmer ([10]) and the references therein). The roughness referred before are also widely focused on, and firstly demonstrated by Massera and Schäffer ([28]) under hypothesis that the original matrix $A(t)$ is bounded. Schäffer ([39]) subsequently eliminated the assumption of boundedness. Coppel ([15]) gave a general elementary proof of roughness if matrix $A(t)$ commutes with the projection $P$. In 1978 Coppel ([16, pp.28-33]) exhibited a simpler proof via the so-called *projected integral inequalities* raised by Hale ([22, pp.110-111]). Later, Naulin and Pinto ([31]) improved the size of perturbation $B(t)$ in Coppel's [16, pp.34-35] without boundedness of $A(t)$ yet. Popescu ([36]) further generalized the results of [16] and [31] to infinite dimensional Banach spaces. Thereafter, the notion of nonuniform exponential dichotomy, roughly speaking dichotomy formula (1.2) involving extra nonuniform constants in exponents, was proposed by Barreira and Valls ([7]), where its roughness was also studied. In 2013 Zhou, Lu and Zhang ([48]) studied the roughness of tempered exponential dichotomy for random difference equations in Banach spaces lack of the so-called *Multiplicative Ergodic Theorem*. Moreover, plenty of works on the roughness of exponential dichotomy could be found in [12, 13, 16, 31, 36, 7] for continuous dynamical systems and in [34, 38, 48, 49] for discrete dynamical systems and references therein. In addition, the corresponding admissibility problem of dichotomy, i.e. admissible functions pair of solutions $x$ and inhomogeneous perturbations $f$, was investigated extensively in [29, 38, 49, 6, 19] and so on.

Although the research on dichotomy involved ODEs ([28, 29, 22, 16, 30, 31, 12, 36, 7, 8, 10, 19]), difference equations ([26, 38, 49, 48]), functional differential equations ([32]), random systems ([2, 20, 14, 27]), skew-product semiflows ([13, 34]), etc., till now there is no result of dichotomy for fractional differential equations (FDEs for short). Fractional derivative started from a letter from L'Hospital to Leibniz about discussing the meaning of a half derivative. From then on, because of better approximation to practical model associated with memory and hereditary phenomena than ODEs and PDEs, FDEs are steadily developed in the aspects of

Physics and Chemistry ([47, 44]), Biology and Medicine ([18, 41]), Engineering and Control Theory ([3, 35]) and Economics and Psychology ([11, 40]), especially in recent decades (see monographs [35, 25, 21, 50, 23, 17]). Traditional definitions of fractional derivative and integral, such as Riemann-Liouville's, Caputo's and Grünwald-Letnikov's ([35, 25]), have no product rule and chain rule of derivative, such that the solutions of FDEs neither fulfil group property nor generate dynamical systems. There is a vast of works on well-posedness ([50]), stability ([25]), Laplace transform method and optimal control ([35]), variational method, attractors and numerical solutions ([21]) and chaos ([5]) of FDEs, but the study on hyperbolicity of FDEs is temporarily in blank state. Until 2014, Khalil et al. ([24]) introduced a new definition of fractional derivative, that is so-called *conformable fractional derivative*, which can almost satisfy all corresponding characteristics analogous to integer derivative. Thus, the solutions of CFDEs (abbreviation of conformable fractional differential equations) also can generate dynamical systems, which makes it possible to consider the hyperbolic behaviors of FDEs. Later, Abdeljawad ([1]) accomplished the definition of left and right conformable fractional derivatives and the variation of constants formula of CFDEs and solved CFDEs via Laplace transform. In 2017 Souahi et al. ([42]) employed Lyapunov direct method to present the stability, asymptotic stability and exponential stability of CFDEs. In 2019 Khan et al. ([43]) further verified the generalized definition and its semigroup and linear properties of conformable derivative and existence and uniqueness of solutions for CFDEs. During the same year, Balci et al. ([4]) displayed the Neimark-Sacker bifurcation and chaotic behavior for a tumor-immune system modelled by a CFDE. In 2020 Xie et al. ([46]) showed an exact solution and difference scheme for a gray model with conformable derivative. Recently, Wu et al. ([45]) revealed the Hyers-Ulam stability of a conformable fractional model.

In this paper we attempt to establish the theory of dichotomy for CFDEs. In order to generalize the hyperbolicity of ODEs to CFDEs, we first modify the definitions of conformable fractional derivative and integral and a conformable exponential function originated from [24, 1]. Subsequently, some preliminaries, e.g. the well-posedness of solutions, operator semigroups and variation of constants formula, are achieved in section 2. In section 3, we provide the definitions of so-called *conformable exponential stability and dichotomy* with respect to CFDEs, whose asymptotic rate is the conformable exponential function. These stability and dichotomy include the classical exponential stability and dichotomy ([16]) in ODEs with integer derivative as special cases. Meanwhile, we develop the conformable fractional integral versions of projected inequalities to prove the existence of conformable exponential dichotomy and corresponding invariant manifolds. In section 4, we discuss the roughness of conformable exponential dichotomy in $\mathbb{R}_+$. In section 5, we additionally study nonuniform conformable exponential stability and dichotomy and their roughness in $\mathbb{R}_+$. Our results extend the works of Hale ([22]), Coppel ([16]), Barreira and Valls ([7]) to CFDEs.

# 2 Linear CFDEs

Throughout this paper, we define the following functions sets:

$$
\begin{aligned}
I(\mathbb{R},\mathbb{R}) &:= \{\varphi : \mathbb{R} \to \mathbb{R} \,|\, \varphi \text{ is a nondecreasing function}\},\\
C(\mathbb{R},\mathbb{R}) &:= \{\varphi : \mathbb{R} \to \mathbb{R} \,|\, \varphi \text{ is a continuous function}\},\\
C_b(\mathbb{R},\mathbb{R}) &:= \{\varphi : \mathbb{R} \to \mathbb{R} \,|\, \varphi \text{ is a continuous and bounded function}\},\\
C_I(\mathbb{R},\mathbb{R}) &:= \{\varphi : \mathbb{R} \to \mathbb{R} \,|\, \varphi \text{ is a continuous and nondecreasing function}\}.
\end{aligned}
$$

Further, set real constants $\alpha \in (0, 1]$ and $t_*, t_0, t^*$ satisfying $t_* < t_0 < t^*$, function $f : (t_*, t^*) \to \mathbb{R}$, and norms

$$
\begin{aligned}
\|x(t)\| &:= \sum_{i=1}^{n} |x_i(t)|, \quad x : [t_0, +\infty) \to \mathbb{R}^n,\\
\|A(t)\| &:= \max\{\sum_{i=1}^{n} |a_{i1}(t)|, \sum_{i=1}^{n} |a_{i2}(t)|, ..., \sum_{i=1}^{n} |a_{in}(t)|\}, \quad A : \mathbb{R} \to \mathbb{R}^{n\times n}.
\end{aligned}
$$

In this section, we focus on the qualitative properties of linear CFDEs and their perturbations. Analogously to the linear ODEs, there also exists fundamental solutions for linear CFDEs. Consider the nonautonomous linear CFDE

$$
\mathcal{T}^\alpha x = A(t)x, \quad (t, x) \in \mathbb{R}^{n+1}, \tag{2.1}
$$

where matrix function $A \in C(\mathbb{R}, \mathbb{R}^{n\times n})$. Our primary purpose is to establish its well-posedness, e.g. existence and uniqueness, continuous dependence on initial data of solutions and continuation of solutions. Before this, as a preliminary, we modify the definition and some properties of conformable fractional derivative and integral raised by Khalil, Horani, Yousef and Sababheh([24]) and Abdeljawad([1]), to make them make more sense.

**Definition 2.1** *The $\alpha$-conformable fractional derivative of $f$ is defined as*

$$
\mathcal{T}^\alpha f(t) := \lim_{\varepsilon\to 0} \frac{f(t + \varepsilon|t|^{1-\alpha}) - f(t)}{\varepsilon}, \quad t \in (t_*, t^*). \tag{2.2}
$$

*In particular, if $\lim_{t\to 0} \mathcal{T}^\alpha f(t)$ exists, then*

$$
\mathcal{T}^\alpha f(0) := \lim_{t\to 0} \mathcal{T}^\alpha f(t).
$$

*Here function $f$ is called as $\alpha$-conformable differentiable, if $\mathcal{T}^\alpha f(t)$ exists.*

Our Definition 2.1 extends the one in [24] to the case of $t \le 0$. And different from the definition in [1], there are same formulae in (2.2) for both $t \le 0$ and $t \ge 0$. Further, we can deduce the following relations between conformable fractional derivative and Newton-Leibniz derivative and between conformable fractional integral and Riemann integral.

**Proposition 2.1** *The $\alpha$-conformable fractional derivative of $f$ can be represented as*

$$\mathcal{T}^{\alpha} f(t) = |t|^{1-\alpha} f'(t), \quad t \in (t_*, t^*).$$

**Proposition 2.2** *The $\alpha$-conformable fractional integral of $f$ is given by*

$$\mathcal{I}_{t_0}^{\alpha} f(t) = \int_{t_0}^{t} |s|^{\alpha-1} f(s) ds, \quad t \in (t_*, t^*).$$

The following special function and fractional integral inequality both will be useful throughout this paper.

**Definition 2.2** *The following special function is called as a conformable exponential function:*

$$E_\alpha(\lambda, t) := \begin{cases} \exp\Big(\lambda \dfrac{t^\alpha}{\alpha}\Big) = \sum\limits_{k=0}^{+\infty} \dfrac{\lambda^k t^{\alpha k}}{\alpha^k k!}, & \lambda \in \mathbb{R}, \quad t \in \mathbb{R}_+, \\ \exp\Big(-\lambda \dfrac{(-t)^\alpha}{\alpha}\Big) = \sum\limits_{k=0}^{+\infty} \dfrac{(-\lambda)^k (-t)^{\alpha k}}{\alpha^k k!}, & \lambda \in \mathbb{R}, \quad t \in \mathbb{R}_-. \end{cases}$$

**Lemma 2.1** *Let functions $a \in I([t_0, t^*), \mathbb{R}_+)$ and $f \in C([t_0, t^*), \mathbb{R}_+)$. Assume that $u : [t_0, t^*) \to \mathbb{R}_+$ satisfies fractional integral inequality*

$$u(t) \leq a(t) + \mathcal{I}_{t_0}^{\alpha} f(t) u(t), \quad t \in [t_0, t^*). \tag{2.3}$$

*Then $u$ can be estimated by*

$$\begin{aligned} u(t) &\leq a(t) e^{\mathcal{I}_{t_0}^{\alpha} f(t)} \\ &\leq a(t) E_\alpha\Big(\sup_{s \in [t_0, t]} f(s), |t|\Big) E_\alpha\Big(\sup_{s \in [t_0, t]} f(s), |t_0|\Big), \quad t \in [t_0, t^*). \end{aligned} \tag{2.4}$$

One can prove Lemma 2.1 easily, and the following results on well-posedness of solutions also can be attained simply. Consider the initial value problem (IVP) as follows

$$\begin{cases} \mathcal{T}^{\alpha} x(t) = f(t, x(t)), & (t, x) \in \mathbb{R}^{n+1}, \\ x(t_0) = x_0. \end{cases} \tag{2.5}$$

Given constants $a, b > 0$ and domains

$$\begin{aligned} D_+ &= \{(t, x) \in \mathbb{R}^{n+1} : t \in [t_0 - a, t_0 + a] \cap \mathbb{R}_+, \|x - x_0\| \leq b\}, \quad t_0 \geq 0, \\ D_- &= \{(t, x) \in \mathbb{R}^{n+1} : t \in [t_0 - a, t_0 + a] \cap \mathbb{R}_-, \|x - x_0\| \leq b\}, \quad t_0 \leq 0, \end{aligned}$$

assume that the function $f$ satisfies:

**(B1)** $f \in C(D_+, \mathbb{R}^n)$ (resp. $f \in C(D_-, \mathbb{R}^n)$);

**(B2)** $f(t,x)$ satisfies Lipschitz condition with respect to $x$ in $D_+$ (resp. $D_-$), i.e., there is a positive constant $L$ such that

$$\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|, \quad (t,x_1),(t,x_2) \in D_+ \text{ (resp. } D_-).$$

**Theorem 2.1** *Suppose that* **(B1)** *and* **(B2)** *hold. Then IVP* (2.5) *has a unique continuous solution in* $I_+ := [t_0 - \delta_+, t_0 + \delta_+] \cap \mathbb{R}_+$ *for* $t_0 \geq 0$ *(resp.* $I_- := [t_0 - \delta_-, t_0 + \delta_-] \cap \mathbb{R}_-$ *for* $t_0 \leq 0$*), where*

$$\delta_+ := \min\left\{a, \frac{b}{M_+} t_0^{1-\alpha}\right\}, \quad M_+ := \max_{(t,x)\in D_+} \|f(t,x)\|,$$
$$\delta_- := \min\left\{a, \frac{b}{M_-} (-t_0)^{1-\alpha}\right\}, \quad M_- := \max_{(t,x)\in D_-} \|f(t,x)\|.$$

Applying Lemma 2.1, the following lemma on continuation of solutions can be naturally proved.

**Lemma 2.2** *All solutions of* (2.1) *have maximal interval* $\mathbb{R}$.

Analogously to linear ODEs, some elementary properties on linear CFDEs (2.1) will be presented as follows, whose proofs will be omitted because of trivia.

**Proposition 2.3** *If* $x_1, x_2 : \mathbb{R} \to \mathbb{R}^n$ *are both solutions of* (2.1)*, then* $a_1x_1 + a_2x_2$ *is also a solution of* (2.1) *for any* $a_1, a_2 \in \mathbb{R}$*. And the set of all solutions of* (2.1) *is an* $n$*-D linear space.*

**Remark 2.1** $n \times n$ *matrix function* $X(t)$*, consisting of* $n$ *linearly independent solutions* $x_1(t), ..., x_n(t)$ *as its columns, is also called a fundamental solution of* (2.1)*. And for different fundamental solutions* $X(t)$ *and* $Y(t)$*, they can be linearly represented by each other, i.e., there exists an invertible linear transformation* $C$ *such that* $Y(t) = X(t)C$ *for all* $t \in \mathbb{R}$*.*

**Proposition 2.4** *The general solution of* (2.1) *associated with initial data* $x_0$ *can be written as*

$$x(t) = X(t)X^{-1}(t_0)x_0, \quad t \in \mathbb{R}, \tag{2.6}$$

*where* $X(t)$ *is any fundamental solution of* (2.1)*.*

**Proposition 2.5** *If $X(t)$ is a fundamental solution of* (2.1), *then*

$$\det X(t) = \det X(t_0) \exp\Big(\mathcal{I}_{t_0}^{\alpha} \mathrm{tr} A(t)\Big), \quad t \in \mathbb{R}.$$

As a special case of (2.1), the linear autonomous system

$$\mathcal{T}^{\alpha} x = Ax, \quad (t, x) \in \mathbb{R}^{n+1}, \tag{2.7}$$

where $A$ is an $n \times n$ real constant matrix, also has the following characteristic similar to linear autonomous ODE.

**Definition 2.3** *The conformable exponent of an $n \times n$ real constant matrix $A$ is defined as*

$$E_{\alpha}(A, 1) := \sum_{k=0}^{+\infty} \frac{A^k}{\alpha^k k!}, \tag{2.8}$$

*and denote $E_{\alpha}(0, 1) = I$ for convention.*

**Proposition 2.6** *The power series in* (2.8) *is convergent for any matrix $A$.*

Recall the Jordan canonical form in ODEs as follows

$$P^{-1}AP := \begin{bmatrix} J_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & J_l \end{bmatrix}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}, \quad i = 1, 2, ..., l, \tag{2.9}$$

where $P$ is an $n \times n$ nonsingular complex matrix and $\lambda_i$ is an eigenvalue of $A$. Thus, the conformable exponent of a matrix can be easily computed as follows.

**Proposition 2.7** *Let $A$ is an $n \times n$ real matrix with Jordan canonical form in* (2.9), *then*

$$E_{\alpha}(A, 1) = PE_{\alpha}(P^{-1}AP, 1)P^{-1} = P \begin{bmatrix} E_{\alpha}(J_1, 1) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & E_{\alpha}(J_l, 1) \end{bmatrix} P^{-1}.$$

Further, one can verify the following formula.

**Proposition 2.8** *If $A = \lambda I + N$, where the nilpotent matrix $N$ is*

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

*then the following expression fulfills:*

$$E_\alpha(A,1) = E_\alpha(\lambda,1)\Big(I + \frac{N}{\alpha} + \frac{N^2}{\alpha^2 2!} + \cdots + \frac{N^{n-1}}{\alpha^{(n-1)}(n-1)!}\Big).$$

Relying on the preliminaries above, one can solve (2.7) as follows.

**Lemma 2.3** *The matrix $E_\alpha(A,t)$ is a fundamental solution of* (2.7) *for all $t \in \mathbb{R}$.*

Both Proposition 2.4 and Lemma 2.3 lead to the following result on general solutions of (2.7).

**Proposition 2.9** *The general solution of* (2.7) *associated with initial data $x_0$ can be expressed as*

$$x(t) = \frac{E_\alpha(A,t)}{E_\alpha(A,t_0)} x_0, \quad t \in \mathbb{R}.$$

Consider the inhomogeneous linear CFDE

$$\mathcal{T}^\alpha x = A(t)x + f(t), \quad (t,x) \in \mathbb{R}^{n+1}, \tag{2.10}$$

where $f \in C(\mathbb{R}, \mathbb{R}^n)$ and matrix function $A \in C(\mathbb{R}, \mathbb{R}^{n\times n})$. One can verify the analogous properties to ODEs as follows.

**Proposition 2.10** *Like ODEs, if both $x_1^*(t)$ and $x_2^*(t)$ are solutions of* (2.10)*, then $x_1^*(t) - x_2^*(t)$ is a solution of* (2.1)*. On the other hand, if $x(t)$ and $x^*(t)$ are solutions of* (2.1) *and* (2.10) *respectively, then $x(t) + x^*(t)$ is also a solution of* (2.10)*.*

These properties can easily lead to the following structure of general solutions for (2.10).

**Proposition 2.11** *If $x^*(t)$ is a solution of* (2.10)*, then general solutions of* (2.10) *associated with initial data $x_0$ can be represented as*

$$x(t) = X(t)X^{-1}(t_0)x_0 + x^*(t), \quad t \in \mathbb{R},$$

*where $X(t)$ is any fundamental solution of* (2.1)*.*

Next, we will give the variation of constants formula for (2.10).

**Theorem 2.2** *Let $X(t)$ is a fundamental matrix of* (2.1)*, then the general solutions of* (2.10) *associated with initial data $x_0$ can be given by*

$$x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\mathcal{I}_{t_0}^{\alpha}X^{-1}(t)f(t), \quad t \in \mathbb{R}. \tag{2.11}$$

*Particularly, if $A(t)$ degenerates into an $n \times n$ real constant matrix $A$, the variation of constants formula* (2.11) *becomes the form*

$$x(t) = \frac{E_\alpha(A,t)}{E_\alpha(A,t_0)}x_0 + E_\alpha(A,t)\mathcal{I}_{t_0}^{\alpha}\frac{f(t)}{E_\alpha(A,t)}, \quad t \in \mathbb{R}.$$

**Proposition 2.12** *If an $n \times n$ real constant matrix $A$ has only eigenvalues with negative real part, then there exist constants $K, \lambda > 0$ such that*

$$\|E_\alpha(A,t)\| \le KE_\alpha(-\lambda,t), \quad t \in \mathbb{R}_+.$$

The proof is similar to the case in ODEs, referred Proposition 2.27 in [9, p.77].

# 3 Stability and conformable exponential dichotomy

In this section, we study the concepts of stability and conformable exponential dichotomy of CFDEs. Before this, Souahi, Makhlouf and Hammami([42]) combined Lyapunov stability and properties of conformable fractional derivative given by Abdeljawad([1]) to raise the concepts of stability, asymptotic stability and fractional exponential stability for the nonlinear system (2.5). For the nonautonomous linear CFDE (2.1), the definitions of uniform stability and uniformly asymptotic stability are more essential.

Based on the definition of stability for CFDEs described in [42], we introduce the following definition of uniformly stability analogous to the corresponding concept of ODEs in e.g. [16, p.1].

**Definition 3.1** *The solution $\hat{x}(t)$ of system (2.5) is said to be*

**(C1)** *uniformly stable, if for any $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) > 0$ such that any solution $x(t)$ of (2.5) satisfies for some $s \geq 0$, the inequality $\|x(s)-\hat{x}(s)\| < \delta$ implies $\|x(t) - \hat{x}(t)\| < \varepsilon$ for all $t \geq s$;*

**(C2)** *attractive, if there exists $\delta_0 > 0$ and $T := T(\varepsilon) > 0$ for any $\varepsilon > 0$ such that for some $s \geq 0$, the inequality $\|x(s) - \hat{x}(s)\| < \delta_0$ implies $\|x(t) - \hat{x}(t)\| < \varepsilon$ for all $t \geq s + T$;*

**(C3)** *uniformly asymptotically stable, if it is uniformly stable and attractive.*

The following definition is on the *conformable exponential stability.*

**Definition 3.2** *The solution $x_* = 0$ of system (2.5) is conformable exponential stable if*

$$\|x(t)\| \leq K \frac{E_\alpha(\lambda, t_0)}{E_\alpha(\lambda, t)} \|x_0\|, \quad t \geq t_0,$$

*where constants $K, \lambda > 0$.*

More generally, we focus on the significant application of Definition 3.1 to linear equation (2.1).

**Proposition 3.1** *Suppose $X(t)$ is a fundamental matrix of (2.1) and $c$ is a real constant, then solution $x_* = 0$ of (2.1) is said to be*

**(D1)** *stable for any $t_0 \in \mathbb{R}$ if and only if there exists $K := K(t_0) > 0$ such that*

$$\|X(t)\| \leq K, \quad t_0 \leq t < +\infty;$$

**(D2)** *uniformly stable for $t_0 \geq c$ if and only if there exists $K := K(c) > 0$ such that*

$$\|X(t)X^{-1}(s)\| \leq K, \quad t_0 \leq s \leq t < +\infty;$$

**(D3)** *asymptotically stable for any $t_0 \in \mathbb{R}$ if and only if $\lim_{t\to+\infty} \|X(t)\| = 0$;*

**(D4)** *uniformly asymptotically stable for $t_0 \geq c$ if and only if there exist $K := K(c) > 0$ and $\lambda := \lambda(c) > 0$ such that*

$$\|X(t)X^{-1}(s)\| \leq K \frac{E_\alpha(\lambda, s)}{E_\alpha(\lambda, t)}, \quad t_0 \leq s \leq t < +\infty. \tag{3.1}$$

*Particularly,* **(D1)-(D4)** *all hold for autonomous system* (2.7)*, if fundamental matrix* $X(t)$ *is replaced by* $E_\alpha(A,t)$*.*

The proof of **(D1)-(D4)** can refer to the Theorem 2.1 in [22, p.84]. In particular, since the conformable exponential stability implies the uniformly asymptotic stability, one can simply verify conclusion **(D4)**. The definitions of the corresponding stabilities above for Caputo FDEs had been proposed in references e.g. [17, p.140].

Next, we shall propose the concept of *conformable exponential dichotomy* for linear CFDE (2.1).

**Definition 3.3** *Suppose that* $X(t)$ *is a fundamental matrix of* (2.1)*. The equation* (2.1) *possesses a conformable exponential dichotomy if there exists a projection matrix* $P$*, i.e.* $P^2 = P$*, and positive constants* $N_i$*,* $\beta_i$ $(i = 1, 2)$ *such that*

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\le N_1 \frac{E_\alpha(\beta_1, s)}{E_\alpha(\beta_1, t)}, \quad t \ge s, \\ \|X(t)(I-P)X^{-1}(s)\| &\le N_2 \frac{E_\alpha(\beta_2, t)}{E_\alpha(\beta_2, s)}, \quad s \ge t. \end{aligned} \tag{3.2}$$

*In particular,* (2.1) *possesses an ordinary dichotomy if* (3.2) *hold with* $\beta_1 = \beta_2 = 0$*.*

Finally, we concern perturbation of nonautonomous linear CFDE (2.1). Consider the perturbed equation

$$\mathcal{T}^\alpha x = A(t)x + f(t,x), \quad (t,x) \in \mathbb{R}^{n+1}, \tag{3.3}$$

where $f \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$ and matrix function $A \in C(\mathbb{R}, \mathbb{R}^{n\times n})$.

The following conclusion give out the projection form of equivalent integral equation and the existence of bounded solutions for equation (3.3).

**Lemma 3.1** *Suppose that function* $f \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$*,* $P$ *is a projection matrix given in Definition 3.3 and equation* (2.1) *possesses a conformable exponential dichotomy. If* $x \in C_b([t_0, +\infty), \mathbb{R}^n)$ *is a solution of* (3.3) *with* $x(t_0) = x_0$ *for constant* $t_0 \in \mathbb{R}_+$*, then*

$$\begin{aligned} x(t) =& X(t)PX^{-1}(t_0)x_0 + X(t)P\mathcal{I}^\alpha_{t_0} X^{-1}(t)f(t,x(t)) \\ &+ X(t)(I-P)\mathcal{I}^\alpha_{+\infty} X^{-1}(t)f(t,x(t)), \quad t \ge t_0. \end{aligned} \tag{3.4}$$

*If* $x \in C_b((-\infty, t_0], \mathbb{R}^n)$ *is a solution of* (3.3) *with* $x(t_0) = x_0$ *for constant* $t_0 \in \mathbb{R}_-$*, then*

$$\begin{aligned} x(t) =& X(t)(I-P)X^{-1}(t_0)x_0 + X(t)(I-P)\mathcal{I}^\alpha_{t_0} X^{-1}(t)f(t,x(t)) \\ &+ X(t)P\mathcal{I}^\alpha_{-\infty} X^{-1}(t)f(t,x(t)), \quad t \le t_0. \end{aligned} \tag{3.5}$$

*Conversely, any bounded solution of* (3.4) *or* (3.5) *is a solution of* (3.3)*.*

**Proof.** For convenience, we only prove (3.4), formula (3.5) can be proved in an analogous manner. Assume $x(t)$ is a bounded solution of (3.3) and $M := \sup_{t\in[t_0,+\infty)} \|x(t)\|$ for $t \geq t_0$. The continuity of $f$ implies that there exists a positive constant $N$ such that $N := \sup_{t\in[t_0,+\infty)} \|f(t,x(t))\|$. By the variation of constants formula (2.11), for any $\tau \geq t_0$, the solution $x(t)$ satisfies

$$\begin{aligned} X(t)(I-P)X^{-1}(t)x(t) =& X(t)(I-P)X^{-1}(\tau)x(\tau) \\ &+ X(t)(I-P)\mathcal{I}_\tau^\alpha X^{-1}(t)f(t,x(t)), \quad t,\tau \geq t_0, \end{aligned} \tag{3.6}$$

where the following estimate can be obtained by (3.2)

$$\begin{aligned} \|X(t)(I-P)X^{-1}(\tau)x(\tau)\| &\leq N_2 \frac{E_\alpha(\beta_2,t)}{E_\alpha(\beta_2,\tau)} \sup_{t\in[t_0,+\infty)} \|x(t)\| \\ &\leq MN_2 \frac{E_\alpha(\beta_2,t)}{E_\alpha(\beta_2,\tau)}, \quad t,\tau \geq t_0. \end{aligned}$$

It yields that

$$\lim_{\tau\to+\infty} \|X(t)(I-P)X^{-1}(\tau)x(\tau)\| = 0, \quad t \geq t_0.$$

On the other hand, in integral equation (3.6) for $t \geq t_0$,

$$\begin{aligned} \|X(t)(I-P)\mathcal{I}_\tau^\alpha X^{-1}(t)f(t,x(t))\| &\leq NN_2 E_\alpha(\beta_2,t)|\mathcal{I}_\tau^\alpha E_\alpha(-\beta_2,t)| \\ &\leq NN_2 E_\alpha(\beta_2,t)\int_t^\tau s^{\alpha-1}\exp\Big(-\beta_2\frac{s^\alpha}{\alpha}\Big)ds \\ &\leq \frac{NN_2E_\alpha(\beta_2,t)}{\beta_2}(E_\alpha(-\beta_2,t)-E_\alpha(-\beta_2,\tau)) \\ &\leq \frac{NN_2}{\beta_2}, \end{aligned}$$

which implies that

$$\|X(t)(I-P)\mathcal{I}_{+\infty}^\alpha X^{-1}(t)f(t,x(t))\| < +\infty, \quad t \geq t_0.$$

It follows from (3.6) that

$$X(t)(I-P)X^{-1}(t)x(t) = X(t)(I-P)\mathcal{I}_{+\infty}^\alpha X^{-1}(t)f(t,x(t)), \quad t \geq t_0. \tag{3.7}$$

From the variation of constants formula (2.11), it also follows that for $t \geq t_0$,

$$X(t)PX^{-1}(t)x(t) = X(t)PX^{-1}(t_0)x(t_0) + X(t)P\mathcal{I}_{t_0}^\alpha X^{-1}(t)f(t,x(t)). \tag{3.8}$$

Since $x(t) = X(t)PX^{-1}(t)x(t) + X(t)(I-P)X^{-1}(t)x(t)$, substituting (3.7) and (3.8) into it, we attain (3.4). And the converse conclusion can be verified by direct calculation to end the proof. $\square$

The following Lemma is the fractional-order version of projected integral inequality.

**Lemma 3.2** *Suppose that $N_i$, $\beta_i$ and $\varepsilon$ are all positive constants for $i=1,2$, and bounded continuous nonnegative solutions $u(t)$ satisfy*

$$\begin{aligned} u(t) \leq & N_1\frac{E_\alpha(\beta_1,t_0)}{E_\alpha(\beta_1,t)}+\frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}\mathcal{I}^\alpha_{t_0}E_\alpha(\beta_1,t)u(t) \\ & -\varepsilon N_2E_\alpha(\beta_2,t)\mathcal{I}^\alpha_{+\infty}\frac{u(t)}{E_\alpha(\beta_2,t)}, \quad t\geq t_0\geq 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} u(t) \leq & N_2\frac{E_\alpha(\beta_2,t)}{E_\alpha(\beta_2,t_0)}-\varepsilon N_2E_\alpha(\beta_2,t)\mathcal{I}^\alpha_{t_0}\frac{u(t)}{E_\alpha(\beta_2,t)} \\ & +\frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}\mathcal{I}^\alpha_{-\infty}E_\alpha(\beta_1,t)u(t), \quad t\leq t_0\leq 0. \end{aligned} \tag{3.10}$$

*Set that*

$$\theta:=\varepsilon\Big(\frac{N_1}{\beta_1}+\frac{N_2}{\beta_2}\Big), \quad K_i:=\frac{N_i}{1-\theta}, \quad \lambda_i:=\beta_i-\frac{\varepsilon N_i}{1-\theta}, \quad i=1,2.$$

*If $\theta<1$, then*

$$u(t)\leq\begin{cases} K_1\dfrac{E_\alpha(\lambda_1,t_0)}{E_\alpha(\lambda_1,t)}, & t\geq t_0,\\ K_2\dfrac{E_\alpha(\lambda_2,t)}{E_\alpha(\lambda_2,t_0)}, & t\leq t_0. \end{cases}$$

**Proof.** Without loss of generality, we only consider inequality (3.9), because inequality (3.10) can be changed into (3.9) through transformations $t\to -t$ and $t_0\to -t_0$. Next, we need to verify $\lim\limits_{t\to+\infty}u(t)=0$. In deed, since $u(t)$ is bounded, let $\sigma:=\limsup\limits_{t\to+\infty}u(t)$. If $\sigma>0$ and for any constant $\vartheta$ satisfying $\theta<\vartheta<1$, there exists $t_1\geq t_0$ such that for any $t\geq t_1$ we have $u(t)\leq\vartheta^{-1}\sigma$. For $t\geq t_1$ we compute

$$\begin{aligned} u(t) \leq & N_1\frac{E_\alpha(\beta_1,t_0)}{E_\alpha(\beta_1,t)}+\frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}\mathcal{I}^\alpha_{t_0}E_\alpha(\beta_1,t_1)u(t_1) \\ & +\vartheta^{-1}\sigma\Big[\frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}\mathcal{I}^\alpha_{t_1}E_\alpha(\beta_1,t)-\varepsilon N_2E_\alpha(\beta_2,t)\mathcal{I}^\alpha_{+\infty}\frac{1}{E_\alpha(\beta_2,t)}\Big] \\ \leq & N_1\frac{E_\alpha(\beta_1,t_0)}{E_\alpha(\beta_1,t)}+\frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}\mathcal{I}^\alpha_{t_0}E_\alpha(\beta_1,t_1)u(t_1)+\vartheta^{-1}\sigma\varepsilon\Big(\frac{N_1}{\beta_1}+\frac{N_2}{\beta_2}\Big). \end{aligned}$$

Since $\theta<\vartheta<1$, the upper limit of the right hand side of the inequality above is less than $\sigma$ as $t\to+\infty$. It follows from the inequality above that

$$\sigma\leq\vartheta^{-1}\sigma\varepsilon\Big(\frac{N_1}{\beta_1}+\frac{N_2}{\beta_2}\Big)<\sigma,$$

that is a contradiction. Hence, $\sigma=0$ and $\lim\limits_{t\to+\infty}u(t)=0$.

Set $v(t) := \sup\limits_{\tau\geq t} u(\tau)$. Obviously, the function $v(t)$ is nonincreasing and for any $t \geq t_0$, there exists $t_2 \geq t$ such that for $t \leq s \leq t_2$, $v(t) = u(t_2) = v(s)$. Replacing $t$ in (3.9) with $t_2$, for $t \geq t_0$ we calculate that

$$\begin{aligned}
v(t) = u(t_2) &\leq N_1 \frac{E_\alpha(\beta_1, t_0)}{E_\alpha(\beta_1, t_2)} + \frac{\varepsilon N_1}{E_\alpha(\beta_1, t_2)} \mathcal{I}^\alpha_{t_0} E_\alpha(\beta_1, t_2) u(t_2) \\
&\quad -\varepsilon N_2 E_\alpha(\beta_2, t_2) \mathcal{I}^\alpha_{+\infty} \frac{u(t_2)}{E_\alpha(\beta_2, t_2)} \\
&\leq N_1 \frac{E_\alpha(\beta_1, t_0)}{E_\alpha(\beta_1, t)} + \frac{\varepsilon N_1}{E_\alpha(\beta_1, t)} \mathcal{I}^\alpha_{t_0} E_\alpha(\beta_1, t_2) u(t_2) \\
&\quad -\varepsilon N_2 E_\alpha(\beta_2, t_2) \mathcal{I}^\alpha_{+\infty} \frac{u(t_2)}{E_\alpha(\beta_2, t_2)} \\
&\leq N_1 \frac{E_\alpha(\beta_1, t_0)}{E_\alpha(\beta_1, t)} + \frac{\varepsilon N_1}{E_\alpha(\beta_1, t)} \mathcal{I}^\alpha_{t_0} E_\alpha(\beta_1, t) u(t) \\
&\quad -v(t) \Big[ \frac{\varepsilon N_1}{E_\alpha(\beta_1, t_2)} \mathcal{I}^\alpha_{t_2} E_\alpha(\beta_1, t) + \varepsilon N_2 E_\alpha(\beta_2, t_2) \mathcal{I}^\alpha_{+\infty} \frac{1}{E_\alpha(\beta_2, t_2)} \Big] \\
&\leq N_1 \frac{E_\alpha(\beta_1, t_0)}{E_\alpha(\beta_1, t)} + \frac{\varepsilon N_1}{E_\alpha(\beta_1, t)} \mathcal{I}^\alpha_{t_0} E_\alpha(\beta_1, t) u(t) + v(t)\varepsilon\Big(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}\Big).
\end{aligned}$$

Put $w(t) := \dfrac{E_\alpha(\beta_1, t)}{E_\alpha(\beta_1, t_0)} v(t)$, then it follows from the definition of $v$ that

$$\begin{aligned}
w(t) &\leq N_1 + \frac{\varepsilon N_1}{E_\alpha(\beta_1, t_0)} \mathcal{I}^\alpha_{t_0} E_\alpha(\beta_1, t) v(t) + \theta w(t) \\
&= N_1 + \varepsilon N_1 \mathcal{I}^\alpha_{t_0} w(t) + \theta w(t), \quad t \geq t_0,
\end{aligned}$$

that is

$$w(t) \leq \frac{N_1}{1-\theta} + \frac{\varepsilon N_1}{1-\theta} \mathcal{I}^\alpha_{t_0} w(t), \quad t \geq t_0.$$

Applying Lemma 2.1 to the inequality above, we attain

$$w(t) \leq \frac{N_1}{1-\theta} \exp(\mathcal{I}^\alpha_{t_0} \frac{\varepsilon N_1}{1-\theta}) = \frac{N_1}{1-\theta} \frac{E_\alpha(\frac{\varepsilon N_1}{1-\theta}, t)}{E_\alpha(\frac{\varepsilon N_1}{1-\theta}, t_0)}, \quad t \geq t_0.$$

Combining with the definitions of $v$ and $w$, we acquire

$$u(t) \leq \frac{N_1}{1-\theta} \frac{E_\alpha(\frac{\varepsilon N_1}{1-\theta}, t)}{E_\alpha(\frac{\varepsilon N_1}{1-\theta}, t_0)} \frac{E_\alpha(\beta_1, t_0)}{E_\alpha(\beta_1, t)} = K_1 \frac{E_\alpha(\lambda_1, t_0)}{E_\alpha(\lambda_1, t)}, \quad t \geq t_0,$$

where $K_1 = \dfrac{N_1}{1-\theta}$ and $\lambda_1 = \beta_1 - \dfrac{\delta N_1}{1-\theta}$. Therefore, Lemma 3.2 is proved. □

As a corollary of Lemma 3.2, we introduce a more useful result in estimate of dichotomy.

**Corollary 3.1** *Suppose that $N_i$, $\beta_i$ and $\varepsilon$ are all positive constants for $i = 1, 2$, and bounded continuous nonnegative solutions $u(t)$ satisfy*

$$\begin{aligned} u(t) \leq & N_2 \frac{E_\alpha(\beta_2, t)}{E_\alpha(\beta_2, s)} + \frac{\varepsilon N_1}{E_\alpha(\beta_1, t)} \mathcal{I}_{t_0}^\alpha E_\alpha(\beta_1, t) u(t) \\ & - \varepsilon N_2 E_\alpha(\beta_2, t) \mathcal{I}_s^\alpha \frac{u(t)}{E_\alpha(\beta_2, t)}, \quad s \geq t \geq t_0 \geq 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} u(t) \leq & N_1 \frac{E_\alpha(\beta_1, s)}{E_\alpha(\beta_1, t)} - \varepsilon N_2 E_\alpha(\beta_2, t) \mathcal{I}_{t_0}^\alpha \frac{u(t)}{E_\alpha(\beta_2, t)} \\ & + \frac{\varepsilon N_1}{E_\alpha(\beta_1, t)} \mathcal{I}_s^\alpha E_\alpha(\beta_1, t) u(t), \quad s \leq t \leq t_0 \leq 0. \end{aligned} \tag{3.12}$$

*If $\theta < 1$, then*

$$u(t) \leq \begin{cases} K_2 \dfrac{E_\alpha(\lambda_2, t)}{E_\alpha(\lambda_2, s)}, & s \geq t \geq t_0, \\ K_1 \dfrac{E_\alpha(\lambda_1, s)}{E_\alpha(\lambda_1, t)}, & s \leq t \leq t_0, \end{cases}$$

*where $\theta$, $K_i$ and $\lambda_i$ were all defined in Lemma 3.2.*

**Proof.** Without loss of generality, we only consider inequality (3.11), because inequality (3.12) can be changed into (3.11) through transformations $t \to -t$, $s \to -s$ and $t_0 \to -t_0$. Let $t_1 := (s^\alpha - t^\alpha + t_0^\alpha)^{1/\alpha}$, then $s \geq t_1 \geq t_0$, because of the fact $s \geq t \geq t_0$. From (3.11) it follows that for $s \geq t_1 \geq t_0 \geq 0$,

$$\begin{aligned} u((s^\alpha - t_1^\alpha + t_0^\alpha)^{1/\alpha}) \leq & N_2 \frac{E_\alpha(\beta_2, t_0)}{E_\alpha(\beta_2, t_1)} \\ & + \varepsilon N_1 \int_{t_0}^{(s^\alpha - t_1^\alpha + t_0^\alpha)^{1/\alpha}} \tau^{\alpha-1} \frac{E_\alpha(\beta_1, \tau) E_\alpha(\beta_1, t_1)}{E_\alpha(\beta_1, s) E_\alpha(\beta_1, t_0)} u(\tau) d\tau \\ & + \varepsilon N_2 \int_{(s^\alpha - t_1^\alpha + t_0^\alpha)^{1/\alpha}}^{s} \tau^{\alpha-1} \frac{E_\alpha(\beta_2, s) E_\alpha(\beta_2, t_0)}{E_\alpha(\beta_2, \tau) E_\alpha(\beta_2, t_1)} u(\tau) d\tau. \end{aligned}$$

Put $v(t_1) := u((s^\alpha - t_1^\alpha + t_0^\alpha)^{1/\alpha})$ then $u(\tau) = v((s^\alpha - \tau^\alpha + t_0^\alpha)^{1/\alpha})$. The inequality above yields that for $s \geq t_1 \geq t_0 \geq 0$,

$$\begin{aligned} v(t_1) \leq & N_2 \frac{E_\alpha(\beta_2, t_0)}{E_\alpha(\beta_2, t_1)} \\ & + \varepsilon N_1 \int_{t_0}^{(s^\alpha - t_1^\alpha + t_0^\alpha)^{1/\alpha}} \tau^{\alpha-1} \frac{E_\alpha(\beta_1, \tau) E_\alpha(\beta_1, t_1)}{E_\alpha(\beta_1, s) E_\alpha(\beta_1, t_0)} v((s^\alpha - \tau^\alpha + t_0^\alpha)^{1/\alpha}) d\tau \\ & + \varepsilon N_2 \int_{(s^\alpha - t_1^\alpha + t_0^\alpha)^{1/\alpha}}^{s} \tau^{\alpha-1} \frac{E_\alpha(\beta_2, s) E_\alpha(\beta_2, t_0)}{E_\alpha(\beta_2, \tau) E_\alpha(\beta_2, t_1)} v((s^\alpha - \tau^\alpha + t_0^\alpha)^{1/\alpha}) d\tau. \end{aligned}$$

Let $\iota := (s^\alpha - \tau^\alpha + t_0^\alpha)^{1/\alpha}$, then

$$\begin{aligned} v(t_1) \leq & N_2 \frac{E_\alpha(\beta_2, t_0)}{E_\alpha(\beta_2, t_1)} + \varepsilon N_2 \int_{t_0}^{t_1} \iota^{\alpha-1} \frac{E_\alpha(\beta_2, \iota)}{E_\alpha(\beta_2, t_1)} v(\iota) d\iota \\ & + \varepsilon N_1 \int_{t_1}^{s} \iota^{\alpha-1} \frac{E_\alpha(\beta_1, t_1)}{E_\alpha(\beta_1, \iota)} v(\iota) d\iota, \quad s \geq t_1 \geq t_0 \geq 0. \end{aligned}$$

The inequality also can be amplified as

$$\begin{aligned} v(t_1) \leq & N_2 \frac{E_\alpha(\beta_2, t_0)}{E_\alpha(\beta_2, t_1)} + \frac{\varepsilon N_2}{E_\alpha(\beta_2, t_1)} \mathcal{I}_{t_0}^\alpha E_\alpha(\beta_2, t_1) v(t_1) \\ & - \varepsilon N_1 E_\alpha(\beta_1, t_1) \mathcal{I}_{+\infty}^\alpha \frac{v(t_1)}{E_\alpha(\beta_1, t_1)}, \quad t_1 \geq t_0 \geq 0. \end{aligned}$$

By the synchronous boundedness of both functions $u$ and $v$, we employ Lemma 3.2 to gain

$$v(t_1) \leq K_2 \frac{E_\alpha(\lambda_2, t_0)}{E_\alpha(\lambda_2, t_1)}, \quad t_1 \geq t_0.$$

It follows from the definition of $v(t_1)$ that

$$u(t) \leq K_2 \frac{E_\alpha(\lambda_2, t)}{E_\alpha(\lambda_2, s)}, \quad s \geq t \geq t_0,$$

where $K_2 = \dfrac{N_2}{1-\theta}$, $\lambda_2 = \beta_2 - \dfrac{\varepsilon N_2}{1-\theta}$ given in Lemma 3.2. Hence, Corollary 3.1 is proved. $\square$

In the end of this section, we demonstrate the relation of invariant manifolds between equation (2.1) and its perturbation (3.3). But before we do that, let us introduce the following notion.

**Definition 3.4** *Let $\Omega$ is any subset of $\mathbb{R}^n$ including zero and $P$ is a projection matrix such that $\mathbb{R}^n = P\mathbb{R}^n \oplus (I-P)\mathbb{R}^n$ and $P^2 = P$. We say $\Omega$ is tangent to $(I-P)\mathbb{R}^n$ (resp. $P\mathbb{R}^n$) at zero, if $\|Px\|/\|(I-P)x\| \to 0$ (resp. $\|(I-P)x\|/\|Px\| \to 0$) as $x \to 0$ in $\Omega$.*

From now on, let $k := \mathcal{R}(I-P)$, where denote $\mathcal{R}(P)$ by the rank of matrix $P$ and assume that

**(E1)** $\zeta \in C_I(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $\zeta(0) = 0$;

**(E2)** $\Lambda(\zeta)$ consists of functions $f \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$ such that

$$\begin{aligned} f(t,0) &= 0, \\ \|f(t,x) - f(t,y)\| &\le \zeta(\sigma)\|x-y\|, \quad \|x\|, \|y\| \le \sigma; \end{aligned}$$

**(E3)** Projection matrix $P$ fulfils $X(t)P = PX(t)$ for all $t \in \mathbb{R}$.

**Theorem 3.1** *Suppose that* **(E1)-(E3)** *hold and denote the unstable and stable manifolds of the hyperbolic equilibrium $x = 0$ of equation* (3.3) *as $U_k := U_k(f)$ and $S_{n-k} := S_{n-k}(f)$ respectively, for any $f \in \Lambda(\zeta)$. Then $U_k$ and $S_{n-k}$ are tangent to $(I-P)\mathbb{R}^n$ and $P\mathbb{R}^n$ at $x = 0$ respectively, where $(I-P)\mathbb{R}^n$ and $P\mathbb{R}^n$ are the unstable and stable invariant subspaces of the hyperbolic equilibrium $x = 0$ of* (2.1)*, respectively. Moreover, there exist positive constants $M$, $\gamma_1$ and $\gamma_2$ such that*

$$\begin{aligned} \|x(t)\| \le & M\frac{E_\alpha(\gamma_1, t_0)}{E_\alpha(\gamma_1, t)}\|x(t_0)\|, \quad t \ge t_0 \ge 0, \quad x(t_0) \in S_{n-k}, \\ \|x(t)\| \le & M\frac{E_\alpha(\gamma_2, t)}{E_\alpha(\gamma_2, t_0)}\|x(t_0)\|, \quad t \le t_0 \le 0, \quad x(t_0) \in U_k. \end{aligned} \tag{3.13}$$

**Remark 3.1** *The hyperbolic equilibrium $x = 0$ of ODE $\dot{x} = A(t)x$ is also the hyperbolic equilibrium of CFDE* (2.1)*. In fact, by Definition 2.1, one can verify*

$$\mathcal{T}^\alpha x(0) = \lim_{t\to 0}\mathcal{T}^\alpha x(t) = \lim_{t\to 0} A(t)x(t) = \lim_{t\to 0}\dot{x}(t) = \dot{x}(0),$$

*which implies the assertion.*

**Proof of Theorem 3.1.** Assume $\lambda$, $N_i$, $\beta_i$ $(i = 1, 2)$ are all given in (3.1) and (3.2) respectively, and the function $\zeta(\sigma)$ $(\sigma \ge 0)$ is given in **(E1)**. Take $\delta$ satisfy

$$(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2})\zeta(\delta) < \frac{1}{2}, \quad N_1 < (\beta_2 + \lambda - 4N_1N_2\zeta(\delta))(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}). \tag{3.14}$$

Choose $x_0$ satisfy $\|x_0\| \le \delta/2N_1$ for $x_0 \in \mathbb{R}^n$, and define $\mathcal{L}(Px_0, \delta)$ is a set of functions $x \in C([t_0, +\infty), \mathbb{R}^n)$, where $\|x\|_\infty := \sup\limits_{t_0 \le t < +\infty}\|x(t)\| \le \delta$ and $t_0 \ge 0$. $\mathcal{L}(Px_0, \delta)$ is a closed bounded subset consisting of the Banach space of all bounded continuous functions mapping $[t_0, +\infty)$ to $\mathbb{R}^n$ with the uniform topology. For any $x \in \mathcal{L}(Px_0, \delta)$, define

$$\begin{aligned}(\mathcal{J}x)(t) :=& X(t)PX^{-1}(t_0)x_0 + X(t)P\mathcal{I}^{\alpha}_{t_0}X^{-1}(t)f(t, x(t)) \\ &+ X(t)(I-P)\mathcal{I}^{\alpha}_{+\infty}X^{-1}(t)f(t, x(t)), \quad t \geq t_0.\end{aligned} \tag{3.15}$$

It is easy to know that $\mathcal{J}x$ is well defined and continuous for $t \geq t_0$. From (3.2), (3.14) and **(E2)**, we calculate that

$$\begin{aligned}\|(\mathcal{J}x)(t)\| \leq\ & N_1\frac{E_\alpha(\beta_1, t_0)}{E_\alpha(\beta_1, t)}\|x_0\| + \frac{N_1}{E_\alpha(\beta_1, t)}\mathcal{I}^{\alpha}_{t_0}E_\alpha(\beta_1, t)\|f(t, x(t))\| \\ & -N_2E_\alpha(\beta_2, t)\mathcal{I}^{\alpha}_{+\infty}\frac{\|f(t, x(t))\|}{E_\alpha(\beta_2, t)} \\ \leq\ & N_1\frac{E_\alpha(\beta_1, t_0)}{E_\alpha(\beta_1, t)}\|x_0\| + \zeta(\delta)\Big(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}\Big)\|x\|_\infty \\ \leq\ & N_1\|x_0\| + \zeta(\delta)\Big(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}\Big)\delta \\ <\ & \frac{\delta}{2} + \frac{\delta}{2} = \delta,\end{aligned}$$

thus $\|\mathcal{J}x\|_\infty < \delta$ and $\mathcal{J} : \mathcal{L}(Px_0, \delta) \to \mathcal{L}(Px_0, \delta)$.

Analogously to the computation above, we obtain

$$\|(\mathcal{J}x)(t) - (\mathcal{J}y)(t)\| \leq \zeta(\delta)\Big(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}\Big)\|x-y\|_\infty \leq \frac{1}{2}\|x-y\|_\infty, \quad t \geq t_0,$$

which implies that $\mathcal{J}$ is a contraction mapping in $\mathcal{L}(Px_0, \delta)$. In fact, there is a unique fixed point $x_*(t, Px_0) \in \mathcal{L}(Px_0, \delta)$ satisfying (3.4). Note that the function $x_*(t, Px_0)$ is continuous with respect to $Px_0$ and $x_*(t, 0) = 0$. Let $x_*(t) := x_*(t, Px_0)$ and $\hat{x}_*(t) := x_*(t, P\hat{x}_0)$, it follows from (3.4), (3.1) and **(E3)** that

$$\begin{aligned}\|x_*(t) - \hat{x}_*(t)\| \leq\ & K\frac{E_\alpha(\lambda, t_0)}{E_\alpha(\lambda, t)}\|Px_0 - P\hat{x}_0\| \\ & +\frac{N_1\zeta(\delta)}{E_\alpha(\beta_1, t)}\mathcal{I}^{\alpha}_{t_0}E_\alpha(\beta_1, t)\|x_*(t) - \hat{x}_*(t)\| \\ & -N_2\zeta(\delta)E_\alpha(\beta_2, t)\mathcal{I}^{\alpha}_{+\infty}\frac{\|x_*(t) - \hat{x}_*(t)\|}{E_\alpha(\beta_2, t)}, \quad t \geq t_0.\end{aligned}$$

By Lemma 3.2, we acquire

$$\|x_*(t, Px_0) - x_*(t, P\hat{x}_0)\| \leq 2K\frac{E_\alpha(\gamma_1, t_0)}{E_\alpha(\gamma_1, t)}\|Px_0 - P\hat{x}_0\|, \quad t \geq t_0, \tag{3.16}$$

where $\gamma_1 = \lambda - \dfrac{N_1\beta_1\beta_2}{N_1\beta_2 + N_2\beta_1}$. Combining the fact $x_*(t, 0) = 0$ and (3.16), one can verify that the first expression of the estimate (3.13) is true. Proceeding analogously to (3.16), the second estimate in (3.13) is also true, when $\gamma_2 = \lambda - \dfrac{N_2\beta_1\beta_2}{N_1\beta_2 + N_2\beta_1}$.

Set $B_{\delta/2N_1}$ is the open ball in $\mathbb{R}^n$ centered at the origin with radius $\delta/2N_1$, and take $S^*_{n-k} := \{x : x = x_*(t_0, Px_0), x_0 \in B_{\delta/2N_1} \cap \mathbb{R}^n\}$. Let $h(Px_0) := x_*(t_0, Px_0)$ for $x_0 \in B_{\delta/2N_1} \cap \mathbb{R}^n$, we observe that $h$ is a continuous mapping from $B_{\delta/2N_1} \cap (P\mathbb{R}^n)$ to $S^*_{n-k}$, then

$$h(Px_0) = Px_0 + X(t_0)(I - P)\mathcal{I}^{\alpha}_{+\infty}X^{-1}(t_0)f(t_0, x_*(t_0, Px_0)).$$

Given $x_0, \hat{x}_0 \in B_{\delta/2N_1} \cap \mathbb{R}^n$, we employ (3.2), (3.14), (3.16) and **(E2)** to obtain

$$\begin{aligned}\|h(Px_0) - h(P\hat{x}_0)\| &\geq \|Px_0 - P\hat{x}_0\| + N_2\zeta(\delta)E_\alpha(\beta_2, t_0)\mathcal{I}^{\alpha}_{+\infty}\frac{\|x_*(t_0) - \hat{x}_*(t_0)\|}{E_\alpha(\beta_2, t_0)}\\ &\geq \|Px_0 - P\hat{x}_0\| \left[1 + E_\alpha(\beta_2 + \gamma_1, t_0)\mathcal{I}^{\alpha}_{+\infty}\frac{2N_1N_2\zeta(\delta)}{E_\alpha(\beta_2 + \gamma_1, t_0)}\right]\\ &\geq \left(1 - \frac{2N_1N_2\zeta(\delta)}{\beta_2 + \gamma_1}\right)\|Px_0 - P\hat{x}_0\| \geq \frac{1}{2}\|Px_0 - P\hat{x}_0\|,\end{aligned}$$

yielding $h$ is a bijective. And since $h^{-1} = P$ is continuous, $h$ is a homeomorphism. Hence, $S^*_{n-k}$ is homeomorphic to the $(n-k)$-D open unit ball in $\mathbb{R}^{n-k}$. If $S^*_{n-k}$ is not a positively invariant set, then we expand $S^*_{n-k}$ into the positively invariant $S_{n-k}$, by absorbing all the positive orbits of the solutions starting from $S^*_{n-k}$. From the uniqueness of the solutions, $S_{n-k}$ is also homeomorphic to the open unit ball in $\mathbb{R}^{n-k}$. In other words, the case $\|Px\| < \delta/2N_1$ for all $x \in S_{n-k}$ implies $S_{n-k} \equiv S^*_{n-k}$. It follows from (3.15), (3.16), **(E2)** and the fact $x_*(t, 0) = 0$ that

$$\begin{aligned}\|(I - P)x_*(t_0, Px_0)\| &\leq -N_2E_\alpha(\beta_2, t_0)\mathcal{I}^{\alpha}_{+\infty}\frac{\|f(t_0, x_*(t_0, Px_0))\|}{E_\alpha(\beta_2, t_0)}\\ &\leq -N_2E_\alpha(\beta_2, t_0)\mathcal{I}^{\alpha}_{+\infty}\frac{\zeta(\|x_*(t_0, Px_0)\|)}{E_\alpha(\beta_2, t_0)}\|x_*(t_0, Px_0)\|\\ &\leq -N_2E_\alpha(\beta_2, t_0)\mathcal{I}^{\alpha}_{+\infty}\frac{\zeta(2N_1\|Px_0\|)}{E_\alpha(\beta_2, t_0)}2N_1\|Px_0\|\\ &\leq \frac{2N_1N_2}{\beta_2}\zeta(2N_1\|Px_0\|)\|Px_0\|.\end{aligned}$$

Since $\|Px_0\| \to 0$ as $\|x_0\| \to 0$, we get $\|(I-P)x_*(t_0, Px_0)\|/\|Px_0\| \to 0$ as $\|x_0\| \to 0$ in $S_{n-k}$. Consequently $S_{n-k}$ is tangent to $P\mathbb{R}^n$ at zero. Similarly, one can construct the set $U_k$ via (3.5), and complete the proof of Theorem 3.1. □

# 4 Roughness of dichotomy

Our focus of this section is the roughness of the conformable exponential dichotomy. That is the preservation of dichotomy for hyperbolic linear systems undergoing small linear perturbation. Consider the perturbed equation of linear CFDE (2.1) as follows

$$\mathcal{T}^\alpha y = [A(t) + B(t)]y, \quad (t, y) \in \mathbb{R}^{n+1}, \tag{4.1}$$

where matrix functions $A \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ and $B \in C_b(\mathbb{R}, \mathbb{R}^{n\times n})$. The following is one of our main results of this paper.

**Theorem 4.1** *Assume that $X(t)$ is a fundamental matrix of* (2.1) *such that $X(0) = I$, and equation* (2.1) *possesses a conformable exponential dichotomy, i.e., estimates* (3.2) *hold in $\mathbb{R}_+$. If $\varepsilon := \sup_{t\geq 0} \|B(t)\|$ is sufficiently small, then perturbed equation* (4.1) *also possesses a conformable exponential dichotomy in $\mathbb{R}_+$.*

**Proof.** We divide the proof of Theorem 4.1 into the following three steps.

Step 1: Finding bounded solutions of equation (4.1). Let matrix function $Y \in C_b(\mathbb{R}_+, \mathbb{R}^{n\times n})$ equipped with norm

$$\|Y\|_\infty := \sup_{t\geq 0} \|Y(t)\|.$$

Define mapping $L : C_b(\mathbb{R}_+, \mathbb{R}^{n\times n}) \to C_b(\mathbb{R}_+, \mathbb{R}^{n\times n})$ as

$$LY(t) = X(t)P + X(t)P\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y(t) + X(t)(I - P)\mathcal{I}_{+\infty}^\alpha X^{-1}(t)B(t)Y(t).$$

It follows from (3.2) that

$$\|LY(t)\| \leq \frac{N_1}{E_\alpha(\beta_1, t)} + \frac{\varepsilon N_1\|Y\|_\infty}{E_\alpha(\beta_1, t)}\mathcal{I}_0^\alpha E_\alpha(\beta_1, t) - E_\alpha(\beta_2, t)\mathcal{I}_{+\infty}^\alpha \frac{\varepsilon N_2\|Y\|_\infty}{E_\alpha(\beta_2, t)}.$$

Observing that $LY(t)$ is bounded and continuous for $t \geq 0$, we obtain

$$\|LY\|_\infty \leq N_1 + \varepsilon\Big(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}\Big)\|Y\|_\infty.$$

Given another $\hat{Y} \in C_b(\mathbb{R}_+, \mathbb{R}^{n\times n})$, analogously we get

$$\|LY - L\hat{Y}\|_\infty \leq \varepsilon\Big(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}\Big)\|Y - \hat{Y}\|_\infty.$$

This yields that the mapping $L$ has a unique $Y_1 \in C_b(\mathbb{R}_+, \mathbb{R}^{n\times n})$ such that

$$\begin{aligned} Y_1(t) =& X(t)P + X(t)P\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_1(t) \\ &+ X(t)(I - P)\mathcal{I}_{+\infty}^\alpha X^{-1}(t)B(t)Y_1(t), \end{aligned} \tag{4.2}$$

if

$$\theta := \varepsilon\Big(\frac{N_1}{\beta_1} + \frac{N_2}{\beta_2}\Big) < 1.$$

Obviously, $Y_1(t)$ is also a matrix solution of (4.1) and differentiable. Post-projecting $P$ on both hands sides of (4.2), we also know that $Y_1(t)P$ is the unique fixed point of $L$, and $Y_1(t)P = Y_1(t)$.

Step 2: Constructing projection matrix. Let $Q := Y_1(0)$, then $QP = Q$. Combining (4.2) with the property $P(I-P) = 0$, and replacing $t$ with $s$, we attain

$$X(t)PX^{-1}(s)Y_1(s) = X(t)P + X(t)P\mathcal{I}_0^\alpha X^{-1}(s)B(s)Y_1(s). \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned} Y_1(t) =& X(t)PX^{-1}(s)Y_1(s) + X(t)P\mathcal{I}_s^\alpha X^{-1}(t)B(t)Y_1(t) \\ &+ X(t)(I-P)\mathcal{I}_{+\infty}^\alpha X^{-1}(t)B(t)Y_1(t), \quad t \geq s \geq 0. \end{aligned} \tag{4.4}$$

Noting (4.3) with $t = s = 0$, we gain $PQ = P$. Post-projecting $Q$ on both hands sides of (4.2) again, we acquire

$$\begin{aligned} Y_1(t)Q = & X(t)P + X(t)P\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_1(t)Q \\ &+X(t)(I-P)\mathcal{I}_{+\infty}^\alpha X^{-1}(t)B(t)Y_1(t)Q, \end{aligned}$$

implying $Y_1(t)Q$ is also a fixed point of $L$. In conclusion,

$$Y_1(t)Q = Y_1(t) = Y_1(t)P.$$

Obviously, $Q$ is a projection when $t = 0$.

Provided $Y(t)$ is a fundamental matrix of (4.1) fulfilling $Y(0) = I$, we derive

$$Y_1(t) = Y(t)Q. \tag{4.5}$$

Set

$$Y_2(t) := Y(t)(I-Q), \tag{4.6}$$

then $Y(t) = Y_1(t) + Y_2(t)$. Relying on the variation of constants formula (2.11), we calculate that

$$\begin{aligned} Y_2(t) &= X(t)X^{-1}(0)Y_2(0) + X(t)\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t) \\ &= X(t)Y(0)(I-Q) + X(t)\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t) \\ &= X(t)(I-Q) + X(t)\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t). \end{aligned} \tag{4.7}$$

Combining (4.7) with the fact $(I-P)(I-Q) = I-Q$, and replacing $t$ with $s$, we acquire

$$X(t)(I-P)X^{-1}(s)Y_2(s) = X(t)(I-Q) + X(t)(I-P)\mathcal{I}_0^\alpha X^{-1}(s)B(s)Y_2(s). \tag{4.8}$$

Subsequently, by (4.7) and (4.8), we receive

$$\begin{aligned} Y_2(t) &= X(t)(I-P)X^{-1}(s)Y_2(s) + X(t)\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t) \\ &\quad -X(t)(I-P)\mathcal{I}_0^\alpha X^{-1}(s)B(s)Y_2(s) \\ &= X(t)(I-P)X^{-1}(s)Y_2(s) + X(t)P\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t) \\ &\quad +X(t)(I-P)\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t) - X(t)(I-P)\mathcal{I}_0^\alpha X^{-1}(s)B(s)Y_2(s) \\ &= X(t)(I-P)X^{-1}(s)Y_2(s) + X(t)P\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t) \\ &\quad +X(t)(I-P)\mathcal{I}_s^\alpha X^{-1}(t)B(t)Y_2(t), \quad s \geq t \geq 0. \end{aligned} \tag{4.9}$$

From (4.4) and (4.9) it follows that for any vector $\xi$,

$$\begin{aligned}\|Y_1(t)\xi\| \le N_1\frac{E_\alpha(\beta_1,s)}{E_\alpha(\beta_1,t)}\|Y_1(s)\xi\| + \frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}|\mathcal{I}_s^\alpha E_\alpha(\beta_1,t)Y_1(t)\xi| \\ -\varepsilon N_2 E_\alpha(\beta_2,t)\Big|\mathcal{I}_{+\infty}^\alpha \frac{Y_1(t)\xi}{E_\alpha(\beta_2,t)}\Big|, \quad t\ge s\ge 0,\end{aligned}$$

and

$$\begin{aligned}\|Y_2(t)\xi\| \le N_2\frac{E_\alpha(\beta_2,t)}{E_\alpha(\beta_2,s)}\|Y_2(s)\xi\| + \frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}|\mathcal{I}_0^\alpha E_\alpha(\beta_1,t)Y_2(t)\xi| \\ -\varepsilon N_2 E_\alpha(\beta_2,t)\Big|\mathcal{I}_s^\alpha \frac{Y_2(t)\xi}{E_\alpha(\beta_2,t)}\Big|, \quad s\ge t\ge 0.\end{aligned}$$

Thus, by Lemma 3.2 and Corollary 3.1, we can know

$$\begin{aligned}\|Y_1(t)\xi\| \le K_1\frac{E_\alpha(\lambda_1,s)}{E_\alpha(\lambda_1,t)}\|Y_1(s)\xi\|, \quad t\ge s\ge 0, \\ \|Y_2(t)\xi\| \le K_2\frac{E_\alpha(\lambda_2,t)}{E_\alpha(\lambda_2,s)}\|Y_2(s)\xi\|, \quad s\ge t\ge 0,\end{aligned} \tag{4.10}$$

where $K_i = \dfrac{N_i}{1-\theta}$, $\lambda_i = \beta_i - \dfrac{\varepsilon N_i}{1-\theta}$ and $\theta = \varepsilon\Big(\dfrac{N_1}{\beta_1}+\dfrac{N_2}{\beta_2}\Big)$ for $i=1,2$.

Step 3: Estimation of fundamental solutions. To prove from (4.10) that the perturbed equation (4.1) also possesses a conformable exponential dichotomy, we only need to exhibit that $Y(t)QY^{-1}(t)$ is bounded. From the facts $(I-P)P=0$, $(I-P)(I-P)=I-P$ and (4.2) it follows that

$$X(t)(I-P)X^{-1}(t)Y_1(t) = X(t)(I-P)\mathcal{I}_{+\infty}^\alpha X^{-1}(t)B(t)Y_1(t).$$

By (4.10), for any vector $\xi$, we calculate that

$$\begin{aligned}\|X(t)(I-P)X^{-1}(t)Y_1(t)\xi\| &\le -\varepsilon N_2 E_\alpha(\beta_2,t)\mathcal{I}_{+\infty}^\alpha \frac{\|Y_1(t)\xi\|}{E_\alpha(\beta_2,t)} \\ &\le -E_\alpha(\lambda_1+\beta_2,t)\|Y_1(t)\xi\|\mathcal{I}_{+\infty}^\alpha \frac{\varepsilon K_1 N_2}{E_\alpha(\lambda_1+\beta_2,t)} \\ &\le \frac{\varepsilon K_1 N_2}{\lambda_1+\beta_2}\|Y_1(t)\xi\|.\end{aligned} \tag{4.11}$$

Analogously, pre-multiplying $X(t)PX^{-1}(t)$ on both hands sides of (4.7), by the property $P(I-Q)=0$, we obtain

$$X(t)PX^{-1}(t)Y_2(t) = X(t)P\mathcal{I}_0^\alpha X^{-1}(t)B(t)Y_2(t).$$

It follows from (4.10) that for any vector $\xi$,

$$\begin{aligned}\|X(t)PX^{-1}(t)Y_2(t)\xi\| &\le \frac{\varepsilon N_1}{E_\alpha(\beta_1,t)}\Big|\mathcal{I}_0^\alpha E_\alpha(\beta_1,t)Y_2(t)\xi\Big| \\ &\le \frac{\varepsilon K_2 N_1}{E_\alpha(\lambda_2+\beta_1,t)}\|Y_2(t)\xi\|\mathcal{I}_0^\alpha E_\alpha(\lambda_2+\beta_1,t) \\ &\le \frac{\varepsilon K_2 N_1}{\lambda_2+\beta_1}\|Y_2(t)\xi\|. \qquad (4.12)\end{aligned}$$

Substituting (4.5) and (4.6) into (4.11) and (4.12) respectively, and replacing $\xi$ by $Y^{-1}(t)\xi$, we acquire

$$\begin{aligned}\|X(t)(I-P)X^{-1}(t)Y_1(t)\xi\| &\le \|X(t)(I-P)X^{-1}(t)Y(t)QY^{-1}(t)\xi\| \\ &\le \frac{\varepsilon K_1 N_2}{\lambda_1+\beta_2}\|Y(t)QY^{-1}(t)\xi\|, \qquad (4.13)\end{aligned}$$

and

$$\begin{aligned}\|X(t)PX^{-1}(t)Y_2(t)\xi\| &\le \|X(t)PX^{-1}(t)Y(t)(I-Q)Y^{-1}(t)\xi\| \\ &\le \frac{\varepsilon K_2 N_1}{\lambda_2+\beta_1}\|Y(t)(I-Q)Y^{-1}(t)\xi\|. \qquad (4.14)\end{aligned}$$

On the other hand, it is evident to derive that

$$\begin{aligned}Y(t)QY^{-1}(t)-X(t)PX^{-1}(t) &= X(t)[P+(I-P)]X^{-1}(t)Y(t)QY^{-1}(t) \\ &\quad -X(t)PX^{-1}(t)Y(t)[Q+(I-Q)]Y^{-1}(t) \\ &= X(t)PX^{-1}(t)Y(t)QY^{-1}(t) \\ &\quad +X(t)(I-P)X^{-1}(t)Y(t)QY^{-1}(t) \\ &\quad -X(t)PX^{-1}(t)Y(t)QY^{-1}(t) \\ &\quad -X(t)PX^{-1}(t)Y(t)(I-Q)Y^{-1}(t) \\ &= X(t)(I-P)X^{-1}(t)Y(t)QY^{-1}(t) \\ &\quad -X(t)PX^{-1}(t)Y(t)(I-Q)Y^{-1}(t). \qquad (4.15)\end{aligned}$$

Combining (4.13) and (4.14) with (4.15), we can obtain

$$\|Y(t)QY^{-1}(t)-X(t)PX^{-1}(t)\| \le \frac{\varepsilon K_1 N_2}{\lambda_1+\beta_2}\mu_1 + \frac{\varepsilon K_2 N_1}{\lambda_2+\beta_1}\mu_2,$$

where $\mu_1(t) := \|Y(t)QY^{-1}(t)\|$, $\mu_2(t) := \|Y(t)(I-Q)Y^{-1}(t)\|$. For convenience, take $N := \max\{N_1, N_2\}$ and $\beta := \min\{\beta_1, \beta_2\}$ such that $\theta \le \hat{\theta} := 2\varepsilon N/\beta$, and it yields

$$\begin{aligned}\mu_1 = \|Y(t)QY^{-1}(t)\| &\le \Big(\frac{\varepsilon K_1 N_2}{\lambda_1+\beta_2}\mu_1 + \frac{\varepsilon K_2 N_1}{\lambda_2+\beta_1}\mu_2\Big) + \|X(t)PX^{-1}(t)\| \\ &\le \Big(\frac{\varepsilon K_1 N_2}{\lambda_1+\beta_2}\mu_1 + \frac{\varepsilon K_2 N_1}{\lambda_2+\beta_1}\mu_2\Big) + N_1 \\ &\le \eta(\mu_1+\mu_2)+N, \qquad (4.16)\end{aligned}$$

where $\eta = \dfrac{\varepsilon N^2}{2\beta - 5\varepsilon N}$. It is obvious that

$$Y(t)QY^{-1}(t) - X(t)PX^{-1}(t) = X(t)(I-P)X^{-1}(t) - Y(t)(I-Q)Y^{-1}(t),$$

then

$$\mu_2 \le \Big(\frac{\varepsilon K_1 N_2}{\lambda_1 + \beta_2}\mu_1 + \frac{\varepsilon K_2 N_1}{\lambda_2 + \beta_1}\mu_2\Big) + N_2 \le \eta(\mu_1 + \mu_2) + N. \tag{4.17}$$

By adding the inequality (4.16) and (4.17), we attain

$$\mu_1 + \mu_2 \le \frac{2N}{1 - 2\eta}.$$

If $\eta < 1/2$, then

$$\mu_1, \mu_2 \le \eta(\mu_1 + \mu_2) + N \le \frac{N}{1 - 2\eta}.$$

Substituting (4.5) and (4.6) into (4.10), and replacing $\xi$ by $Y^{-1}(s)\xi$, we gain

$$\begin{aligned}
\|Y(t)QY^{-1}(s)\xi\| &\le K_1 \frac{E_\alpha(\lambda_1, s)}{E_\alpha(\lambda_1, t)}\|Y(s)QY^{-1}(s)\xi\| \\
&\le \frac{K_1 N}{1 - 2\eta}\frac{E_\alpha(\lambda_1, s)}{E_\alpha(\lambda_1, t)}\|\xi\|, \quad t \ge s \ge 0, \\
\|Y(t)(I-Q)Y^{-1}(s)\xi\| &\le K_2 \frac{E_\alpha(\lambda_2, t)}{E_\alpha(\lambda_2, s)}\|Y(s)(I-Q)Y^{-1}(s)\xi\| \\
&\le \frac{K_2 N}{1 - 2\eta}\frac{E_\alpha(\lambda_2, t)}{E_\alpha(\lambda_2, s)}\|\xi\|, \quad s \ge t \ge 0.
\end{aligned}$$

For the arbitrariness of vector $\xi$, we obtain the conformable exponential dichotomy as follows

$$\begin{aligned}
\|Y(t)QY^{-1}(s)\| &\le \frac{K_1 N}{1 - 2\eta}\frac{E_\alpha(\lambda_1, s)}{E_\alpha(\lambda_1, t)}, \quad t \ge s \ge 0, \\
\|Y(t)(I-Q)Y^{-1}(s)\| &\le \frac{K_2 N}{1 - 2\eta}\frac{E_\alpha(\lambda_2, t)}{E_\alpha(\lambda_2, s)}, \quad s \ge t \ge 0.
\end{aligned}$$

Therefore, Theorem 4.1 is proved completely. □

Finally, we present the concrete constants of estimates in Theorem 4.1. Like the condition $N = \max\{N_1, N_2\}$ and $\beta = \min\{\beta_1, \beta_2\}$ such that $\theta \le \hat{\theta} = 2\varepsilon N/\beta$ holds, let $N \ge 1$ and $\hat{\theta} < 2/5N$ such that $\eta = \dfrac{\varepsilon N^2}{2\beta - 5\varepsilon N} < \dfrac{N}{10N - 5} < \dfrac{1}{2}$. Thus, by elementary calculation, we can obtain the following brief statement.

**Corollary 4.1** *Suppose that equation* (2.1) *possesses the conformable exponential dichotomy* (3.2) *in* $\mathbb{R}_+$. *If*

$$\varepsilon := \sup_{t\in\mathbb{R}_+} \|B(t)\| < \frac{\beta}{5N^2},$$

*then perturbed equation* (4.1) *also possesses the following conformable exponential dichotomy:*

$$\|Y(t)QY^{-1}(s)\| \leq \frac{25N^2}{9}\frac{E_\alpha(\beta-3\varepsilon N, s)}{E_\alpha(\beta-3\varepsilon N, t)}, \quad t \geq s \geq 0,$$
$$\|Y(t)(I-Q)Y^{-1}(s)\| \leq \frac{25N^2}{9}\frac{E_\alpha(\beta-3\varepsilon N, t)}{E_\alpha(\beta-3\varepsilon N, s)}, \quad s \geq t \geq 0,$$

*where* $Y(t)$ *is a fundamental matrix of* (4.1) *such that* $Y(0) = I$, *and both projection matrices* $Q$ *and* $P$ *have the same rank. Moreover,*

$$\|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)\| \leq \frac{2\varepsilon N^3}{2\beta - 5\varepsilon N - 2\varepsilon N^2}, \quad t \geq 0.$$

# 5 Nonuniform dichotomy

This section is a continuation of studies for the conformable exponential dichotomy. More precisely, we concern *nonuniform conformable exponential dichotomy.* Let $\mathcal{B}(Z)$ consist of all bounded linear operators in Banach space $Z$. Consider nonautonomous linear CFDE on $Z$

$$\mathcal{T}^\alpha x = A(t)x, \quad (t, x) \in J \times Z, \tag{5.1}$$

where linear operator $A \in C(J, \mathcal{B}(Z))$ for some interval $J \subset \mathbb{R}$ and $\mathcal{B}(Z)$ is also a Banach space with the norm $\|A\| := \sup_{x\in Z, \|x\|=1} \|Ax\|$ for all $A \in \mathcal{B}(Z)$. Let $T(t, s)$ be a family of evolution operators satisfying $x(t) = T(t, s)x(s)$ for $t \geq s$ and $t, s \in J$, where $x(t)$ is any solution of (5.1). $T(t, s)$ further satisfies:

**(F1)** $T(t, t) = \mathrm{Id}$ (abbreviation of identity) for $t \in J$;

**(F2)** $T(t, s)T(s, \tau) = T(t, \tau)$ for $t, s, \tau \in J$;

**(F3)** the evolution operator $T(t, s)$ is invertible and $T^{-1}(t, s) = T(s, t)$ for $t, s \in J$.

First, we introduce the notions of nonuniform asymptotical stability and nonuniform conformable exponential dichotomy.

**Definition 5.1** *Equation* (5.1) *is said to be nonuniform asymptotically stable in* $J$ *if there exist constants* $\hat{N}, \hat{\beta} > 0$ *and* $\epsilon \geq 0$ *such that*

$$\|T(t,s)\| \leq \hat{N}\frac{E_\alpha(\hat{\beta},s)}{E_\alpha(\hat{\beta},t)}E_\alpha(\epsilon,|s|), \quad t \geq s, \quad t,s \in J. \tag{5.2}$$

*In particular,* (5.1) *is uniformly asymptotically stable like* (3.1) *if* (5.2) *hold with* $\epsilon = 0$.

**Definition 5.2** *Equation* (5.1) *is said to admit a nonuniform conformable exponential dichotomy in* $J$ *if there exist projections* $P : J \to \mathcal{B}(Z)$ *such that*

$$T(t,s)P(s) = P(t)T(t,s), \quad t \geq s, \quad t,s \in J, \tag{5.3}$$

*and constants* $\hat{N}_i, \hat{\beta}_i > 0$ $(i = 1,2)$ *and* $\epsilon \geq 0$ *such that for* $t,s \in J$,

$$\begin{aligned} \|T(t,s)P(s)\| &\leq \hat{N}_1\frac{E_\alpha(\hat{\beta}_1,s)}{E_\alpha(\hat{\beta}_1,t)}E_\alpha(\epsilon,|s|), \quad t \geq s, \\ \|T(t,s)(\mathrm{Id} - P(s))\| &\leq \hat{N}_2\frac{E_\alpha(\hat{\beta}_2,t)}{E_\alpha(\hat{\beta}_2,s)}E_\alpha(\epsilon,|s|), \quad s \geq t. \end{aligned} \tag{5.4}$$

*In particular,* (5.1) *admits a uniform conformable exponential dichotomy like* (3.2) *if* (5.4) *hold with* $\epsilon = 0$.

All results in this section are presented in $\mathbb{R}_+$, and denote

$$\mathcal{I}_s^\alpha f(t,\cdot) := \int_s^t \tau^{\alpha-1} f(t,\tau)d\tau.$$

Consider the linear perturbation of (5.1) as follows

$$\mathcal{T}^\alpha x = [A(t) + B(t)]x, \quad (t,x) \in \mathbb{R}_+ \times Z, \tag{5.5}$$

where linear operators $A \in C(\mathbb{R}_+, \mathcal{B}(Z))$ and $B \in C_b(\mathbb{R}_+, \mathcal{B}(Z))$. The following theorem gives out the roughness of nonuniform asymptotical stability.

**Theorem 5.1** *Assume that equation* (5.1) *admits nonuniform asymptotical stability in* $\mathbb{R}_+$, *and there exists constant* $\delta$ *such that* $\|B(t)\| \leq \delta/E_\alpha(\epsilon,t)$ *for* $t \in \mathbb{R}_+$. *If* $\theta := \delta\hat{N}/\hat{\beta} < 1$, *then equation* (5.5) *also admits nonuniform asymptotical stability in* $\mathbb{R}_+$, *that is,*

$$\|U(t,s)\| \leq \frac{\hat{N}}{1-\theta}\frac{E_\alpha(\gamma,s)}{E_\alpha(\gamma,t)}E_\alpha(\epsilon,s), \quad t \geq s, \quad t,s \in \mathbb{R}_+, \tag{5.6}$$

*where* $\gamma = \hat{\beta} - \dfrac{\delta\hat{N}}{1-\theta}$ *and* $U(t,s)$ *denotes the evolution operator associated to* (5.5).

**Proof.** Consider the space

$$W := \{U(t,s)_{t\geq s} \in \mathcal{B}(Z) : U \text{ is continuous and } \|U\|_\alpha < \infty,\ (t,s) \in \mathbb{R}_+^2\},$$

equipped with $\alpha$-weighted norm

$$\|U\|_\alpha := \sup\left\{\frac{\|U(t,s)\|}{E_\alpha(\epsilon,s)} : t \geq s, (t,s) \in \mathbb{R}_+^2\right\}. \tag{5.7}$$

It is easy to verify that $W$ is a Banach space. In $W$ define an operator $\mathcal{J}$ by

$$(\mathcal{J}U)(t,s) = T(t,s) + \mathcal{I}_s^\alpha T(t,\cdot)B(\cdot)U(\cdot,s).$$

It follows from (5.2) that

$$\begin{aligned}
\|(\mathcal{J}U)(t,s)\| &\leq \|T(t,s)\| + \mathcal{I}_s^\alpha \|T(t,\cdot)\|\|B(\cdot)\|\|U(\cdot,s)\| \\
&\leq \hat{N}\frac{E_\alpha(\hat{\beta},s)}{E_\alpha(\hat{\beta},t)}E_\alpha(\epsilon,s) + \frac{\delta\hat{N}\|U\|_\alpha E_\alpha(\epsilon,s)}{E_\alpha(\hat{\beta},t)}\mathcal{I}_s^\alpha E_\alpha(\hat{\beta},t) \\
&\leq \hat{N}E_\alpha(\epsilon,s) + \frac{\delta\hat{N}}{\hat{\beta}}\|U\|_\alpha E_\alpha(\epsilon,s).
\end{aligned}$$

And by (5.7) we obtain

$$\|\mathcal{J}U\|_\alpha \leq \hat{N} + \frac{\delta\hat{N}}{\hat{\beta}}\|U\|_\alpha < \infty,$$

which yields that the operator $\mathcal{J} : W \to W$ is well defined. Analogously to the computation above, we have

$$\|\mathcal{J}U_1 - \mathcal{J}U_2\|_\alpha \leq \frac{\delta\hat{N}}{\hat{\beta}}\|U_1 - U_2\|_\alpha, \quad U_1, U_2 \in W,$$

which implies that $\mathcal{J}$ is a contraction since $\delta < \hat{\beta}/\hat{N}$. So there exists a unique $U \in W$ satisfying $\mathcal{J}U = U$, and one can verify that it is a solution of (5.5). We apply Lemma 3.2 with condition $\theta := \delta\hat{N}/\hat{\beta} < 1$ to the estimation of $\|U(t,s)\|$. And inequality (5.6) is true. □

Subsequently, our purpose is to establish roughness of nonuniform conformable exponential dichotomy in $\mathbb{R}_+$. A preliminary theorem and the main theorem of roughness are both stated as follows.

**Theorem 5.2** *Assume that equation* (5.1) *admits a nonuniform conformable exponential dichotomy* (5.4) *in* $\mathbb{R}_+$, *and there exists constant* $\delta$ *such that* $\|B(t)\| \leq \delta/E_\alpha(\epsilon,t)$ *for* $t \in \mathbb{R}_+$. *If*

$$\theta := \delta\left(\frac{\hat{N}_1}{\hat{\beta}_1} + \frac{\hat{N}_2}{\hat{\beta}_2}\right) < 1, \quad \epsilon < \min\{\hat{\beta}_1, \hat{\beta}_2\}, \tag{5.8}$$

*then there exist projections* $\hat{P}:\mathbb{R}_+\to\mathcal{B}(Z)$ *such that*

$$\hat{T}(t,s)\hat{P}(s)=\hat{P}(t)\hat{T}(t,s),\quad t\geq s,\quad t,s\in\mathbb{R}_+, \tag{5.9}$$

*and constants* $K_i,\lambda_i>0$ $(i=1,2)$ *and* $\epsilon\geq 0$ *such that*

$$\begin{aligned}\|\hat{T}(t,s)|\mathrm{Im}\hat{P}(s)\|&\leq K_1\frac{E_\alpha(\lambda_1,s)}{E_\alpha(\lambda_1,t)}E_\alpha(\epsilon,s),\quad t\geq s\geq 0,\\ \|\hat{T}(t,s)|\mathrm{Im}(\mathrm{Id}-\hat{P}(s))\|&\leq K_2\frac{E_\alpha(\lambda_2,t)}{E_\alpha(\lambda_2,s)}E_\alpha(\epsilon,s),\quad s\geq t\geq 0,\end{aligned} \tag{5.10}$$

*where* $K_i=\dfrac{\hat{N}_i}{1-\theta}$, $\lambda_i=\hat{\beta}_i-\dfrac{\delta\hat{N}_i}{1-\theta}$ $(i=1,2)$, *and* $\hat{T}(t,s)$ *is the evolution operator associated to equation* (5.5).

**Theorem 5.3** *Assume that equation* (5.1) *admits a nonuniform conformable exponential dichotomy* (5.4) *in* $\mathbb{R}_+$ *under condition* (5.8). *If* $\delta$ *is sufficiently small such that* $\|B(t)\|\leq\delta/E_\alpha(2\epsilon,t)$ *for* $t\in\mathbb{R}_+$, *then equation* (5.5) *also admits a nonuniform conformable exponential dichotomy in* $\mathbb{R}_+$.

**Proof of Theorem 5.2.** We divide the proof into the following several steps.

Step 1: Construction of bounded solutions for (5.5). Recall space $W$ in Theorem 5.1, then the following lemma gives out the existence of bounded solution.

**Lemma 5.1** *For each* $t,s\in\mathbb{R}_+$, *equation* (5.5) *has a unique solution* $U\in W$ *such that*

$$\begin{aligned}U(t,s)=&T(t,s)P(s)+\mathcal{I}_s^\alpha T(t,\cdot)P(\cdot)B(\cdot)U(\cdot,s)\\ &+\mathcal{I}_{+\infty}^\alpha T(t,\cdot)(\mathrm{Id}-P(\cdot))B(\cdot)U(\cdot,s),\quad t\geq s.\end{aligned} \tag{5.11}$$

**Proof.** Clearly, if the function $U(t,s)_{t\geq s}$ satisfies (5.11), then it is a solution of (5.5). We must demonstrate that the operator $L$ defined by

$$\begin{aligned}(LU)(t,s)=&\,T(t,s)P(s)+\mathcal{I}_s^\alpha T(t,\cdot)P(\cdot)B(\cdot)U(\cdot,s)\\ &+\mathcal{I}_{+\infty}^\alpha T(t,\cdot)(\mathrm{Id}-P(\cdot))B(\cdot)U(\cdot,s),\quad t\geq s,\end{aligned}$$

has a unique fixed point in $W$. It follows from (5.4) that

$$\begin{aligned}\|(LU)(t,s)\|\leq&\,\|T(t,s)P(s)\|+\mathcal{I}_s^\alpha\|T(t,\cdot)P(\cdot)\|\|B(\cdot)\|\|U(\cdot,s)\|\\ &-\mathcal{I}_{+\infty}^\alpha\|T(t,\cdot)(\mathrm{Id}-P(\cdot))\|\|B(\cdot)\|\|U(\cdot,s)\|\\ \leq&\,\hat{N}_1\frac{E_\alpha(\hat{\beta}_1,s)}{E_\alpha(\hat{\beta}_1,t)}E_\alpha(\epsilon,s)+\delta\left(\frac{\hat{N}_1}{\hat{\beta}_1}+\frac{\hat{N}_2}{\hat{\beta}_2}\right)\|U\|_\alpha E_\alpha(\epsilon,s).\end{aligned}$$

Combining (5.7) with (5.8), we obtain

$$\|LU\|_\alpha \le \hat{N}_1 + \theta\|U\|_\alpha < \infty,$$

this implies that the operator $L : W \to W$ is well defined. Analogously to the computation above, we have

$$\|LU_1 - LU_2\|_\alpha \le \theta\|U_1 - U_2\|_\alpha, \quad U_1, U_2 \in W,$$

which shows that $L$ is a contraction since $\theta < 1$. Then there exists a unique $U \in W$ such that $LU = U$. Therefore, Lemma 5.1 is proved. $\square$

Now we explain that the bounded solutions exhibit the following property.

**Lemma 5.2** *For each $t \ge \tau \ge s$ in $\mathbb{R}_+$,*

$$U(t,\tau)U(\tau,s) = U(t,s).$$

**Proof.** From (5.11) and (5.3), for some $\tau \in \mathbb{R}_+$ we can calculate that

$$\begin{aligned} U(t,\tau)U(\tau,s) = {} & T(t,s)P(s) + \mathcal{I}_s^\alpha T(t,\tau)P(\tau)B(\tau)U(\tau,s) \\ & + \mathcal{I}_\tau^\alpha T(t,\cdot)P(\cdot)B(\cdot)U(\cdot,\tau)U(\tau,s) \\ & + \mathcal{I}_{+\infty}^\alpha T(t,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)U(\cdot,\tau)U(\tau,s), \quad t \ge \tau \ge s. \end{aligned}$$

Let $H(t,\tau) := U(t,\tau)U(\tau,s) - U(t,s)$ for $t \ge \tau \ge s$, this yields

$$H(t,\tau) = \mathcal{I}_\tau^\alpha T(t,\cdot)P(\cdot)B(\cdot)H(\cdot,s) + \mathcal{I}_{+\infty}^\alpha T(t,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)H(\cdot,s). \quad (5.12)$$

Define operator $\mathcal{K}$ as

$$(\mathcal{K}\hat{H})(t,\tau) := \mathcal{I}_\tau^\alpha T(t,\cdot)P(\cdot)B(\cdot)\hat{H}(\cdot,s) + \mathcal{I}_{+\infty}^\alpha T(t,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)\hat{H}(\cdot,s),$$

for any $\hat{H} \in W$ and $t \ge \tau$. It follows from the identity above and (5.4) that

$$\begin{aligned} \|(\mathcal{K}\hat{H})(t,\tau))\| & \le \mathcal{I}_\tau^\alpha \|T(t,\cdot)P(\cdot)\|\|B(\cdot)\|\|\hat{H}(\cdot,s)\| \\ & \quad - \mathcal{I}_{+\infty}^\alpha \|T(t,\cdot)(\mathrm{Id} - P(\cdot))\|\|B(\cdot)\|\|\hat{H}(\cdot,s)\| \\ & \le \delta\left(\frac{\hat{N}_1}{\hat{\beta}_1} + \frac{\hat{N}_2}{\hat{\beta}_2}\right)\|\hat{H}\|_\alpha E_\alpha(\epsilon,s). \end{aligned}$$

By (5.7), we have

$$\|\mathcal{K}\hat{H}\|_\alpha \le \theta\|\hat{H}\|_\alpha < \infty,$$

then $\mathcal{K}: W \to W$ is well defined for $t \geq \tau$. Similarly to the calculation above, we attain

$$\|\mathcal{K}\hat{H}_1 - \mathcal{K}\hat{H}_2\|_\alpha \leq \theta \|\hat{H}_1 - \hat{H}_2\|_\alpha, \quad \hat{H}_1, \hat{H}_2 \in W.$$

Because of hypothesis (5.8), $\mathcal{K}$ is a contraction. Thus, there is a unique $\hat{H} \in W$ such that $\mathcal{K}\hat{H} = \hat{H}$. On the other hand, we know that $0 \in W$ satisfies (5.12) and $\mathcal{K}0 = 0$. By Lemma 5.1, we assert $H = \hat{H} = 0$ for $t \geq \tau \geq s$ in $\mathbb{R}_+$. Therefore, Lemma 5.2 is proved. □

Step 2: Establishment of projections $\hat{P}(t)$ in (5.9). Given constant $\iota \in \mathbb{R}_+$, for any $t \geq \iota$ in $\mathbb{R}_+$, we consider the following linear operator

$$\hat{P}(t) := \hat{T}(t,\iota)U(\iota,\iota)\hat{T}(\iota,t), \tag{5.13}$$

where $\hat{T}(t,s)$ is the evolution operator associated to (5.5). Clearly, the operator $\hat{P}(t)$ may depend on $\iota$, and $U(\iota,\iota)U(\iota,\iota) = U(\iota,\iota)$ by Lemma 5.2. The following lemma illustrates the commutativity of projections $\hat{P}(t)$ as formula (5.9).

**Lemma 5.3** *For any $t \in \mathbb{R}_+$, the operator $\hat{P}(t)$ is a projection satisfying* (5.9).

**Proof.** By the details above and **(F1)-(F2)**, we derive

$$\begin{aligned}\hat{P}(t)\hat{P}(t) &= \hat{T}(t,\iota)U(\iota,\iota)\hat{T}(\iota,t)\hat{T}(t,\iota)U(\iota,\iota)\hat{T}(\iota,t)\\ &= \hat{T}(t,\iota)U(\iota,\iota)U(\iota,\iota)\hat{T}(\iota,t) = \hat{P}(t),\end{aligned}$$

then $\hat{P}(t)$ is a projection. Furthermore, for $t \geq s$ we can calculate that

$$\hat{T}(t,s)\hat{P}(s) = \hat{T}(t,s)\hat{T}(s,\iota)U(\iota,\iota)\hat{T}(\iota,t)\hat{T}(t,s) = \hat{P}(t)\hat{T}(t,s).$$

This completes the proof of Lemma 5.3. □

Step 3: Characterization of bounded solutions. The following two lemmas propose the nonuniform projection integral equation and its property respectively.

**Lemma 5.4** *For some $s \in \mathbb{R}_+$, if $z \in C_b([s,+\infty), Z)$ is a solution of* (5.5) *with $z(s) = z_s$, then*

$$z(t) = T(t,s)P(s)z_s + \mathcal{I}_s^\alpha T(t,\cdot)P(\cdot)B(\cdot)z(\cdot) + \mathcal{I}_{+\infty}^\alpha T(t,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)z(\cdot).$$

The proof of this lemma is similar to the method of Lemma 3.1 when $\epsilon < \min\{\hat{\beta}_1, \hat{\beta}_2\}$ holds.

**Lemma 5.5** *For some $s \in \mathbb{R}_+$, if the function $\hat{P}(\cdot)\hat{T}(\cdot, s) \in C_b([s, +\infty), \mathcal{B}(Z))$, then*

$$\begin{aligned}\hat{P}(t)\hat{T}(t,s) =& T(t,s)P(s)\hat{P}(s) + \mathcal{I}_s^\alpha T(t,\cdot)P(\cdot)B(\cdot)\hat{P}(\cdot)\hat{T}(\cdot,s) \\ &+ \mathcal{I}_{+\infty}^\alpha T(t,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)\hat{P}(\cdot)\hat{T}(\cdot,s).\end{aligned} \tag{5.14}$$

**Proof.** For a given $\iota \in \mathbb{R}_+$, it follows from Lemma 5.1 that the function $U(t,\iota)\xi$ is a solution of (5.5) with initial value $U(\iota,\iota)\xi$ for any $\xi \in Z$ and $t \geq \iota$. By (5.13) and (5.9), we gain $U(t,\iota) = \hat{T}(t,\iota)U(\iota,\iota)$, and

$$\begin{aligned}\hat{P}(t)\hat{T}(t,s) &= \hat{T}(t,s)\hat{P}(s) = \hat{T}(t,s)\hat{T}(s,\iota)U(\iota,\iota)\hat{T}(\iota,s) \\ &= \hat{T}(t,\iota)U(\iota,\iota)\hat{T}(\iota,s) = U(t,\iota)\hat{T}(\iota,s).\end{aligned}$$

Thus, the equation (5.5) has solution in the form of $U(t,\iota)\xi$ as follows

$$z(t) = \hat{P}(t)\hat{T}(t,s)\xi = U(t,\iota)\hat{T}(\iota,s)\xi, \quad \xi \in Z.$$

Observing that the above solution is bounded for $t \geq s$, and

$$z(s) = U(s,\iota)\hat{T}(\iota,s)\xi = \hat{P}(s)\hat{T}(s,s)\xi = \hat{P}(s)\xi,$$

we employ Lemma 5.4 to complete the proof of Lemma 5.5. □

The following Lemma is the projected integral inequality in the case of nonuniform conformable exponential dichotomy, and the method of its proof can be referred to the Lemma 3.2 and Corollary 3.1.

**Lemma 5.6** *Given $s \in \mathbb{R}_+$. Assume that the functions $u \in C_b([s, +\infty), \mathbb{R}_+)$ and $v \in C_b([0, s], \mathbb{R}_+)$ respectively satisfy the following inequalities*

$$\begin{aligned}u(t) \leq& \hat{N}_1 \frac{E_\alpha(\hat{\beta}_1, s)}{E_\alpha(\hat{\beta}_1, t)} E_\alpha(\epsilon, s) u_s + \frac{\delta \hat{N}_1}{E_\alpha(\hat{\beta}_1, t)} \mathcal{I}_s^\alpha E_\alpha(\hat{\beta}_1, t) u(t) \\ &- \delta \hat{N}_2 E_\alpha(\hat{\beta}_2, t) \mathcal{I}_{+\infty}^\alpha \frac{u(t)}{E_\alpha(\hat{\beta}_2, t)}, \quad t \geq s \geq 0,\end{aligned} \tag{5.15}$$

$$\begin{aligned}v(t) \leq& \hat{N}_2 \frac{E_\alpha(\hat{\beta}_2, t)}{E_\alpha(\hat{\beta}_2, s)} E_\alpha(\epsilon, s) v_s + \frac{\delta \hat{N}_1}{E_\alpha(\hat{\beta}_1, t)} \mathcal{I}_0^\alpha E_\alpha(\hat{\beta}_1, t) v(t) \\ &- \delta \hat{N}_2 E_\alpha(\hat{\beta}_2, t) \mathcal{I}_s^\alpha \frac{v(t)}{E_\alpha(\hat{\beta}_2, t)}, \quad s \geq t \geq 0,\end{aligned} \tag{5.16}$$

*where $u_s := u(s)$ and $v_s := v(s)$. If*

$$\theta := \delta\Big(\frac{\hat{N}_1}{\hat{\beta}_1} + \frac{\hat{N}_2}{\hat{\beta}_2}\Big) < 1,$$

*then there exist positive constants $K_i$ and $\lambda_i (i = 1, 2)$ such that*

$$u(t) \leq K_1 \frac{E_\alpha(\lambda_1, s)}{E_\alpha(\lambda_1, t)} E_\alpha(\epsilon, s) u_s, \quad t \geq s \geq 0,$$
$$v(t) \leq K_2 \frac{E_\alpha(\lambda_2, t)}{E_\alpha(\lambda_2, s)} E_\alpha(\epsilon, s) v_s, \quad s \geq t \geq 0,$$

*where* $K_i = \dfrac{\hat{N}_i}{1-\theta}$, $\lambda_i = \hat{\beta}_i - \dfrac{\delta \hat{N}_i}{1-\theta}$.

Step 4: Norm bounds of evolution operator. We verify that the norms of the operators $\hat{T}(t,s)|\mathrm{Im}\hat{P}(s)$ and $\hat{T}(t,s)|\mathrm{Im}(\mathrm{Id} - \hat{P}(s))$ are bounded.

**Lemma 5.7** *For any $t \geq s$ in $\mathbb{R}_+$, the first inequality in* (5.10) *holds.*

**Proof.** Given $\xi \in Z$, and for $t \geq s \geq 0$, assume that

$$u(t) := \|\hat{P}(t)\hat{T}(t,s)\xi\|,$$

then $u_s = \|\hat{P}(s)\xi\|$. By Lemma 5.5, we know that $u(t)$ is bounded and satisfies (5.15). It follows from Lemma 5.6 that

$$\|\hat{P}(t)\hat{T}(t,s)\xi\| \leq K_1 \frac{E_\alpha(\lambda_1, s)}{E_\alpha(\lambda_1, t)} E_\alpha(\epsilon, s) \|\hat{P}(s)\xi\|, \quad t \geq s \geq 0,$$

where $K_1$ and $\lambda_1$ are given in Lemma 5.6. Again by Lemma 5.3, we gain

$$\hat{P}(t)\hat{T}(t,s) = \hat{T}(t,s)\hat{P}(s) = \hat{T}(t,s)\hat{P}(s)\hat{P}(s).$$

Taking $\mu := \hat{P}(s)\xi$, it yields that

$$\|\hat{T}(t,s)\hat{P}(s)\mu\| \leq K_1 \frac{E_\alpha(\lambda_1, s)}{E_\alpha(\lambda_1, t)} E_\alpha(\epsilon, s) \|\mu\|, \quad t \geq s \geq 0.$$

Therefore, we can obtain the desired inequality. □

**Lemma 5.8** *For any $s \geq t$ in $\mathbb{R}_+$, the second inequality in* (5.10) *holds.*

**Proof.** By analogy with Lemma 5.5, we need to attain an equation for $(\mathrm{Id} - \hat{P}(t))\hat{T}(t,s)$ via Lemma 5.3. Actually, from the variation of constants formula (2.11), we have

$$\hat{T}(t,s) = T(t,s) + \mathcal{I}_s^\alpha T(t,\cdot)B(\cdot)\hat{T}(\cdot,s).$$

Let function $w(t) := \hat{T}(t,\iota)(\mathrm{Id} - \hat{P}(\iota))$ for some $\iota \in \mathbb{R}_+$, then

$$w(t) = T(t,\iota)(\mathrm{Id} - \hat{P}(\iota)) + \mathcal{I}_{\iota}^{\alpha} T(t,\cdot)B(\cdot)w(\cdot). \tag{5.17}$$

From (5.11) and (5.13) with $t = s = \iota$, we calculate that

$$\hat{P}(\iota) = U(\iota,\iota) = P(\iota) + \mathcal{I}_{+\infty}^{\alpha} T(\iota,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)U(\cdot,\iota).$$

Pre-projecting $P(\iota)$ on both hands sides of the above identity, we acquire $P(\iota)\hat{P}(\iota) = P(\iota)$, and

$$(\mathrm{Id} - P(\iota))(\mathrm{Id} - \hat{P}(\iota)) = \mathrm{Id} - \hat{P}(\iota). \tag{5.18}$$

Combining (5.17) with (5.18), and replacing $t$ with $s$, we derive

$$\begin{aligned} T(t,s)(\mathrm{Id} - P(s))w(s) &= T(t,\iota)(\mathrm{Id} - P(\iota))(\mathrm{Id} - \hat{P}(\iota)) \\ &\quad + \mathcal{I}_{\iota}^{\alpha} T(t,s)(\mathrm{Id} - P(s))B(s)w(s) \\ &= T(t,\iota)(\mathrm{Id} - \hat{P}(\iota)) + \mathcal{I}_{\iota}^{\alpha} T(t,s)(\mathrm{Id} - P(s))B(s)w(s). \end{aligned}$$

It follows from (5.17) and the identity above that

$$\begin{aligned} w(t) &= T(t,s)(\mathrm{Id} - P(s))w(s) + \mathcal{I}_{\iota}^{\alpha} T(t,\cdot)B(\cdot)w(\cdot) \\ &\quad - \mathcal{I}_{\iota}^{\alpha} T(t,s)(\mathrm{Id} - P(s))B(s)w(s) \\ &= T(t,s)(\mathrm{Id} - P(s))w(s) + \mathcal{I}_{\iota}^{\alpha} T(t,\cdot)P(\cdot)B(\cdot)w(\cdot) \\ &\quad + \mathcal{I}_{s}^{\alpha} T(t,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)w(\cdot). \end{aligned} \tag{5.19}$$

On the other hand, by Lemma 5.3, we attain

$$(\mathrm{Id} - \hat{P}(t))\hat{T}(t,s) = \hat{T}(t,s)(\mathrm{Id} - \hat{P}(s)). \tag{5.20}$$

Recalling the function $w(\tau)$, we get $w(\tau)\hat{T}(\iota,s) = (\mathrm{Id} - \hat{P}(\tau))\hat{T}(\tau,s)$. Post-multiplying $\hat{T}(\iota,s)$ on both hands sides of (5.19), this implies

$$\begin{aligned} (\mathrm{Id} - \hat{P}(t))\hat{T}(t,s) =& T(t,s)(\mathrm{Id} - P(s))(\mathrm{Id} - \hat{P}(s)) \\ &+ \mathcal{I}_{\iota}^{\alpha} T(t,\cdot)P(\cdot)B(\cdot)(\mathrm{Id} - \hat{P}(\cdot))\hat{T}(\cdot,s) \\ &+ \mathcal{I}_{s}^{\alpha} T(t,\cdot)(\mathrm{Id} - P(\cdot))B(\cdot)(\mathrm{Id} - \hat{P}(\cdot))\hat{T}(\cdot,s). \end{aligned} \tag{5.21}$$

Fixed $\xi \in Z$, we consider $v(t) := \|\hat{T}(t,s)(\mathrm{Id} - \hat{P}(s))\xi\|$ for $s \geq t \geq 0$ and $v_s = \|(\mathrm{Id} - \hat{P}(s))\xi\|$. According to (5.19) and (5.20), it is well known that the function $v(t)$ satisfies the inequality (5.16). Employing Lemma 5.6 and the similar proof to Lemma 5.7, we easily acquire desired inequality and complete the proof. □

In conclusion, Lemmas 5.3, 5.7 and 5.8 all derive Theorem 5.2 together. □

The following Lemma will help to prove Theorem 5.3.

**Lemma 5.9** *For any $t \in \mathbb{R}_+$, if constant $\delta$ described as in Theorem 5.3 is small enough, then*

$$\|\hat{P}(t)\| \leq 4\hat{N}E_\alpha(\epsilon, t), \quad \|\mathrm{Id} - \hat{P}(t)\| \leq 4\hat{N}E_\alpha(\epsilon, t). \tag{5.22}$$

**Proof.** Replacing $s$ by $t$ and pre-multiplying $(\mathrm{Id} - P(t))$ on both hands sides of (5.14), we have

$$(\mathrm{Id} - P(t))\hat{P}(t) = \mathcal{I}^\alpha_{+\infty} T(t, \cdot)(\mathrm{Id} - P(\cdot))B(\cdot)\hat{P}(\cdot)\hat{T}(\cdot, t). \tag{5.23}$$

It follows from Lemmas 5.7 and 5.3 that for $\tau \geq t \geq 0$,

$$\|\hat{P}(\tau)\hat{T}(\tau, t)\| = \|\hat{T}(\tau, t)\hat{P}(t)\hat{P}(t)\| \leq K_1 \frac{E_\alpha(\lambda_1, t)}{E_\alpha(\lambda_1, \tau)} E_\alpha(\epsilon, t)\|\hat{P}(t)\|. \tag{5.24}$$

By (5.23) and (5.4) we calculate that

$$\begin{aligned} \|(\mathrm{Id} - P(t))\hat{P}(t)\| &\leq -\mathcal{I}^\alpha_{+\infty}\|T(t, \cdot)(\mathrm{Id} - P(\cdot))\|\|B(\cdot)\|\|\hat{P}(\cdot)\hat{T}(\cdot, t)\| \\ &\leq -E_\alpha(\hat{\beta}_2 + \lambda_1 + \epsilon, t)\|\hat{P}(t)\|\mathcal{I}^\alpha_{+\infty} \frac{\delta K_1 \hat{N}_2}{E_\alpha(\hat{\beta}_2 + \lambda_1 + \epsilon, t)} \\ &\leq \frac{\delta K_1 \hat{N}_2}{\hat{\beta}_2 + \lambda_1 - \epsilon}\|\hat{P}(t)\|, \end{aligned} \tag{5.25}$$

where constant $\epsilon$ was chosen as satisfying $\epsilon < \min\{\hat{\beta}_1, \hat{\beta}_2\}$ in order to guarantee the above denominator $\hat{\beta}_2 + \lambda_1 - \epsilon > 0$. Analogously to (5.23), replacing $t$ with $s$ and pre-multiplying $P(t)$ on both hands sides of (5.21), we attain

$$P(t)(\mathrm{Id} - \hat{P}(t)) = \mathcal{I}^\alpha_\iota T(t, \cdot)P(\cdot)B(\cdot)(\mathrm{Id} - \hat{P}(\cdot))\hat{T}(\cdot, t). \tag{5.26}$$

Using Lemma 5.8, for $t \geq \tau \geq 0$ this implies

$$\|(\mathrm{Id} - \hat{P}(\tau))\hat{T}(\tau, t)\| \leq K_2 \frac{E_\alpha(\lambda_2, \tau)}{E_\alpha(\lambda_2, t)} E_\alpha(\epsilon, t)\|\mathrm{Id} - \hat{P}(t)\|. \tag{5.27}$$

From (5.26) and (5.4) one can compute that

$$\begin{aligned} \|P(t)(\mathrm{Id} - \hat{P}(t))\| &\leq \mathcal{I}^\alpha_\iota \|T(t, \cdot)P(\cdot)\|\|B(\cdot)\|\|(\mathrm{Id} - \hat{P}(\cdot))\hat{T}(\cdot, t)\| \\ &\leq \frac{\delta K_2 \hat{N}_1}{E_\alpha(\hat{\beta}_1 + \lambda_2 - \epsilon, t)}\|\mathrm{Id} - \hat{P}(t)\|\mathcal{I}^\alpha_\iota E_\alpha(\hat{\beta}_1 + \lambda_2 - \epsilon, t) \\ &\leq \frac{\delta K_2 \hat{N}_1}{\hat{\beta}_1 + \lambda_2 - \epsilon}\|\mathrm{Id} - \hat{P}(t)\|, \end{aligned} \tag{5.28}$$

where the chosen constant $\epsilon < \min\{\hat{\beta}_1, \hat{\beta}_2\}$ similarly. Obviously,

$$\hat{P}(t) - P(t) = (\mathrm{Id} - P(t))\hat{P}(t) - P(t)(\mathrm{Id} - \hat{P}(t)).$$

Taking $\hat{N} := \max\{\hat{N}_1, \hat{N}_2\}$ and $\hat{\beta} := \min\{\hat{\beta}_1, \hat{\beta}_2\}$ and combining (5.25) with (5.28), we gain

$$\begin{aligned}\|\hat{P}(t) - P(t)\| &\leq \frac{\delta K_1 \hat{N}_2}{\hat{\beta}_2 + \lambda_1 - \epsilon}\|\hat{P}(t)\| + \frac{\delta K_2 \hat{N}_1}{\hat{\beta}_1 + \lambda_2 - \epsilon}\|\mathrm{Id} - \hat{P}(t)\| \\ &\leq \hat{\eta}(\|\hat{P}(t)\| + \|\mathrm{Id} - \hat{P}(t)\|), \end{aligned} \tag{5.29}$$

where

$$\hat{\eta} = \frac{\delta \hat{N}^2 \hat{\beta}}{2\hat{\beta}^2 - 5\delta \hat{N}\hat{\beta} - \epsilon(\hat{\beta} - 2\delta\hat{N})}.$$

Moreover, by (5.4) with $t = s$, it is easy to obtain that

$$\|P(t)\| \leq \hat{N} E_\alpha(\epsilon, t), \quad \|Q(t)\| \leq \hat{N} E_\alpha(\epsilon, t).$$

From (5.29), this yields

$$\begin{aligned}\|\hat{P}(t)\| &\leq \|\hat{P}(t) - P(t)\| + \|P(t)\| \\ &\leq \hat{\eta}(\|\hat{P}(t)\| + \|\mathrm{Id} - \hat{P}(t)\|) + \hat{N} E_\alpha(\epsilon, t).\end{aligned}$$

Since $\|(\mathrm{Id} - \hat{P}(t)) - (\mathrm{Id} - P(t))\| = \|\hat{P}(t) - P(t)\|$, we also derive

$$\begin{aligned}\|(\mathrm{Id} - \hat{P}(t))\| &\leq \|\hat{P}(t) - P(t)\| + \|\mathrm{Id} - P(t)\| \\ &\leq \hat{\eta}(\|\hat{P}(t)\| + \|\mathrm{Id} - \hat{P}(t)\|) + \hat{N} E_\alpha(\epsilon, t).\end{aligned}$$

They together imply that

$$\|\hat{P}(t)\| + \|\mathrm{Id} - \hat{P}(t)\| \leq 2\hat{\eta}(\|\hat{P}(t)\| + \|\mathrm{Id} - \hat{P}(t)\|) + 2\hat{N} E_\alpha(\epsilon, t),$$

and

$$\|\hat{P}(t)\| + \|\mathrm{Id} - \hat{P}(t)\| \leq \frac{2\hat{N} E_\alpha(\epsilon, t)}{1 - 2\hat{\eta}}.$$

Choose $\hat{\eta} < 1/4$, then

$$\|\hat{P}(t)\| + \|\mathrm{Id} - \hat{P}(t)\| \leq 4\hat{N} E_\alpha(\epsilon, t),$$

yielding Lemma 5.9. □

Finally, we end this paper with the proof of roughness for nonuniform conformable exponential dichotomy.

**Proof of Theorem 5.3.** From (5.24) and (5.22), we show that

$$\begin{aligned}\|\hat{P}(\tau)\hat{T}(\tau, t)\| &\leq \frac{\hat{N}\hat{\beta}}{\hat{\beta} - 2\delta\hat{N}} \frac{E_\alpha(\hat{\lambda}, t)}{E_\alpha(\hat{\lambda}, \tau)} E_\alpha(\epsilon, t)\|\hat{P}(t)\| \\ &\leq \frac{4\hat{N}^2\hat{\beta}}{\hat{\beta} - 2\delta\hat{N}} \frac{E_\alpha(\hat{\lambda}, t)}{E_\alpha(\hat{\lambda}, \tau)} E_\alpha(2\epsilon, t), \quad \tau \geq t \geq 0,\end{aligned}$$

where $\hat{\lambda} = \hat{\beta} - \dfrac{\delta \hat{N} \hat{\beta}}{\hat{\beta} - 2\delta \hat{N}}$. Analogously, it follows from (5.27) and (5.22) that

$$\|(\mathrm{Id} - \hat{P}(\tau))\hat{T}(\tau, t)\| \leq \frac{4\hat{N}^2\hat{\beta}}{\hat{\beta} - 2\delta\hat{N}} \frac{E_\alpha(\hat{\lambda}, \tau)}{E_\alpha(\hat{\lambda}, t)} E_\alpha(2\epsilon, t), \quad t \geq \tau \geq 0.$$

Therefore, we can acquire the desired inequalities like (5.4), and the proof is completed. □

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