A new iteration algorithm with relaxation factor for the Cauchy problem of time-fractional diffusion equation*

Zhenping Li[1,3] , Xiangtuan Xiong[2,†]

[1] *Department of Mathematics and Physics, Luoyang Institute of Science and Technology, Luoyang, 471023, China*

[2] *Department of Mathematics, Northwest Normal University, Lanzhou, 730070, China*

[3] *Henan Key Laboratory of Green Building Materials Manufacturing and Intelligent Equipment, Luoyang, 471023, China*

Abstract

In this paper, we investigate a Cauchy problem for the time-fractional diffusion equation by using a new simplified iteration algorithm. The algorithm arises from the well-known Kozlov-Maz'ya iterative scheme for solving the Cauchy problem of classical ill-posed boundary value problems. A relaxation factor in the algorithm is introduced. Based on Fourier analysis, we give a choice rule for the relaxation factor. The iteration algorithm combined with a stopping criterion is given for stable numerical reconstruction of the solution. For illustration, several numerical experiments are constructed to demonstrate the feasibility and efficiency of the proposed method.

Key words: time-fractional diffusion equation; Cauchy problem; ill-posedness; iterative algorithm; stopping criterion

Mathematics Subject Classification: 35R25, 47A52, 35R30

1 Introduction

Partial differential equations with fractional-order derivative arose from the studies of continuous random walk and Lévy motion [1–4] and high-frequency financial data [5,6]. Among these studies, the modeling of advection and dispersion phenomena in groundwater hydrology to simulate the transport of passive tracers carried by fluid flow in a porous medium resulted in a partial differential equation with fractional-order derivative [7–11]. For economical and safe control of plants, there is a great demand for a good estimation of the

*The project is supported by the National Natural Science Foundation of China (No. 11661072) and the Natural Science Foundation of Northwest Normal University (No. NWNU-LKQN-17-5).

†Corresponding author. Email: xiangtuanxion@nwnu.edu.cn; xiongxt@gmail.com

concentration and flow of pollutants from only spatially observed data during advection and dispersion processes. In general, fluid flow and diffusion phenomena are governed by a fractional diffusion equation. If the initial concentration distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable from solving a well-posed forward problem [12, 13]. In real life, however, the boundary conditions can be lost, and the distribution data can only be collected at a particular time. This is usually referred to as ill-posed backward determination problem, which is, in nature, "unstable" because the unknown solution and its derivatives have to be determined from indirect observable data that contains measurement error. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the problem and the ill-conditioning of the resultant discretized matrix.

Anomalous diffusion has been found in a broad variety of physics and engineering disciplines, for example, electron transportation [14], dissipation [15], and heat conduction [16]. It is well known that the continuous-time random walk is a microscopic model for anomalous diffusion. By an argument similar to the derivation of the classical diffusion equation from the random walk model, one can derive time-fractional diffusion models [17,18]. In recent years, the study of the fractional derivative anomalous diffusion equation has attracted attention from many researchers, such as [20–28].

In many practical real-life problems, diffusion is anomalous, but the boundary data can only be measured on a part of the boundary. This leads to the Cauchy problem of the fractional diffusion equation, which is by nature ill-posed, refer to [19] for more details. To the knowledge of the authors, there is still a lack of full understanding of inverse problems for fractional differential equations. Recent works can be found in [29–37].

The Cauchy problem of the classical heat diffusion equation arises from many physical and engineering disciplines. It is well known that the Cauchy problem of heat diffusion equation is ill-posed in the Hadamard sense that a small measurement error in the boundary data can induce enormous error in the solution. Under an additional condition, a continuous dependence of the solution on the Cauchy data can be obtained. This is called conditional stability [38]. Due to the severe ill-posedness of the problem, it is difficult, if not impossible, to solve the Cauchy problem of heat diffusion equation by using classical numerical methods without using some kinds of regularization strategies [39]. Theoretical concepts and computational implementation related to the Cauchy problem of heat diffusion equation have been discussed by many authors. See, for instance, [40, 41] for computational aspects and [42] for theoretical aspects. To the authors' knowledge, most of the literature on the Cauchy problem of time-fractional diffusion equation considered the convergence rate for regularization methods using the Fourier technique. The work studied the problem in a bounded semi-infinite domain, and by using the analytic solution, one can construct the regularization solution. However, in this paper, we consider the Cauchy problem in a bounded finite domain. The difference from most similar problems is that the analytic solution is not available now, and the regularization methods cannot be constructed via the analytic solution. In order to overcome this difficulty, we will apply the iterative method with a relaxation factor. Although many iterative procedures are presented for solving many ill-posed problems, e.g., [43, 44], the relaxation factor in the iterative algorithms is seldom discussed from the viewpoint of convergence of algorithms.

It is well known that iterative regularization methods stabilize the ill-posed problems by stopping the iterative process at the optimum point. This is the semi-convergence of iterative methods for solving ill-posed problems. In these methods, the number of iterations performed plays the role of a regularization parameter. In most iterative methods for solving ill-posed problems of mathematical physics equations, a least squares method is applied [45], and a corresponding variational functional should be given, the gradient of the functional should be calculated, and the computation of the gradient of a given functional is usually complex. In this paper, we will not give any functional and will not compute the corresponding gradient. The used idea in this paper is from the Kozlov-Maz'ya iteration [43]. The Kozlov-Maz'ya method is very efficient for solving the ill-posed Cauchy problem of partial differential equations with integer-order derivatives. And this method has been extended to solve other types of ill-posed problems [44]. Since the formula of integration by parts for the integer-order differentiation does not exist for the fractional-order differentiation, this leads to some theoretical results for classical differential equations that are not adapted to the fractional differential equations. Due to this fact, to the authors' knowledge, this method has not been applied to solving the Cauchy problem of fractional differential equations yet. For describing this iterative method more clearly, we only restrict ourselves to solving the Cauchy problem in the one-dimensional case. We will propose a similar but simpler algorithm with a heuristic stopping rule and a relaxation factor for solving the Cauchy problem of the fractional diffusion equation. The method with Fourier analysis for choosing the relaxation factor is considered. Numerical results show that our newly proposed stopping rule with the relaxation factor works well. The contribution of this paper consists of two points:

1. A simplified Kozlov-Maz'ya iteration with a relaxation factor is tried to solve the Cauchy problem of fractional diffusion equations. For this problem, the algorithm is simpler than the classical Kozlov-Maz'ya iteration, and the cost of the computation is less.

2. A theoretical analysis on the relaxation factor is conducted. Based on Fourier analysis, a choice rule for the relaxation factor is given.

The paper is organized as follows. In Section 2, we consider two similar Cauchy problems for the time-fractional diffusion equation, and in Section 3, we give the rules for selecting the relaxation factor of the iterative algorithm using the Fourier transformation technique. In the last section, several numerical examples are constructed to demonstrate the feasibility and efficiency of the proposed method.

2 Mathematical formulation of the Cauchy problem

In most work on the Cauchy problem of fractional diffusion equation, the authors consider the following Cauchy problem for the time-fractional diffusion equation in the bounded semi-infinite domain $\Omega_1 := \{(x,t)|0 < x < a, t > 0\}$ and in the bounded finite domain $\Omega_2 := \{(x,t)|0 < x < a, 0 < t < T\}$:

$$\begin{array}{l} \frac{\partial^\beta u}{\partial t^\beta} - u_{xx} = 0, \\ u(0,t) = \phi(t), \\ u_x(0,t) = h(t), \\ u(x,0) = 0, \end{array} \tag{2.1}$$

where the time fractional derivative $\frac{\partial^\beta u}{\partial t^\beta}$ is the Caputo fractional derivative of order β ($0 < \beta \leq 1$) defined in [46],

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\beta},\ 0 < \beta < 1, \tag{2.2}$$

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial u(x,t)}{\partial t},\ \beta = 1. \tag{2.3}$$

Problem I. *The Cauchy problem defined in Ω_1 is to recover the unknown solutions $L(t) := u(a,t)$ from the given data pairs $\phi(t)$, $h(t)$.*

Most work on this problem only considered the homogeneous initial condition. This can be explained in the following:

Remark 2.1. *The following problem with non-homogeneous initial condition*

$$\begin{array}{l} \frac{\partial^\beta u^{(1)}}{\partial t^\beta} - u^{(1)}_{xx} = 0, \\ u^{(1)}(0,t) = \phi(t), \\ u^{(1)}_x(0,t) = h(t), \\ u^{(1)}(x,0) = u_0(x) \end{array} \tag{2.4}$$

can be split into Problem I and the following problem

$$\begin{array}{l} \frac{\partial^\beta u^{(2)}}{\partial t^\beta} - u^{(2)}_{xx} = 0, \\ u^{(2)}(0,t) = 0, \\ u^{(2)}_x(0,t) = 0, \\ u^{(2)}(x,0) = u_0(x). \end{array} \tag{2.5}$$

The above problem (2.5) is a well-posed problem. Therefore, in (2.1), the homogeneous initial condition is not important for the inverse problem.

In this paper, we consider seeking the solutions in two different domains Ω_1 and Ω_2 for problem (2.1). In the case of Ω_1, we can take the Laplace transform and find the analytic solution. By using the Laplace transform and the Fourier transform [47], the solution of Problem I in the frequency domain can be obtained,

$$\hat{u}(x,\xi) = \cosh(x\eta)\hat{\phi}(\xi) + \frac{\sinh(x\eta)}{\eta}\hat{h}(\xi), \tag{2.6}$$

where ξ is the variable of Fourier transform with respect to t,

$$\eta := (i\xi)^{\frac{\beta}{2}} = |\xi|^{\frac{\beta}{2}}(\cos(\frac{\beta\pi}{4}) + i\,\mathrm{sign}(\xi)\sin(\frac{\beta\pi}{4})). \tag{2.7}$$

The inverse Fourier transform on (2.6) gives $u(x,t)$.

We note here that, for fixed $0 < x \leq a$, the values of $|\cosh(x\eta)|$ and $|\frac{\sinh(x\eta)}{\eta}|$ in (2.6) are unbounded as $|\eta| \to \infty$. Therefore, the problem is ill-posed, and some regularization methods should be applied.

So far, for Problem I, there has been a lot of literature considering the theoretic result of regularization errors. But if the time t is finite, the analysis of regularization errors is difficult. In most practical situations, the time variable t is finite, and we consider the following problem:

Problem II. *The Cauchy problem (2.1) in the bounded finite domain* $\Omega_2 := \{(x,t)|0 < x < a, 0 < t < T\}$, *where* T *is a positive constant. Reconstruct the solution* $u(a,t)$ *from the Cauchy data pairs.*

However, we could not find an explicit expression for the solution via Fourier transformation in this case, and we cannot construct a regularization solution or give the error analysis. Noting that this problem is similar to Problem I, we can analyze the regularization methods for solving this problem based on the methods for Problem I because the solution of Problem I can be expressed explicitly. Now we describe an iterative algorithm with a relaxation factor for solving Problem II.

3 A simplified iterative algorithm

It is well-known that the Kozlov-Maz'ya iteration for solving the Cauchy problem of the classical partial differential equations is equivalent to the Landweber method [48]. For clarifying the idea of this method, we consider the Cauchy problem of classical heat equation in Ω_2:

$$\begin{array}{l} \frac{\partial u}{\partial t} - u_{xx} = 0, \\ u(0,t) = \phi(t), \\ u_x(0,t) = h(t), \\ u(x,0) = 0. \end{array} \tag{3.1}$$

Here we want to seek the solution $f(t) := u(a,t)$. The Kozlov-Maz'ya iteration is given as follows:

Step 1. Given an initial guess $f_0(t)$ and solve the forward problem to obtain $v(0,t)$ and $v_x(a,t)$:

$$\begin{array}{l} \frac{\partial v}{\partial t} - v_{xx} = 0, \\ v(a,t) = f_0(t), \\ v_x(0,t) = h(t), \\ v(x,0) = 0. \end{array} \tag{3.2}$$

We define the mapping $K : f_0(t) \to v(0,t)$, i.e., $Kf_0(t) = v(0,t)$, where K also depends on $h(t)$.

Step 2. Based on the obtained data $v_x(a,t)$ from Step 1, then solve the forward problem to obtain $f_1(t) := w(a,t)$:

$$\begin{array}{l} \frac{\partial w}{\partial t} - w_{xx} = 0, \\ w(0,t) = \phi(t), \\ w_x(a,t) = v_x(a,t), \\ w(x,0) = 0. \end{array} \tag{3.3}$$

Let $z(x,t) = w(x,t) - v(x,t)$, then $z(x,t)$ satisfies

$$\begin{array}{l} \frac{\partial z}{\partial t} - z_{xx} = 0, \\ z(0,t) = \phi(t) - Kf_0(t) := r(t), \\ z_x(a,t) = 0, \\ z(x,0) = 0. \end{array} \tag{3.4}$$

We define the mapping $K^* : z(0,t) \to z(a,t)$, i.e., $K^*(\phi(t) - Kf_0(t)) = z(a,t) = f_1(t) - f_0(t)$. Therefore, This formula becomes

$$f_1(t) = f_0(t) + K^*(\phi(t) - Kf_0(t)).$$

Using the iteration index k, we have for $k = 1, 2, \cdots$

$$f_k(t) = f_{k-1}(t) + K^*(\phi(t) - Kf_{k-1}(t)).$$

This is the famous Landweber method for solving ill-posed operator equations.

In this method, for solving the inverse problems, one has to solve two forward problems (3.2) and (3.3). If the space dimension becomes higher, then the cost of computation is very expensive. For this reason, we propose a simplified version of the Kozlov-Maz'ya iteration. In this version, we use Dirichlet data in the forward problem and use the Neumann data for fitting, and only one forward problem needs to be solved. Thus, the cost of computation is much less than the classical version.

The iterative algorithm for solving (2.1) in Ω_2 can be constructed as follows:

- Step 1. Specify an initial guess T_0 for the temperature $w(a,t)$ at the boundary $x = a$.

- Step 2. If $T_k(t)$ has been constructed, solve the well-posed forward problem

$$\begin{array}{ll} \frac{\partial^\beta w^{(k)}}{\partial t^\beta} - (w^{(k)})_{xx} = 0, & 0 < x < a, 0 < t < T, \\ w^{(k)}(0,t) = \phi(t), & 0 < t < T, \\ w^{(k)}(a,t) = T_k(t), & 0 < t < T, \\ w^{(k)}(x,0) = 0, & 0 < x < a \end{array} \tag{3.5}$$

  to determine the k-th approximation $w_x^{(k)}(0,t)$.

- Step 3. Construct $T_{k+1}(t)$, which is given by

$$T_{k+1}(t) = T_k(t) - c(w_x^{(k)}(0,t) - h(t)), \tag{3.6}$$

  where c is a relaxation factor that makes the algorithm convergent.

- Step 4. Repeat Steps 2 and 3 until a prescribed stopping criterion is satisfied.

Obviously, we only need to solve the forward problem (3.5) in each iteration, which is different from the classical method. In Step 3, we construct the iteration scheme by using the Neumann data $w_x^{(k)}(0,t)$ to fit the data $h(t)$.

3.1 Analysis on the relaxation factor

In order to show the convergence of the algorithm, we turn to Problem I for the sake of theoretical analysis. For Problem I, we write the key steps:

- If $\tilde{T}_k(t)$ has been constructed, solve the well-posed forward problem

$$\begin{array}{ll} \frac{\partial^\beta u^{(k)}}{\partial t^\beta} - (u^{(k)})_{xx} = 0, & 0 < x < a,\ t > 0, \\ u^{(k)}(0,t) = \tilde{\phi}(t), & t > 0, \\ u^{(k)}(a,t) = \tilde{T}_k(t), & t > 0, \\ u^{(k)}(x,0) = 0, & 0 < x < a \end{array} \tag{3.7}$$

to determine the k-th approximation $u_x^{(k)}(0,t)$.

- Construct $\tilde{T}_{k+1}(t)$, which is given by

$$\tilde{T}_{k+1}(t) = \tilde{T}_k(t) - c(u_x^{(k)}(0,t) - \tilde{h}(t)), \tag{3.8}$$

where c is a positive parameter.

Let us define the error function $e^{(k)}(x,t) := u^{(k)}(x,t) - u(x,t)$, where $u(x,t)$ is the solution of the forward problem:

$$\begin{array}{ll} \frac{\partial^\beta u}{\partial t^\beta} - u_{xx} = 0, & 0 < x < a,\ t > 0, \\ u(0,t) = \tilde{\phi}(t), & t > 0, \\ u(a,t) = \tilde{L}(t), & t > 0, \\ u(x,0) = 0, & 0 < x < a. \end{array} \tag{3.9}$$

Due to (3.7) and (3.9), we have for $e^{(k)}(x,t)$:

$$\begin{array}{ll} \frac{\partial^\beta e^{(k)}}{\partial t^\beta} - (e^{(k)})_{xx} = 0, & 0 < x < a,\ t > 0, \\ e^{(k)}(0,t) = 0, & t > 0, \\ e^{(k)}(a,t) = \varepsilon_k, & t > 0, \\ e^{(k)}(x,0) = 0, & 0 < x < a. \end{array} \tag{3.10}$$

where $\varepsilon_k := \tilde{T}_k(t) - \tilde{L}(t)$.

Noting that $e_x^{(k)}(0,t) := u_x^{(k)}(0,t) - u_x(0,t) = u_x^{(k)}(0,t) - \tilde{h}(t)$, (3.8) becomes

$$\varepsilon_{k+1} = \varepsilon_k - c e_x^{(k)}(0,t). \tag{3.11}$$

Using the Fourier transform in (3.10) with respect to t, we can obtain the solution of (3.10) in the frequency domain:

$$\hat{e}^{(k)}(x,\xi) = \frac{\sinh(\eta x)}{\sinh(\eta)} \hat{\varepsilon}_k(\xi). \tag{3.12}$$

From (3.8), we have

$$\hat{e}_x^{(k)}(0,\xi) = \eta \frac{1}{\sinh(\eta)} \hat{\varepsilon}_k(\xi). \tag{3.13}$$

Consider (3.11) in the frequency domain, we have

$$\hat{\varepsilon}_{k+1}(\xi) = \hat{\varepsilon}_k(\xi) - c\hat{e}_x^{(k)}(0,\xi) = (1 - c\frac{\eta}{\sinh(\eta)})\hat{\varepsilon}_k(\xi). \tag{3.14}$$

In order to guarantee the convergence of the algorithm, we should select the parameter c, which should satisfy $|1 - c\frac{\eta}{\sinh(\eta)}| < 1$. For example, if $0 < c < 1$, we have $|1 - c\frac{\eta}{\sinh(\eta)}| < 1$. In the forthcoming section, we set $0 < c < 1$ for computation.

3.2 Solve the forward problem

Now we come back to Problem II. For the proposed iteration method, how to solve the forward problem (3.5) is the key problem. For convenience, we drop the index k in (3.5).

Before conducting the numerical experiment, we first give the finite difference scheme for solving the forward problem.

We concentrate on the following forward problem for the given data $\phi(t)$ and $T(t)$:

$$\begin{array}{ll} \frac{\partial^\beta w}{\partial t^\beta} - w_{xx} = 0, & 0 < x < a, 0 < t < T, \\ w(0,t) = \phi(t), & 0 < t < T, \\ w(a,t) = T(t), & 0 < t < T, \\ w(x,0) = 0, & 0 < x < a. \end{array} \tag{3.15}$$

Using an unconditionally stable implicit finite difference scheme, we can obtain the data $w_x(0,t)$. Let $t_n = (n-1)\tau$ be the grid points at the t-axis, $n = 1, 2, \cdots, s+1$ with time step length $\tau = T/s$, $x_j = (j-1)h$, $j = 1, 2, \cdots, m+1$ with spatial step length $h = a/m$ be the spatial grid points; w_j^n be the difference approximation to $w(x_j, t_n)$. The finite difference scheme for the solution is given as follows [49]:

$$-\frac{1}{h^2} w_{j-1}^2 + (\sigma_{\beta,\tau} + \frac{2}{h^2}) w_j^2 - \frac{1}{h^2} w_{j+1}^2 = \sigma_{\beta,\tau} w_j^1, \ j = 2, 3, \cdots, m, \tag{3.16}$$

and for $n = 3, \cdots, s+1$,

$$-\frac{1}{h^2} w_{j-1}^n + (\sigma_{\beta,\tau} + \frac{2}{h^2}) w_j^n - \frac{1}{h^2} w_{j+1}^n = \sigma_{\beta,\tau} w_j^{n-1} - \sigma_{\beta,\tau} \sum_{i=3}^{n} \upsilon_{i-1,\beta} (w_j^{n-i+2} - w_j^{n-i+1}), \ j = 2, 3, \cdots, m, \tag{3.17}$$

with initial condition

$$w_j^1 = 0, \ j = 1, 2, \cdots, m+1 \tag{3.18}$$

and boundary condition

$$w_1^n = \phi((n-1)\tau), \ w_{m+1}^n = T((n-1)\tau), \ n = 2, \cdots, s+1, \tag{3.19}$$

where $\sigma_{\beta,\tau} = \frac{1}{\Gamma(1-\beta)(1-\beta)\tau^\beta}$, $\upsilon_{i,\beta} = i^{1-\beta} - (i-1)^{1-\beta}$, $i = 2, 3, \cdots$. The Neumann data for the Cauchy problem can be constructed by the approximation:

$$w_x(0, t_n) \approx \frac{w_2^n - w_1^n}{h}, \ n = 1, 2, \cdots, s+1. \tag{3.20}$$

4 Numerical results and discussion

4.1 Experiment setting

Since an explicit exact solution for the inverse problem II is difficult to construct, we solve the following forward problem with the exact data $L(t)$, which will be reconstructed in the corresponding inverse problem

$$\begin{array}{ll} \frac{\partial^\beta w}{\partial t^\beta} - w_{xx} = 0, & 0 < x < a, 0 < t < T, \\ w(0,t) = \phi(t), & 0 < t < T, \\ w(a,t) = L(t), & 0 < t < T, \\ w(x,0) = 0, & 0 < x < a \end{array} \tag{4.1}$$

to obtain the exact data $w_x(0,t) = h(t)$. Now for the inverse problem II, we have the Cauchy data $\phi(t)$, $h(t)$, and the exact solution $w(a,t) := L(t)$. In computation, we can obtain the Cauchy data $\phi(t)$, $h(t)$ in the form of vectors denoted as Φ, H. The noisy Cauchy data of Φ_δ and H_δ are generated as:

$$\Phi_\delta = \Phi + (\max \Phi) \cdot \ \sigma_1 \ rand(size(\Phi)), \ H_\delta = H + (\max H) \cdot \ \sigma_2 \ rand(size(H)), \quad (4.2)$$

where

$$\Phi = (\phi(t_1), \cdots, \phi(t_{s+1}))^T, \quad (4.3)$$

$$H = (h(t_1), \cdots, h(t_{s+1}))^T, \quad (4.4)$$

$$\delta = RMSE(\Phi_\delta - \Phi) + RMSE(H_\delta - H), \quad (4.5)$$

in which $\max \Phi$, $\max H$ denotes the maximum of the vectors Φ, H, respectively, σ_1 indicates the error level of Φ, σ_2 denotes the error level of H, and $RMSE$ denotes the root mean square error between two vectors W and W^* defined by

$$RMSE(W - W^*) = \sqrt{\frac{1}{s+1}\sum_{j=1}^{s+1}(W(t_j) - W^*(t_j))^2}, \quad (4.6)$$

where $rand(size(\cdot))$ is a random number generated by MATLAB between $[-1, 1]$.

Numerical implementation is performed using MATLAB in IEEE double precision. In the numerical examples, we fix $T = 1$, $a = 1$, $h = \frac{1}{100}$, and $\tau = \frac{1}{50}$. For each noise level σ_1, σ_2, the value of δ can be computed by (4.5). For Example 1 and Example 2, we fix the factor $c = 0.3$. However, in Example 3, we change the factor c.

4.2 The initial guess

An arbitrary function $T_0(t)$ may be selected as an initial guess for the boundary value $w(a,t)$ of the Cauchy problem II. However, in order to improve the rate of convergence of the iterative algorithm, we have selected a function that ensures the solution of the problem satisfies the compatibility conditions $w(a,0) = w(x,0) = 0$.

In all of numerical examples, the initial guess is given by a constant function

$$T_0(t) = 0. \quad (4.7)$$

4.3 The convergence error

In order to investigate the convergence of the described algorithm, at every iteration, we evaluate the convergence error defined by

$$e_b = \|w^{(k)}(t) - L(t)\|_{L^2[0,T]}, \quad (4.8)$$

which can be computed by the notation of RMSE defined by (4.6). $w^{(k)}(t)$ is the computation value after k iterations, and $L(t)$ is the exact value defined by $L(t) := w(a,t)$. However, the error e_b can only be computed if the exact solution $w(a,t)$ is known, and such a solution is not available in practice. Therefore, a stopping criterion must be developed in order to cease the iterative process when convergence has been achieved.

4.4 The stopping criterion

In general, iterative regularization methods stabilize the ill-posed problems by stopping the iterative process at the optimum point. In these methods, the number of iterations performed plays the role of a regularization parameter. Therefore, a stopping criterion is needed for the iterative algorithm considered in this paper.

In order to obtain a stopping criterion, let us review the famous iterative method, i.e., the Landweber iterative method for solving ill-posed problems.

Consider linear ill-posed problems:

$$Ax = y, \tag{4.9}$$

where $A : H_1 \to H_2$ is a linear, injective, closed operator between infinite-dimensional Hilbert spaces H_1 and H_2 with non-closed range $R(A)$. Suppose that $y^\delta \in H_2$ is the noisy data with

$$\|y - y^\delta\| \le \delta$$

and known noise level δ.

Landweber [50] suggested rewriting the equation $Ax = y$ in the form $x = (I - \lambda A^* A)x + \lambda A^* y$ for some $\lambda > 0$ and iterating this, i.e., computing

$$x^0 := 0,\ x^k = (I - \lambda A^* A)x^{k-1} + \lambda A^* y \tag{4.10}$$

for $k = 1, 2, \cdots$.

Let $r > 1$ be a fixed number. Stop the algorithm at the first occurrence of k with $\|Ax^{k,\delta} - y^\delta\| \le r\delta$. Under the stopping rule, the Landweber iteration method is convergent [39].

Now following the idea from the above stopping rule, we have

$$\|w_x^{(k)}(0,t) - h(t)\|_{L^2[0,T]} \le r\delta,\ \ r > 1, \tag{4.11}$$

where $w_x^{(k)}(0,t)$ is the computation value after k iterations, and $h(t)$ is the exact value defined by $h(t) := w_x(0,t)$.

4.5 Numerical examples

Example 1. First we solve the forward problem (4.1) with $\beta = 0.5$, $w(0,t) := \phi(t) = 1 - e^{-2t}$ and $w(a,t) := L(t) = 3\sin(2\pi t)$ by using the implicit finite difference method. We get the exact data $h(t) := w_x(0,t)$. Figure 1 displays the input data $w_x(0,t)$. Now we use the iterative algorithm with the initial guess $T_0(t) = 0$ for reconstructing the solution $w(a,t)$. In practice, the noise level $\sigma_2 \gg \sigma_1$, it is reasonable to only consider the noise level σ_2 and set $\sigma_1 = 0$. In the iterative algorithm for inverse problem II, we add random noise $\sigma_2 = 10^{-1}$ to the exact data $h(t)$. According to (4.5), we have $\delta = 0.10$.

(a) First we investigate the convergence error. We plot the convergence error (4.8) vs the number of iterations, see Figure 2. The error e_b initially decreases rapidly and then it slows down and tends to a small positive constant of order $2 * 10^{-1}$.

(b) As we point out before, it is impossible for one to plot the convergence error defined by (4.8) because the exact solution $L(t)$ is not known in practice. Now we test the stopping

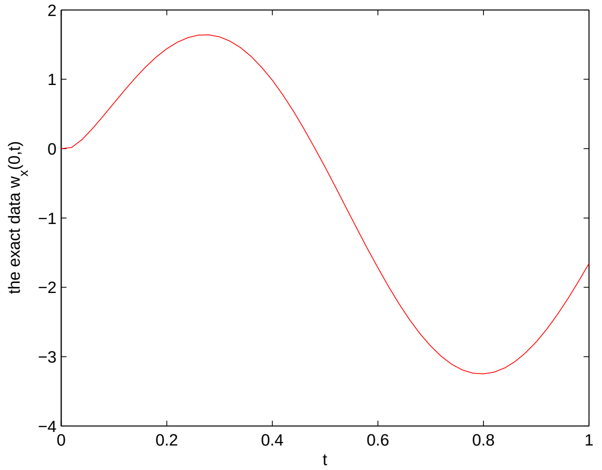


Figure 1: The input data $w_x(0,t)$ for Example 1.

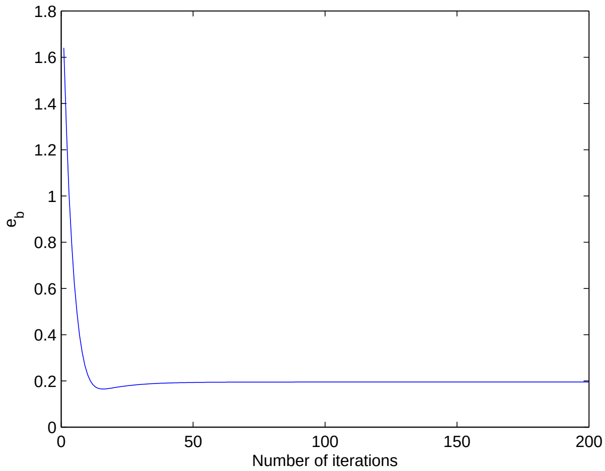
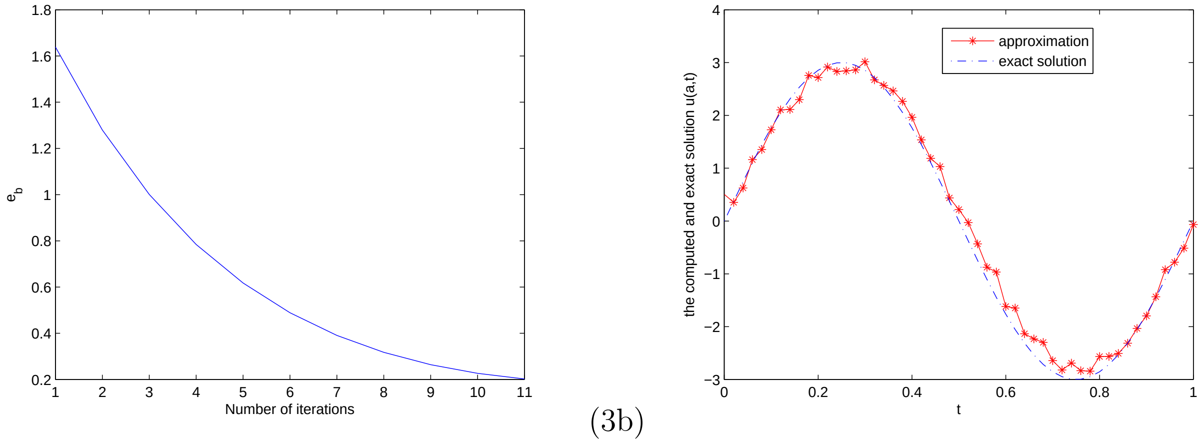
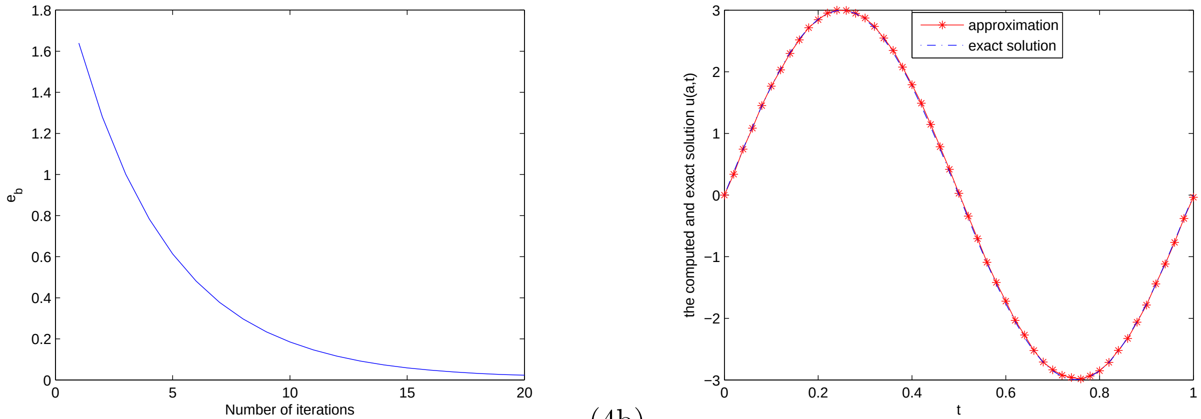


Figure 2: The convergence error vs number of iterations for Example 1.



(3a) (3b)

Figure 3: (3a) The convergence error during the iterative process for Example 1; (3b) Reconstruction result with RMSE=0.23 and iterative step $k = 11$ for Example 1.



(4a) (4b)

Figure 4: (4a) The convergence error during the iterative process for Example 1; (4b) Reconstruction result with RMSE=0.023 and iterative step $k = 20$ for Example 1.

rule (4.11), where we take $r = 2$. The results are shown in Figure 3, where (3a) shows the convergence error during the iterative process until (4.11) is satisfied, (3b) shows the result of reconstruction by the iterative method.

From Figure 3, we can see that the stopping rule (4.11) is effective, and the iterative stop point is almost optimum.

(c) If the noise level $\sigma_2 = 10^{-2}$ and the other parameters are the same as (b), using the stopping rule (4.11), the reconstruction result is shown in Figure 4.

Example 2. Consider the following forward problem

$$\begin{array}{ll} \frac{\partial^\beta w}{\partial t^\beta} - w_{xx} = 0, & 0 < x < a, 0 < t < T, \\ w(0,t) = 0, & 0 < t < T, \\ w(a,t) = L(t), & 0 < t < T, \\ w(x,0) = 0, & 0 < x < a, \end{array} \tag{4.12}$$

where $L(t)$ is given by

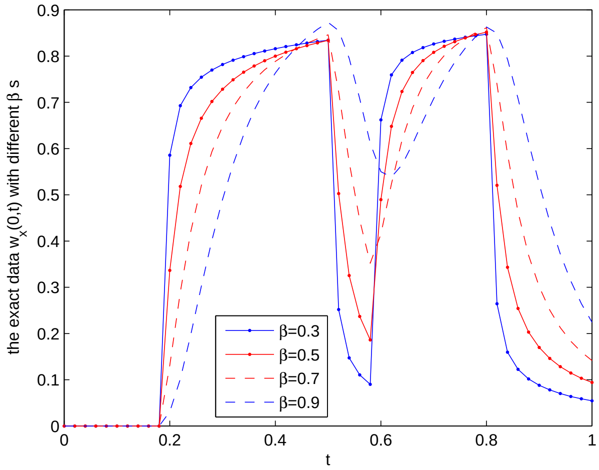


Figure 5: The exact input data $w_x(0,t)$ vs β for Example 2.

$$L(t) = \begin{cases} 1, & 0.2 \leq t \leq 0.5 \,\text{and}\, 0.6 \leq t \leq 0.8, \\ 0, & \text{else}. \end{cases} \tag{4.13}$$

The results displayed in Figures 5-7 are:

Figure 5 shows the input data $w_x(0,t)$ with different β.

Figure 6 shows the result for the convergence error with $\beta = 0.9$, $\sigma_2 = 10^{-2}$.

From Figure 6, we see that the number of iterations performed plays the role of a regularization parameter. Therefore, a stopping rule should be employed.

Figure 7 shows the result with $\beta = 0.7$ and $\sigma_2 = 10^{-2}$. Fig.(7a) shows the convergence error during the iterative process until the stopping rule (4.11) is satisfied with $r = 2$. Fig.(7b) shows the approximation result. For smaller β, we can obtain better approximation results. This is because the smaller β is, the less ill-posed Problem II is, which has been pointed out in Section 2, see (2.7).

Example 3. Let $w(0,t) = 1 - e^{-t^2}$ and

$$L(t) := w(a,t) = \begin{cases} 2t, & if \quad 0 \leq t \leq 0.5, \\ 2(1-t), & if \quad 0.5 \leq t \leq 1. \end{cases}$$

In this example, we change the factor c.

The results displayed in Figures 8 - 10 are:

Figure 8 shows the input data $w_x(0,t)$ for different β.

Figure 9 shows the result for the convergence error with $\beta = 0.9$, $\sigma_2 = 10^{-2}$, $c = 0.1$.

Figure 10 shows the result with $\beta = 0.9$ and $\sigma_2 = 10^{-1}$, $\delta = 0.02$, $c = 0.1$. Fig.(10a) shows the convergence error during the iterative process until the stopping rule (4.11) is satisfied with $r = 2$. Fig.(10b) shows the approximation result.

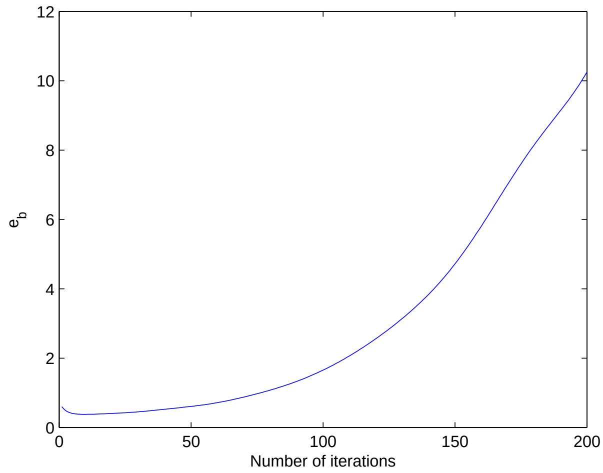
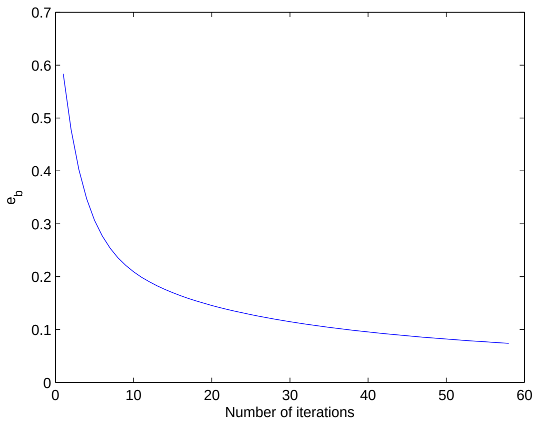


Figure 6: The convergence error vs the number of iterations for Example 2.

(7a)



(7b)

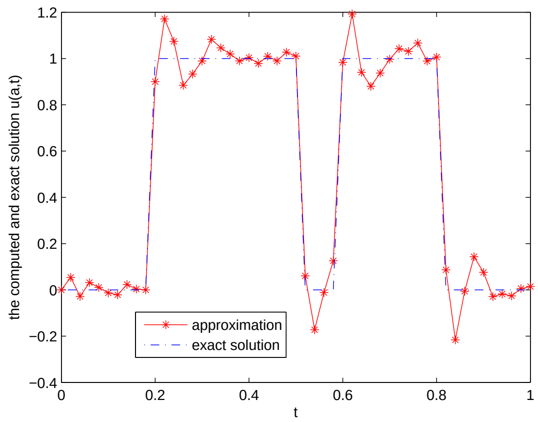


Figure 7: (7a) The convergence error during the iterative process for Example 2; (7b) Reconstruction result with RMSE=0.07 and iterative step $k = 50$ for Example 2.

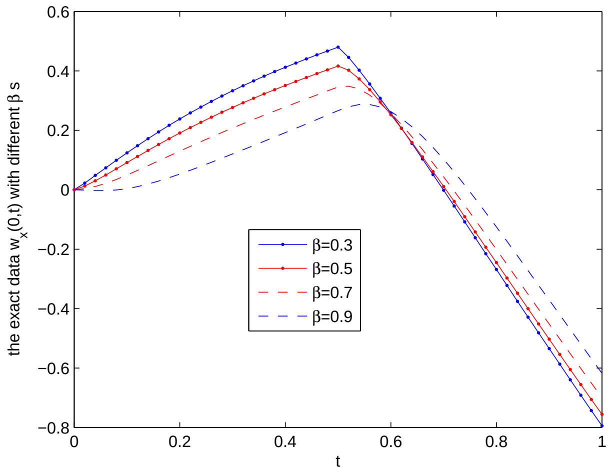


Figure 8: The exact input data $w_x(0,t)$ vs β for Example 3.

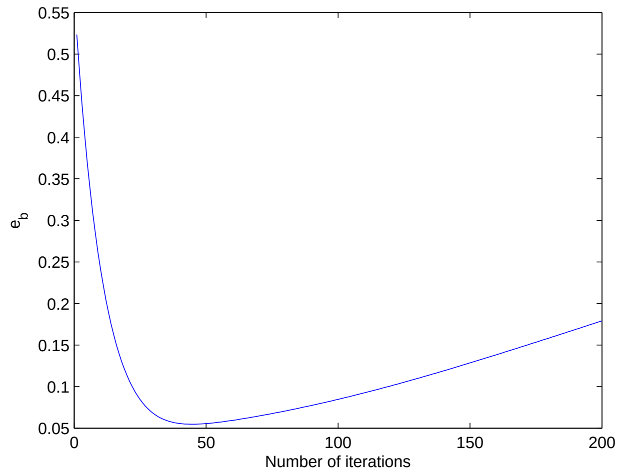
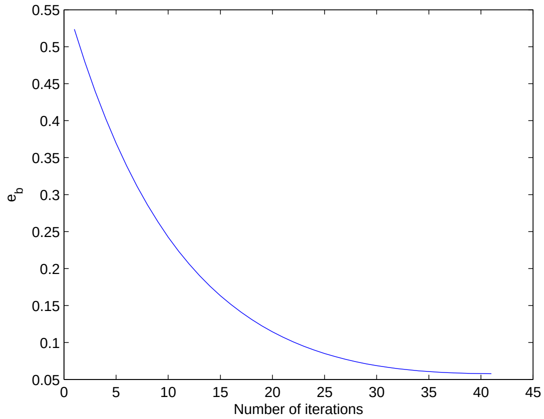
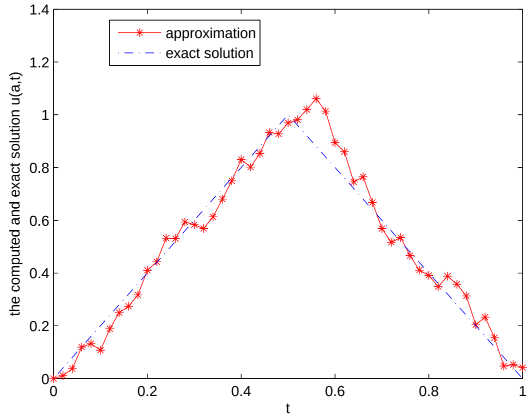


Figure 9: The convergence error vs the number of iterations for Example 3.



(10a)



(10b)

Figure 10: (10a) The convergence error during the iterative process for Example 3; (10b) Reconstruction result with RMSE=0.06 and iterative step $k = 40$ for Example 3.

5 Conclusion

In this paper, based on the Kozlov-Maz'ya iteration method, we give a simplified iterative algorithm with a relaxation factor for solving the Cauchy problem for the fractional diffusion equation. This simplified version only needs to solve one forward problem in each iterative step and can make the cost of computation less. We give the stopping rule for iterative steps with the relaxation factor. We find out that the iterative method has a semi-convergence effect for this ill-posed problem. However, some detailed analysis for the convergence should be done in the future work. Numerical experiments show that the newly proposed stopping rule with the relaxation factor is effective for solving the inverse problem. This method will be extended to the two-dimensional case in the future.

Acknowledgements

The authors would like to thank the reviewers and editors for pointing out the valuable comments.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] E. Barkai, R. Metzler, J. Klafter, From continuous time random walks to the fractional Fokker-Planck equation, Phys. Rev., E 61 (2000) 132-138.

[2] A.S. Chaves, A fractional diffusion equation to describe Lévy flights, Phys. Lett., A 239 (1998) 13-16.

[3] G.M. Zaslavsky, Fractional kinetic-equation for Hamiltonian chaos, Phys. D, 76 (1994) 110-122.

[4] M.M. Meerschaert, H.P. Scheffer, Semi-stable Lévy Motion, Frac. Calc. Appl. Anal., 5 (2002) 27-54.

[5] M. Raberto, E. Scalas, F. Mainardi, Waiting-times and returns in high-frequency financial data: an empirical study, Phys. A, 314 (2002) 749-755.

[6] L. Sabatelli, S. Keating, J. Dudley, P. Richmond, Waiting time distributions in financial markets, Eur. Phys. J., B 27 (2002) 273-275.

[7] A. Blumen, G. Zumofen, J. Klafter, Transport aspects in anomalous diffusion: Lévy walks, Phys. Rev., A 40 (1989) 3964-3973.

[8] J.P. Bouchaud, A. Georges, Anomalous diffusion in disordered media statistical mechanisms, models and physical applications., Phys. Reports, 195(4-5) (1990) 127-293.

[9] Y. Hatano, N. Hatano, H. Amano, Aerosol migration near Chernobyl: Long-term data and modeling, Atmos. Environ., 32 (1998) 2587-2594.

[10] E.E. Adams, L.W. Gelhar, Field-study of dispersion in a heterogeneous aquifer, Water Resour. Res., 28 (1992) 3293-3307.

[11] Y. Zhang, M. M. Meerschaert, B. Baeumer, Particle tracking for time-fractional diffusion, Phy. Rev., E 78(3) (2008) 036705.

[12] M.M. Meerschaert, J. Mortensen, S.W. Wheatcraft, Fractional vector calculus for fractional advection-dispersion, Phys. A, 367 (2006) 181-190.

[13] M.M. Meerschaert, D. Benson, H.P. Scheffer, B. Baeumer, Stochastic solution of space-time fractional diffusion equations, Phys. Rev. E, 65 (2002) 1103-1106.

[14] H. Scher, E.W. Montroll, Anomalous transit-time dispersion in amorphous, Phys. Rev., B12(6) (1975) 2455-2477.

[15] T.L. Szabo, J. Wu, A model for longitudinal and shear wave propagation in viscoelastic media, J. Acoust. Soc. Am., 107(5) (2000) 2437-2446.

[16] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, P. Paradisi, Discrete random walk models for space-time fractional diffusion, Chem. Phys., 284 (2002) 521-541.

[17] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Reports, 339 (2000) 1-77.

[18] I.M. Sokolov, J. Klafter, A. Blumen, Fractional kinetics, Phys. Today, 55 (2002) 48-54.

[19] J. Hadamard, Lectures on Cauchys Problem in Linear Partial Differential Equations, Dover Publications, New York (1953).

[20] X.T. Xiong, Q. Zhou, Y.C. Hon, An inverse problem for fractional diffusion equation in 2-dimensional case: Stability analysis and regularization, J. Math. Anal. Appl., 393 (2012) 185-199.

[21] A.N. Bondarenko, D.S. Ivaschenko, Numerical methods for solving inverse problems for time fractional diffusion equation with variable coefficient, J. Inverse Ill-pose. P., 17(5) (2009) 419-440.

[22] J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for one-dimensional fractional diffusion equation, Inverse Probl., 25 (2009) 115002 (16pp).

[23] D.A. Murio, Stable numerical solution of a fractional-diffusion inverse heat conduction problem, Comput. Math. Appl., 53 (2007) 1492-1501.

[24] G.H. Zheng, T. Wei, Spectral regularization method for a Caucy problem of the time fractional advection-dispersion equation, J. Comput. Appl. Math., 233 (2010) 2631-2640.

[25] X. Xu, J. Cheng, M. Yamamoto, Carleman estimate for a fractional diffusion equation with half order and application, Appl. Anal., 90(9) (2011) 1355-1371.

[26] H.D.N. Duy, T.N. Huy, L.L. Dinh and G.Q.T. Le, Inverse problem for nonlinear backward space-fractional diffusion equation, J. Inverse Ill-Posed Probl., 25(4) (2016) 423-443.

[27] A.H. Salehi Shayegan and A. Zakeri, A numerical method for determining a quasi solution of a backward time-fractional diffusion equation, Inverse Probl. Sci. En., 26(8) (2018) 1130-1154.

[28] H. Zhang and X. Zhang, Generalized Tikhonov method for the final value problem of time-fractional diffusion equation, Int. J. Comput. Math., 94 (2017) 66-78.

[29] A.B. Kostin and S.I. Piskarev, Inverse source problem for the abstract fractional differential equation, J. Inverse Ill-Posed Probl., 29(2) (2021) 267-281.

[30] F. Yang, Q.X. Sun and X.X. Li, Three Landweber iterative methods for solving the initial value problem of time-fractional diffusion wave equation on spherically symmetric domain, Inverse Probl. Sci. En., 29(3) (2021) 1-51.

[31] F. Yang, Q.C. Wang and X.X. Li, Unknown source identification problem for space-time fractional diffusion equation: optimal error bound analysis and regularization method, Inverse Probl. Sci. En., 29(2) (2021) 1-45.

[32] H.W. Zhang and X.J. Zhang, Generalized Tikhonov method for the final value problem of time-fractional diffusion equation, Int. J. Comput. Math., 94(1) (2017) 66-78.

[33] S. Alimov and R. Ashurov, Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation, J. Inverse Ill-Posed Probl., 28(5) (2020) 651-658.

[34] S.Z. Jiang and T. Wei, Recovering a time-dependent potential function in a time fractional diffusion equation by using a nonlinear condition, Inverse Probl. Sc. En., 29(2) (2020) 174-195.

[35] S.Z. Jiang and Y.J. Wu, Recovering a time-dependent potential function in a multi-term time fractional diffusion equation by using a nonlinear condition, J. Inverse Ill-Posed Probl., 29(2) (2021) 233-248.

[36] S.P. Yang , X.T. Xiong and Y.Z. Han, A modified fractional Landweber method for a backward problem for the inhomogeneous timefractional diffusion equation in a cylinder, Int. J. Comput. Math., 97 (2020) 2375-2393.

[37] S. Nematia, P.M. Lima, S. Sedaghat, Legendre wavelet collocation method combined with the GaussCJacobi quadrature for solving fractional delay-type integro-differential equations, Appl. Numer. Math., 149 (2020) 99-112.

[38] J. Cheng, M. Yamamoto, One new strategy for a priori choice of regularizing parameters in Tikhonov's regularization, Inverse Probl., 16 (2000) L31.

[39] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Kluwer Academic Publisher, Dordrecht Boston London (1996).

[40] Y.C. Hon, T. Wei, Backus-Gilbert algorithm for the Cauchy problem of laplace equation, Inverse Probl., 17 (2001) 261-271.

[41] Y.C. Hon, T. Wei, The method of fundamental solutions for solving multidimensional inverse heat conduction problems, Comp. Model. Eng. Sci., 7(2) (2005) 119-132.

[42] X.T. Xiong, Regularization theory and algorithm for some inverse problems for parabolic differential equations. PhD dissertation, Lanzhou University (2007).

[43] V.A. Kozlov, V.G. Mazya, On iterative procedures for solving ill-posed boundary value problems that preserve differential equations, Leningrad Math. J., 5 (1990) 1207-1228.

[44] M. Jourhmane, N.S. Mera, An iterative algorithm for the backward heat conduction problem based on variable relaxation factors, Inverse Problems in Engng, 10(4) (2002) 293-308.

[45] T. Wei, X. L. Li and Y. S. Li, An inverse time-dependent source problem for a time-fractional diffusion equation, Inverse Probl., 32 (2016), no. 8, 085003.

[46] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego (1999).

[47] X.T. Xiong, J.X. Wang, M. Li, An optimal method for fractional heat conduction problem backward in time, Appl. Anal., 91 (2012) 823-840.

[48] G. Bastay , V.A. Kozlov and B.O. Turesson, Iterative methods for an inverse heat conduction problem, J. Inverse Ill-Posed Probl., 9(4) (2001) 375-388.

[49] D.A. Murio, Implicit finite defference approximation for time fractional diffusion equations, Comput. Math. Appl., 56 (2008) 1138-1145.

[50] L. Landweber, An iteration formula for Fredholm integral equations of the first kind, Am. J. Math., 73 (1951) 615-624.