# Limit cycles bifurcated from a kind of piecewise smooth generalized Abel equation via the first order Melnikov analysis

Cheng Wang[a], Fei Guan[b], Qianqian Zhao[c,*]

[a]School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210023, P. R. China

[b]College of Statistics and Mathematics, Hebei University of Economics and Business, Shijiazhuang 050061, P. R. China

[c]College of Statistics and Mathematics, Hebei University of Economics and Business, Shijiazhuang 050061, P. R. China

**Abstract**

The study of the existence and distribution of limit cycles for generalized Abel equations comes from the famous Small-Pugh problem, which has been extended to non-smooth case. In this paper, we consider a kind of piecewise smooth generalized Abel equation with the separation line $t = 0$. We are interested in its number of nontrivial limit cycles which are bifurcated from the periodic annulus of unperturbed equation. Under the first order Melnikov analysis, we show that the upper bound of this kind nontrivial limit cycles is $2(m+1)$ if $p$ is odd, and $m+1$ if $p$ is even. The upper bound in both cases can be reached separately.

*Keywords:* Piecewise smooth, Generalized Abel equation, Nontrivial limit cycles, Bifurcation, Melnikov function
*2020 MSC:* 34C07, 34C23, 34C25

## 1. Introduction and main result

Many practical problems can be modeled as following scalar differential equation

$$\dot{x} := \frac{dx}{dt} = \sum_{i=0}^{k} a_i(t)x^i, \tag{1.1}$$

where $a_i(t) \in C^\infty([0,1])$, $i = 0, 1, \cdots, k$, such as relativistic dissipative cosmological model in [13], harvesting model in [2], tracking control problem in [4], glioblastoma growth in [12], etc. This kind equation is usually called *generalized Abel equation*.

*Corresponding author
*Email addresses:* mathwc@126.com (Cheng Wang), guanfei@hueb.edu.cn (Fei Guan), zqqyly@hueb.edu.cn (Qianqian Zhao )

Let $u(t;\rho)$ be the solution of (1.1) determined by $u(0;\rho) = \rho$, then it is periodic if $u(1;\rho) = \rho$. An isolated periodic solution is called a *limit cycle*. Determining upper bounds of limit cycles for generalized Abel equations is famous Smale-Pugh problem, see [26]. Although this problem is the one in low dimensional dynamical systems, it is still open so far, see problems $5-7$ in [5].

For equation (1.1), when $k = 1$, it is a linear equation and it has 1 limit cycle, see [20, 22]. When $k = 2$, it is well-known Ricatti equation, and it has 2 limit cycles, see [20, 23, 24]. In the case of $k = 3$, it is called Abel equation, and in [20, 25], Lins-Neto and Panov showed that the number of Abel equation is not bounded. Under condition $k \geq 3$, the number of limit cycles may become bounded if some additional constraints are added to the coefficients $a_i(t)$, $i = 0, 1, \cdots, k$, see [1, 6, 8, 15, 18, 14].

In addition to its own importance, the generalized Abel equation can be used to study a class of planar polynomial differential systems in the following:

$$\begin{aligned} \dot{x} &= ax - y + P_n(x,y), \\ \dot{y} &= x + ay + Q_n(x,y), \end{aligned} \tag{1.2}$$

where $P_n$ and $Q_n$ are homogeneous polynomials of degree $n \geq 2$, and $a \in \mathbb{R}$. By a similar-to-polar change of coordinates, which was first introduced by Cherkas in [3], (1.2) can be reduced to (1.1). Some papers found the limit cycles of (1.2) by studying the limit cycles of generalized Abel equation, see [15] and the references therein. In recent years, there have been many other interesting results concerning smooth or piecewise smooth generalized equations, see [10, 9, 13, 30, 16, 27, 21, 17, 29, 28] and the references therein.

In this paper, we focus on the following type of generalized Abel equation:

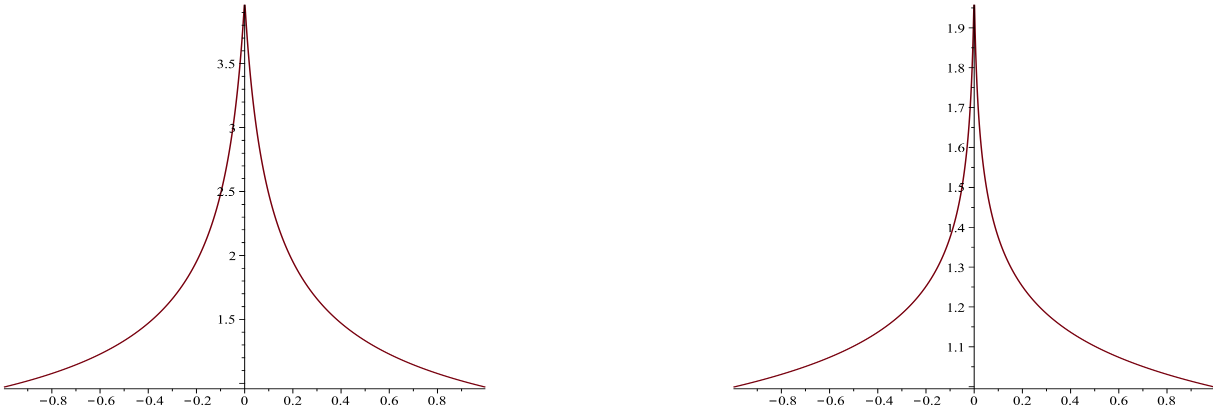
$$\frac{dx}{dt} = \begin{cases} \dfrac{x^p}{p-1} + G^-(x,t), \ -1 \leq t \leq 0, x \in R, \\ -\dfrac{x^p}{p-1} + G^+(x,t), \ 0 \leq t \leq 1, x \in R, \end{cases}$$

where $p \in \mathbb{Z}^+ \setminus \{1\}$, $|\varepsilon| \ll 1$, $G^-(x,t) = \sum_{i=1}^{+\infty} \varepsilon^i (A_i^-(t)x^p + B_i^-(t)x^{2p-1})$, $G^+(x,t) = \sum_{i=1}^{+\infty} \varepsilon^i (A_i^+(t)x^p + B_i^+(t)x^{2p-1})$, $A_i^\pm(t) = \sum_{j=0}^{n} a_{ij}^\pm t^j$, $B_i^\pm(t) = \sum_{j=0}^{m} b_{ij}^\pm t^j$, $i = 1, 2, 3, \cdots$. Notice that (1.3) is non-smooth at time $t = 0$. Let $x_\varepsilon(t;\rho)$ be the solution of (1.3) determined by $x_\varepsilon(0;\rho) = \rho$. When $t > 0$, the flow $x_\varepsilon(t;\rho)$ starts from point $B(0,\rho)$ and goes to $C(1, x_\varepsilon(1;\rho))$, and when $t < 0$, it starts from $B$ and goes backwards to $A(-1, x_\varepsilon(-1;\rho))$. Hence, $x_\varepsilon(t;\rho)$ defines a map $A \to B \to C$, see Figure 2.

Similar to (1.1), we call that $x_\varepsilon(t;\rho)$ is *periodic* if $x_\varepsilon(-1;\rho) = x_\varepsilon(1;\rho)$. The isolated periodic solution of equation (1.3) is defined as its *limit cycle*. Further more, if a limit cycle is not equal to zero, then it is defined as a *nontrivial limit cycle*. The origin is called a *center* if there is a periodic annulus around it.

When $\varepsilon = 0$, we get $x_0(t;\rho)$ by direct calculation

$$x_0(t;\rho) = \begin{cases} \left(\dfrac{1}{-t+\frac{1}{\rho^{p-1}}}\right)^{\frac{1}{p-1}}, & -1 \le t \le 0, \rho \ne 0, \\ \left(\dfrac{1}{t+\frac{1}{\rho^{p-1}}}\right)^{\frac{1}{p-1}}, & 0 \le t \le 1, \rho \ne 0, \\ 0, & \rho = 0. \end{cases} \tag{1.3}$$



(a) Graph of $x_0(t;4)$ when $p=3$.

(b) Graph of $x_0(t;2)$ when $p=8$.

Figure 1: Graphs of $x_0(t;\rho)$ with different $\rho$ and $p$.

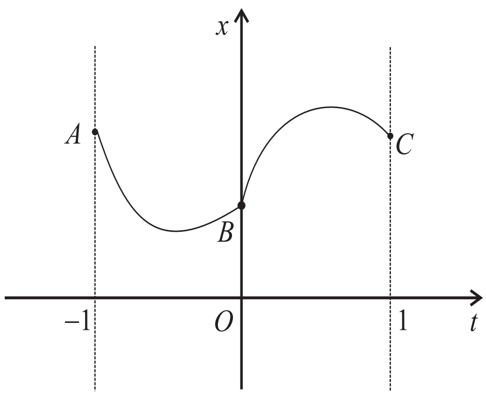



Figure 2: Illustration of flow $x_\varepsilon(t;\rho)$ ($A \to B \to C$).

If $p$ is odd (resp. even), there exists an interval $\rho \in (-\infty, +\infty)$ (resp. $\rho \in (-1, +\infty)$) such that $x_0(-1;\rho) = x_0(1;\rho)$. In other words, the unperturbed equation of (1.3) has a periodic annulus $U_1$ when $p$ is odd, and $U_2$ when even, where

$$U_1 = [-1, 1] \times (-\infty, +\infty),$$

$$U_2 = [-1, 1] \times (-1, +\infty).$$

Obviously, the origin is a center whether $p$ is odd or even.

When $\varepsilon \neq 0$, let

$$\Delta(t; \rho) = x_\varepsilon(1; \rho) - x_\varepsilon(-1; \rho).$$

Obviously, the solution $x_\varepsilon(t; \rho)$ is a limit cycle if and only if $\Delta(t; \rho) = 0$. Expand $x_\varepsilon(t; \rho)$ at $\varepsilon = 0$:

$$x_\varepsilon(t; \rho) = \sum_{i=0}^{+\infty} \frac{1}{i!} \frac{\partial^i x_\varepsilon}{\partial \varepsilon^i}\bigg|_{\varepsilon=0} \varepsilon^i = \sum_{i=0}^{+\infty} S_i(t; \rho)\varepsilon^i,$$

where $\frac{1}{i!} \frac{\partial^i x_\varepsilon}{\partial \varepsilon^i}\big|_{\varepsilon=0} \triangleq S_i(t; \rho)$. Define

$$M_i(\rho) = S_i(1; \rho) - S_i(-1; \rho),$$

where $i = 1, 2, 3, \cdots$. Then,

$$\begin{aligned}
\Delta(t; \rho) &= x_\varepsilon(1; \rho) - \varepsilon(-1; \rho) \\
&= x_0(1; \rho) - x_0(-1; \rho) + \sum_{i=1}^{+\infty} \left[S_i(1; \rho) - S_i(-1; \rho)\right] \varepsilon^i \\
&= \sum_{i=1}^{+\infty} M_i(\rho)\varepsilon^i.
\end{aligned}$$

We call $M_i(\rho)$ the *i-th order Melnikov function of equation* (1.3).

It is worth noticing that, based on Theorem 3.3 in [11], when $M_1(\rho) \not\equiv 0$, the number of limit cycles of (1.3) is determined by the number of zeros of $M_1(\rho)$.

By studying the first order Melnikov function and using the properties of Chebyshev systems, we get the main result:

**Theorem 1.1.** *For equation* (1.3)*, if $M_1(\rho) \not\equiv 0$, then the maximum number of nontrivial limit cycles bifurcated from $U_1$ is $2(m + 1)$ if $p$ is odd, and bifurcated from $U_2$ is $m + 1$ if $p$ is even. The upper bound in both cases can be reached separately.*

The paper is organized as follows: some preliminaries are presented in Section 2, and the main result is proved in Section 3.

## 2. Preliminaries

In this section, firstly, we show the definition of ECT-system and also one of its properties, more details see [16, 7, 19]. Secondly, two ECT-systems are proved.

**Definition 2.1.** Let $\Phi_0, \Phi_1, \cdots, \Phi_n$ be analytic functions on an open interval $I$ of $\mathbb{R}$. The order set $(\Phi_0, \Phi_1, \cdots, \Phi_n)$ is an **extended complete Chebyshev system (ECT-system)** on $I$, if for all $k = 0, 1, \cdots, n$, any non-trivial linear combination

$$\lambda_0\Phi_0 + \lambda_1\Phi_1 + \cdots + \lambda_k\Phi_k$$

has at most $k$ isolated zeros on $I$ counted with multiplicities and such upper bound is sharp.

From the Proposition in [19], we can get the following Lemma easily:

**Lemma 2.1.** *The order set of functions* $(1, \Phi_1, \Phi_2, \cdots, \Phi_n)$ *is an ECT-system on an interval* $I$*, if and only if* $(\Phi_1', \Phi_2', \cdots, \Phi_n')$ *is an ECT-system on* $I$*.*

We obtain the next Lemma based on the main result in [7]:

**Lemma 2.2.** *The family of analytic functions*

$$I_k(y) := \int_0^1 \frac{t^k}{(1-yt)^2}\mathrm{d}t,$$

*for* $k = 0, 1, 2, \cdots, n$*, is an ECT-system both on* $(-\infty, 1)$ *and* $(-\infty, 0)$ *respectively.*

**Theorem 2.3.** *Let*

$$\phi_0(y) = 1, \phi_1(y) = y\int_0^1 \frac{1}{1-yt}\mathrm{d}t, \phi_2(y) = y\int_0^1 \frac{t}{1-yt}\mathrm{d}t, \cdots, \phi_{m+1}(y) = y\int_0^1 \frac{t^m}{1-yt}\mathrm{d}t,$$

*then* $(\phi_0(y), \phi_1(y), \cdots, \phi_{m+1}(y))$ *is an ECT-system both on* $(-\infty, 1)$ *and* $(-\infty, 0)$ *respectively.*

*Proof.* By direct calculation, we get that

$$\frac{\mathrm{d}\phi_{k+1}(y)}{\mathrm{d}y} = \int_0^1 \frac{t^k}{1-yt}\mathrm{d}t + y\int_0^1 \frac{t^{k+1}}{(1-yt)^2}\mathrm{d}t = I_k(y),$$

where $k = 0, 1, 2, \cdots, m$. Based on Lemma 2.1 and 2.2, the Theorem holds. □

## 3. Proof of Theorem 1.1

When $-1 \le t \le 0$,

$$\begin{aligned}
&\partial_t S_1(t;\rho) = \partial_t\partial_\varepsilon x_\varepsilon(t;\rho)\,|_{\varepsilon=0} = \partial_\varepsilon\partial_t x_\varepsilon(t;\rho)\,|_{\varepsilon=0} \\
=&\partial_\varepsilon\left(\frac{x_\varepsilon^p(t;\rho)}{p-1} + \sum_{i=0}^{+\infty}\varepsilon^i\left[A_i^-(t)x_\varepsilon^p(t;\rho) + B_i^-(t)x_\varepsilon^{2p-1}(t;\rho)\right]\right)_{\varepsilon=0} \\
=&\frac{p}{p-1}x_0^{p-1}(t;\rho)S_1(t;\rho) + A_1^-(t)x_0^p(t;\rho) + B_1^-(t)x_0^{2p-1}(t;\rho).
\end{aligned}$$

Also because

$$\partial_t\left(\frac{S_1(t;\rho)}{x_0^p(t;\rho)}\right) = \frac{\partial_t S_1(t;\rho)}{x_0^p(t;\rho)} - \frac{p}{p-1}\frac{S_1(t;\rho)}{x_0(t;\rho)},$$

we get

$$\partial_t\left(\frac{S_1(t;\rho)}{x_0^p(t;\rho)}\right) = A_1^-(t) + B_1^-(t)x_0^{p-1}(t;\rho).$$

Furthermore,

$$S_1(-1;\rho) = x_0^p(-1;\rho)\int_0^{-1}\left(A_1^-(t) + B_1^-(t)x_0^{p-1}(t;\rho)\right)\mathrm{d}t.$$

When $0 \le t \le 1$, we get $S_1(1;\rho)$ similarly:

$$S_1(1;\rho) = x_0^p(1;\rho)\int_0^{1}\left(A_1^+(t) + B_1^+(t)x_0^{p-1}(t;\rho)\right)\mathrm{d}t.$$

Let $y = -\rho^{p-1}$, $a_{1i}^+ + (-1)^i a_{1i}^- = a_i$, $b_{1j}^+ + (-1)^j b_{1j}^- = b_j$, $i = 0, 1, \cdots, n$, $j = 0, 1, \cdots, m$, then,

$$\begin{aligned}
M_1(\rho) &= S_1(1;\rho) - S_1(-1;\rho) \\
&= x_0^p(1;\rho)\int_0^1\left[\left[A_1^+(t) + A_1^-(-t)\right] + \left[B_1^+(t) + B_1^-(-t)\right]\frac{1}{t-\frac{1}{y}}\right]\mathrm{d}t \\
&= x_0^p(1;\rho)\left[\sum_{i=0}^{n+1}\frac{a_i}{i+1} - y\int_0^1 (b_0 + b_1 t + b_2 t + \cdots + b_m t^m)\frac{1}{1-yt}\mathrm{d}t\right] \\
&= x_0^p(1;\rho)\left[\sum_{i=0}^{n+1}\frac{a_i}{i+1} - b_0\phi_1(y) - b_1\phi_2(y) - \cdots - b_m\phi_{m+1}(y)\right],
\end{aligned}$$

Based on Theorem 2.3, $(\phi_0(y), \phi_1(y), \cdots, \phi_{m+1}(y))$ is an ECT-system both on $(-\infty, 1)$ and $(-\infty, 0)$ respectively. Furthermore, obviously, $\sum_{i=0}^{n+1}\frac{a_i}{i+1}$, $-b_0$, $-b_1$, $\cdots$, $-b_m$ are independent. So, the number of zeros of $M_1(y)$ is at most $m+1$ both on $(-\infty, 1)$ and $(-\infty, 0)$.

If $p$ is odd, then $y = -\rho^{p-1} < 0$, the number of zeros of $M_1(\rho)$ is at most $2(m+1)$ on $(-\infty, +\infty)$, which can be reached. If $p$ is even, then $y = -\rho^{p-1} \in (-\infty, 1)$, the number of zeros of $M_1(\rho)$ is at most $m+1$ on $(-1, +\infty)$, which can be reached, too.

According to Theorem 3.3 in [11], we get that Theorem 1.1 holds.


## Acknowledgments

The first author is partially supported by the National Natural Science Foundation of China (No. 12001265), the Natural Science Foundation of Jiangsu Province, P. R. China (No. BK20200829). The second author is funded by Key Program of Hebei University of Economics and Business (2024ZD12). The third author is partially funded by Science Research Project of Hebei Education Department (NO. QN2023168), the National Natural Science Foundation of China (NO. 12301195).


## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

## References


[1] M. J. Álvarez, A. Gasull, and H. Giacomini. A new uniqueness criterion for the number of periodic orbits of Abel equations. *J. Differential Equations*, 234(1):161–176, 2007.

[2] D. M. Benardete, V. W. Noonburg, and B. Pollina. Qualitative tools for studying periodic solutions and bifurcations as applied to the periodically harvested logistic equation. *Amer. Math. Monthly*, 115(3):202–219, 2008.

[3] L. A. Cherkas. The number of limit cycles of a certain second order autonumous system. *Differencial nye Uravnenija*, (no. 5,):944–946, 960, 1976.

[4] E. Fossas, J. M. Olm, and H. Sira-Ramírez. Iterative approximation of limit cycles for a class of Abel equations. *Phys. D*, 237(23):3159–3164, 2008.

[5] A. Gasull. Some open problems in low dimensional dynamical systems. *SeMA J.*, 78(3):233–269, 2021.

[6] A. Gasull and A. Guillamon. Limit cycles for generalized Abel equations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 16(12):3737–3745, 2006.

[7] A. Gasull, C. Li, and J. Torregrosa. A new Chebyshev family with applications to Abel equations. *J. Differential Equations*, 252(2):1635–1641, 2012.

[8] A. Gasull and J. Llibre. Limit cycles for a class of Abel equations. *SIAM J. Math. Anal.*, 21(5):1235–1244, 1990.

[9] M. Han. On the maximum number of periodic solutions of piecewise smooth periodic equations by average method. *J. Appl. Anal. Comput.*, 7(2):788–794, 2017.

[10] M. Han and L. Sheng. Bifurcation of limit cycles in piecewise smooth systems via Melnikov function. *J. Appl. Anal. Comput.*, 5(4):809–815, 2015.

[11] M. Han and J. Yang. The maximum number of zeros of functions with parameters and application to differential equations. *J. Nonlinear Model. Anal.*, 3(1):13–34, 2021.

[12] T. Harko and M. K. Mak. Travelling wave solutions of the reaction-diffusion mathematical model of glioblastoma growth: an Abel equation based approach. *Math. Biosci. Eng.*, 12(1):41–69, 2015.

[13] J. Huang and J. Li. On the number of limit cycles in piecewise smooth generalized Abel equations with two asymmetric zones. *Nonlinear Anal. Real World Appl.*, 66:Paper No. 103551, 17, 2022.

[14] J. Huang and H. Liang. A geometric criterion for equation $\dot{x} = \sum_{i=0}^{m} a_i(t)x^i$ having at most $m$ isolated periodic solutions. *J. Differential Equations*, 268(10):6230–6250, 2020.

[15] J. Huang, H. Liang, and J. Llibre. Non-existence and uniqueness of limit cycles for planar polynomial differential systems with homogeneous nonlinearities. *J. Differential Equations*, 265(9):3888–3913, 2018.

[16] J. Huang and Z. Peng. Bifurcation of a kind of piecewise smooth generalized Abel equation via first and second order analyses. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 30(16):2050247, 20, 2020.

[17] J. Huang, J. Torregrosa, and J. Villadelprat. On the number of limit cycles in generalized Abel equations. *SIAM J. Appl. Dyn. Syst.*, 19(4):2343–2370, 2020.

[18] J. Huang and Y. Zhao. Periodic solutions for equation $\dot{x} = A(t)x^m + B(t)x^n + C(t)x^l$ with $A(t)$ and $B(t)$ changing signs. *J. Differential Equations*, 253(1):73–99, 2012.

[19] M. Krusemeyer. The Teaching of Mathematics: Why Does the Wronskian Work? *Amer. Math. Monthly*, 95(1):46–49, 1988.

[20] A. Lins Neto. On the number of solutions of the equation $dx/dt = \sum_{j=0}^{n} a_j(t)x^j$, $0 \leq t \leq 1$, for which $x(0) = x(1)$. *Invent. Math.*, 59(1):67–76, 1980.

[21] C. J. Liu and S. Q. Wang. On the center problem for generalized Abel equations. *Acta Math. Sin. (Engl. Ser.)*, 39(12):2329–2337, 2023.

[22] N. G. Lloyd. The number of periodic solutions of the equation $\dot{z} = z^N + p_1(t)z^{N-1} + ... + p_N(t)$. *Proc. London Math. Soc. (3)*, 27:667–700, 1973.

[23] N. G. Lloyd. On a class of differential equations of Riccati type. *J. London Math. Soc. (2)*, 10:1–10, 1975.

[24] N. G. Lloyd. A note on the number of limit cycles in certain two-dimensional systems. *J. London Math. Soc. (2)*, 20(2):277–286, 1979.

[25] A. A. Panov. Variety of Poincaré mappings for cubic equations with variable coefficients. *Funktsional. Anal. i Prilozhen.*, 33(4):84–88, 1999.

[26] S. Smale. Mathematical problems for the next century. In *Mathematics: frontiers and perspectives*, pages 271–294. Amer. Math. Soc., Providence, RI, 2000.
[27] R. Tian and Y. Zhao. On the number of limit cycles in piecewise smooth generalized Abel equations with many separation lines. *Nonlinear Anal. Real World Appl.*, 80:Paper No. 104151, 2024.
[28] Y. Tian, X. Shang, and M. Han. Bifurcation of limit cycles in a piecewise smooth near-integrable system. *J. Math. Anal. Appl.*, 504(2):Paper No. 125578, 11, 2021.
[29] X. Yu, J. Huang, and C. Liu. Maximum number of limit cycles for Abel equation having coefficients with linear trigonometric functions. *J. Differential Equations*, 410:301–318, 2024.
[30] Q. Zhao, J. Yu, and C. Wang. Nontrivial limit cycles in a kind of piecewise smooth generalized Abel equation. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 32(14):Paper No. 2250216, 18, 2022.