New exploration on approximate controllability of damped elastic beam systems in Banach spaces *

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Abstract: This article mainly studies the existence and approximate controllability of mild solutions for a class of Volterra-Fredholm type integral-differential damped elastic beam systems in Banach spaces. Firstly, the existence of mild solutions was obtained using Banach fixed point theorem and operator semigroup theory. Secondly, we formalized and proved the sufficient conditions for the approximate controllability of our desired problem. To test the results of approximate controllability, we used sequence method without assuming that the corresponding linear system is approximately controllable. Finally, an example is given to illustrate the theory results.

Keywords: Approximate controllability, Damped elastic beam systems, Mild solutions, Sequence method

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1 Introduction

As a significant and autonomous field within modern engineering research, the study of elastic beams finds extensive applications across various disciplines including mechanics,

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material sciences, physics, and geology. Moreover, in specific contexts, these beams play an almost irreplaceable role. Consequently, the investigation of beam vibration equations has increasingly garnered substantial attention and keen interest among scholars across these fields.

In 1744, Leonhard Euler conducted a study on the lateral vibrations of beams and presented the vibration functions and frequency equations under In 1751, while addressing a similar issue, Daniel Bernoulli formulated the vibration equation for beams, which became known as the Euler-Bernoulli beam equation

$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\Big(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\Big)=0,\quad 0<x<1,\ t>0.$$

The equation at hand represents the fundamental vibration equation for beams, where $\rho(x)$ denotes the mass density of the beam, E stands for the modulus of elasticity, and $I(x)$ represents the moment of inertia of the beam's cross-section. Over the ensuing decades, numerous scholars have conducted extensive research on the vibration equations of elastic beams. With the advancement of science and technology, coupled with the rapid development of aerospace engineering, the vibration equations of spacecraft beams, modeled mathematically through structural damping, have gradually come into focus. Beginning in 1981, Chen and Russel [16] were the first to introduce the damped elastic systems

$$\begin{cases} \ddot{u}(t)+\rho B\dot{u}(t)+Au(t)=0, \\ u(0)=u_0,\ \dot{u}(0)=u_1, \end{cases}$$

where $A: D(A)\subset E\to E$, $B: D(B)\subset E\to E$ are densely defined closed linear operators on Banach space E, $\rho>0$ is a constant.

This issue has garnered significant attention and interest among scholars, becoming one of the quintessential research subjects in the field of evolution equations. Last several years, numerous scholars have employed nonlinear analysis methods and techniques, for example operator semigroup theory, fixed point theorems, and monotone iterative methods, to conduct thorough research on certain nonlinear structural damping elastic beam systems. These investigations have yielded meaningful results, as detailed in [16, 18, 25–30, 34, 39, 40, 43, 44, 53, 54, 58, 84, 85] and their references.

In 1960, Kalman [36] introduced the concept of controllability for the first time. This notion is fundamental in the study and design of control systems, where many dynamic systems are engineered to allow control to affect only certain parts of the system state. However, in practical industrial operations, it is often the case that only a small subset of the dynamic system's full state is observable. Consequently, assessing the feasibility of controlling the entire state of a dynamic system is of critical importance. This has led to the emergence of the concepts of exact and approximate controllability.

Controllability is one of the fundamental concepts in mathematical control theory, which is extensively applied across numerous fields of science and technology. In finite-dimensional spaces, the controllability of linear and nonlinear systems, represented by ordinary differential equations, has been extensively studied by various authors. In Banach spaces, the concept of infinite-dimensional systems has been somewhat broadened, irrespective of whether common impulsive effects are included. For a thorough investigation of this matter, readers are referred to the pertinent literature [1–8, 10–15, 19–24, 35, 47–52, 55–57, 59, 63, 64, 67, 68, 74–83, 87, 89].

It is widely believed that achieving precise controllability of abstract semilinear control systems in infinite dimensional space is challenging, because it requires the controllability operator to be surjective. Therefore, it is necessary to explore a weaker concept of controllability, namely approximate controllability. Mathematical control theory forms a part of application oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. Roughly speaking, there have been two main lines of work in control theory, which sometimes seemed to proceed in very different directions, but which are, in fact, complementary. One of these is based on the idea that a good model of the object to be controlled is available and that one wants to somehow optimize its behavior. The other main line of work is based on the constraints imposed by uncertainty about the model or about the environment in which the object operates. In 1983, Zhou [88] established sufficient conditions for approximate controllability of semi linear abstract equations, applicable to infinite and finite dimensions. Subsequently, Mahmudov [60, 62] studied the approximate controllability of abstract semilinear deterministic and stochastic control systems under the natural assumption of approximate controllability of related linear control systems. In 2008, Mahmudov [60, 62] studied the approximate controllability of abstract evolution equations in Hilbert space. Recently, the author discussed the existence and exact controllability of semilinear measure driven equations in [14, 15, 17].

Currently, there are two methods to explore approximate controllability issues. On the one hand, multiple authors have constructed controls through conjugation problems and achieved controllable results, as detailed in [31–33, 46] and their references. Currently, in [45], the author has constructed control through conjugation problems and obtained controllability results for Volterra-Fredholm type systems in Banach spaces.

On the other hand, some researchers have used sequence methods to verify the approximate controllability of semilinear differential systems. Zhou [88] established the sufficient conditions for existence and approximate controllability of solutions to semilinear abstract equations without time delay using sequence method. Recently, Shukla et al. [71] combined the strong cosine family with the sine family to study the approximate controllability of semilinear systems with state delays, using the sequence method. After that, the authors of [69] used the same approach to investigate the approximate controllability of differential equation involving neutral function and delay. The authors of [65] investigated the approximate controllability of nonlinear differential systems of second order involving stochastic differential systems and of McKean-Vlasov type by using the sequence approach. In application, the authors of [90] studied the collision dynamics of three-solitons in an optical communication system with third-order dispersion and nonlinearity. In [71], the authors derived necessary requirements for the approximate controllability of semilinear delay differential systems. They explored the approximate controllability results for the given system using the sequential method. For more details, see the articles [37, 41, 42, 46, 71, 88] and references therein.

However, to the best of our knowledge, there is no result on approximate controllability of damped elastic beam systems, using techniques as in [37, 46, 71, 88]. Inspired by the ideas and methods of the above approaches, this paper aims to exploring the existence and approximate controllability of mild solutions for damped elastic beam systems

$$\begin{cases} \ddot{u}(t) + \rho A\dot{u}(t) + A^2u(t) = F(t, u(t), (Gu)(t), (Hu)(t)) + Bv(t), \quad t \in [0, a], \\ u(0) = u_0, \quad \dot{u}(0) = u_1, \end{cases} \tag{1.1}$$

where u'' and u' are the first and second order partial derivatives of u with respect to t, $\rho \geq 2$ is the damping coefficient, $J = [0, a], a > 0$, $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$ are densely defined closed (possibly unbounded) linear operators on a Banach space E. The control function v takes its values in the space $L^2(J, U)$ where U is a Banach space. Additionally, B is a linear bounded operator from U to E. The functions $F : J \times E \times E \times E \to E$, $f : J \times J \times E \to E$ and $g : J \times J \times E \to E$ are nonlinear. Also, the functions F, f, and g are Carathéodory continuous. The operator G and H are specified by

$$(Gu)(t) = \int_0^t f(t, s, u(s))ds,$$

$$(Hu)(t) = \int_0^a g(t, s, u(s))ds.$$

The intention of the current manuscript is to explore the approximate controllability of damped elastic beam systems involving Volterra-Fredholm type integro-differential systems. Meanwhile, the existence and uniqueness of mild solutions of the given system is verified by employing the Banach fixed point theorem combined with semigroup operators. No one has used the sequence method to study the approximate controllability of the damped elastic beam systems involving Volterra-Fredholm type integro-differential system, so we have used the sequence method here.

The primary contributions of this paper are as follows:

1. This article explores the approximate controllability of damped elastic beam systems, without the assumption of corresponding linear systems being approximately controllability. Furthermore, by integrating the Banach fixed point theorem with semigroup operators, it verifies the existence and uniqueness of mild solutions for the system.

2. The merit of the approximate sequence method lies in its flexibility. It does not require the corresponding linear control system to be approximately controllable, nor does it require defining a Gammer control function to transform the control problem into a fixed-point problem for the operator.

3. Due to the previous research on the approximate controllability of Volterra-Fredholm type damping elastic beam systems without using sequential methods, we adopt this method in our current study, which differs from the results in [13].

The structure of this paper is as follows: The second section presents preliminary details; the third section utilizes the Banach fixed point theorem to elucidate the existence and uniqueness of mild solutions for system (1.1). The fourth section demonstrates our results on the approximate controllability of system (1.1) through a sequence method. The final section illustrates the application of the obtained results through practical examples. The conclusion section provides a summary of this paper.

2 Preliminaries

Let E and U be two real Banach spaces, with norms $\|\cdot\|$ and $\|\cdot\|_U$ respectively. Denote by $C(J,E)$ the Banach space of all continuous functions from the interval J to E with norm

$$\|u\|_C = \sup_{t\in J} \|u(t)\|, \quad u \in C(J,E),$$

Furthermore, let $L^2(J,U)$ be the Banach space of all U-valued Bochner square integrable functions defined on J with norm

$$\|u\|_{L^2(J,U)} = \Big(\int_0^a \|u(t)\|_U^2 dt\Big)^{\frac{1}{2}}, \quad u \in L^2(J,U).$$

Throughout the article, we assume that the following conditions:

(A1) Suppose that $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a C_0 semigroup $\mathcal{T}(t)(t \geq 0)$ in E.

In accordance with Definition 3 and Lemma 2.2 from [25, 30, 53], we delineate the mild solution to problem (1.1) as follows.

Definition 2.1. *Function $u \in C(J, E)$ is referred to as a mild solution to problem (1.1) if $u(\cdot)$ satisfies*

$$u(t) = \mathcal{T}_2(t)u_0 + \int_0^t \mathcal{T}_2(t-s)\mathcal{T}_1(s)u_1 ds + \int_0^t \int_0^s \mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau) \times [F(\tau, u(\tau), (Gu)(\tau), (Hu)(\tau)) + (Bv)(\tau)]d\tau ds, \ t \in J, \tag{2.1}$$

where C_0-semigroups $\mathcal{T}_i(t)(t \geq 0)(i = 1, 2)$ satisfy

$$\mathcal{T}_i(t) = \mathcal{T}(\sigma_i t)(i = 1, 2), \quad t \geq 0, \tag{2.2}$$

$$\sigma_1 + \sigma_2 = \rho, \ \sigma_1\sigma_2 = 1, \quad 0 < \sigma_1 \leq \sigma_2, \quad \rho \geq 2. \tag{2.3}$$

In view of lemma 2.7 in [86], if $\mathcal{T}(t)(t \geq 0)$ is a C_0-semigroup, then $\mathcal{T}_i(t)(t \geq 0)(i = 1, 2)$ are also C_0-semigroup for $t > 0$, such that

$$\| \mathcal{T}_i(t) \|_{\mathcal{L}(E)} \leq M e^{\sigma_i t}(i = 1, 2). \tag{2.4}$$

From (2.4), we know that

$$M_i = \sup_{t \in \mathbb{R}^+} \|\mathcal{T}_i(t)\|_{\mathcal{L}(E)} \tag{2.5}$$

is a finite number.

3 Mild solutions

In this section, we employ the Banach fixed point theorem to demonstrate the existence and uniqueness of mild solutions for system (1.1). Throughout the paper, we impose the following hypotheses:

(A2) Function $F : J \times E \times E \times E \to E$ be continuous, $\exists \ P_1 > 0$, for $\forall \ v_i, y_i, z_i \in E$, $i = 1, 2$, $t \in J$ such that

$$\|F(t, v_1, y_1, z_1) - F(t, v_2, y_2, z_2)\| \leq P_1\Big(\|v_1 - v_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|\Big).$$

Moreover, $\exists \ P_2 > 0$ such that

$$\sup_{t \in J} \|F(t, 0, 0, 0)\| \leq P_2.$$

(A3) The function f satisfies the condition that $\exists\ L, L_f > 0$, $\forall u, v \in E,\quad t \in J$ such that

$$\|f(t,s,u) - f(t,s,v)\| \leq L\|u - v\|,$$

$$\sup_{t,s\in J} \|f(t,s,0)\| \leq L_f.$$

(A4) The function g satisfies the condition that $\exists\ \ N, N_g > 0$, $\forall u, v \in E,\quad t \in J$ such that

$$\|g(t,s,u) - g(t,s,v)\| \leq N\|u - v\|,$$

$$\sup_{t,s\in J} \|g(t,s,0)\| \leq N_g.$$

Theorem 3.1. *Assuming that conditions (A1)-(A4) are satisfied, then the system (1.1) possesses a unique mild solution on J provided that $M_1 M_2 P_1 a^2(1 + La + Na) < 1$.*

Proof. Define the operator $Q : C(J, E) \to C(J, E)$, which is given

$$(Qu)(t) = \mathcal{T}_2(t)u_0 + \int_0^t \mathcal{T}_2(t-s)\mathcal{T}_1(s)u_1 ds + \int_0^t \int_0^s \mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau) \times [F(\tau, u(\tau), (Gu)(\tau), (Hu)(\tau)) + (Bv)(\tau)]d\tau ds, \quad t \in J. \tag{3.1}$$

Through direct calculation, we know that Q is clearly defined on $C(J, E)$. According to Definition 2.1, it can be easily seen that the mild solution of the system (1.1) on J is equivalent to the fixed point of the operator Q defined by (3.1). Next, we will use the Banach fixed point theorem to prove that the operator Q has a fixed point.

Let $B_R = \{u \in C(J, E) : \|u\| \leq R, t \in J\}$, where R is a positive constant.

Step 1. we prove that $QB_R \subseteq B_R$. To prove this, then, for each $u \in B_R$, according to

(A3)-(A5), it can be concluded that

$$
\begin{aligned}
\|(Qu)(t)\| &\le \|\mathcal{T}_2(t)u_0\| + \int_0^t \|\mathcal{T}_2(t-s)\mathcal{T}_1(s)u_1\|ds \\
&+ \int_0^t \int_0^s \|\mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau)\| \cdot \|[Bv(\tau) + F(\tau, u(\tau)), (Gu)(\tau), (Hu)(\tau)]\| d\tau ds \\
&\le M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2 \int_0^t \int_0^s \|(Bv)(s)\| d\tau ds \\
&+ M_1M_2 \int_0^t \int_0^s \{\|F(\tau, u(\tau), (Gu)(\tau), (Hu)(\tau)) - F(\tau, 0, 0, 0)\| \\
&+ \|F(\tau, 0, 0, 0)\|\} d\tau ds \\
&\le M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1 \int_0^t \int_0^s \Big\{\|u(\tau)\| + \int_0^\tau \|f(\tau, \eta, u(\eta))\| d\eta + \int_0^a \|g(\tau, \eta, u(\eta))\| d\eta\Big\} d\tau ds \\
&\le M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1 \int_0^t \int_0^s \Big\{\|u(\tau)\| + \int_0^\tau (\|f(\tau, \eta, u(\eta)) - f(\tau, \eta, 0)\| + \|f(\tau, \eta, 0)\|) d\eta \\
&+ \int_0^a (\|g(\tau, \eta, u(\eta)) - g(\tau, \eta, 0)\| + \|g(\tau, \eta, 0)\|) d\eta\Big\} d\tau ds \\
&\le M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1 \int_0^t \int_0^s \Big\{\|u(\tau)\| + La\|u(\eta)\| + L_fa + Na\|u(\eta)\| + N_ga\Big\} d\tau ds \\
&\le M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1a^3(L_f + N_g) + M_1M_2P_1a^2(1 + La + Na)R \le R,
\end{aligned}
$$

if we choose

$$
\begin{aligned}
R \ge [&M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1a^3(L_f + N_g)] \times [1 - M_1M_2P_1a^2(1 + La + Na)]^{-1},
\end{aligned}
$$

then it means that $QB_R \subseteq B_R$.

Step 2. We show that $Q : B_R \to B_R$ is a contraction. In fact, $u_1, u_2 \in B_R, \forall t \in J$, we obtain

$$
\begin{aligned}
\|(Qu_1)(t)-(Qu_2)(t)\| \leq & \int_0^t \int_0^s \|\mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau)\| \\
& \times \|F(\tau, u_1(\tau), (Gu_1)(\tau), (Hu_1)(\tau)) \\
& - F(\tau, u_2(\tau), (Gu_2)(\tau), (Hu_2)(\tau))\| d\tau ds \\
& \leq M_1 M_2 \int_0^t \int_0^s \|F(\tau, u_1(\tau), (Gu_1)(\tau), (Hu_1)(\tau)) \\
& - F(\tau, u_2(\tau), (Gu_2)(\tau), (Hu_2)(\tau))\| d\tau ds \\
& \leq M_1 M_2 P_1 \int_0^t \int_0^s \Big\{ \|u_1(s)-u_2(s)\| \\
& + \| \int_0^\tau [f(s,\eta,u_1(\eta)) - f(\tau,\eta,u(\tau))] d\eta \| \\
& + \| \int_0^a [g(\tau,\eta,u_1(\eta)) - g(\tau,\eta,u_2(\eta))] d\eta \| \Big\} d\tau ds \\
& \leq M_1 M_2 P_1 (1+La+Na) \int_0^t \int_0^s \|u_1-u_2\| d\tau ds \\
& \leq \frac{M_1 M_2 P_1 (1+La+Na) a^2}{2} \|u_1-u_2\|. \qquad (3.2)
\end{aligned}
$$

In view of (3.1), (3.2), and induction on n, we have

$$\|(Q^n u_1)(t)-(Q^n u_2)(t)\| \leq \frac{\Big[M_1 M_2 P_1 (1+La+Na) a^2\Big]^n}{(2n)!} \|u_1-u_2\|.$$

Hence

$$\|Q^n u_1 - Q^n u_2\| \leq \frac{\Big[M_1 M_2 P_1 (1+La+Na) a^2\Big]^n}{(2n)!} \|u_1-u_2\|.$$

Since

$$\frac{\Big[M_1 M_2 P_1 (1+La+Na) a^2\Big]^n}{(2n)!} \longrightarrow 0, \quad as\ n \longrightarrow \infty.$$

Therefore, for n large enough $\frac{\left[M_1 M_2 P_1 (1+La+Na) a^2\right]^n}{(2n)!} < 1$, according to the Banach fixed point theorem, the operator Q has a unique fixed point $u \in B_R$, which is the mild solution of the system (1.1) in J. □

4 Approximate controllability

For any $u \in C(J,E)$, the last stages of u is mentioned as $\xi_a = u(a)$ at time a.

We define a continuous linear bounded operator $\mathbb{L}$ from $L^2(J,E)$ into E as

$$\mathbb{L}p = \int_0^a \int_0^s \mathcal{T}_2(a-s)\mathcal{T}_1(s-\tau)p(\tau)d\tau ds$$

for $p(\cdot) \in L^2(J,E)$.

Definition 4.1. *Let $u(t;F,v)$ be a mild solution of the system (1.1) related to F and $v \in L^2(J,U)$. Then the set*

$$L_a(F) = \{(u(a);v) : v \in L^2(J,U)\} \subset E.$$

Definition 4.2. *If $L_a(F)$ is dense in E, then system (1.1) is considered approximately controllable on interval J, meaning $\overline{L_a(F)} = E$. That is, for $\forall\ \epsilon > 0$, $\xi_a \in D(A)$, $\exists\ v \in L^2(J,U)$, we have*

$$\left\|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds - \mathbb{L}F(\cdot, u_\epsilon(\cdot), (Gu_\epsilon)(\cdot), (Hu_\epsilon)(\cdot)) - \mathbb{L}Bv_\epsilon\right\| < \epsilon.$$

To this purpose, we need the following hypothesis:

(A5) For each $p(\cdot) \in L^2(J,E)$, $\exists\ \ q \in \overline{R(B)}$ with $\mathbb{L}p = \mathbb{L}q$. For $\forall\ \epsilon > 0$, $p(\cdot) \in L^2(J,E)$, $\exists\ \ v(\cdot) \in L^2(J,U)$ such that

$$\|\mathbb{L}p - \mathbb{L}Bv\| < \epsilon.$$

(A6) $\|Bv(\cdot)\|_{L^2(J,E)} \leq \lambda \|p(\cdot)\|_{L^2(J,E)}$, λ is a positive constant independent of $p(\cdot)$.

Lemma 4.1. *If the hypothesis (A1)-(A4) hold, then the $(\varphi v)(\cdot)$ with*

$$\|(\varphi v)(t)\| \leq K e^{M_1 M_2 P_1 a^2(1+La+Na)},$$

and $K = M_2[\|u_0\| + aM_1\|u_1\| + M_1\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1 P_1 a(L_f a + N_g a) + M_1 P_2 a^2]$.

Let $v_1(\cdot)$ and $v_1(\cdot)$ be in $L^2(J,U)$, then we have

$$\|u_1 - u_2\|_{L^2(J,E)} \leq M_1 M_2 a^2 e^{M_1 M_2 P_1 a^2(1+La+Na)} \|Bu_1 - Bu_2\|_{L^2(J,E)},$$

where $u_n(t) = (\varphi v_n)(t), n = 1, 2$.

Proof. The solution mapping $(\varphi v)(t) = u(t)$ is described as

$$u(t) = \mathcal{T}_2(t)u_0 + \int_0^t \mathcal{T}_2(t-s)\mathcal{T}_1(s)u_1 ds$$
$$+ \int_0^t \int_0^s \mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau)[F(\tau, u(\tau), (Gu)(\tau), (Hu)(\tau)) + Bv(\tau)]d\tau ds.$$

For $t \in J$, we have

$$\begin{aligned}
\|u(t)\| &\leq \|\mathcal{T}_2(t)u_0\| + \int_0^t \|\mathcal{T}_2(t-s)\mathcal{T}_1(s)u_1\|ds \\
&+ \int_0^t \int_0^s \|\mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau)[Bv(\tau) + F(\tau, u(\tau), (Gu)(\tau), (Hu)(\tau))]\|d\tau ds \\
&\leq M_2\|u_0\| + aM_1M_2\|u_1\| + \int_0^t \int_0^s \|\mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau)\| \cdot \Big\|Bv(t)\Big\|d\tau ds \\
&+ \int_0^t \int_0^s \|\mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau)\| \cdot \Big\|F(\tau, u(\tau), (Gu)(\tau), (Hu)(\tau)) \\
&- F(\tau, 0, 0, 0) + F(\tau, 0, 0, 0)\Big\|d\tau ds \\
&\leq M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1 \int_0^t \int_0^s \Big\{\|u(\tau)\| + \int_0^\tau \|f(\tau, \eta, u(\eta))\|d\eta + \int_0^a \|g(\tau, \eta, u(\eta))\|d\eta\Big\}d\tau ds \\
&\leq M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1 \int_0^t \int_0^s \Big\{\|u(\tau)\| + \int_0^\tau (\|f(\tau, \eta, u(\eta)) - f(\tau, \eta, 0)\| + \|f(\tau, \eta, 0)\|)d\eta \\
&+ \int_0^a (\|g(\tau, \eta, u(\eta)) - g(\tau, \eta, 0)\| + \|g(\tau, \eta, 0)\|)d\eta\Big\}d\tau ds \\
&\leq M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1 \int_0^t \int_0^s \Big\{\|u(\tau)\| + La\|u(\eta)\| + L_fa + Na\|u(\eta)\| + N_ga\Big\}d\tau ds \\
&\leq M_2\|u_0\| + aM_1M_2\|u_1\| + M_1M_2\sqrt{a^3}\|Bv\|_{L^2(J,E)} + M_1M_2P_2a^2 \\
&+ M_1M_2P_1a \int_0^t \Big\{\|u(\tau)\| + La\|u(\eta)\| + L_fa + Na\|u(\eta)\| + N_ga\Big\}ds \\
&\leq K + M_1M_2P_1a(1 + La + Na) \int_0^t \|u(s)\|ds,
\end{aligned}$$

and $K = M_2[\|u_0\| + aM_1\|u_1\| + M_1P_1a^2(L_fa + N_ga) + M_1P_2a^2 + M_1\sqrt{a^3}\|Bv\|_{L^2(J,E)}]$.

In view of Grömwall's inequality, we get

$$\|(\varphi v)(t)\| = \|u(t)\| \leq Ke^{M_1M_2P_1a^2(1+La+Na)}.$$

Taking $u_1(\cdot), u_2(\cdot) \in E$ and $v_1(\cdot), v_2(\cdot) \in L^2(J, U)$, then

$$
\begin{aligned}
\|u_1(t) - u_2(t)\| \le & \int_0^t \int_0^s \|\mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau)\| \cdot \|F(s, u_1(s), (Gu_1)(s), (Hu_1)(s)) \\
& - F(s, u_2(s), (Gu_2)(s), (Hu_2)(s))\| d\tau ds \\
& + \int_0^t \int_0^s \|T_2(t-s)T_1(s-\tau)\| \cdot \|Bv_1(s) - Bv_2(s)\| ds \\
& \le M_1 M_2 P_1 \int_0^t \int_0^s \Big\{ \|u_1(s) - u_2(s)\| \\
& + \| \int_0^\tau [f(\tau, \eta, u_1(\eta)) - f(\tau, \eta, u_2(\eta))] d\eta \| \\
& + \| \int_0^a [g(\tau, \eta, u_1(\eta)) - g(\tau, \eta, u_2(\eta))] d\eta] \| \Big\} d\eta ds \\
& + M_1 M_2 \sqrt{a^3} \|Bv_1(s) - Bv_2(s)\|_{L^2(J,E)} \\
& \le M_1 M_2 \sqrt{a^3} \|Bv_1(s) - Bv_2(s)\|_{L^2(J,E)} \\
& + M_1 M_2 P_1 a(1 + La + Na) \int_0^t \|u_1(s) - u_2(s)\| ds.
\end{aligned}
$$

In view of Grönwall's inequality, we get

$$
\|u_1(t) - u_2(t)\| \le M_1 M_2 \sqrt{a^3} e^{M_1 M_2 P_1 a^2 (1+La+Na)} \|Bv_1 - Bv_2\|_{L^2(J,E)}.
$$

Hence, we get

$$
\begin{aligned}
\|u_1 - u_2\|_{L^2(J,E)} &= \Big(\int_0^a \|u_1(s) - u_2(s)\|^2 ds \Big)^{\frac{1}{2}} \\
&\le M_1 M_2 a^2 e^{M_1 M_2 P_1 a^2 (1+La+Na)} \|Bv_1 - Bv_2\|_{L^2(J,E)}.
\end{aligned}
$$

The following assumptions are needed to prove our results:

(A7) The constant λ satisfies $P_1(1 + La + Na) M_1 M_2 \lambda e^{M_1 M_2 P_1 a^2 (1+La+Na)} a^2 < 1$.

We aim to construct an approximation sequence to find an alternative equivalent condition that postulates the linear system's approximate controllability, focusing on the study of the approximate controllability of mild solutions for damped elastic beam system (1.1).

Theorem 4.1. *If the conditions (A1)-(A7) hold. Then the system (1.1) is approximately controllable on J.*

Proof. **Step 1.** We are verify to prove $D(A) \subset \overline{K_a(F)}$. Since the field $D(A)$ is dense in E. To achieve this, we must prove that for $\forall\ \epsilon > 0$, $\xi_a \in D(A)$, $\exists\ \ v_\epsilon(\cdot) \in L^2(J, U)$, and

$$
\Big\| \xi_a - \mathcal{T}_2(a) u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s) u_1 ds - \mathbb{L}F(\cdot, u_\epsilon(\cdot), (Gu_\epsilon)(\cdot), (Hu_\epsilon)(\cdot)) - \mathbb{L}Bv_\epsilon \Big\| < \epsilon.
$$

where $u_\epsilon(t) = (\varphi v_\epsilon)(t)$ with

$$u_\epsilon(t) = \mathcal{T}_2(a)u_0 + \int_0^a \mathcal{T}_2(a-s)T_1(s)u_1 ds + \int_0^t \int_0^s \mathcal{T}_2(t-s)\mathcal{T}_1(s-\tau) \times [Bv_\epsilon(\tau) + F(\tau, u_\epsilon(\tau)), (Gu_\epsilon)(\tau), (Hu_\epsilon)(\tau)] d\tau ds.$$

As $\xi_a \in D(A)$, then $\exists\ \ p \in C^1(J,E))$ with

$$\mathbb{L}p = \xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds.$$

Step 2. We establish a sequence recursively as follows:
For $\forall \epsilon > 0$, $v_1(\cdot) \in L^2(J,U)$. By (A5), $\exists\ \ v_2(\cdot) \in L^2(J,U)$, we have

$$\|\mathbb{L}(p - F(\cdot, u_1(\cdot; v_1), (Gu_1)(\cdot; v_1), (Hu_1)(\cdot; v_1))) - \mathbb{L}Bv_2\| < \frac{\epsilon}{2^2}.$$

Hence,

$$\Big\|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds - \mathbb{L}F(\cdot, u_1(\cdot; v_1), (Gu_1)(\cdot; v_1), (Hu_1)(\cdot; v_1))) - \mathbb{L}Bv_2\Big\| < \frac{\epsilon}{2^2}, \tag{4.1}$$

where $u_1(t) = (\varphi v_1(t))$, $t \in J$. For $v_2(\cdot) \in L^2(J,U)$ thus obtained, we determine $w_2(\cdot) \in L^2(J,U)$ by hypotheses (A4) and (A5), we get

$$\|\mathbb{L}(F(\cdot, u_2(\cdot; v_2), (Gu_2)(\cdot; v_2), (Hu_2)(\cdot; v_2))) - F(\cdot, u_1(\cdot; v_1), (Gu_1)(\cdot; v_1), (Hu_1)(\cdot; v_1))) - \mathbb{L}Bw_2\| < \frac{\epsilon}{2^3}, \tag{4.2}$$

and the assumption (A6), we have

$$\begin{aligned}\|Bw_2\|_{L^2(J,E)} &\le \lambda\|F(\cdot, u_2(\cdot), (Gu_2)(\cdot), (Hu_2)(\cdot)) - F(\cdot, u_1(\cdot), (Gu_1)(\cdot), (Hu_1)(\cdot))\|_{L^2(J,E)} \\ &\le \lambda\Big(\int_0^t \|F(s, u_2(s), (Gu_2)(s), (Hu_2)(s)) - F(s, u_1(s), (Gu_1)(s), (Hu_1)(s))\|^2 ds\Big)^{\frac{1}{2}} \\ &\le \lambda P_1(1 + La + Na)\Big(\int_0^t \|u_2(s) - u_1(s)\|^2 ds\Big)^{\frac{1}{2}} \\ &\le \lambda P_1(1 + La + Na)\|v_1 - v_2\|_{L^2(J,E)}.\end{aligned}$$

Thus, and Lemma 4.1, we have

$$\|Bw_2\|_{L^2(J,E)} \le \lambda P_1(1 + La + Na)M_1 M_2 a^2 e^{M_1 M_2 P_1 a^2(1+La+Na)}\|Bu_1 - Bu_2\|_{L^2(J,E)},$$

and $u_n(t) = (\varphi v_n)(t), n = 1, 2$. Now, we define $v_3 = v_2 - w_2$, $v_3 \in L^2(J, U)$, and which has the following properties:

$$
\begin{aligned}
&\|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds - \mathbb{L}F(\cdot, u_2(\cdot; v_2), (Gu_2)(\cdot; v_2), (Hu_2)(\cdot; v_2)) - \mathbb{L}Bv_3\| \\
&\quad \le \|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds - \mathbb{L}F(\cdot, u_1(\cdot; v_1), (Gu_1)(\cdot; v_1), (Hu_1)(\cdot; v_1)) \\
&\quad - \mathbb{L}Bv_2 + \mathbb{L}Bw_2 - \mathbb{L}[F(\cdot, u_2(\cdot; v_2), (Gu_2)(\cdot; v_2), (Hu_2)(\cdot; v_2)) \\
&\quad - F(\cdot, u_1(\cdot; v_1), (Gu_1)(\cdot; v_1), (Hu_1)(\cdot; v_1))]\| \\
&\quad \le \|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds - \mathbb{L}F(\cdot, u_1(\cdot; v_1), (Gu_1)(\cdot; v_1), (Hu_1)(\cdot; v_1)) \\
&\quad - \mathbb{L}Bv_2\| + \|\mathbb{L}Bw_2 - \mathbb{L}[F(\cdot, u_2(\cdot; v_2), (Gu_2)(\cdot; v_2), (Hu_2)(\cdot; v_2)) \\
&\quad - F(\cdot, u_1(\cdot; v_1), (Gu_1)(\cdot; v_1), (Hu_1)(\cdot; v_1))]\|.
\end{aligned}
$$

Combining (4.1) with (4.2), we get that

$$
\begin{aligned}
&\left\|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds - \mathbb{L}F(\cdot, u_2(\cdot; v_2), (Gu_2)(\cdot; v_2), (Hu_2)(\cdot; v_2)) - \mathbb{L}Bv_3\right\| \\
&\quad < \Big(\frac{1}{2^2} + \frac{1}{2^3}\Big)\epsilon.
\end{aligned}
$$

The mathematical induction method means that $\exists\ v_n(\cdot) \in L^2(J, U)$, we obtain

$$
\begin{aligned}
&\Big\|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds \\
&\quad - \mathbb{L}F(\cdot, u_n(\cdot; v_n), (Gu_n)(\cdot; v_n), (Hu_n)(\cdot; v_n)) - \mathbb{L}Bv_{n+1}\Big\| \\
&\quad < \Big(\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}}\Big)\epsilon,
\end{aligned} \tag{4.3}
$$

and $u_n(t) = (\varphi v_n)(t), n = 1, 2, \ldots,\ t \in J$, we have

$$
\|Bv_{n+1} - Bv_n\|_{L^2(J,E)} = \lambda P_1(1 + La + Na)M_1M_2a^2 e^{M_1M_2P_1a^2(1+La+Na)}\|Bv_n - Bv_{n-1}\|_{L^2(J,E)}.
$$

Obviously, from (A7), the sequence $\{Bv_n : n = 1, 2, \ldots\}$ is a Cauchy sequence in Banach space $L^2(J, E)$ and $\exists\ v^* \in L^2(J, E)$, we have

$$
\lim_{n\to\infty} Bv_n = v^* \in L^2(J, E).
$$

Thus, for $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$
\|\mathbb{L}Bv_{N+1} - \mathbb{L}Bv_N\| < \frac{\epsilon}{2}, \tag{4.4}
$$

and hence

$$\begin{aligned}
&\Big\|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)T_1(s)u_1 ds \\
&\quad - \mathbb{L}F(\cdot, u_N(\cdot; v_N), (Gu_N)(\cdot; v_N), (Hu_N)(\cdot; v_N)) - \mathbb{L}Bv_N\Big\| \\
&\quad \leq \|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds \\
&\quad - \mathbb{L}F(\cdot, u_N(\cdot; v_N), (Gu_N)(\cdot; v_N), (Hu_N)(\cdot; v_N)) - \mathbb{L}Bv_{N+1}\| \\
&\quad + \|\mathbb{L}Bv_{N+1} - \mathbb{L}Bv_N\|,
\end{aligned}$$

and $u_N(t) = (\varphi v_N)(t)$, $t \in J$. By (4.3), (4.4), we get that

$$\begin{aligned}
&\|\xi_a - \mathcal{T}_2(a)u_0 - \int_0^a \mathcal{T}_2(a-s)\mathcal{T}_1(s)u_1 ds \\
&\quad - \mathbb{L}F(\cdot, u_N(\cdot; v_N), (Gu_N)(\cdot; v_N), (Hu_N)(\cdot; v_N)) - \mathbb{L}Bv_N\| \\
&\quad < \Big(\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}}\Big)\epsilon + \frac{\epsilon}{2} \\
&\quad \leq \epsilon.
\end{aligned}$$

As $N \to \infty$, we get $\xi_a \in \overline{K_a(F)}$, the system (1.1) is approximately controllable on J. □

Theorem 4.2. *Assuming the range of operator B is dense in $L^2(J,E)$. Then, under assumptions (A1)-(A4), the system (1.1) is approximate controllable.*

Proof. Since the range of the operator B is dense in $L^2(J,E)$, for $\forall\ p(\cdot) \in L^2(J,E)$ and $\overline{\delta} > 0$, $\exists\ \ Bv(\cdot) \in R(B)$, $v(\cdot) \in L^2(J,U)$, we have

$$\|Bv(\cdot) - p(\cdot)\|_{L^2(J,E)} < \overline{\delta}\|p(\cdot)\|_{L^2(J,E)}. \tag{4.5}$$

Now, we have

$$\begin{aligned}
\|\mathbb{L}p - \mathbb{L}Bv\| &\leq M_1M_2 \int_0^a \int_0^s \|p(\tau) - Bv(\tau)\| d\tau ds \\
&\leq M_1M_2\sqrt{a^3}\|p(\cdot) - Bv(\cdot)\|_{L^2(J,E)} \\
&\leq M_1M_2\sqrt{a^3}\overline{\delta}\|p(\cdot)\|_{L^2(J,E)} \\
&< \epsilon.
\end{aligned}$$

Thus, from Eq. (4.5), we have

$$\begin{aligned}
\|Bv(\cdot)\|_{L^2(J,E)} &= \|Bv(\cdot) - p(\cdot) + p(\cdot)\|_{L^2(J,E)} \\
&\leq \|Bv(\cdot) - p(\cdot)\|_{L^2(J,E)} + \|p(\cdot)\|_{L^2(J,E)} \\
&\leq \overline{\delta}\|p(\cdot)\|_{L^2(J,E)} + \|p(\cdot)\|_{L^2(J,E)} \\
&\leq (\overline{\delta} + 1)\|p(\cdot)\|_{L^2(J,E)}.
\end{aligned}$$

This means that if we choose $\overline{\delta} > 0$ in a way that validates (A7), then conditions (A5) and (A6) are satisfied. Then, the approximate controllability of system (1.1) is derived from Theorem 4.1. ■

Remark 4.1. *Currently, the majority of articles assume that the corresponding linear control systems are approximately controllable. By defining the Gammer control function, the control problem is transformed into an operator's fixed point problem, which is then studied using fixed point theorems to investigate the system's approximate controllability. In this project, we eschew these methods. Notably, the approximation sequence method has proven effective in investigating other issues, particularly those of integer order.*

5 Example

Consider the following damped elastic beam system of the form

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} + 6\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} + \frac{\partial^4 u(x,t)}{\partial x^4} = \frac{t^2 \sin(2\pi t)}{4(1+|u(x,t)|)} + \frac{1}{5}e^{-t}\sin\left(\int_0^t (t-s)u(x,s)ds\right) \\ +\frac{1}{6}e^{-t}\cos\left(\int_0^1 e^{-|t-s|}u(x,s)ds\right) + \kappa v(x,t), & (x,t) \in \Omega \times [0,1], \\ u|_{\partial\Omega} = 0,\ \Delta u|_{\partial\Omega} = 0, & t \in [0,1], \\ u(x,0) = u_0, \\ \frac{\partial u(x,0)}{\partial t} = u_1, & x \in \Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary $\partial\Omega$ and Δ is the Laplace operator.

Let Banach space $E = L^2(\Omega)$ with L^2-norm $\|\cdot\|_2$. The operator $A : D(A) \subset E \to E$ by

$$Au = -\frac{\partial^2 u}{\partial x^2}, \quad u \in D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

From [9], we know that $-A$ generates C_0-semigroup $\mathcal{T}(t)(t \geq 0)$, which satisfied

$$\|\mathcal{T}(t)\| \leq e^{-\delta t},\ \delta > 0,\ t \geq 0.$$

Thus, $-\sigma_1 A$ and $-\sigma_2 A$ generate C_0 semigroups $\mathcal{T}_1(t)(\geq 0)$ and $\mathcal{T}_2(t)(t \geq 0)$ respectively, which satisfy

$$\|\mathcal{T}_i(t)\| = \|\mathcal{T}(\sigma_i t)\| \leq e^{-\delta\sigma_i t}, \quad t \geq 0, \quad i = 1, 2,$$

where $\sigma_1 = 3 + 2\sqrt{2},\ \sigma_2 = 3 - 2\sqrt{2}$ defined by (2.3). Then we obtain that

$$M_i := \sup_{0 \leq s \leq t \leq 1} \|\mathcal{T}_i(t)\|_{\mathcal{L}(E)} = 1,$$

then assumption (A1) holds.

For $u \in L^2(\Omega)$, we set $u(t) = u(\cdot, t)$ and

$$f(t,s,u(s)) = (t-s)u(\cdot,s), \quad g(t,s,u(t,s)) = e^{-|t-s|}u(\cdot,s),$$

$$(Gu)(t) = \int_0^t (t-s)u(\cdot,s)ds, \quad (Hu)(t) = \int_0^1 e^{-|t-s|}u(\cdot,s)ds,$$

$$F(t,u(t),(Gu)(t),(Hu)(t)) = \frac{t^2\sin(2\pi t)}{4(1+|u(\cdot,t)|)} + \frac{1}{5}e^{-t}\sin\big((Gu)(t)\big) + \frac{1}{6}e^{-t}\cos\big((Hu)(t)\big).$$

Let $B : U := E \to E$ with $Bv(t) = \kappa v(\cdot,t)$, system (5.1) can be transformed into system (1.1).

Theorem 5.1. *The system (5.1) has a mild solution $u \in C(\Omega \times J, E)$.*

Proof. In view of the nonlinear term F, we know that $F(t,u,v,w)$ is continuous about the variables u, v, w with constant $P_1 = \max\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}\} = \frac{1}{4}$. And we get that assumption (A2) is satisfied with positive constant $P_2 = \frac{1}{4}$. From the fact $L = L_f = N = N_g = 1$, one can easily to verify that (A3)-(A4) holds. Thus, all the assumptions of Theorem 3.1 are satisfied. Therefore, by Theorem 3.1, the conclusion holds. □

Theorem 5.2. *If the following conditions*

(H5) For each $p(\cdot) \in L^2(J,E)$, $\exists\ q \in \overline{R(B)}$ with $\mathbb{L}p = \mathbb{L}q$. Hence it follows that for $\forall\ \epsilon > 0$ and $p(\cdot) \in L^2(J,E)$, $\exists\ v(\cdot) \in L^2(J,U)$ such that

$$\|\mathbb{L}p - \mathbb{L}Bv\| < \epsilon,$$

where $\mathbb{L}p = \int_0^a \int_0^s \mathcal{T}_2(a-s)\mathcal{T}_1(s-\tau)p(\tau)d\tau ds$ for $p(\cdot) \in L^2(J,E)$.

(H6) $\|Bv(\cdot)\|_{L^2(J,E)} \le \lambda \|p(\cdot)\|_{L^2(J,E)}$, where λ is a positive constant independent of $p(\cdot)$.

(H7) The constant λ satisfies $P_1(1+La+Na)M_1M_2\lambda e^{M_1M_2P_1a^2(1+La+Na)}a^2 < 1$

hold, then the system (5.1) is approximate controllability.

Proof. In view of nonlinear term F, we know that $F(t,u,v,w)$ is continuous about the variables u, v, w with constant $P_1 = \max\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}\} = \frac{1}{4}$. Thus (A2) is satisfied with $P_2 = \frac{1}{4}$. From t $L = L_f = N = N_g = 1$, we verify that (A3)-(A4) holds. And combine with (A1), (H5)-(H7) hold. In view of Theorem 4.1, our conclusion holds. □

6 Conclusions

This paper concentrated on the approximate controllability of damped elastic beam systems with initial conditions involving Volterra-Fredholm type integro-differential system. Firstly, the existence of mild solutions for the proposed system was investigated by using the Banach fixed point combined with semigroup operators. Also, by using the sequential method, approximate control results were obtained. In future work, based on the results of this paper, we will investigate the approximate controllability of Volterra-Fredholm type integro-differential third order dispersion system involving non local conditions and and control delay. Moreover, we also study this class of problem related to mathematical control problem such as controllability and optimal control. Especially, we will further consider approximate controllability of Atangana-Baleanu fractional neutral delay integro-differential stochastic systems and Atangana–Baleanuneutral fractional stochastic hemivariationaolf inequality by utilizing the sequential methods described in this article.

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My manuscript has no associate data.

Conflict of interest

The author declares that he has no conflict of interest.

References

[1] Abada, N., Benchohra, M., & Hammouche, H. (2009). Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions. *Journal of Differential Equations*, 246, 3834-3863.

[2] Arthi, G., & Park, J. (2014). On controllability of second-order impulsive neutral integro-differential systems with infinite delay. *IMA J. Math. Control Inf*, 1-19.

[3] Arthi, G., & Balachandran, K. (2013). Controllability results for damped second-order impulsive neutral integro-differential systems with nonlocal conditions. *J. Control Theory Appl*, 11, 186-192.

[4] Arthi, G., & Balachandran, K. (2012). Controllability of damped second-order neutral functional differential systems with impulses. *Taiwanese Journal of Mathematics*, 16, 89-106.

[5] Acquistapace P. (1988). Evolution operators and strong solution of abstract parabolic equations. Differential Integral Equations. 1:433-457.

[6] Acquistapace P.,& Terreni B.(1987). A unified approach to abstract linear parabolic equations. Rend Semin Mat Univ Padova. 78:47-107.

[7] Alkhazzan, A., Jiang, P., Baleanu, D., Khan, H., & Khan, A. (2018). Stability and existence results for a class of nonlinear fractional differential equations with singularity, *Math Meth Appl Sci.*, 1–14.

[8] Ansari, K. J., Ilyas, A. F., Shah, K., Khan, A., & Abdeljawad, T. (2023). On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative, *Waves in Random and Complex Media.*, 1-20.

[9] Amann, H. (1978). Periodic solutions of semilinear parabolic equations, In: L. Cesari, R. Kannan, R. Weinberger(Eds.), Nonlinear Anal., A Collection of Papers in Honor of Erich H. Rothe, New York: Academic Press, 1-29.

[10] Bedi, P., Kumar, A., & Khan, A. (2021). Controllability of neutral impulsive fractional differential equations with Atangana-Baleanu-Caputo derivatives, *Chaos, Solitons and Fractals.*, 150, 111153.

[11] Benchohra, M., Gorniewicz, L., Ntouyas, S. K., & Ouahab, A. (2004). Controllability results for impulsive functional differential inclusions. *Reports on Mathematical Physics*, 54, 211-228.

[12] Balachandran, K., & Sakthivel, R. (2001). Controllability of integro-differential systems in Banach spaces. *Applied Mathematics and Computation,* 118, 63-71.

[13] Chen, P., Zhang, X.,& Li, Y. (2020). Approximate controllability of non-autonomous evolution system with nonlocal conditions. *J. Dyn. Control. Syst*, 26, 1-16.

[14] Cao, Y.,& Sun, J. (2015). Existence of solutions for semilinear measure driven equations. *J. Math. Anal. Appl*, 425, 621-631.

[15] Cao, Y.,& Sun, J. (2017). Controllability of measure driven evolution systems with nonlocal conditions. *Appl. Math. Comput*, 299, 119-126.

[16] Chen, G., & Russell, D. L. (1982). A mathematical model for linear elastic systems with structural damping, *Quart. Appl. Math.*, 39, 433-454.

[17] Cao, Y., & Sun, J. (2018). Approximate controllability of semilinear measure driven systems. *Mathematische. Nachrichten*, 291, 1979-1988.

[18] Chen, G., & Russell. D. L. (1982). A mathematical model for linear elastic systems with structural damping. *Quart. Appl. Math*, 39(1982): 433-454.

[19] Diagana, T. (2018). Semilinear evolution eqautions and their applications. Springer Nature Switzerland AG.

[20] Djaout, A., Benbachir, M., Lakrib, M., Matar, M. M., Khan, A., & Abdeljawad, T. (2022). Solvability and stability analysis of a coupled system involving generalized fractional derivatives. *AIMS Mathematics.*, 8(4), 7817–7839.

[21] Dineshkumar, C., Nisar, K. S., Udhayakumar, R., & Vijayakumar, V. (2022). A discussion on approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions. *Asian J Control.*, 24, 2378–2378.

[22] Dineshkumar, C., & Udhayakumar, R. (2020). Results on approximate controllability of nondensely defined fractional neutral stochastic differential systems. *Numer Methods Partial Differential Eq.*, 1–27.

[23] Dineshkumar, C., Udhayakumar, R., Vijayakumar, V., & Anurag Shukla, Kottakkaran Sooppy Nisarc. (2023). New discussion regarding approximate controllability for Sobolev-type fractional stochastic hemivariational inequalities of order $r \in (1, 2)$. *Communications in Nonlinear Science and Numerical Simulation.* 116, 106891.

[24] Dineshkumar, C., Udhayakumar, R., Vijayakumar, K. S., Nisar, V., Shukla, A., Aty, A.H. A., Mahmoud, M. E., & Mahmoud, M. (2022). A note on existence and approximate controllability outcomes of Atangana-Baleanu neutral fractional stochastic hemivariational inequality. *Results in Physics*, 38, 105647.

[25] Diagana, T. (2017). Well-posedness for some damped elastic systems in Banach spaces. *Applied Mathematics Letters*, 71(2017) 74-80.

[26] Diagana, T. (2017). Well-posedness for some damped elastic systems in Banach spaces. *Appl. Math. Lett.*, 71, 74-80.

[27] Fan, H., & Li, Y. (2014). Monotone iterative technique for the elastic systems with structural damping in Banach spaces. *Comput. Math. Appl.*, 68, 384-391.

[28] Fan, H., & Li, Y. (2014). Analyticity and exponential stability of semigroups for the elastic systems with structural damping in Banach spaces. *J. Math. Anal. Appl.*, 410, 316-322.

[29] Fan, H., & Gao, F. (2014). Asymptotic stability of solutions to elastis systems with structural damping. *Electron. J. Differ. Eq.*, 245, 1-9.

[30] Fan, H., Li, Y., & Chen, P. (2013). Existence of mild solutions for the elastic systems with structural damping in Banach spaces. *Abstract and Applied Analysis*, Article ID 746893, 1-6.

[31] Fu, X. (2017). Approximate controllability of semilinear non-autonomous evolution systems with state-dependent delay. *Evol Equ Control Theory*, 6, 517-534.

[32] Fu, X., & Huang, R. Approximate controllability of semilinear non-autonomous evolutionary systems with nonlocal conditions. *Autom Remote Control*, 77, 428-442.

[33] Fu X, T. & Zhang Y.(2013). Exact null controllability of non-autonomous functional evolution systems with nonlocal conditions. *Acta Math Sci Ser B Engl Ed*, 33(840), 747-757.

[34] Graber, P. J. & Lasiecka, I. (2014). Analyticity and Gevrey class regularity for a strongly damped wave equation with hyperbolic dynamic boundary conditions. *Semigroup Forum*, 88(2), 333-365.

[35] George, R. K. (1995). Approximate controllability of non-autonomous semilinear systems. *Nonlinear Anal.*, 24:1377-1393.

[36] Kalman, R. E. (1963). Controllablity of linear dynamical systems. *Contrib Diff Equ*, 1, 190-213.

[37] Gautam, P., Shukla, A., Johnson M., & Vijayakumar, V. (2024). Approximate controllability of third order dispersion systems. *Bull. Sci. math.*, 191, 103394.

[38] Ge, F. D., Zhou, H. C.,& Kou, C. H. (2016). Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique. *Appl. Math. Comput*, 275, 107-120.

[39] Gou, H., & Li, Y. (2020). Mixed monotone iterative technique for damped elastic systems in Banach spaces. *J. Pseudo-Differ. Oper. Appl.*, 11, 917-933.

[40] Gou, H., & Li, Y. (2020). A Study on Damped Elastic Systems in Banach Spaces. *Numer. Func. Anal. Opt.*, 41, 542-570.

[41] Gou, H., & Li, Y. (2020). A study on impulsive fractional hybrid evolution equations using sequence method. *Comput. Appl. Math.*, 39, 1–31.

[42] Gou, H., & Li, Y. (2022). A study on approximate controllability of non-autonomous evolution system with nonlocal conditions using sequence method. *Optimization*, 71(16), 4763–4783.

[43] Huang, F. (1985). On the holomorphic property of the semigroup associated with linear elastic systems with structural damping. *Acta Math. Sci.(Chinese)*, 5, 271-277.

[44] Huang, F., & Liu, K. (1988). Holomiphic property and exponential stability of the semigroup associated with linear elastic systems with damping. *Ann. Diff. Eqs.*, 4(4), 411-424.

[45] Hernández, E., & Regan, D. O. (2009). Controllability of Volterra-Fredholm type systems in Banach spaces. *J. Franklin Inst*, 346, 95-101.

[46] Jeong, J. M., Ju, E. Y., & Cho, S. H. (2016). Control problems for semilinear second order equations with cosine families. *Advances in Difference Equations*, 2016, 125.

[47] Ji, S. (2014). Approximate controllability of semilinear nonlocal fractional differential systems via an approximating method. *Appl. Math. Comput*, 236, 43-53.

[48] Khan, A., Khan, H., Gómez-Aguilar, J.F., & Abdeljawa, T. (2019). Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel. *Chaos, Solitons and Fractals*, 127, 422–427.

[49] Khan, A., Alshehri, H. M., Gómez-Aguilar, J.F., Khan, Z. A., & Anaya, G. F. (2021). A predator-prey model involving variable-order fractional differential equations with Mittag-Leffler kernel. *Advances in Difference Equations.*, 2021, 183.

[50] Khan, H., Tunc, C., & Khan, A. (2020). Stability results and existence theorems for nonlinear delay fractional differential equations with φ_P^*-operator. *Journal of Applied Analysis and Computation.*, 10(2), 584–597.

[51] Khan, A., Syam, M. I., Zada, A. & Khan, H. (2018). Stability analysis of nonlinear fractional differential equations with Caputo and Riemann-Liouville derivatives. *Eur. Phys. J. Plus.*, 133, 264.

[52] Khan, A., Khan, Z. A., Abdeljawad, T., & Khan, H. (2022). Analytical analysis of fractional-order sequential hybrid system with numerical application. *Advances in Continuous and Discrete Models.*, 2022, 12.

[53] Luong, V.T., & Tung, N.T. (2017). Decay Mild Solutions for Elastic Systems with Structural Damping Involving Nonlocal Conditions. *Mathematics.* , 50, 55-67.

[54] Luong, V.T., & Tung, N.T. (2020). Exponential decay for elastic systems with structural damping and infinite delay. *Appl. Anal.*, 99, 13-28.

[55] Liu, Z., Motreanu, D., & Zeng, S. (2021). Generalized penalty and regularization method for differential variational hemivariationak inequalities. *SIAM J. Optim.*, 31(2), 1158-1183.

[56] Li, X., Liu, Z., & Papageorgiou, N. S. (2023). Solvability and pullback attractor for a class of differential hemivariational inequalities with its applications. *Nonlinearity.*, 36, 1323-1348.

[57] Liu, Y., Liu, Z., & Papageorgiou, N. S. (2023). Sensitivity analysis of optimal control problems driven by dynamic history-dependent variational-hemivariational inequalities. *Journal of Differential Equations.*, 342, 559-595.

[58] Liu, K., & Liu, Z. (1997). Analyticity and Differentiability of Semigroups Associated with Elastic Systems with Damping and Gyroscopic Forces. *J. Differ. Equations.*, 141, 340-355.

[59] Liu, Z., Lv, J., & Sakthivel, R. (2014). Approximate controllability of fractional functional evolution inclusions with delay in Hilbert spaces. *IMA J. Math. Control Inform*, 31(3), 363-383.

[60] Mahmudov, N. I. (2003). Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. *SIAM J Control Optim*, 42, 1604-1622.

[61] Mahmudov, N. I. (2008). Approximate controllability of evolution systems with nonlocal conditions. *Nonlinear Anal*, 68, 536-546.

[62] Mahmudov, N. I.,& Zorlu, S. (2014) On the approximate controllability of fractional evolution equations with compact analytic semigroup. *J Comput. Appl. Math.* , 259, 194-204.

[63] Mokkedem, F. Z.,& Fu, X. (2016). Approximate controllability for a semilinear evolution system with infinite delay. *J. Dyn. Control. Syst*, 22, 71-89.

[64] Ma, Y. K., Dineshkumar, C., Vijayakumar, V., Udhayakumar, R., Shukla, A., Nisar, K. S. (2023). Approximate controllability of Atangana-Baleanu fractional neutral delay integro differential stochastic systems with nonlocal conditions. *Ain Shams Engineering Journal*, 14(3), 101882.

[65] Muthukumar, P., & Balasubramaniam, P. (2010). Approximate controllability of second-order damped McKean-Vlasov stochastic evolution equations. *Comput. Math. Appl.*, 60(10) 2788–2796.

[66] Pazy, A. (1983). Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York.

[67] Pang, X., Li, X., & Liu, Z. (2023). Decay mild solutions of Hilfer fractional differential variational-hemivariational inequalities. *Nonlinear Analysis: Real World Applications.*, 71, 103834.

[68] Patel, R., Vijayakumar, V., Jadon, S. D., & Shukla. A. (2023). An analysis on the existence of mild solution and optimal control for semilinear thermoelastic system. *Numerical Functional Analysis and Optimization*, 44(14), 1570-1582,

[69] Park, J.Y., & Kang, S.N. (2001) Approximate controllability of neutral functional differential system with unbounded delay *Int. J. Math. Math. Sci.*, 26(12), 737–744.

[70] Sakthivel, R., Anthoni, S. M., & Kim, J. H. (2005). Existence and controllability result for semilinear evolution integro differential Systems. *Mathematical and Computer Modelling*, 41, 1005-1011.

[71] Shukla, A., Sukavanam, N., & Pandey, D. N. (2015). Approximate controllability of semilinear system with state delay using sequence method. *Journal of The Franklin Institute*, 352, 5380-5392.

[72] Sakthivel, R., & Anandhi, E. (2010). Approximate controllability of impulsive differential equations with state-dependent delay. *Int. J. Control*, 83, 387-493.

[73] Sakthivel, R.,& Anandhi, E. R. (2009). Approximate controllability of impulsive differential equations with state-dependent delay. *International Journal of Control*, 83(2), 387-393.

[74] Shah, K., Abdalla, B., Abdeljawad, T., & Gul, R. (2023). Analysis of multipoint impulsive problem of fractional-order differential equations. *Boundary Value Problems.*, 2023(1), 1-17.

[75] Sher, M., Shah, K., Fečkan, M., & Khan, R. A. (2020). Qualitative Analysis of Multi-Terms Fractional Order Delay Differential Equations via the Topological Degree Theory. *Mathematics.*, 8(2), 218.

[76] Shah, K., Sher, M., & Abdeljawad, T. (2020). Study of evolution problem under Mittag–Leffler type fractional order derivative. *Alexandria Engineering Journal.*, 59(5), 3945-3951.

[77] Sakthivel, R. (2013). Approximate controllability of nonlinear fractional dynamical systems. *Commun. Nonlinear Sci. Numer. Simul*, 18, 3498-3508.

[78] Sakthivel, R.,& Ren, Y. (2013). Approximate controllability of fractional differential equations with state-dependent delay. *Results Math*, 63, 949-963.

[79] Shen, L.,& Sun, J. (2013). Approximate controllability of abstract stochastic impulsive systems wih multiple time-varying delays. *Int. J. Robust Nonlinear Control*, 63, 827-838.

[80] Singh, A., Vijayakumar, V., Shukla, A., & Chauhan, S, (2022). A Note on Asymptotic Stability of Semilinear Thermoelastic System. *Qualitative Theory of Dynamical Systems.* 21, 75.

[81] Tajadodi, H., Khan, A., Aguilar, J. F. G., Khan, H. (2021). Optimal control problems with Atangana-Baleanu fractional derivative. *Optim Control Appl Meth.*, 42, 42:96–109.

[82] Vijayakumar, V., Udhayakumar, R., Zhou, Y., & Sakthivel, N. (2020). Approximate controllability results for Sobolev-type delay differential system of fractional order without uniqueness. *Numer Methods Partial Differential Eq.*, 1–20.

[83] Vijayakumar, V., Ravichandran, C., Murugesu, R., & Trujillo, J.J. (2014). Controllability results for a class of fractional semilinear integro-differential inclusions via resolvent operators. *Applied Mathematics and Computation.*, 247, 152–161.

[84] Wei, S. (2019). Global existence of mild solutions for the elastic system with structural damping. *Ann. Appl. Math.*, 35 , 180-188.

[85] Wei, M., & Li, Y. (2023). Existence and global asymptotic behavior of mild solutions for damped elastic systems with delay and nonlocal conditions. *J. Anal. Appl. Comput.*, 13(2) , 874-892.

[86] Wei, M., Li, Y., & Li, Q. (2022). Positive mild solutions for damped elastic systems with delay and nonlocal conditions in ordered Banach space. *Qualitative Theory of Dynamical Systems.*, 21, 128.

[87] Yan, Z.,& Lu, F. (2015). On approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay. *Appl. Anal*, 94, 1235-1258.

[88] Zhou, H. X. (1983). Approximate controllability for a class of semilinear abstract equations. *SIAM J. Control Optim*, 21(4), 551-565.

[89] Zhou, Y., Vijayakumar, V., Ravichandran, C., & Murugesu, R. (2017). Controllability results for fractioanl order neutral functional differential inclusions with infinite delay. *Fixed Point Theory.*, 18(2), 773-798.

[90] Zhou, Q., Huang, Z., Sun, Y., Triki, H., Liu, W., & Biswas, A. (2023). Collision dynamics of three-solitons in an optical communication system with third-order dispersion and nonlinearity. *Nonlinear Dyn.*, 111, 5757–5765.