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INTUITIONISTIC FUZZY STABILITY OF AN EULER-LAGRANGE TYPE QUARTIC FUNCTIONAL EQUATION

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Abstract In this paper, we investigate the Hyers-Ulam stability of the following Euler-Lagrange type quartic functional equation

$$\begin{aligned} f(ax+y)+f(x+ay)+\frac{1}{2}a(a-1)^2f(x-y) \\ =(a^2-1)^2(f(x)+f(y))+\frac{1}{2}a(a+1)^2f(x+y) \end{aligned}$$

in intuitionistic fuzzy normed spaces, where $a \neq 0, a \neq \pm 1$. Furthermore, we investigate intuitionistic fuzzy continuity through the existence of a certain solution of a fuzzy stability problem for approximately Euler-Lagrange quartic functional equation.

Keywords Hyers-Ulam stability, Euler-Lagrange type quartic functional equation, intuitionistic fuzzy normed space, intuitionistic fuzzy continuity

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1. Introduction

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In 1984, Katsaras [17] introduced an idea of a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In the same year Wu and Fang [36] introduced a notion fuzzy normed apace to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological vector spaces. In 1992, Felbin [9] introduced an alternative definition of a fuzzy norm on a vector space with an associated metric of Kaleva and Seikkala type [14]. Some mathematics have define fuzzy normed on a vector form various point of view [19,29,37]. In particular, Bang and Samanta [4] following Cheng and Mordeson [7] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric of Kramosil and Michalek type [18]. They established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [5]. The notion of intuitionistic fuzzy set introduced by Atanassov [3] has been used extensively in many areas of mathematics and sciences. The notion of intuitionistic

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fuzzy norm [23, 28, 33] is also useful one to deal with the inexactness and vagueness arising in modeling. There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modeling the inexactness through the intuitionistic fuzzy norm. The stability problem for Pexiderized quadratic functional equation, Jensen functional equation, cubic functional equation, functional equations associated with inner product spaces, and additive functional equation was considered in [21, 22, 24, 27, 35], respectively, in the intuitionistic fuzzy normed spaces.

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows. It is said that a functional equation G is stable if each function g satisfying the equation G-approximately is near to the true solution of G. The first problem concerning group homomorphisms was raised by Ulam [34] in 1940. Hyers developed Ulam's question [12]. He posed the following theorem:
Suppose that U and V are Banach spaces and ρ is a mapping from U to V such that the following inequality satisfies for some $\delta > 0$ and for all $u, v \in U$,

$$\|\rho(u+v) - \rho(u) - \rho(v)\| \leq \delta.$$

Then there is only one additive mapping $T : U \to V$ such that

$$\|T(u) - \rho(u)\| \leq \delta$$

for all $u \in U$. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mapping and by Rassias [32] for linear mapping by considering an unbounded Cauchy difference. The result of Rassias has provided a lot of influence during the last three decades in the development of generalization of Hyers-Ulam stability concept. Furthermore, in 1994, Găvruţa [10] provided a further generalization of Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Recently several stability results have been obtained for various equations and mappings with more general domains and ranges have been investigated by a number of authors and there are many interesting results concerning this problem and applications [2, 6, 8, 16, 25, 40, 41]. In particular, Ger [11] introduced the Euler-Lagrange quadratic functional equation

$$f(ax + by) + f(ax - by) = a^2 f(x) + b^2 f(y),$$

for fixed reals a, b with $a \neq 0$, $b \neq 0$. Zhang *et al.* [42] introduced and studied the Euler-Lagrange quadratic functional equation for the set-valued version of the Euler-Lagrange quadratic functional equation. Rassias [31] introduced the following Euler-Lagrange type quadratic functional equation

$$f(ax + by) + f(ax - by) = (a^2 + b^2)(f(x) + f(y)),$$

for fixed reals a, b with $a \neq 0$, $b \neq 0$. Jun and Kim [13] proved the Hyers-Ulam stability of the following Euler-Lagrange type cubic functional equation

$$f(ax + by) + f(bx + ay) = (a + b)(a - b)^2(f(x) + f(y)) + ab(a + b)f(x + y),$$

where $a \neq 0$, $b \neq 0$, $a \pm b \neq 0$, for all $x, y \in X$. Lee et al. [20] considered the following functional equation

$$f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y)$$

and established the general solution and the Hyers-Ulam stability for this functional equation. This functional equation is called quartic functional equation and every solution of this quartic equation is said to be a quartic function. Obviously, the function $f(x) = x^4$ satisfies the quartic functional equation. Kang [15] investigated the solution and the Hyers-Ulam stability for the quartic functional equation

$$f(ax+by)+f(ax-by) = a^2b^2(f(x+y)+f(x-y))+2a^2(a^2-b^2)f(x)-2b^2(a^2-b^2)f(y),$$

where $a, b \in \mathbb{Z} \setminus \{0\}$ with $a \neq \pm 1, \pm b$.

The following generalized Euler-Lagrange radical multifarious functional equation was introduced and studied by Ramdoss *et al.* [30]:

$$\begin{aligned} & s(\alpha z + w) + s(\alpha z - w) + s(z + \alpha w) + s(z - \alpha w) \\ & \quad - (\alpha + \alpha^2)[s(z + w) + s(z - w)] \\ & \quad = -2\left(\alpha + \alpha^2 - \alpha^p - 1\right) s(z) - (1 + (-1)^p)\left(\alpha + \alpha^2 - \alpha^p - 1\right) s(w) \\ & \quad + \sum_{k \in 2\mathbb{N}}^{p-2(p:\ even);\ p-1(p:\ odd)} 2pC_k \left(\alpha^{p-k} + \alpha^k - (\alpha + \alpha^2\right) s\left(\sqrt[p]{z^{p-k} w^k}\right) \end{aligned} \tag{1.1}$$

for $\alpha \neq 0, \pm 1$ and $p, k \in \mathbb{N}, p > k, p > 2$. Several Euler-Lagrange type functional equations have been investigated by numerous mathematicians [38, 39, 43].

In this paper, we consider the following Euler-Lagrange type quartic functional equation

$$\begin{aligned} & f(ax + y) + f(x + ay) + \frac{1}{2}a(a - 1)^2 f(x - y) \\ & \quad = (a^2 - 1)^2(f(x) + f(y)) + \frac{1}{2}a(a + 1)^2 f(x + y) \end{aligned} \tag{1.2}$$

for a fixed integer $a \neq 0, \pm 1$ and all $x, y \in X$. We determine some stability results concerning the above Euler-Lagrange quartic functional equation in the setting of intuitionistic fuzzy normed spaces (IFNS). Further, we study intuitionistic fuzzy continuity through the existence of a certain solution of a fuzzy stability problem for approximately Euler-Lagrange type quartic functional equation.

Throughout this paper, assume that the symbol $\mathbb{N}$ denotes the set of all natural numbers.

2. Preliminaries

In this section we recall some notations and basic definitions used in this paper.

Definition 2.1. A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0, 1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.2. A binary operation $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(i) $\diamond$ is associative and commutative,

(ii) $*$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 2.3. The five-tuple $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be intuitionistic fuzzy normed spaces (for short, INFS) if $\mathcal{X}$ is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $\mathcal{X} \times (0, \infty)$ satisfying the following conditions: For every $x, y \in \mathcal{X}$ and $s, t > 0$,
(i) $\mu(x,t) + \nu(x,t) \leq 1$,
(ii) $\mu(x,t) > 0$,
(iii) $\mu(x,t) = 1$ if and only if $x = 0$,
(iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
(vi) $\mu(x,.) : (0,\infty) \to [0,1]$ is continuous,
(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(viii) $\nu(x,t) < 1$,
(ix) $\nu(x,t) = 0$ if and only if $x = 0$,
(x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (v) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,
(xi) $\nu(x,.) : (0,\infty) \to [0,1]$ is continuous,
(xii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.
In this case (μ, ν) is called an intuitionistic fuzzy norm.

For simplicity in notation, we denote the intuitionistic fuzzy normed space by $(\mathcal{X}, \mu, \nu)$ instead of $(\mathcal{X}, \mu, \nu, *, \diamond)$. For example, let $(\mathcal{X}, \|.\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathcal{X}$ and every $t \in \mathbb{R}$, consider

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|}, & t > 0 \\ 0, & t \leq 0, \end{cases}$$

then $(\mathcal{X}, \mu, \nu)$ is an IFNS.

The concept of convergence and Cauchy sequences in the setting of IFNS were introduced by Saadati and Park [33] and further studied by Mursaleen and Mohiuddine [26].

Definition 2.4. Let $(\mathcal{X}, \mu, \nu)$ be an IFNS. Then a sequence $\{x_n\}$ is said to be:

(i) convergent to $L \in \mathcal{X}$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - L, t) > 1 - \varepsilon$ and $\nu(x_n - L, t) < \varepsilon$ for all $n \geq n_0$. In this case, we write $(\mu, \nu) - \lim x_n = L$ or $x_n \to L$ as $n \to \infty$.

(ii) a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, t) > 1 - \varepsilon$ and $\nu(x_n - x_m, t) < \varepsilon$ for all $n, m \geq n_0$.

Definition 2.5. An IFNS $(\mathcal{X}, \mu, \nu)$ is said to be complete if each intuitionistic fuzzy Cauchy sequence in $(\mathcal{X}, \mu, \nu)$ is intuitionistic fuzzy convergent in $(\mathcal{X}, \mu, \nu)$. In this case, $(\mathcal{X}, \mu, \nu)$ is called an intuitionistic fuzzy Banach space.

3. Approximation on intuitionistic fuzzy normed spaces

Theorem 3.1. *Let $\mathcal{X}$ be a linear space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be a mapping such that, for some $|\alpha| > |a|^4$,*

$$\mu'(\varphi(\frac{x}{a}, 0), t) \geq \mu'(\varphi(x, 0), |\alpha| t),$$
$$\nu'(\varphi(\frac{x}{a}, 0), t) \leq \nu'(\varphi(x, 0), |\alpha| t)$$

and let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and a φ-approximately quartic mapping in the sense that

$$\begin{aligned}\mu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ &- \frac{1}{2}a(a+1)^2 f(x+y), t) \geq \mu'(\varphi(x,y), t), \\ \nu(f(ax+y) &+ f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ &- \frac{1}{2}a(a+1)^2 f(x+y), t) \leq \nu'(\varphi(x,y), t) \end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Then there exists a unique quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}\mu(\mathcal{Q}(x) - f(x), t) &\geq \mu'(\varphi(x, 0), \frac{(|\alpha| - |a|^4)t}{2}), \\ \nu(\mathcal{Q}(x) - f(x), t) &\leq \nu'(\varphi(x, 0), \frac{(|\alpha| - |a|^4)t}{2})\end{aligned} \tag{3.2}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.1), we get

$$\begin{aligned}\mu(f(ax) - a^4 f(x), t) &\geq \mu'(\varphi(x, 0), t), \\ \nu(f(ax) - a^4 f(x), t) &\leq \nu'(\varphi(x, 0), t)\end{aligned} \tag{3.3}$$

for all $x \in \mathcal{X}$ and all $t > 0$, which implies that

$$\begin{aligned}\mu(a^4 f(\frac{x}{a}) - f(x), t) &\geq \mu'(\varphi(x, 0), |\alpha| t), \\ \nu(a^4 f(\frac{x}{a}) - f(x), t) &\leq \nu'(\varphi(x, 0), |\alpha| t).\end{aligned} \tag{3.4}$$

Replacing x by $\frac{x}{a^n}$ in (3.4), we obtain

$$\begin{aligned}\mu(a^{4(n+1)} f(\frac{x}{a^{n+1}}) - a^{3n} f(\frac{x}{a^n}), |a|^{4n} t) &\geq \mu'(\varphi(x, 0), |\alpha|^{n+1} t), \\ \nu(a^{4(n+1)} f(\frac{x}{a^{n+1}}) - a^{3n} f(\frac{x}{a^n}), |a|^{4n} t) &\leq \nu'(\varphi(x, 0), |\alpha|^{n+1} t).\end{aligned} \tag{3.5}$$

Replacing t by $\frac{t}{|\alpha|^{n+1}}$ in (3.5), we get

$$\begin{aligned}
&\mu(a^{4(n+1)}f(\frac{x}{a^{n+1}})-a^{3n}f(\frac{x}{a^{n}}),\frac{|a|^{4n}}{|\alpha|^{n+1}}t)\geq\mu'(\varphi(x,0),t),\\
&\nu(a^{4(n+1)}f(\frac{x}{a^{n+1}})-a^{3n}f(\frac{x}{a^{n}}),\frac{|a|^{4n}}{|\alpha|^{n+1}}t)\leq\nu'(\varphi(x,0),t).
\end{aligned}\tag{3.6}$$

It follows from

$$a^{4n}f(\frac{x}{a^{n}})-f(x)=\sum_{j=0}^{n-1}(a^{4(j+1)}f(\frac{x}{a^{j+1}})-a^{3j}f(\frac{x}{a^{j}}))$$

and (3.6) that

$$\begin{aligned}
\mu(a^{4n}f(\frac{x}{a^{n}})-f(x),\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t)&\geq\prod_{j=0}^{n-1}\mu(a^{4(j+1)}f(\frac{x}{a^{j+1}})-a^{3j}f(\frac{x}{a^{j}}),\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\\
&\geq\mu'(\varphi(x,0),t),\qquad\qquad(3.7)\\
\nu(a^{4n}f(\frac{x}{a^{n}})-f(x),\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t)&\leq\coprod_{j=0}^{n-1}\nu(a^{4(j+1)}f(\frac{x}{a^{j+1}})-a^{3j}f(\frac{x}{a^{j}}),\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\\
&\leq\nu'(\varphi(x,0),t),
\end{aligned}$$

for all $x\in\mathcal{X}$, all $t>0$ and $n>0$, where

$$\prod_{j=0}^{n-1}\omega_j=\omega_0*\omega_2*...*\omega_{n-1},\qquad\coprod_{j=0}^{n-1}\omega_j=\omega_0\diamond\omega_2\diamond...\diamond\omega_{n-1}.$$

Replacing x by $\frac{x}{a^m}$ in (3.7), we obtain

$$\begin{aligned}
&\mu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^{m}}),\sum_{j=0}^{n-1}\frac{|a|^{4(j+m)}}{|\alpha|^{j+m+1}}t)\geq\mu'(\varphi(x,0),t),\\
&\nu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^{m}}),\sum_{j=0}^{n-1}\frac{|a|^{4(j+m)}}{|\alpha|^{j+m+1}}t)\leq\nu'(\varphi(x,0),t).
\end{aligned}$$

Hence

$$\begin{aligned}
&\mu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^{m}}),\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\geq\mu'(\varphi(x,0),t),\\
&\nu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^{m}}),\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}t)\leq\nu'(\varphi(x,0),t)
\end{aligned}$$

for all $x\in\mathcal{X}$, all $t>0$, $n\geq0$ and $m\geq0$. Thus

$$\mu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^{m}}),t)\geq\mu'(\varphi(x,0),\frac{t}{\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}}),$$

$$\nu(a^{4(n+m)}f(\frac{x}{a^{n+m}})-a^{4m}f(\frac{x}{a^m}),t)\leq\nu'(\varphi(x,0),\frac{t}{\sum_{j=m}^{n+m-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}}) \tag{3.8}$$

for all $x\in\mathcal{X}$, all $t>0$, $n\geq 0$ and $m\geq 0$. Since $|\alpha|>|a|^4$ and $\Sigma_{j=0}^{\infty}\frac{|a|^{4j}}{|\alpha|^{j}}<\infty$, the sequence $a^{4n}f(\frac{x}{a^n})$ is a Cauchy sequence in $(\mathcal{Y},\mu,\nu)$. Since $(\mathcal{Y},\mu,\nu)$ is complete, this sequence converges to some point $\mathcal{Q}(x)\in\mathcal{Y}$. Fix $x\in\mathcal{X}$ and put $m=0$ in (3.8). Then we obtain

$$\mu(a^{4n}f(\frac{x}{a^n})-f(x),t)\geq\mu'(\varphi(x,0),\frac{t}{\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}}),$$
$$\nu(a^{4n}f(\frac{x}{a^n})-f(x),t)\leq\nu'(\varphi(x,0),\frac{t}{\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}})$$

for all $x\in\mathcal{X}$, all $t>0$ and $n\geq 0$. Thus we get

$$\begin{aligned}\mu(\mathcal{Q}(x)-f(x),t)&\geq\mu(\mathcal{Q}(x)-a^{4n}f(\frac{x}{a^n}),\frac{t}{2})*\mu(a^{4n}f(\frac{x}{a^n})-f(x),\frac{t}{2})\\&\geq\mu'(\varphi(x,0),\frac{t}{2\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}}),\\\nu(\mathcal{Q}(x)-f(x),t)&\leq\nu(\mathcal{Q}(x)-a^{4n}f(\frac{x}{a^n}),\frac{t}{2})\diamond\nu(a^{4n}f(\frac{x}{a^n})-f(x),\frac{t}{2})\\&\leq\nu'(\varphi(x,0),\frac{t}{2\sum_{j=0}^{n-1}\frac{|a|^{4j}}{|\alpha|^{j+1}}})\end{aligned}$$

for large n. By taking the limit as $n\to\infty$, we obtain

$$\mu(\mathcal{Q}(x)-f(x),t)\geq\mu'(\varphi(x,0),\frac{(|\alpha|-|a|^4)t}{2}),$$
$$\nu(\mathcal{Q}(x)-f(x),t)\leq\nu'(\varphi(x,0),\frac{(|\alpha|-|a|^4)t}{2})$$

for all $x\in\mathcal{X}$ and all $t>0$. Replacing x and y by $\frac{x}{a^n}$ and $\frac{y}{a^n}$ in (3.1), respectively, we obtain

$$\begin{aligned}&\mu(a^{4n}f(\frac{ax+y}{a^n})+f(x+ay)+\frac{1}{2}a(a-1)^2a^{4n}f(\frac{x-y}{a^n})\\&-(a^2-1)^2a^{4n}(f(\frac{x}{a^n})+f(\frac{y}{a^n}))-\frac{1}{2}a(a+1)^2a^{4n}f(\frac{x+y}{a^n}),t)\geq\mu'(\varphi(\frac{x}{a^n},\frac{y}{a^n}),\frac{t}{|a|^{4n}}),\\&\nu(a^{4n}f(\frac{ax+y}{a^n})+f(x+ay)+\frac{1}{2}a(a-1)^2a^{4n}f(\frac{x-y}{a^n})\\&-(a^2-1)^2a^{4n}(f(\frac{x}{a^n})+f(\frac{y}{a^n}))-\frac{1}{2}a(a+1)^2a^{4n}f(\frac{x+y}{a^n}),t)\leq\nu'(\varphi(\frac{x}{a^n},\frac{y}{a^n}),\frac{t}{|a|^{4n}})\end{aligned}$$

for all $x\in\mathcal{X}$ and all $t>0$. Note that

$$\lim_{n\to\infty}\mu'(a^{4n}\varphi(\frac{x}{a^n},\frac{y}{a^n}),t)=\lim_{n\to\infty}\mu'(\varphi(x,y),\frac{|\alpha|^nt}{|a|^{4n}})=1,$$
$$\lim_{n\to\infty}\nu'(a^{4n}\varphi(\frac{x}{a^n},\frac{y}{a^n}),t)=\lim_{n\to\infty}\nu'(\varphi(x,y),\frac{|\alpha|^nt}{|a|^{4n}})=0$$

for all $x \in \mathcal{X}$ and all $t > 0$. We observe that $\mathcal{Q}$ fulfills (1.2). Therefore, $\mathcal{Q}$ is an Euler-Lagrange type quartic mapping.

It is left to show that the quartic mapping $\mathcal{Q}$ is unique. Assume that there is another Euler-Lagrange type quartic mapping $\mathcal{Q}' : \mathcal{X} \to \mathcal{Y}$ satisfying (3.2). For each $x \in \mathcal{X}$, clearly $a^{4n}\mathcal{Q}(\frac{x}{a^n}) = \mathcal{Q}(x)$ and $a^{4n}\mathcal{Q}'(\frac{x}{a^n}) = \mathcal{Q}'(x)$ for all $n \in \mathbb{N}$. It follows from (3.3) that

$$\begin{aligned}\mu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) &= \mu(a^{4n}\mathcal{Q}(\frac{x}{a^n}) - a^{4n}\mathcal{Q}'(\frac{x}{a^n}), t)\\ &= \mu(\mathcal{Q}(\frac{x}{a^n}) - \mathcal{Q}'(\frac{x}{a^n}), \frac{t}{|a|^{4n}})\\ &= \mu(\mathcal{Q}(\frac{x}{a^n}) - f(\frac{x}{a^n}), \frac{t}{2|a|^{4n}})\\ &* \mu(f(\frac{x}{a^n}) - \mathcal{Q}'(\frac{x}{a^n}) - \frac{t}{2|a|^{4n}})\\ &\geq \mu'(\varphi(x, 0), \frac{|\alpha|^n(|\alpha| - |a|^4)t}{2|a|^{4n}}),\end{aligned}$$

and similarly

$$\nu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) \leq \nu'(\varphi(x, 0), \frac{\alpha^n(|\alpha| - |k|^4)t}{2|a|^{4n}})$$

for all $x \in \mathcal{X}$ and all $t > 0$. Since $|\alpha| > |a|^4$, we get

$$\begin{aligned}\lim_{n\to\infty} \mu'(\varphi(x, 0), \frac{|\alpha|^n(|\alpha| - |a|^4)t}{2|a|^{4n}}) &= 1,\\ \lim_{n\to\infty} \nu'(\varphi(x, 0), \frac{|\alpha|^n(|\alpha| - |a|^4)t}{2|a|^{4n}}) &= 0.\end{aligned}$$

Therefore, $\mu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) = 1$ and $\nu(\mathcal{Q}(x) - \mathcal{Q}'(x), t) = 0$ for all $x \in \mathcal{X}$ and all $t > 0$, that is, the mapping $\mathcal{Q}(x)$ is unique, as desired. □

Theorem 3.2. *Let $\mathcal{X}$ be a linear space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be a mapping such that, for some $0 < |\alpha| < |a|^4$,*

$$\begin{aligned}\mu'(\varphi(ax, 0), t) &\geq \mu'(\alpha\varphi(x, 0), t),\\ \nu'(\varphi(ax, 0), t) &\leq \nu'(\alpha\varphi(x, 0), t).\end{aligned}$$

Let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and a φ-approximately quartic mapping in the sense that

$$\begin{aligned}&\mu(f(ax + y) + f(x + ay) + \frac{1}{2}a(a-1)^2 f(x - y) - (a^2 - 1)^2(f(x) + f(y))\\ &\quad - \frac{1}{2}a(a+1)^2 f(x + y), t) \geq \mu'(\varphi(x, y), t),\\ &\nu(f(ax + y) + f(x + ay) + \frac{1}{2}a(a-1)^2 f(x - y) - (a^2 - 1)^2(f(x) + f(y))\\ &\quad - \frac{1}{2}a(a+1)^2 f(x + y), t) \leq \nu'(\varphi(x, y), t)\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Then there exists a unique quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned} &\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\varphi(x, 0), \frac{|a|^4 - |\alpha|}{2} t), \\ &\nu(\mathcal{Q}(x) - f(x), t) \nu \nu'(\varphi(x, 0), \frac{|a|^4 - |\alpha|}{2} t) \end{aligned} \tag{3.9}$$

for all $x \in \mathcal{X}$ and all $t > 0$.

Proof. The techniques are similar to that of Theorem 3.1. Hence we present a sketch of the proof. Letting $y = 0$ in (3.9), we get

$$\begin{aligned} &\mu(\frac{f(ax)}{a^4} - f(x), t) \geq \mu'(\varphi(x, 0), t), \\ &\nu(\frac{f(ax)}{a^4} - f(x), t) \leq \nu'(\varphi(x, 0), t) \end{aligned} \tag{3.10}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Replacing x by $a^n x$ in (3.10), we obtain

$$\begin{aligned} &\mu(\frac{f(a^{(n+1)}x)}{a^4} - f(a^n x), t) \geq \mu'(\varphi(x, 0), \frac{t}{|\alpha|^n}), \\ &\nu(\frac{f(a^{(n+1)}x)}{a^4} - f(a^n x), t) \leq \nu'(\varphi(x, 0), \frac{t}{|\alpha|^n}) \end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. For positive integers n and m,

$$\begin{aligned} &\mu(\frac{f(a^{(n+m)}x)}{a^{4(n+m)}} - \frac{f(a^m x)}{a^{4m}}, t) \geq \mu'(\varphi(x, 0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{|\alpha|^j}{a^{4(j+1)}}}), \\ &\nu(\frac{f(a^{(n+m)}x)}{a^{4(n+m)}} - \frac{f(a^m x)}{a^{4m}}, t) \leq \nu'(\varphi(x, 0) \frac{t}{\sum_{j=m}^{n+m-1} \frac{|\alpha|^j}{a^{4(j+1)}}}), \end{aligned} \tag{3.11}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Hence $\{\frac{f(a^n x)}{a^{4n}}\}$ is a Cauchy sequence in intuitionistic fuzzy Banach space. Therefore, there is a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by $\mathcal{Q}(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{4n}}$ and hence (3.11) with $m = 0$ implies

$$\begin{aligned} &\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\varphi(x, 0), \frac{|a|^4 - |\alpha|}{2} t), \\ &\nu(\mathcal{Q}(x) - f(x), t) \nu \nu'(\varphi(x, 0), \frac{|a|^4 - |\alpha|}{2} t) \end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. This complete the proof. □

4. Intuitionistic fuzzy continuity

In this section we establish intuitionistic fuzzy continuity by using continuous approximately quartic mappings.

Definition 4.1. Let $f : \mathbb{R} \to \mathcal{X}$ be a mapping, where $\mathbb{R}$ is endowed with the Euclidean topology and $\mathcal{X}$ is an intuitionistic fuzzy normed space equipped with intuitionistic fuzzy norm (μ, ν). Then f is called intuitionistic fuzzy continuous at a point $t_0 \in \mathbb{R}$ if for all $\varepsilon > 0$ and all $0 < \alpha < 1$ there exists $\delta > 0$ such that for each t with $0 < |t - t_0| < \delta$

$$\mu(f(tx) - f(t_0x), \varepsilon) \geq \alpha,$$
$$\nu(f(tx) - f(t_0x), \varepsilon) \leq 1 - \alpha.$$

Theorem 4.1. *Let $\mathcal{X}$ be a normed space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0) = 0$ and an (r, s)-approximately quartic mapping in the sense that for some r, s and some $z_0 \in \mathcal{Z}$*

$$\begin{aligned}\mu(f(ax+y) + f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ - \frac{1}{2}a(a+1)^2 f(x+y), t) \geq \mu'((\|x\|^r + \|y\|^s)z_0, t), \\ \nu(f(ax+y) + f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y)) \\ - \frac{1}{2}a(a+1)^2 f(x+y), t) \leq \nu'((\|x\|^r + \|y\|^s)z_0, t)\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. If $r, s < 4$, then there exists a unique quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\|x\|^r z_0, \frac{(|a|^4 - |a|^r)t}{2}), \\ \nu(\mathcal{Q}(x) - f(x), t) \leq \nu'(\|x\|^r z_0, \frac{(|a|^4 - |a|^r)t}{2})\end{aligned} \tag{4.1}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Furthermore, if for some $x \in \mathcal{X}$ and all $n \in \mathbb{N}$, the mapping $\mathcal{H} : \mathbb{R} \to \mathcal{Y}$ defined by $\mathcal{H}(t) = f(a^n tx)$ is intuitionistic fuzzy continuous. Then the mapping $t \to \mathcal{Q}(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is intuitionistic fuzzy continuous.

Proof. Define $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ by $\varphi(x, y) = (\|x\|^r + \|y\|^s)z_0$. By Theorem 3.2, there exists a unique quartic mapping $\mathcal{Q}$ satisfying (4.1). We have

$$\begin{aligned}&\mu(\mathcal{Q}(x) - \frac{f(a^n x)}{a^{4n}}, t) = \mu(\frac{\mathcal{Q}(a^n x)}{a^{4n}} - \frac{f(a^n x)}{a^{4n}}, t) \\ &= \mu(\mathcal{Q}(a^n x) - f(a^n x), |a|^{4n}t) \geq \mu'(|a|^{4rn}\|x\|^r z_0, \frac{|a|^{4n}(|a|^4 - |a|^r)t}{2}) \\ &\geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4 - |a|^r)t}{2|a|^{rn}}), \\ &\nu(\mathcal{Q}(x) - \frac{f(a^n x)}{a^{4n}}, t) = \nu(\frac{\mathcal{Q}(a^n x)}{a^{4n}} - \frac{f(a^n x)}{a^{4n}}, t), \\ &= \nu(\mathcal{Q}(a^n x) - f(a^n x), |a|^{4n}t) \leq \nu'(|a|^{4rn}\|x\|^r z_0, \frac{|a|^{4n}(|a|^4 - |a|^r)t}{2}), \\ &\leq \nu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4 - |a|^r)t}{2|a|^{rn}})\end{aligned} \tag{4.2}$$

for all $x \in \mathcal{X}$, all $t > 0$ and $n \in \mathbb{N}$. Fix $x \in \mathcal{X}$ and $t_0 \in \mathbb{R}$. Given $\varepsilon > 0$ and $0 < \alpha < 1$. By (4.2), we obtain

$$\mu(\mathcal{Q}(tx) - \frac{f(a^n tx)}{a^{4n}}, t) \geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4 - |a|^r)t}{2|t|^r|a|^{rn}})$$

$$\geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|1+t_0|^r|a|^{rn}}),$$

$$\mu(\mathcal{Q}(tx)-\frac{f(a^n tx)}{a^{4n}}, t) \geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|t|^r|a|^{rn}})$$

$$\geq \mu'(\|x\|^r z_0, \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|1+t_0|^r|a|^{rn}}) \tag{4.3}$$

for all $t \in \mathbb{R}$ and all $|t-t_0|<1$. Since $r<3$, we have $\lim_{n\to\infty} \frac{|a|^{4n}(|a|^4-|a|^r)t}{2|1+t_0|^r|a|^{rn}} = \infty$, and hence there exists $n_0 \in \mathbb{N}$ such that

$$\mu(\mathcal{Q}(tx)-\frac{f(a^{n_0} tx)}{a^{4n_0}}, \frac{\varepsilon}{3}) \geq \alpha,$$

$$\nu(\mathcal{Q}(tx)-\frac{f(a^{n_0} tx)}{a^{4n_0}}, \frac{\varepsilon}{3}) \leq 1-\alpha$$

for all $t \in \mathbb{R}$ and all $|t-t_0|<1$. By the intuitionistic fuzzy continuity of the mapping $s \to f(a^{n_0}tx)$, there exists $\delta<1$ such that for each t with $0<|t-t_0|<\delta$, we have

$$\mu(\frac{f(a^{n_0} tx)}{a^{4n_0}}-\frac{f(a^{n_0} t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \geq \alpha,$$

$$\nu(\frac{f(a^{n_0} tx)}{a^{4n_0}}-\frac{f(a^{n_0} t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \leq 1-\alpha.$$

It follows that

$$\mu(\mathcal{Q}(tx)-\mathcal{Q}(t_0x), \varepsilon) \geq \mu(\mathcal{Q}(tx)-\frac{f(a^{n_0} tx)}{a^{4n_0}}, \frac{\varepsilon}{3})*$$

$$\mu(\frac{f(a^{n_0} tx)}{a^{4n_0}}-\frac{f(a^{n_0} t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3})*\mu(\frac{f(a^{n_0} t_0x)}{a^{4n_0}}-\mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \geq \alpha$$

$$\mu(\mathcal{Q}(tx)-\mathcal{Q}(t_0x), \varepsilon) \geq \mu(\mathcal{Q}(tx)-\frac{f(a^{n_0} tx)}{a^{4n_0}}, \frac{\varepsilon}{3})*$$

$$\mu(\frac{f(a^{n_0} tx)}{a^{4n_0}}-\frac{f(a^{n_0} t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3})*\mu(\frac{f(a^{n_0} t_0x)}{a^{4n_0}}-\mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \geq \alpha,$$

$$\nu(\mathcal{Q}(tx)-\mathcal{Q}(t_0x), \varepsilon) \leq \nu(\mathcal{Q}(tx)-\frac{f(a^{n_0} tx)}{a^{4n_0}}, \frac{\varepsilon}{3})\diamond$$

$$\nu(\frac{f(a^{n_0} tx)}{a^{4n_0}}-\frac{f(a^{n_0} t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3})\diamond\nu(\frac{f(a^{n_0} t_0x)}{a^{4n_0}}-\mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \leq \alpha$$

$$\nu(\mathcal{Q}(tx)-\mathcal{Q}(t_0x), \varepsilon) \leq \nu(\mathcal{Q}(tx)-\frac{f(a^{n_0} tx)}{a^{4n_0}}, \frac{\varepsilon}{3})\diamond$$

$$\nu(\frac{f(a^{n_0} tx)}{a^{4n_0}}-\frac{f(a^{n_0} t_0x)}{a^{4n_0}}, \frac{\varepsilon}{3})\diamond\nu(\frac{f(a^{n_0} t_0x)}{a^{4n_0}}-\mathcal{Q}(t_0x), \frac{\varepsilon}{3}) \leq 1-\alpha$$

for all $t \in \mathbb{R}$ and all $0<|t-t_0|<\delta$. Therefore, the mapping $t \to \mathcal{Q}(tx)$ is intuitionistic fuzzy continuous. □

Theorem 4.2. *Let $\mathcal{X}$ be a normed space and let $(\mathcal{Z}, \mu', \nu')$ be an IFNS. Let $(\mathcal{Y}, \mu, \nu)$ be an intuitionistic fuzzy Banach space and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(0)=0$ and a (r,s)-approximately quartic mapping in the sense that for some r, s and some $z_0 \in \mathcal{Z}$*

$$\mu(f(ax+y)+f(x+ay)+\frac{1}{2}a(a-1)^2f(x-y)-(a^2-1)^2(f(x)+f(y))$$

$$- \frac{1}{2}a(a+1)^2 f(x+y), t) \geq \mu'((\|x\|^r + \|y\|^s)z_0, t),$$

$$\nu(f(ax+y) + f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - (a^2-1)^2(f(x)+f(y))$$
$$- \frac{1}{2}a(a+1)^2 f(x+y), t) \leq \nu'((\|x\|^r + \|y\|^s)z_0, t)$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. If $r, s < 4$, then there exists a unique quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\mu(\mathcal{Q}(x) - f(x), t) \geq \mu'(\|x\|^r z_0, \frac{(|a|^r - |a|^4)t}{2}),$$
$$\nu(\mathcal{Q}(x) - f(x), t) \leq \nu'(\varphi(x, 0), \frac{(|a|^r - |a|^4)t}{2})$$

for all $x \in \mathcal{X}$ and all $t > 0$. Furthermore, if for some $x \in \mathcal{X}$ and all $n \in \mathbb{N}$, the mapping $\mathcal{H} : \mathbb{R} \to \mathcal{Y}$ defined by $\mathcal{H}(t) = f(a^n tx)$ is intuitionistic fuzzy continuous. Then the mapping $t \to \mathcal{Q}(tx)$ from $\mathbb{R}$ to $\mathcal{Y}$ is intuitionistic fuzzy continuous.

Proof. Define $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ by $\varphi(x, y) = (\|x\|^r + \|y\|^s)z_0$ for all $x \in \mathcal{X}$. Since $r > 4$, we have $\alpha = |a|^r > |a|^4$. By Theorem 3.1, there exists a unique quartic mapping $\mathcal{Q}$ satisfying (4.3).

The rest of the proof can be done by the same line as in Theorem 4.1. □

5. Conclusion

We have investigated the Hyers-Ulam stability of the Euler-Lagrange type quartic functional equation (1.2) in intuitionistic fuzzy normed spaces. Furthermore, we have studied intuitionistic fuzzy continuity through the existence of a certain solution of a fuzzy stability problem for approximately Euler-Lagrange quartic functional equation. In future work, we will apply the results to understand the Hyers-Ulam stability of several Euler-Lagrange type functional equtions in fuzzy Hilbert C^*-modules anf Hilbert C^*-modules to study the automatic contuinity.

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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