Haar Wavelet Method with Caputo derivative for Solution of a System of Fractional Integro-Differential equations

Rohul Amin[1*], Raheem Ullah[1], Imran Khan[1], Wojciech Sumelka[2]

[1] *Department of Mathematics, University of Peshawar, Khyber Pakhtunkhwa, Pakistan.*

[2] *Institute of Structural Analysis, Poznan University of Technology, Piotrowo 5 Street, Poznan 60-965, Poland*

Abstract

In this paper, a numerical method based on Haar wavelet with Caputo derivative is developed for the solution of a system of fractional integro-differential equations (FIDEs). The solution of these equations is difficult due to the non-local nature of fractional derivatives and integrals. Different numerical and analytical methods have been developed to overcome these challenges. We develop numerical scheme for solution of different types of systems of FIDEs. The proposed method is then applied to different test problems to demonstrate its robustness and effectiveness. The experiential error analysis is carried out for all test problems. These experiments involve the calculation and analysis of different error norms, such as the maximum absolute error and root mean square error. The numerical experiment shows that increasing the collocation points the errors reduces significantly. The results show that the present numerical scheme is a precise and efficient technique for solving such systems of FIDEs.

Keywords: Fractional IDEs, collocation method, maximum absolute and root mean square errors, simulation.

*Corresponding author.

1. Introduction

Fractional integro-differential equations (FIDEs) have gained increasing attention in recent years due to their applications in various fields, such as physics, biology, economics, and engineering. FIDEs are a differential equation (DE) type that involves fractional-order derivatives and function integrals. They can be expressed as a combination of integer-order DEs and fractional-order DEs, where the fractional-order derivatives are noninteger powers of the differential operator. The integrals in the equations add another layer of complexity, making them difficult to solve analytically. These equations have applications in physics, engineering, biology, and finance, to name a few [1]. A system of FIDEs is a set of multiple DEs containing fractional-order derivatives and integrals. The equations in a system are typically interconnected, meaning that the solution for one equation depends on the solutions of the other equations in the system. These equations are used to model complex dynamic systems, such as those found in physics, engineering, and biology. A system of FIDEs, as FIDEs itself, has many applications in engineering [2], quantum mechanics [3], and many other fields of science [4].

Fractional differential equations (FDEs) with non-integer order can describe many physical phenomena. Applications of FDEs can be found in physics, chemistry, biology, engineering, and finance [4]. One of the main advantages of using FDEs is that they can provide a more accurate description of real-world phenomena than ordinary differential equations (ODEs). For example, modeling of viscoelastic materials, diffusion phenomena in porous media, and control systems with delay can be improved through FDEs. Solving FDEs is more challenging than solving ODEs, and analytical solutions for FDEs are available for only a few of them. Fractional calculus and FDEs have many applications in various fields, and their study is still a very active area of research in mathematics and other sciences. Many definitions are found in the literature of FD, such as Riemann-Liouvlle FD, Caputo FD, etc. [1].

Solving a system of FIDEs can be challenging and often requires advanced mathematical techniques, such as numerical simulations and iterative methods. Several numerical schemes have been developed in the last decade to solve FDEs. Momani and Qaralleh [5] used a domain decomposition method (ADM) to solve a FIDE system. The ADM is used for both linear and non-linear IDEs. Mahdy [6] used the least squares method to solve FIDEs. Li and Sun used the block pulse matrix [7] to examine the iterative solution of the FDEs.Mohammed [8] solved FIDEs by shifting the Chebyshev polynomial scheme and using the least squares approach. Two methods for approximating a function by a polynomial that minimizes the sum of the squares of the difference between the function and the polynomial are the least squares approach and the shifted Chebyshev polynomial. Shifted Chebyshev polynomials are the orthogonal polynomials often used in this method for functions with singularities and nonuniform data. In order to determine the numerical solution of FIDEs, Ali *et al.* [9] worked on hybrid Bernstein and block pulse wavelet approaches.

For the solution of FIDEs, Baofeng Li [10] employed a generalized hat function approach. Weakly singular kernels were used together with the Chebyshev wavelet approach by Bargamadi et al. [11] to obtain the solution of FIDEs. Asgari [18] used operational matrices of triangular function. They also checked the order of convergence and stability of the method. Derakhshan [19] used operational matrices for the solution of coupled systems of FIDEs. They reduced the given FIDEs to algebraic equations to obtainthe Chebyshev unknown coefficients.

This is the structure of the remaining portion of the paper. The paper fundamental ideas are presented in Section 2. Section 3 illustrates the detailed numerical scheme for the FIDE system. Section 4 accomplishes the application of the proposed scheme on various test problems, demonstrating the effectiveness of the method. Finally, Section 5 presents the concluding remarks regarding the study.

2. Preliminaries

The Caputo FD with order α for $f(t)$ is [1]:

$$D^{\alpha}w(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\eta)d}{\eta}(t-\eta)^{1-n+\alpha}, \quad \alpha > 0, \tag{1}$$

where $n \in \mathbb{N}$, $n-1 < \alpha < n$, $t > 0$.

2.1. Haar Wavelet

For Haar family, the scaling function on $[0, 1)$ is $H_1(x) = 1$.

In this series, the other terms are written as [12]:

$$H_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\rho_1, \rho_2), \\ -1 & \text{for} \quad x \in [\rho_2, \rho_3), \\ 0 & \text{elsewhere}, \end{cases} \tag{2}$$

where $\rho_1 = \frac{\zeta}{d}$, $\rho_2 = \frac{1/2+\zeta}{d}$, and $\rho_3 = \frac{1+\zeta}{d}$, $d = 2^j$, $j = 0, 1, \ldots, J$, $\zeta = 0, 1, \ldots, d-1$. Formula $i = d + \zeta + 1$ is used to obtain the value of the index i. References [13, 14, 15, 16] include recent research based on the HWC technique.

2.2. Function approximation

The sum of the Haar series in the interval can represent any function $f(x)$ in $L_2[a, b]$ because the Haar wavelet functions are orthogonal to one another.

$$f(x) = \sum_{i=1}^{\infty} a_i \ h_i(x),$$

such that the coefficients a_i are constant, $h_i(x)$ is a Haar function. For approximation, the aforementioned up to a limited number of terms, an infinite series is ended. using the formula

$$f(x) = \sum_{i=1}^{N} a_i h_i(x),$$

We use the symbol $\mathcal{R}_{i,1}$ for integration of the Haar function.

$$\mathcal{R}_{i,1}(x) = \int_0^1 \mathbf{h}_i(x) dx. \tag{3}$$

Furthermore,

$$\mathcal{R}_{i,1}(t) = \begin{cases} x - \rho_1, \ \text{at } x \in [\rho_1, \rho_2) \\ \rho_3 - x, \ \text{at } x \in [\rho_2, \rho_3), \\ 0 \text{ elsewhere.} \end{cases} \tag{4}$$

3. Numerical scheme for the systems of fractional IDEs

The forthcoming section elaborates on the Haar wavelet collocation method (HWCM) that is developed for solution of systems of fractional IDEs. To proceeds let us consider the following system of fractional IDEs

$$D^{\alpha}\mathbf{W(t)} = a(t)w(t) + \int_0^t K_{ij}(t,\zeta)w(\zeta)d\zeta + \int_0^1 M_{ij}(t,\zeta)w(\zeta)d\zeta + F(t), \qquad (5)$$

with initial conditions (ICs) $w_1(0) = \lambda_1$, $w_2(0) = \lambda_2$, $w_3(0) = \lambda_3$,

where

$$\mathbf{W(t)} = \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix},$$

is vector function, $K = [k_{ij}]3 \times 3$ and $M = [m_{ij}]3 \times 3$ are smooth function and

$$F(t) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

putting the values in equation (5), which become

$$\begin{pmatrix} D^{\alpha}w_1(t) \\ D^{\beta}w_2(t) \\ D^{\gamma}w_3(t) \end{pmatrix} = a(t)\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} + \begin{pmatrix} \int_0^t k_{11}(t,\zeta) & \int_0^t k_{12}(t,\zeta) & \int_0^t k_{13}(t,\zeta) \\ \int_0^t k_{21}(t,\zeta) & \int_0^t k_{22}(t,\zeta) & \int_0^t k_{23}(t,\zeta) \\ \int_0^t k_{31}(t,\zeta) & \int_0^t k_{32}(t,\zeta) & \int_0^t k_{33}(t,\zeta) \end{pmatrix}$$

$$\begin{pmatrix} w_1(\zeta) \\ w_2(\zeta) \\ w_3(\zeta) \end{pmatrix} d\zeta$$

$$+ \begin{pmatrix} \int_0^t m_{11}(t,\zeta) & \int_0^t m_{12}(t,\zeta) & \int_0^t m_{13}(t,\zeta) \\ \int_0^t m_{21}(t,\zeta) & \int_0^t m_{22}(t,\zeta) & \int_0^t m_{23}(t,\zeta) \\ \int_0^t m_{31}(t,\zeta) & \int_0^t m_{32}(t,\zeta) & \int_0^t m_{33}(t,\zeta) \end{pmatrix} \begin{pmatrix} w_1(\zeta) \\ w_2(\zeta) \\ w_3(\zeta) \end{pmatrix} d\zeta + \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}.$$

After some simplification, we have

$$D^{\alpha}w_1(t) \quad = \quad a(t)w_1(t) + \int_0^t k_{11}(t,\vartheta)w_1(\zeta)d\zeta + \int_0^t k_{12}(t,\zeta)w_2(\zeta)d\zeta +$$

$$\int_0^t k_{13}(t,\zeta)w_3(\zeta)d\zeta + \int_0^1 m_{11}(t,\zeta)w_1(\zeta)d\zeta + \int_0^1 m_{12}(t,\zeta)w_1(\zeta)d\zeta +$$

$$\int_0^1 m_{13}(t,\zeta)w_1(\zeta)d\zeta + f_1(t), \tag{6}$$

$$D^\beta w_2(t) = a(t)w_1(t) + \int_0^t k_{21}(t,\zeta)w_1(\zeta)d\zeta + \int_0^t k_{22}(t,\zeta)w_2(\zeta)d\zeta +$$

$$\int_0^t k_{23}(t,\zeta)w_3(\zeta)d\zeta + \int_0^1 m_{21}(t,\zeta)w_1(\zeta)d\zeta + \int_0^1 m_{22}(t,\zeta)w_2(\zeta)d\zeta +$$

$$\int_0^1 m_{23}(t,\zeta)w_3(\zeta)d\zeta + f_2(t), \tag{7}$$

$$D^\gamma w_3(t) = a(t)w_1(t) + \int_0^t k_{31}(t,\zeta)w_1(\zeta)d\zeta + \int_0^t k_{32}(t,\zeta)w_2(\zeta)d\zeta +$$

$$\int_0^t k_{33}(t,\zeta)w_3(\zeta)d\zeta + \int_0^1 m_{31}(t,\zeta)y_2(\zeta)d\zeta + \int_0^1 m_{31}(t,\zeta)y_2(\zeta)d\zeta +$$

$$\int_0^1 m_{31}(t,\zeta)y_2(\zeta)d\zeta + f_3(t), \tag{8}$$

where D^α is FD in Caputo sense, $w_1(t)$, $w_2(t)$, $w_3(t)$ is unknown function, $ax(t)$, f_1, f_2, f_3, are known functions and the initial condition are $w_1(0) = \lambda_1$ $w_2(0) = \lambda_2$ and $w_3(0) = \lambda_3$.

Applying Caputo definition to Eq. (5), we have

$$\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{w_1^{(n)}(\zeta)d\zeta}{(t-\vartheta)^{\alpha-n+1}} = a(t)w_1(t) + \int_0^t k_{11}(t,\zeta)w_1(\zeta)d\zeta + \int_0^t k_{12}(t,\zeta)w_2(\zeta)d\zeta$$

$$+ \int_0^t k_{13}(t,\zeta)w_3(\zeta)d\zeta + \int_0^1 m_{11}(t,\zeta)w_1(\zeta)d\zeta +$$

$$\int_0^1 m_{12}(t,\zeta)w_1(\zeta)d\zeta + \int_0^1 m_{13}(t,\zeta)w_1(\zeta)d\zeta + f_1(t),$$

if we take $0 < \alpha < 1$ then $n = 1$ similarly if we take $1 < \alpha < 2$ then $n = 2$. The numerical scheme is derived for the case when $n = 1$, so Eq. (7) becomes,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w_1'(\zeta)d\zeta}{(t-\vartheta)^{\alpha-1+1}} = a(t)w_1(t) + \int_0^t k_{11}(t,\zeta)w_1(\zeta)d\zeta + \int_0^t k_{12}(t,\zeta)w_2(\zeta)d\zeta$$

$$+\int_0^t k_{13}(t,\zeta)w_3(\zeta)d\zeta+\int_0^1 m_{11}(t,\zeta)w_1(\zeta)d\zeta+$$

$$\int_0^1 m_{12}(t,\zeta)w_1(\zeta)d\zeta+\int_0^1 m_{13}(t,\zeta)w_1(\zeta)d\zeta+f_1(t),$$

After simplification, we get

$$\begin{aligned}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{w_1'(\zeta)d\zeta}{(t-\vartheta)^\alpha} &= a(t)w_1(t)+\int_0^t k_{11}(t,\zeta)w_1(\zeta)d\zeta+\int_0^t k_{12}(t,\zeta)w_2(\zeta)d\zeta\\ &+\int_0^t k_{13}(t,\zeta)w_3(\zeta)d\zeta+\int_0^1 m_{11}(t,\zeta)w_1(\zeta)d\zeta+\\ &\int_0^1 m_{12}(t,\zeta)w_1(\zeta)d\zeta+\int_0^1 m_{13}(t,\zeta)w_1(\zeta)d\zeta+f_1(t),\end{aligned}$$

Now let

$$w_1'(t)=\sum_{i=1}^N \xi_i h_i(t),\quad w_2'(t)=\sum_{i=1}^N \rho_i h_i(t),\quad w_3'(t)=\sum_{i=1}^N \varphi_i h_i(t). \tag{9}$$

Further, integrating and make use of the ICs, we get

$$w_1(t)=\lambda_1+\sum_{i=1}^N \xi_i \mathcal{R}_{(i,1)}(t), \tag{10}$$

$$w_2(t)=\lambda_2+\sum_{i=1}^N \rho_i \mathcal{R}_{(i,1)}(t),\quad w_3(t)=\lambda_3+\sum_{i=1}^N \rho_i \mathcal{R}_{(i,1)}(t),$$

$$\begin{aligned}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\sum_{i=1}^N \xi_i h_i(\zeta)d\zeta}{(t-\vartheta)^\alpha} &= a(t)(\lambda_1+a(t)\xi_i h_i(t))+\int_0^t k_{11}(t,\zeta)(\lambda_1+\xi_i h_i(\zeta))d\zeta\\ &+\int_0^t k_{12}(t,\zeta)(\lambda_2+\rho_i h_i(\zeta))d\zeta+\int_0^t k_{13}(t,\zeta)(\lambda_3+\varphi_i h_i(\zeta))d\zeta\\ &+\int_0^1 m_{11}(t,\zeta)(\lambda_1+\xi_i h_i(\zeta))d\zeta+\int_0^1 m_{12}(t,\zeta)(\lambda_2+\rho_i h_i(\zeta))d\zeta+\\ &\int_0^1 m_{13}(t,\zeta)(\lambda_3+\varphi_i h_i(\zeta))d\zeta+\int_0^1 m_{13}(t,\zeta)\varphi_i h_i(\zeta)d\zeta+f_1(t).\end{aligned}$$

After simplification, we obtain

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\sum_{i=1}^N a_i h_i(\zeta)d\zeta}{(t-\vartheta)^\alpha} = a(t)\lambda_1+a(t)\xi_i h_i(t)+\int_0^t k_{11}(t,\zeta)\lambda_1 d\zeta+\int_0^t k_{11}(t,\zeta)\xi_i h_i(\zeta)d\zeta$$

$$+\int_0^t k_{12}(t,\zeta)\lambda_2 d\zeta+\int_0^t k_{12}(t,\zeta)\rho_i h_i(\zeta)d\zeta+\int_0^t k_{13}(t,\zeta)\lambda_3 d\zeta$$

$$+\int_0^t k_{13}(t,\zeta)\varphi_i h_i(\zeta)d\zeta+\int_0^1 m_{11}(t,\zeta)\lambda_1 d\zeta+\int_0^1 m_{11}(t,\zeta)\xi_i h_i(\zeta)d\zeta$$

$$+\int_0^1 m_{12}(t,\zeta)\lambda_2 d\zeta+\int_0^1 m_{12}(t,\zeta)\rho_i h_i(\zeta)d\zeta+\int_0^1 m_{13}(t,\zeta)\lambda_3$$

$$+\int_0^1 m_{13}(t,\zeta)\varphi_i h_i(\zeta)d\zeta+f_1(t),$$

Now taking common $\sum_{i=1}^{N}\xi_i$, we have

$$\sum_{i=1}^{N}\xi_i\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{h_i(\zeta)d\zeta}{(t-\vartheta)^{\alpha}}-a(t)\mathcal{R}_{(i,1)}(t)+\int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta+\int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta\Big)$$

$$=a(t)\lambda_1+\int_0^t k_{11}(t,\zeta)d\zeta+\int_0^t k_{12}(t,\zeta)\lambda_2 d\zeta+\int_0^t k_{12}(t,\zeta)\sum_{i=1}^{N}\rho_i h_i(\zeta))d\zeta+\int_0^t k_{13}(t,\zeta)\lambda_3 d\zeta$$

$$+\int_0^t k_{13}(t,\zeta)\sum_{i=1}^{N}\varphi_i\mathcal{R}_{(i,1)}(\zeta)d\zeta+\int_0^1 m_{11}(t,\zeta)\lambda_1 d\zeta+\int_0^1 m_{12}(t,\zeta)d\zeta+\int_0^1 m_{12}(t,\zeta)d\zeta$$

$$+\sum_{i=1}^{N}\rho_i\mathcal{R}_{(i,1)}(\zeta)d\zeta+\int_0^1 m_{13}(t,\zeta)\lambda_3 d\zeta+\int_0^1 m_{13}(t,\zeta)d\zeta\sum_{i=1}^{N}\varphi_i\mathcal{R}_{(i,1)}(\zeta)d\zeta+f_1(t) \tag{11}$$

putting collocation point in the above equation, we get

$$\begin{aligned}\sum_{i=1}^{N}\xi_i G(i,j) &= a(t_j)\lambda_1+\int_0^t k_{11}(t,\zeta)d\zeta+\int_0^t k_{12}t,\vartheta\lambda_2 d\zeta+\int_0^t k_{12}(t,\zeta)\sum_{i=1}^{N}\rho_i h_i(\zeta))d\zeta\\ &+\int_0^t k_{13}(t,\zeta)\lambda_3 d\zeta+\int_0^t k_{13}(t,\zeta)\sum_{i=1}^{N}\varphi_i\mathcal{R}_{(i,1)}(\zeta)d\zeta+\int_0^1 m_{11}(t,\zeta)\lambda_1 d\zeta\\ &+\int_0^1 m_{12}t,\zeta d\zeta+\int_0^1 m_{12}(t,\zeta)d\zeta+\sum_{i=1}^{N}\rho_i\mathcal{R}_{(i,1)}(\zeta)d\zeta+\int_0^1 m_{13}(t,\zeta)\lambda_3 d\zeta\\ &+\int_0^1 m_{13}(t,\zeta)d\zeta\sum_{i=1}^{N}\varphi_i\mathcal{R}_{(i,1)}(\zeta)d\zeta+f_1(t). \end{aligned} \tag{12}$$

where,

$$G(i,j) = \Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{h_i(\zeta)d\zeta}{(t-\vartheta)^{\alpha}}-a(t)\mathcal{R}_{(i,1)}(t)+\int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta$$

$$+\int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta).$$

3.1. Evaluation of $G(i,j)$ via Lepik approach

Now, to find the value of $G(i,j)$, we use the method of Lepik [17]. We will discuss the following cases.

Case-1 For $t_j < 0$, Since $h_i(t_j)$=$\mathcal{R}_{(i,1)}(t_j)$=0. $G(i,j) = 0$.

Case-2 For $t_j \in [\alpha, \beta)$, then Eq. (15) becomes

$$G(i,j) = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^{\alpha} \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^{\alpha}} + \int_{\alpha}^{t_j} \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^{\alpha}}\right)$$
$$- a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta).$$

After applying the value of $h_i(t_j)$ and simplifying, we get

$$\begin{aligned} G(i,j) &= \frac{1}{\Gamma(1-\alpha)}\left(\int_{\alpha}^{t_j} \frac{dr}{(t_j-\vartheta)^{\alpha}}\right) - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta \\ &\quad + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta \\ &\quad + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta. \end{aligned}$$

Case-3 For $t_j \in [\beta, \gamma)$, then Eq. (15) becomes

$$\begin{aligned} G(i,j) &= \frac{1}{\Gamma(1-\alpha)}(\int_0^{\alpha} \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^{\alpha}} + \int_{\alpha}^{\beta} \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^{\alpha}} + \int_{\beta}^{t_j} \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^{\alpha}} - a(t)\mathcal{R}_{(i,1)}(t) \\ &\quad + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta). \end{aligned} \tag{13}$$

After applying the value of $h_i(t_j)$ and simplifying, we get

$$\begin{aligned} G(i,j) &= \frac{1}{\Gamma(1-\alpha)}(\int_{\alpha}^{\beta} \frac{dr}{(t_j-\vartheta)^{\alpha}} - \int_{\beta}^{t_j} \frac{dr}{(t_j-\vartheta)^{\alpha}} - a(t)\mathcal{R}_{(i,1)}(t) \\ &\quad + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta). \end{aligned} \tag{14}$$

simplification, we obtain the expression

$$G(i,j) = \frac{1}{\Gamma(1-\alpha)}(\frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - \frac{(t_j-\beta)^{1-\alpha}}{1-\alpha} - \frac{(t_j-\beta)^{1-\alpha}}{1-\alpha} - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta)$$

$$= \frac{1}{\Gamma(1-\alpha)}(\frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - \frac{2(t_j-\beta)^{1-\alpha}}{1-\alpha} - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta). \quad (15)$$

Case-4 For $t_j \in [\gamma, 1)$, then Eq. (15) becomes

$$G(i,j) = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^\alpha \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^\alpha} + \int_\alpha^\beta \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^\alpha} + \int_\beta^\gamma \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^\alpha} + \int_\gamma^{t_j} \frac{h_i(\zeta)d\zeta}{(t_j-\vartheta)^\alpha}\right) - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta).$$

After applying the value of $h_i(t_j)$ and simplifying, we get

$$G(i,j) = \frac{1}{\Gamma(1-\alpha)}\left(\int_\alpha^\beta \frac{dr}{(t_j-\vartheta)^\alpha} - \int_\beta^\gamma \frac{dr}{(t_j-\vartheta)^\alpha}\right) - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta). \quad (16)$$

simplification, we obtain the expression

$$G(i,j) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - \frac{(t_j-\beta)^{1-\alpha}}{1-\alpha} - \frac{(t_j-\beta)^{1-\alpha}}{1-\alpha} + \frac{(t_j-\gamma)^{1-\alpha}}{1-\alpha}\right) - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta).$$

So that

$$G(i,j) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - \frac{2(t_j-\beta)^{1-\alpha}}{1-\alpha} + \frac{(t_j-\gamma)^{1-\alpha}}{1-\alpha}\right) - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta). \quad (17)$$

Thus

$$G(i,j) = \begin{cases} 0 & \text{if } t_j < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \\ \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta) \\ & \text{if } t_j \in [\alpha, \beta), \\ \frac{1}{\Gamma(1-\alpha)} \left(\frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - \frac{2(t_j-\beta)^{1-\alpha}}{1-\alpha} \right) - a(t)\mathcal{R}_{(i,1)}(t) + \\ \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta) \\ & \text{if } t_j \in [\beta, \gamma), \\ \frac{1}{\Gamma(1-\alpha)} \left(\frac{(t_j-\alpha)^{1-\alpha}}{1-\alpha} - \frac{2(t_j-\beta)^{1-\alpha}}{1-\alpha} + \frac{(t_j-\gamma)^{1-\alpha}}{1-\alpha} \right) \\ -a(t)\mathcal{R}_{(i,1)}(t) + \int_0^t k_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta + \int_0^t m_{11}(t,\zeta)\mathcal{R}_{(i,1)}(\zeta)d\zeta) \\ & \text{if } t_j \in [\gamma, 1). \end{cases}$$

Now putting the value of $G(i,j)$ in Eq. (15) results in $N \times N$ system of linear algebraic equations. Furthermore, when solving this system by the Gauss elimination method, we obtain the unknown $\xi_i' s$ for $i = 1, 2, 3...N$ Haar coefficients. Putting the values of $a_i' s$ for $i = 1, 2, 3...N$in Eq. (10) we obtain the approximate solution that is required Eq. (5).

The following steps summarise the whole algorithm:

Algorithm 1: To evaluate the numerical solution of a system of fractional IDEs

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1. Apply Caputo fractional derivative to system (5).
2. Approximate the highest order ordinary derivative by the Haar function, and the integration method yields the expression for other order derivative.
3. Substituting collocation points to the system (5) , one obtains a system of algebraic equations.
4. Gauss elimination scheme is used to find the unknown coefficients.
5. Solution at collocation points is obtained using these coefficients.

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4. Numerical experiments

The section presents various numerical experiments that illustrate the precision and importance of using the HWCM. Experimental error analysis is carried out, and errors are tabulated for different collocation points. These experiments involve the calculation and analysis of different error norms, one of which is the maximum absolute error L_∞, which is defined as

$$L_\infty = \|\hat{w} - w\|_{max} = \max_{1\leq i\leq N} \mid \hat{w}_i - w_i \mid .$$

Root mean square error(M_{cp}) is defined as,

$$M_{cp} = \sqrt{\frac{\sum_{i=1}^{N}(\hat{w}_i - w_i)^2}{N}}.$$

The application of the proposed scheme to different test problems is demonstrated below.

Test problem 1

Consider the following linear FIDE system.

$$D^{\frac{3}{4}}y_1(t) = \frac{-1}{20} - \frac{t}{12} + \frac{4t^{\frac{1}{4}}(-32t^2 + 15)}{15\Gamma(0.25)} + \int_0^1 (t+s)(y_1(s) + y_2(s))ds, \qquad (18)$$

$$D^{\frac{3}{4}}y_2(t) = \frac{-13\sqrt{t}}{60} + \frac{4}{5\Gamma(0.24)}t^{\frac{1}{4}}(-5+8t) + \int_0^1 \sqrt{ts^2}(y_1(s) - y_2(s))ds, \qquad (19)$$

where ICs are $y_1(0) = 0$, $y_2(0) = 0$, and $y_1(t) = t - t^3$ and $y_2(t) = t^2 - t$ are the exact solutions.

The graphical comparison of exact and approximate solutions y_1 and y_2 for test problem 1 is demonstrated in figures 1 and 2, respectively. It is observed that the approximate solution is in excellent agreement with the exact solution. Moreover, the MAE and M_{cp} for problem 1 are presented in tables 1 and 2, respectively. The errors in both cases are observed to be reduced when the collocation points increase.

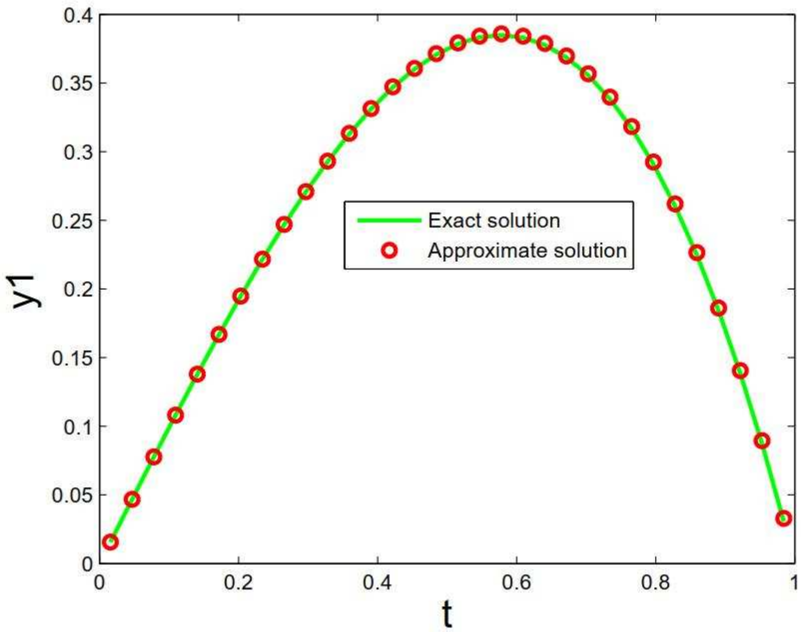

Figure 1: A comparative analysis of y_1 exact versus approximate solutions for the IDEs in test problem 1 at $N = 32$

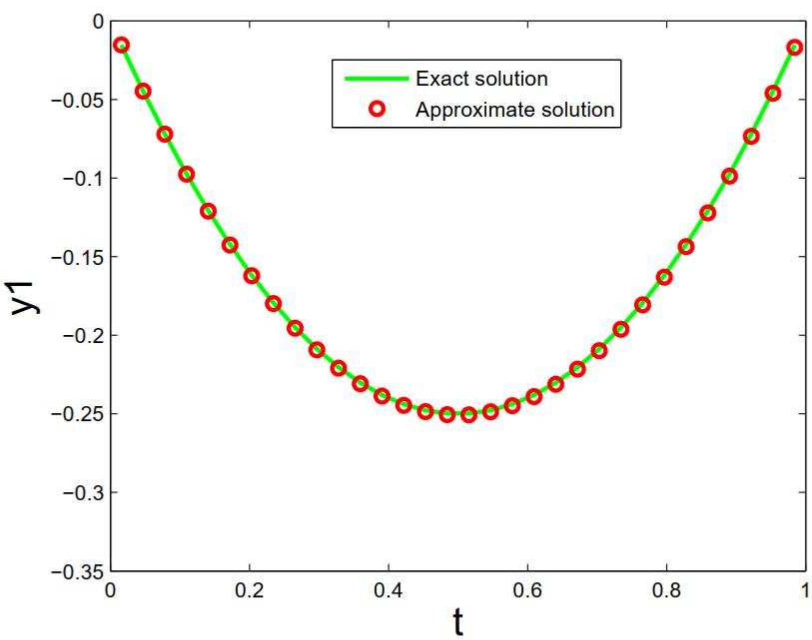

Figure 2: A comparative analysis of y_2 exact versus approximate solutions for the IDEs in test problem 1

J	$N = 2^{J+1}$	L_∞	
		$y_1(t)$	$y_2(t)$
1	$\mathbf{2^2}$	7.109728×10^{-03}	1.004610×10^{-02}
2	$\mathbf{2^3}$	6.365602×10^{-03}	6.478174×10^{-03}
3	$\mathbf{2^4}$	4.848868×10^{-03}	3.225315×10^{-03}
4	$\mathbf{2^5}$	2.355479×10^{-03}	1.469555×10^{-03}
5	$\mathbf{2^6}$	1.064701×10^{-03}	6.438438×10^{-04}
6	$\mathbf{2^7}$	4.650564×10^{-04}	2.767033×10^{-04}
7	$\mathbf{2^8}$	1.996172×10^{-04}	1.177372×10^{-04}
8	$\mathbf{2^9}$	8.489205×10^{-05}	4.983052×10^{-05}
9	$\mathbf{2^{10}}$	3.592118×10^{-05}	2.102871×10^{-05}

Table 1: MAE errors of test problem 1.

J	$N = 2^{J+1}$	M_{cp}	
		$y_1(t)$	$y_2(t)$
1	$\mathbf{2^2}$	4.143352×10^{-03}	7.361072×10^{-03}
2	$\mathbf{2^3}$	3.978996×10^{-03}	3.882518×10^{-03}
3	$\mathbf{2^4}$	2.289060×10^{-03}	1.954836×10^{-03}
4	$\mathbf{2^5}$	1.109005×10^{-03}	9.057392×10^{-04}
5	$\mathbf{2^6}$	5.009045×10^{-04}	4.011089×10^{-04}
6	$\mathbf{2^7}$	2.188161×10^{-04}	1.734762×10^{-04}
7	$\mathbf{2^8}$	9.395948×10^{-05}	7.408489×10^{-05}
8	$\mathbf{2^9}$	3.997497×10^{-05}	3.142193×10^{-05}
9	$\mathbf{2^{10}}$	1.692073×10^{-05}	1.327657×10^{-05}

Table 2: Root mean square error M_{cp} of test problem 1.

Test problem 2

Consider system FIDEs.

$$D^{\frac{4}{5}}y_1(t) = \frac{67t}{80} + \frac{25y^{\frac{6}{5}}(-11+5t)}{33\Gamma(\frac{1}{5})} + \int_0^1 2yt(y_1(s)+y_2(s))ds, \tag{20}$$

$$D^{\frac{4}{5}}y_2(t) = -\frac{83}{160} + \frac{15t}{4}(15t^{\frac{1}{5}}) - \frac{17t}{24} - \int_0^1 (t+s)(y_1(s)-y_2(s))ds, \tag{21}$$

where, ICs are $y_1(0) = 0$, $y_2(0) = 0$, and $y_1(t) = t^3 - t^2$, $y_2(t) = \frac{15}{8}t^2$ are the exact solutions.

The graphical comparison of exact and approximate solutions y_1 and y_2 for test problem 2 is shown in figures 3 and 4, respectively. One can observe an excellent agreement with the exact solution using the proposed scheme. The MAE and M_{cp} for problem 2 are presented in tables 3 and 4, respectively. With the increase in collocation points, the errors in both cases were reduced significantly.

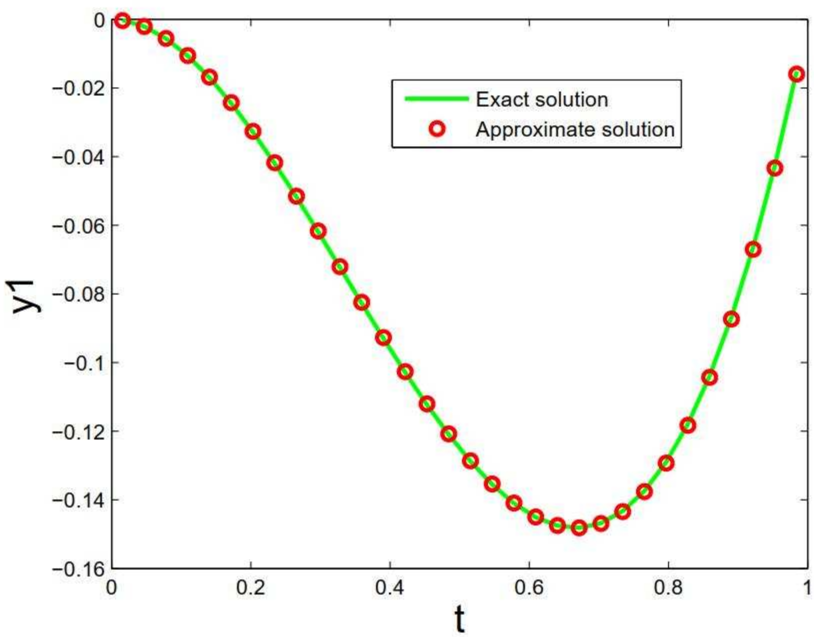

Figure 3: A comparative analysis of y_1 exact versus approximate solutions for the IDEs in test problem 2 at $N = 32$

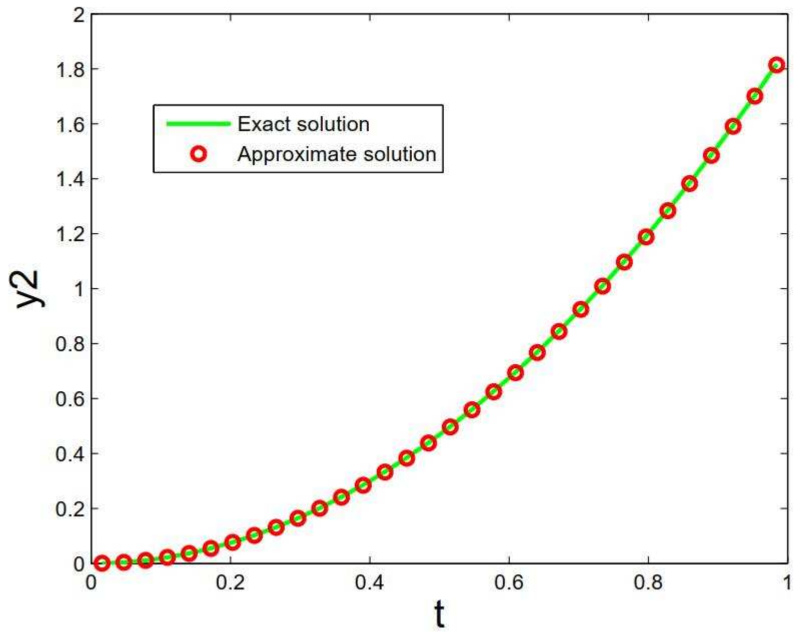

Figure 4: A comparative analysis of y_2 exact versus approximate solutions for the IDEs in test problem 2 at $N = 32$

J	$N = 2^{J+1}$	L_∞	
		$y_1(t)$	$y_2(t)$
1	$\mathbf{2^2}$	7.930871×10^{-03}	1.953125×10^{-02}
2	$\mathbf{2^3}$	2.293442×10^{-03}	1.098183×10^{-02}
3	$\mathbf{2^4}$	1.407094×10^{-03}	5.878679×10^{-03}
4	$\mathbf{2^5}$	8.318226×10^{-04}	2.810980×10^{-03}
5	$\mathbf{2^6}$	4.152454×10^{-04}	1.281897×10^{-03}
6	$\mathbf{2^7}$	1.934644×10^{-04}	5.715671×10^{-04}
7	$\mathbf{2^8}$	8.724058×10^{-05}	2.519718×10^{-04}
8	$\mathbf{2^9}$	3.869604×10^{-05}	1.104265×10^{-04}
9	$\mathbf{2^{10}}$	1.701606×10^{-05}	4.824320×10^{-05}

Table 3: MAE errors of test problem 2.

J	$N=2^{J+1}$	M_{cp}	
		$y_1(t)$	$y_2(t)$
1	$\mathbf{2^2}$	5.455769 $\times 10^{-03}$	1.254088 $\times 10^{-02}$
2	$\mathbf{2^3}$	1.279855 $\times 10^{-03}$	6.432173 $\times 10^{-03}$
3	$\mathbf{2^4}$	6.098773 $\times 10^{-04}$	3.468816 $\times 10^{-03}$
4	$\mathbf{2^5}$	3.230182 $\times 10^{-04}$	1.693191 $\times 10^{-03}$
5	$\mathbf{2^6}$	1.574979 $\times 10^{-04}$	7.821156 $\times 10^{-04}$
6	$\mathbf{2^7}$	7.285949 $\times 10^{-05}$	3.512593 $\times 10^{-04}$
7	$\mathbf{2^8}$	3.277316 $\times 10^{-05}$	1.554746 $\times 10^{-04}$
8	$\mathbf{2^9}$	1.452330 $\times 10^{-05}$	6.828914 $\times 10^{-05}$
9	$\mathbf{2^{10}}$	6.384297 $\times 10^{-06}$	2.987143 $\times 10^{-05}$

Table 4: Root mean squares errors M_{cp} of test problem 2.

Test problem 3

Consider system of mixed FIDEs

$$D^{\frac{3}{4}}y_1(t) = \frac{t^4}{4} + \frac{32t^{\frac{5}{4}}}{5\Gamma(\frac{1}{4})} - \frac{t}{6} - \int_0^t (y_2(t))dt + \int_0^1 2t(y_1(t) - y_2(t))dt, \tag{22}$$

$$D^{\frac{3}{4}}y_2(t) = \frac{t^3}{3} + \frac{384t^{\frac{9}{4}}}{45\Gamma(0.25)} - \int_0^t y_1(t)dt - \int_0^1 2t(y_1(t) + y_2(t))dt, \tag{23}$$

where ICs are $y_1(0) = 0$, $y_2(0) = 0$, where $y_1(t) = t^2$, $y_2(t) = t^3$ are the exact solutions.

The graphical comparison of exact and approximate solutions y_1 and y_2 using the present scheme for test problem 3 is depicted in figures 5 and 6, respectively. MAE and M_{cp} errors for problem 3 are presented in tables 5 and 6, respectively. With the increase in collocation points, the errors in both cases reduce significantly.

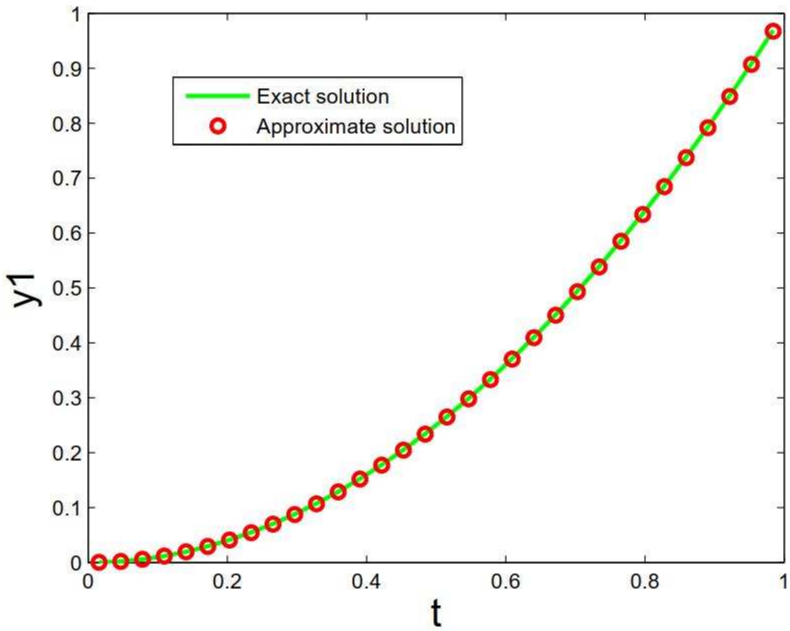

Figure 5: A comparative analysis of y_1 exact versus approximate solutions for the IDEs in test problem 3 at $N = 32$

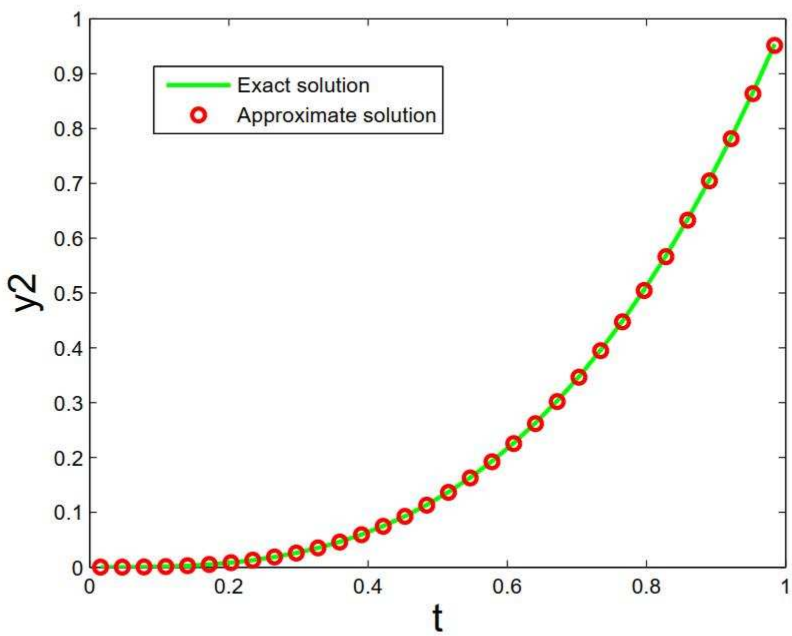

Figure 6: A comparative analysis of y_2 exact versus approximate solutions for the IDEs in test problem 3 at $N = 32$

J	$N = 2^{J+1}$	L_∞	
		$y_1(t)$	$y_2(t)$
1	$\mathbf{2^2}$	1.004610 $\times 10^{-02}$	7.109728 $\times 10^{-03}$
2	$\mathbf{2^3}$	6.478174 $\times 10^{-03}$	8.365602 $\times 10^{-03}$
3	$\mathbf{2^4}$	3.225315 $\times 10^{-03}$	4.848868 $\times 10^{-03}$
4	$\mathbf{2^5}$	1.469555 $\times 10^{-03}$	2.355479 $\times 10^{-03}$
5	$\mathbf{2^6}$	6.438438 $\times 10^{-04}$	1.064701 $\times 10^{-03}$
6	$\mathbf{2^7}$	2.767033 $\times 10^{-04}$	4.650564 $\times 10^{-04}$
7	$\mathbf{2^8}$	1.996172 $\times 10^{-04}$	1.177372 $\times 10^{-04}$
8	$\mathbf{2^9}$	4.983052 $\times 10^{-05}$	8.489205 $\times 10^{-05}$
9	$\mathbf{2^{10}}$	3.592118 $\times 10^{-05}$	2.102871 $\times 10^{-05}$

Table 5: MAE errors of test problem 3.

J	$N = 2^{J+1}$	M_{cp}	
		$y_1(t)$	$y_2(t)$
1	$\mathbf{2^2}$	7.361072 $\times 10^{-03}$	4.143352 $\times 10^{-03}$
2	$\mathbf{2^3}$	3.882518 $\times 10^{-03}$	3.978997 $\times 10^{-03}$
3	$\mathbf{2^4}$	1.954836 $\times 10^{-03}$	2.289060 $\times 10^{-03}$
4	$\mathbf{2^5}$	9.057392 $\times 10^{-04}$	1.109005 $\times 10^{-03}$
5	$\mathbf{2^6}$	4.011010 $\times 10^{-04}$	5.009045 $\times 10^{-04}$
6	$\mathbf{2^7}$	1.734762 $\times 10^{-04}$	2.188161 $\times 10^{-04}$
7	$\mathbf{2^8}$	7.408489 $\times 10^{-05}$	9.395947 $\times 10^{-05}$
8	$\mathbf{2^9}$	3.142193 $\times 10^{-05}$	3.997497 $\times 10^{-05}$
9	$\mathbf{2^{10}}$	1.327657 $\times 10^{-05}$	1.692073 $\times 10^{-05}$

Table 6: Root mean squares errors M_{cp} of test problem 3.

5. Conclusion

The article discussed the effectiveness and accuracy of the HWCM for solving a system of FIDEs. The study used numerical results to validate the efficiency of the

HWCM and demonstrate its potential applicability in various scientific and engineering fields that involve fractional calculus. The article concluded that HWCM is an effective and accurate technique for solving a system of FIDEs. This means that the method can produce accurate solutions to FIDEs, which is a significant problem in many scientific and engineering fields. The study also suggests that the HWCM can be used as a viable alternative to solve complex FIDEs containing multiple variables. The efficiency of the method was validated on the basis of numerical results obtained through simulations. The simulations were conducted on various FIDEs with different complexity levels and the results showed that the HWCM could produce accurate solutions in all cases. This implies that the HWCM can be relied on to provide accurate solutions to FIDEs regardless of their complexity level. We can extend the proposed method for solution of nonlinear system of FIDEs and higher order FIDEs. In summary, the article concludes that the Haar wavelet method HWCM is an effective and accurate technique for solving a system of FIDEs. The numerical results validate its efficiency and demonstrate its potential applicability in various fields of science and engineering that involve fractional calculus. The study also indicates that the HWCM can be considered as a viable alternative for solving complex FIDEs that contain multiple variables.

Competing interests

No Competing interests.

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