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SOME PROPERTIES OF CENTER MANIFOLDS OF DIFFERENTIAL SYSTEMS

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Abstract We present some symmetrical properties of center manifolds of differential systems under certain symmetrical conditions. These properties are fundamental to study local behavior of orbits, including stability of singular points, bifurcation of periodic solutions and homoclinic orbits of the reduced equations.

Keywords Center manifold, symmetrical property, invariance.

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1. Introduction on center manifolds

As we know, the theory of center manifolds plays an important role in the study of differential systems. It is a valid tool to reduce the dimension of the phase space. See [1–8] for general theory of center manifolds and applications of the theory. For a given system, using its center manifold we can obtain a reduced system under certain conditions. If we know more information about the center manifold we can find more properties of the reduced system and therefore make a more precise analysis on local behavior of orbits. In this section we list two well-known theorems on the existence and uniqueness of center manifolds.

Consider a differential system of the form

$$\dot{x} = Ax + f(x), \ x \in \mathbb{R}^n, \tag{1.1}$$

where $n \geq 1$, A is an $n \times n$ matrix, and $f \in C^k(\mathbb{R}^n)$ for some $k \geq 1$ with

$$f(0) = 0, \quad Df(0) = 0. \tag{1.2}$$

Before stating center manifold theorems, we first introduce some notations. Let

$$x = (x_1, x_2, \cdots, x_n), \ \|x\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}}, \ f(x) = (f_1(x), f_2(x), \cdots, f_n(x)).$$

Define

$$\|f\|_k = \max_{\substack{0 \leq j \leq k \\ 1 \leq i, \, l \leq n}} \sup_{x \in \mathbb{R}^n} \Big| \frac{\partial^j f_i(x)}{\partial^j x_l} \Big|$$

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and

$$C_b^k(\mathbb{R}^n) = \{f \in C^k(\mathbb{R}^n) \,|\, \|f\|_k < \infty\}.$$

Also, let

$$\|Df\|_0 = \max_{1\le i,\, l\le n} \sup_{x\in\mathbb{R}^n} \Big|\frac{\partial f_i(x)}{\partial x_l}\Big|.$$

To state center manifold theorems clearly, we let further

$$Ax = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad u \in \mathbb{R}^{n_1}, \quad v \in \mathbb{R}^{n_2}, \tag{1.3}$$

where $n = n_1 + n_2$, A_1 is an $n_1 \times n_1$ matrix with eigenvalues having zero real part and A_2 an $n_2 \times n_2$ matrix with each eigenvalue having nonzero real part. Then by Theorem 1.1 and Theorem 2.1 in Chapter one of [1], we have the following theorem which is called the global center manifold theorem.

Theorem 1.1. *Suppose that* (1.2) *and* (1.3) *hold. Let* $f \in C_b^k(\mathbb{R}^n)$, $k \ge 1$. *Then there is a number* $\delta_k > 0$ *such that if* $\|Df\|_0 < \delta_k$ *then* (1.1) *has a unique global center manifold* W^c *of class* C^k *which is invariant and has the form*

$$W^c = \{x = (u, v) \in \mathbb{R}^n \,|\, v = \psi(u),\, u \in \mathbb{R}^{n_1}\}, \tag{1.4}$$

where

$$\psi \in C^k,\ \psi(0) = 0,\ D\psi(0) = 0,\ Lip(\psi) < 1 \text{ and } \sup_{u\in\mathbb{R}^{n_1}} \|\psi(u)\| < \infty. \tag{1.5}$$

We remark that the uniqueness of W^c of the form (1.4) means that the function ψ satisfying (1.5) is unique.

By Theorem 3.2 of Chapter one in [1], we have the following local manifold theorem.

Theorem 1.2. *Let* (1.2) *and* (1.3) *hold. Suppose* $f \in C^k(\mathbb{R}^n)$ *with* $k \ge 1$. *Then* (1.1) *has a local* C^k *center manifold* W_l^c *of the form*

$$W_l^c = \{x = (u, v) \,|\, v = \psi(u),\, u \in V\}, \tag{1.6}$$

where V *is an open neighborhood of the origin in* $\mathbb{R}^{n_1}$, ψ *is a* C^k *function on* V, *and*

$$\psi(0) = 0, \quad D\psi(0) = 0. \tag{1.7}$$

As we know (see [1]), Theorem 1.2 can be obtained by applying Theorem 1.1 to a system of the form

$$\dot{x} = Ax + f(x)\varphi\Big(\frac{x}{\rho}\Big), \quad x \in \mathbb{R}^n,$$

where $\rho > 0$ is a small constant, φ is a C^∞ cut-off function from $\mathbb{R}^n$ to $[0, 1]$ satisfying

$$\varphi(x) = 1\,(= 0,\ \text{resp.}) \text{ for } \|x\| \le 1\,(\|x\| \ge 2,\ \text{resp.}).$$

In section 2 we present some results on symmetrical properties of center manifolds and the reduced systems under certain symmetrical conditions supposed for a given system of the form (1.1).

2. Main results and proof

Consider system (1.1). Let (1.2) and (1.3) hold. Then (1.1) can be rewritten as

$$\begin{aligned}\dot{u} &= A_1 u + \tilde{f}(u,v),\\ \dot{v} &= A_2 v + \tilde{g}(u,v),\end{aligned} \quad (u,v) \in \mathbb{R}^n.$$

First, we have

Theorem 2.1. *Consider system* (1.1)*, satisfying* (1.2) *and* (1.3)*. Then we have*

(a) If

$$\begin{aligned}\tilde{f}(-u,v) &= -\tilde{f}(u,v),\\ \tilde{g}(-u,v) &= \tilde{g}(u,v)\end{aligned} \tag{2.1}$$

for $(u,v) \in \mathbb{R}^n$*, then* (1.1) *has a local* C^k *center manifold* W_l^c *of the form* (1.6)*, where* ψ *is a* C^k *function on* V *satisfying* (1.7) *and*

$$\psi(-u) = \psi(u) \text{ for } \pm u \in V. \tag{2.2}$$

(b) If $f(-x) = -f(x)$ *or equivalently*

$$\begin{aligned}\tilde{f}(-u,-v) &= -\tilde{f}(u,v),\\ \tilde{g}(-u,-v) &= -\tilde{g}(u,v),\end{aligned} \tag{2.3}$$

then (1.1) *has a local* C^k *center manifold* W_l^c *of the form* (1.6)*, where* ψ *is a* C^k *function on* V *satisfying* (1.7) *and*

$$\psi(-u) = -\psi(u) \text{ for } \pm u \in V. \tag{2.4}$$

Proof. Let $\varphi_0 : (-\infty, +\infty) \to [0,1]$ be a C^∞ function satisfying $\varphi_0(x) = 1$ ($=0$, resp.) for $|x| \le 1$ ($|x| \ge 2$, resp.). Based on φ_0 we introduce a C^∞ function φ on $\mathbb{R}^n$ as follows

$$\varphi(x) = \varphi_0(\|x\|), \quad x \in \mathbb{R}^n, \tag{2.5}$$

where $\|x\|$ is the Euclidan norm of x.

Consider a system of the form

$$\dot{x} = Ax + f_\rho(x), \quad x \in \mathbb{R}^n, \tag{2.6}$$

where

$$f_\rho(x) = f(x)\varphi\Big(\frac{x}{\rho}\Big), \quad x \in \mathbb{R}^n, \tag{2.7}$$

and $\rho > 0$ is a constant.

By Lemma 3.1 in [1] and (1.2) we have $f_\rho \in C_b^k(\mathbb{R}^n)$ and $\|Df_\rho\|_0 < \delta_k$ as $\rho > 0$ is sufficiently small, where δ_k is the constant in Theorem 1.1. Then by Theorem 1.1, (2.6) has a unique global center manifold W^c of the form (1.4), where ψ satisfies (1.5).

Let

$$f_\rho(x) = (\tilde{f}_\rho(u,v), \tilde{g}_\rho(u,v)), \quad (u,v) \in \mathbb{R}^n,$$

where

$$\tilde{f}_\rho(u,v) = \tilde{f}(u,v)\varphi\Big(\frac{x}{\rho}\Big),\ \tilde{g}_\rho(u,v) = \tilde{g}(u,v)\varphi\Big(\frac{x}{\rho}\Big).$$

Note that (2.6) can be rewritten as

$$\begin{aligned}\dot{u} &= A_1 u + \tilde{f}_\rho(u,v),\\ \dot{v} &= A_2 v + \tilde{g}_\rho(u,v).\end{aligned} \tag{2.8}$$

If (2.1) is satisfied, then by (2.5) and (2.7)

$$\tilde{f}_\rho(-u,v) = -\tilde{f}_\rho(u,v),\ \tilde{g}_\rho(-u,v) = \tilde{g}(u,v)$$

for all $(u,v) \in \mathbb{R}^n$, which implies that (2.8) is invariant under the change $(u,v) \to (-u,v)$.

Note that the manifold W^c becomes

$$\tilde{W}^c = \{x = (u,v) \in \mathbb{R}^n \mid v = \psi(-u),\ u \in \mathbb{R}^{n_1}\}$$

under the change $(u,v) \to (-u,v)$. On the other hand, $\tilde{W}^c$ is also a global center manifold of (2.8). Thus, the uniqueness of global center manifold implies that

$$\psi(u) = \psi(-u) \text{ for } u \in \mathbb{R}^{n_1}.$$

Obviously, there exists $\varepsilon_0 = \varepsilon_0(\rho) > 0$ such that $\|(u,\psi(u))\| < \rho$ for $|u| < \varepsilon_0$. Then we can take

$$W_l^c = \{(u,v) \mid v = \psi(u),\ |u| < \varepsilon_0\},$$

and the conclusion (a) follows. The conclusion (b) can be obtained in the same way since under (2.3) the system (2.8) is invariant under the change $(u,v) \to (-u,-v)$. □

As we know, the flow of (1.1) on the manifold W_l^c is determined by the following reduced system

$$\dot{u} = A_1 u + \tilde{f}(u,\psi(u)) \equiv f_l(u),\ |u| < \varepsilon_0. \tag{2.9}$$

If (2.1) or (2.3) holds, then by (2.2) and (2.4) we have $f_l(-u) = -f_l(u)$ for $|u|$ small. Thus, (2.9) is centrally symmetric with respect to the origin.

By the invariance of W_l^c we have for $|u|$ small

$$A_2\psi(u) + \tilde{g}(u,\psi(u)) = \psi'(u)[A_1 u + \tilde{f}(u,\psi(u))],$$

which can be used to compute expansions of $\psi(u)$ at $u = 0$.

For example, by Theorem 2.1, the system

$$\dot{x} = x^3 + 2xy, \quad \dot{y} = y + x^2$$

has a local center manifold of the form

$$y = -x^2 + 2x^4 + O(x^6) = \psi(x).$$

The corresponding reduced system is

$$\dot{x} = x^3 + 2x\psi(x) = -x^3 + O(x^5).$$

Similarly, the system

$$\begin{aligned}\dot{x} &= y + xz,\\ \dot{y} &= -x + yz,\\ \dot{z} &= -z + xy + x^2\end{aligned}$$

has a local center manifold of the form

$$z = \frac{1}{5}(4x^2 - xy + y^2) + O(|x, y|^4),$$

which is even in (x, y).

Now we generalize Theorem 2.1 to a more general case by a similar proof.

Let further

$$A_1 u = \begin{pmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_j \in \mathbb{R}^{m_j}, \quad j = 1, 2, \tag{2.10}$$

where $m_1 + m_2 = n_1$, $u = (u_1, u_2)^T$. Then we can rewrite (1.1) as

$$\begin{aligned} \dot{u_1} &= B_1 u_1 + \tilde{f}_1(u_1, u_2, v), \\ \dot{u_2} &= B_2 u_2 + \tilde{f}_2(u_1, u_2, v), \\ \dot{v} &= A_2 v + \tilde{g}(u_1, u_2, v), \end{aligned} \tag{2.11}$$

where $(u_1, u_2, v) \in G \subset \mathbb{R}^n$ with G the domain of (1.1). By a very similar way to the proof of Theorem 2.1 we can prove the following theorem.

Theorem 2.2. *Consider system* (1.1)*, where the function f is defined on an open set G of the form*

$$G = \{x \in \mathbb{R}^n \mid \|x\| < \varepsilon_0\},\ \varepsilon_0 > 0.$$

Let (1.2), (1.3) *and* (2.10) *hold.*
(a) If (2.11) *satisfies that*

$$\begin{aligned} \tilde{f}_1(-u_1, u_2, v) &= -\tilde{f}_1(u_1, u_2, v), \\ \tilde{f}_2(-u_1, u_2, v) &= \tilde{f}_2(u_1, u_2, v), \\ \tilde{g}(-u_1, u_2, v) &= \tilde{g}(u_1, u_2, v) \end{aligned}$$

for $(u_1, u_2, v) \in G$, then (1.1) *has a local C^k center manifold W_l^c of the form* (1.6)*, where ψ is a C^k function on V satisfying* (1.7) *and*

$$\psi(-u_1, u_2) = \psi(u_1, u_2) \text{ for } (\pm u_1, u_2) \in V.$$

(b) If (2.11) *satisfies that*

$$\begin{aligned} \tilde{f}_1(-u_1, u_2, -v) &= -\tilde{f}_1(u_1, u_2, v), \\ \tilde{f}_2(-u_1, u_2, -v) &= \tilde{f}_2(u_1, u_2, v), \\ \tilde{g}(-u_1, u_2, -v) &= -\tilde{g}(u_1, u_2, v) \end{aligned}$$

for $(u_1, u_2, v) \in G$, then (1.1) *has a local C^k center manifold W_l^c of the form* (1.6)*, where ψ is a C^k function on V satisfying* (1.7) *and*

$$\psi(-u_1, u_2) = -\psi(u_1, u_2) \text{ for } (\pm u_1, u_2) \in V.$$

Under the conditions of the above theorem, instead of (2.7) we define the function f_ρ in (2.6) as

$$f_\rho(x) = \begin{cases} f(x)\varphi(\frac{x}{\rho}), & \|x\| < \varepsilon_0, \\ 0, & \|x\| \geq \varepsilon_0. \end{cases}$$

Then as before, when $\rho \in (0, \varepsilon_0/2)$ is sufficiently small, we have

$$f_\rho \in C_b^k(\mathbb{R}^n), \ \|Df_\rho\|_0 < \delta_k.$$

Hence, Theorem 2.2 can be obtained by applying Theorem 1.1 in the same way as Theorem 2.1.

The above theorem can be used to study center manifolds of differential systems with parameters. For simplicity, consider a three dimensional system of the form

$$\begin{aligned} \dot{x} &= y + f_1(x, y, z, \varepsilon), \\ \dot{y} &= -a_n x^{2n+1} + f_2(x, y, z, \varepsilon), \\ \dot{z} &= \lambda z + f_3(x, y, z, \varepsilon), \end{aligned} \tag{2.12}$$

where $\lambda \neq 0$ is a constant, $a_n = 1\,(\neq 0)$ for $n = 0\,(n \geq 1)$, f_1, f_2 and f_3 are C^∞ functions for $(x, y, z, \varepsilon) \in G \times U$ with $G \subset \mathbb{R}^3$ containing the origin in $\mathbb{R}^3$ and $U \subset \mathbb{R}^m$ a neighborhood of $\varepsilon = 0$ in $\mathbb{R}^m$ for some $m \geq 1$. Further, suppose

$$f_j(x, y, z, 0) = O(|x, y, z|^2), \quad j = 1, 2, 3. \tag{2.13}$$

By adding the equation $\dot{\varepsilon} = 0$ to (2.12), taking $u_1 = (x, y)$, $u_2 = \varepsilon$, $v = z$ and applying Theorem 2.2 to the resulting system, we obtain immediately

Theorem 2.3. *Let* (2.12) *satisfy* (2.13).
(a) If

$$\begin{aligned} f_1(-x, -y, z, \varepsilon) &= -f_1(x, y, z, \varepsilon), \\ f_2(-x, -y, z, \varepsilon) &= -f_2(x, y, z, \varepsilon), \\ f_3(-x, -y, z, \varepsilon) &= f_3(x, y, z, \varepsilon) \end{aligned} \tag{2.14}$$

for $|x| + |y| + |z| + |\varepsilon|$ *small, then* (2.12) *has a local* C^k *center manifold* W_ε^c *of the form*

$$W_\varepsilon^c = \{(x, y, z) \mid z = \psi(x, y, \varepsilon),\ x^2 + y^2 < \delta\} \tag{2.15}$$

for $|\varepsilon| < \delta$ *with* $\delta > 0$ *a small constant, where* $\psi \in C^k$ *and satisfies*

$$\psi(x, y, \varepsilon) = O(|\varepsilon| + |x, y|^2), \quad \psi(-x, -y, \varepsilon) = \psi(x, y, \varepsilon).$$

(b) If

$$f_j(-x, -y, -z, \varepsilon) = -f_j(x, y, z, \varepsilon), \quad j = 1, 2, 3 \tag{2.16}$$

for $|x| + |y| + |z| + |\varepsilon|$ *small, then* (2.12) *has a local* C^k *center manifold* W_ε^c *of the form* (2.15) *for* $|\varepsilon| < \delta$ *with* $\delta > 0$ *a small constant, where* $\psi \in C^k$ *and satisfies*

$$\psi(x, y, \varepsilon) = O(|\varepsilon||x, y| + |x, y|^3), \quad \psi(-x, -y, \varepsilon) = -\psi(x, y, \varepsilon).$$

(c) Under (2.14) *or* (2.16) *the reduced system of* (2.12) *is of the form*

$$\begin{aligned} \dot{x} &= y + f_1(x, y, \psi(x, y, \varepsilon), \varepsilon), \\ \dot{y} &= -a_n x^{2n+1} + f_2(x, y, \psi(x, y, \varepsilon), \varepsilon) \end{aligned}$$

and is centrally symmetric with respect to the origin.

In fact, when (2.16) holds, one has

$$f_j(0,0,0,\varepsilon) = 0, \quad j = 1,2,3,$$

which ensure $\psi(0,0,\varepsilon) = 0$ since the singular point at the origin must lie on the local center manifold W_ε^c. When (2.14) holds, system (2.12) has a singular point $(0,0,z_0(\varepsilon))$ near the origin. Then one can make a change of variables $(x,y,z) \to (x,y,z-z_0(\varepsilon))$ to move the singular point to the origin before applying Theorem 1.1.

The third conclusion of Theorem 2.3 provides an important property of the reduced system of (2.12) which is really useful in the study of limit cycle bifurcation near the origin for (2.12).

Finally we remark that the function ψ in (2.15) satisfies the equation

$$\lambda\psi + f_3(x,y,\psi,\varepsilon) = \psi_x(y + f_1(x,y,\psi,\varepsilon)) + \psi_y(-a_n x^{2n+1} + f_2(x,y,\psi,\varepsilon)),$$

which can be used to compute expansions of ψ in (x,y) near $(x,y) = (0,0)$ for $|\varepsilon|$ small. In particular, it follows $\psi(0,0,\varepsilon) = 0$ if $f_j(0,0,0,\varepsilon) = 0$ for $j = 1,2,3$.

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