Multiple positive solutions of fractional Schrödinger-Poisson system with strong singularities and double critical exponents

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Abstract

This research focuses on analyzing a specific type of fractional Schrödinger-Poisson system that contains strong singular terms and double critical exponents. By employing the critical point theory for nonsmooth functionals and mountain pass theorem, it is demonstrated that there are two positive solutions to this system.
Keywords: strong singularity, double critical exponents, critical point theory, nonsmooth functionals.
Mathematics Subject Classification: 35J10, 35J20, 35J60.

1 Introduction and main results

Consider the multiplicity of positive solutions to the following fractional Schrödinger-Poisson system with strong singularities and double critical exponents

$$\begin{cases} (-\Delta)^s u - \phi|u|^{2_s^*-3}u = |u|^{2_s^*-2}u + \dfrac{\lambda}{u}, & \text{in } \Omega, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a domain that has a well-defined smooth boundary, $s \in (0,1)$, $N > 2s$, $2_s^* = \frac{2N}{N-2s}$, $\lambda > 0$.

*Corresponding author. Email address: jiafengzhang@163.com. Supported by the National Natural Science Foundation of China (No.12461024); the Natural Science Research Project of Department of Education of Guizhou Province (No. QJJ2023012, QJJ2023061, QJJ2023062); the Natural Science Research Project of Guizhou Minzu University (No. GZMUZK[2022]YB06)

Above system comes from the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta u = K(u)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V,\ K,\ f$ satisfy some suitable assumptions. The Schrödinger-Poisson system is a fundamental equation in quantum mechanics introduced by the Austrian physicist Schrödinger in 1926. The equation is a second-order partial differential equation that combines the concept of matter waves with wave equations. It is used to describe the movement of microscopic particles. The system has strong physical implications and is widely used in quantum mechanical models and semiconductor theory, see [1, 2, 3, 4, 5] and references therein for more physical background.

The following Schrödinger-Poisson system with singularities has been extensively investigated,

$$\begin{cases} -\Delta u + \eta\phi u = f(x)u^{-\gamma} + g(x,u), & x \in \Omega, \\ -\Delta\phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

where $\gamma > 0,\ \eta = \pm 1,\ f,\ g$ satisfy some suitable assumptions. When $\eta = 1,\ f(x) = \mu > 0,\ g(x,u) = 0$, Zhang in [6] considered the system (1.2), it is proved that (1.2) has a unique positive solution for any $\mu > 0$ by the variational method. When $\eta = -1,\ f(x) = \mu > 0,\ g(x,u) = 0$, it is proved by using Nehari method that there are two positive solutions for any $\mu > 0$. When $\gamma > 1,\ \eta = 1,\ g(x,u) = 0$, Yu in [8] considered a class of Schrödinger-Poisson systems with strong singularities, by employed variational method to establish sufficient requirements for the existence and uniqueness of positive weak solutions for system (1.2). For more exciting results, see [9, 10, 11] and the references therein.

Recently, the existence and multiplicity of nontrivial and ground state solutions of nonlocal elliptic equations have been received attention

$$\begin{cases} -\Delta u + \phi u = \mu|u|^{p-2}u + f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\mu > 0,\ p \in (2,6),\ f$ satisfy some assumptions. When $\mu = -\alpha,\ \alpha > 0,\ p \in (2, \frac{12}{5})$ and f satisfies some assumptions, Wang in [35] studied the existence and nonexistence of nontrivial solution of a class of Schrödinger-Poisson systems with zero mass potential. By the Nehari-Pohozaev method and variational method, it was proved that there is a nontrivial solution to the system (1.3). When $\alpha = 1,\ f(u) = |u|^{q-2}u,\ q \in (p,6)$, necessary and sufficient condition for the existence of nontrivial radial solutions was established. When $\phi(x) = \frac{1}{4\pi|x|} * |u|^2,\ f(u) = |u|u,\ \mu > 0,\ 3 < p < 6$, Lei in [36] studied the existence of a ground state solution of the Schrödinger-Poisson-Slater type with Coulomb-Sobolev critical growth, and obtained a ground state solution of the equation by the Nehari-Pohozaev method and compactness arguments.

Godoy [37] studied the existence and uniqueness of weak solution to the following singular elliptic problem with mixed boundary conditions,

$$\begin{cases} -\Delta u = g(\cdot\,, u), & \text{in } \Omega, \\ u = \tau, & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial v} = \eta, & \text{on } \Gamma_2, \\ u > 0, & \text{in } \Omega, \end{cases} \tag{1.4}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^n$ such that $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$, Γ_1 and Γ_2 are disjoint closed subsets of $\partial\Omega$, $0 \leq \tau \in W^{\frac{1}{2},2}(\Gamma_1)$, $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$ and $g : \Omega \times (0, \infty) \to \mathbb{R}$ is a nonnegative Carathéodory funcation. The existence and uniqueness of a positive weak solution to the problem were proved using the sub-supersolution theorem and the Hopf boundary lemma.

In recent years, many scholars have focused on the following fractional Schrödinger-Poisson systems

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda\phi u = h(x, u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $0 < s \leq t < 1$, $\lambda > 0$ is a real parameter, and V, h satisfy some assumptions. System (1.5) has a strong physical background and it is one of the main objects of fractional quantum mechanics. It appears in some fields such as nonlinear optics, plasma physics and condensed matter physics. Among others, the fractional Laplace operator describes various phenomena in applied sciences, such as flame propagation, free boundary barrier problem, or Hamilton-Jacobi equation with critical fractional diffusion, see [29, 30, 31]. For work on nonlinear systems of fractional Laplace operators, they also appear in problems of fractional quantum mechanics, physics, chemistry, optimization, etc, see [32, 33, 34] and references therein for more physical background. Consequently, solvability and multiplicity of fractional Schrödinger-Poisson systems have received much attention, see literature [12, 13, 14, 15, 16] and references therein.

Guo [38] considered the following mixed order conformally invariant system with Hartree-type nonlinearity,

$$\begin{cases} (-\Delta)^s u(x) = \Big(\dfrac{1}{|x|^\sigma} * v^{\frac{2N-\sigma}{N-2}}\Big) v^{\frac{N+2s-\sigma}{N-2}(x)}, & \text{in } \mathbb{R}^N, \\ (-\Delta) v(x) = u^{\frac{N+2}{N-2s}}(x), & \text{in } \mathbb{R}^N, \\ u \geq 0, v \geq 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $0 < \alpha \leq 2$, $N \geq 3$, $\sigma \in (0, N)$, $0 < s =: m + \frac{\alpha}{2} < +\infty$, m is a integer. They obtained the new results by classifying all the nonnegative nontrivial classical solutions of the system. Firstly, the equivalence of PDEs system and IEs system was proved. Then, using the method of moving sphere, the classification of the nonnegative solutions of the system (1.6) was given. When $\frac{N}{2} < s =: m + \frac{\alpha}{2} < +\infty$, the Liouville-type theorems results of the system (1.6) were proved for the critical and supercritical-orders respectively.

Meanwhile, fractional Schrödinger-Poisson systems with singular terms

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda\phi u = \mu f(x)u^{-\gamma} + h(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases} \tag{1.7}$$

have been received attention from many authors, where $0 < s \le t < 1,\ \mu > 0,\ 0 < \gamma < 1,\ \lambda > 0$ is a real parameter, and $V,\ f,\ h$ satisfy certain presumptions. Yu [17] studied system (1.7) with $0 < \gamma < 1$ on condition that $\lambda > 0,\ \mu = 1,\ h(x,u) = 0$ and $0 < s \le t < 1,\ 4s + 2t > 3,\ f$ and V satisfy the following assumptions:

(V_1): $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) > V_0 > 0$, where V_0 is a constant.

(V_2): meas $\{x \in \mathbb{R}^3 : -\infty < V(x) \le h\} < +\infty$ for all $h \in \mathbb{R}$.

(f_1): $f \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ is a nonnegative function.

The variational method is used to verify the existence, monotonicity, and uniqueness of the positive solution. When $0 < \gamma < 1,\ V(x) = 0,\ \lambda = 1,\ \mu > 0,\ h(x,u) = g(x)u^{2^*_\alpha - 2}u,\ s = \alpha,\ t = \frac{s}{2},\ s \in (0,N),\ \alpha \in (0,1)$ and $2\alpha < N < 4\alpha$, f and g satisfy the following assumptions:

(H_1): $f(x), g(x) \ge 0$ on Ω.

(H_2): There exist $z \in \Omega$ and $\rho > N$, such that $g(z) = \max_{x\in\Omega} g(x) = 1$ and $g(z) - g(x) = O(|z - x|^\rho)$.

Fan [18] investigated the existence and multiplicity of positive solutions in system (1.7) using Nehari method.

Lei [7] looked into the following nonlocalization Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \phi u^{2^*-2} = \dfrac{\lambda}{u^\gamma}, & \text{in } \Omega, \\ -\Delta\phi = u^{2^*-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N (N \ge 3)$, $\gamma \in (0,1),\ 2^* = \frac{2N}{N-2}$ and $\lambda > 0$ is a numerical value that belongs to the set of real numbers. Two positive solutions of the system (1.8) were obtained by the variational and perturbation methods.

Recently, the following fractional Schrödinger-Poisson system with double critical exponents has also been studied

$$\begin{cases} (-\Delta)^s u + V(x)u - \phi|u|^{2^*_s-3}u = |u|^{2^*_s-2}u + \lambda h(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^s\phi = |u|^{2^*_s-1}, & x \in \mathbb{R}^3, \end{cases} \tag{1.9}$$

where $s \in (0,1),\ 2^*_s = \frac{6}{3-2s},\ \lambda > 0$ is a real parameter, h and V satisfy some suitable hypothesis. When $s \in (0,1),\ \lambda > 0,\ h(x,u) = f(x,u),\ f$ and V satisfy some suitable hypothesis, Jiang and Liao [19] proved that the system (1.9) has at least two positive solutions by applying the variational methods and mountain pass theorem. When

$s \in (\frac{3}{4}, 1)$, $\lambda = 1$, $h(x,u) = g(u)$, g, V satisfy some suitable hypothesis, He [20] studied system (1.9) and obtained a positive solution for this system by the variational method. More results are in [21, 22] and references therein.

This paper is particularly inspired by [23] and [19]. Up to now, there are no research results on system (1.1). The difficult aspect of this paper is that the strong singularities lead to the non-integrability of the functional and the double critical terms lead to the lack of compactness. Therefore, it is difficult to find a solution to system (1.1). The non-integrability of the singular terms is solved by utilizing auxiliary equation and the critical point theory for nonsmooth functionals, and the lack of compactness is solved by utilizing the Brézis-Lieb Lemma. The proof establishes the existence and multiplicity of positive solutions for a specific category of fractional Schrödinger-Poisson systems that contain strong singular terms and possess double critical exponents.

The primary outcomes of our study are as follows.

Theorem 1.1. *Assume that $N \geq 3$, $\lambda > 0$ and $s \in (0,1)$, then there exists $\lambda_* > 0$ such that for $0 < \lambda < \lambda_*$, system (1.1) has at least two positive solutions.*

Remark 1.2. As far as we know, system (1.1) has not been researched thus far. In contrast to [23], we extend to fractional Schrödinger-Poisson systems with double critical exponents. In contrast to [19], we consider the case with a strong singularities terms in a bounded domain.

2 Variational framework

We provide the variational framework for the system (1.1). Let $L^P(\Omega)$ be the standard Lebesgue space with the norm $|u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$. For each $s \in (0,1)$, $D^{s,2}(\Omega)$ is the completion of the set $C_0^\infty(\Omega)$. This set consists of infinitely differentiable functions $u : \mathbb{R}^N \to \mathbb{R}$ with compact support, and the completeness is defined with respect to the following norm:

$$[u]_s^2 = \iint_E \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy,$$

where $E = (\mathbb{R}^N \times \mathbb{R}^N \setminus C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \setminus \Omega$. Equivalently, $\{D^{s,2}(\Omega) = u \in L^{2_s^*}(\Omega) : [u]_s < \infty\}$. $\|u\|_{H^s} = \sqrt{|u|_2^2 + [u]_s^2}$ is a norm of the fractional space $H^s(\Omega) = \{u \in L^2(\Omega) : [u]_s < \infty\}$. We shall be operating within the specified fractional Sobolev space.

$$H = \{u \in H^s(\Omega) \text{ with } u = 0 \text{ a.e in } \mathbb{R}^N \setminus \Omega\}.$$

Then, the norm of H is defined as $\|u\|^2 = [u]_s^2$. It is widely recognized that the function space $D^{s,2}(\Omega)$ is continuously embedded into $L^{2_s^*}(\Omega)$, and there exists the best Sobolev constant $S > 0$ such that

$$S = \inf_{u \in D^{s,2}(\Omega) \setminus \{0\}} \frac{\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx}{(\int_\Omega |u|^{2_s^*} dx)^{\frac{2}{2_s^*}}}. \tag{2.1}$$

The uniqueness of the solution $\phi = \phi_u \in D^{s,2}(\Omega)$ for the second equation of system (1.1) is established using the Lax-Milgram theorem.

The following conclusions can be drawn from the [27] and [28].

Lemma 2.1. *The solution ϕ_u satisfies the following properties:*
(i) $\phi_u \geq 0$ for all $u \in H$;
(ii) $\phi_{tu} = t^{2_s^*-1}\phi_u$ for all $t > 0$ and $u \in H$;
(iii) For each $u \in H$, one has $\|\phi_u\|_{D^{s,2}} \leq S^{-\frac{1}{2}}|u|_{2_s^*}^{2_s^*-1}$ and $\int_\Omega \phi_u|u|^{2_s^*-1}dx \leq S^{-1}|u|_{2_s^*}^{2(2_s^*-1)}$;
(iv) if $\{u_n\} \subset H$ and $u \in H$ are such that $u_n \rightharpoonup u$ in H and $u_n(x) \to u(x)$ a.e. in Ω as $n \to \infty$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\Omega)$. Moreover,

$$\int_\Omega \phi_{u_n}|u_n|^{2_s^*-1}dx - \int_\Omega \phi_{u_n-u}|u_n - u|^{2_s^*-1}dx = \int_\Omega \phi_u|u|^{2_s^*-1}dx + o_n(1).$$

The system (1.1) can be converted into the following fractional Schrödinger equation,

$$\begin{cases} (-\Delta)^s u - \phi_u|u|^{2_s^*-3}u = |u|^{2_s^*-2}u + \dfrac{\lambda}{u}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

The energy functional corresponding to problem (2.2) is precisely defined as

$$I(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2dx - \frac{1}{2(2_s^*-1)}\int_\Omega \phi_u|u|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}dx - \lambda\int_\Omega ln|u|dx$$

for $u \in H$. The singular term is known to cause I to not be differentiable on H. Hence, determining the local minimizer and mountain pass type solutions of issue (2.2) is a challenging task. We take into consideration the following problem in order to first discover a local minimizer solution,

$$\begin{cases} (-\Delta)^s u = \dfrac{\lambda}{u}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

According to Corollary 3.3 of [24], we know that problem (2.3) has a unique positive solution ω_λ with $\omega_\lambda \geq c\phi_1$ (where ϕ_1 is a specific function that corresponds to the lowest eigenvalue, denoted as λ_1, of the problem $(-\Delta)^s\phi + \lambda\phi = 0, \phi|_{\partial\Omega}$). Furthermore, if t belongs to the interval $(0, 1)$, they have demonstrated that

$$\int_\Omega \frac{1}{\phi_1^t}dx < +\infty. \tag{2.4}$$

To analyze problem (2.2), we establish the function $f : \Omega \times \mathbb{R} \to [0, +\infty)$ as

$$f(x,t) = \begin{cases} \dfrac{1}{t}, & \text{if } x \in \Omega \text{ and } t > \omega_\lambda(x), \\ \frac{1}{\omega_\lambda}, & \text{if } x \in \Omega \text{ and } t \leq \omega_\lambda(x). \end{cases}$$

Take into account the following auxiliary problem

$$\begin{cases} (-\Delta)^s u - \phi_u |u|^{2_s^*-3}u = |u|^{2_s^*-2}u + \lambda f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

The functional of problem (2.5) has the following variational structure

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)}\int_\Omega \phi_u |u|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}dx - \lambda\int_\Omega F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,t)dt$. A solution of (2.5) refers to a function u belonging to the set H, such that for each ϕ in H that satisfies

$$\int_\Omega (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi dx - \int_\Omega \phi_u |u|^{2_s^*-3}u\varphi dx - \int_\Omega |u|^{2_s^*-2}u\varphi dx - \lambda\int_\Omega f(x,u)\varphi dx = 0. \tag{2.6}$$

Therefore, we use the theory of critical points of nonsmooth functional to find the critical point of J and thus prove the existence of a positive solution to problem (2.5).

3 Some Relevant Lemmas

Lemma 3.1. *Given that $\{u_n\}$ is bounded in H and $u_n \rightharpoonup u$ in H, we can conclude that*

$$\lim_{n\to\infty}\int_\Omega F(x,u_n)dx = \int_\Omega F(x,u)dx. \tag{3.1}$$

Proof. Since $F(x,u) = \int_0^u f(x,t)dt \le 1$ one has $f(x,u) = \frac{1}{\omega_\lambda}$ when $u < \omega_\lambda$. $f(x,u) = \frac{1}{u}$ when $u > \omega_\lambda$; see that $ln|x| \le |x|$. Then

$$\begin{aligned} F(x,u) = &\int_0^{\omega_\lambda}\frac{1}{\omega_\lambda}dx + \int_{\omega_\lambda}^u \frac{1}{t}dt \\ =&1 + lnu + 2ln\frac{1}{\sqrt{\omega_\lambda}} \\ \le&1 + u + \frac{2}{\sqrt{\omega_\lambda}}. \end{aligned}$$

Based on the facts provided above, it can be inferred

$$F(x,u) \le 1 + u + \frac{2}{\sqrt{\omega_\lambda}}, \text{ for } u \in H. \tag{3.2}$$

From [24], we have $\sqrt{\omega_\lambda} \ge c\sqrt{\phi_1}$. According to equation (2.4), it is true that

$$\int_\Omega \frac{1}{\sqrt{\omega_\lambda}}dx \le c\int_\Omega \frac{1}{\sqrt{\phi_1}}dx < +\infty. \tag{3.3}$$

According to the dominated convergence theorem (3.1) is true. □

First, we review some concepts of critical point theory based on nonsmooth functionals. $f : X \to \mathbb{R}$ is a continuous functional in X. Let (X, d) be a complete metric space. Indicate by $|df|(u)$ the supremun of δ in $[0, \infty)$ such that there exists a positive value r, neighborhood U of $u \in X$, and a continuous map $\sigma : U \times [0, r]$ satisfying

$$\begin{cases} f(\sigma(v,t)) \leq f(v) - \delta t, & (v,t) \in U \times [o,r], \\ d(\sigma(v,t), v) \leq t, & (v,t) \in U \times [o,r]. \end{cases} \tag{3.4}$$

A sequence $\{u_n\}$ of elements in X is referred to a Palais-Smale sequence of the functional f if the absolute value of the derivative of f evaluated at u_n, denoted as $|df|(u_n)$, tends to zero as n approaches infinity, and if the values of $f(u_n)$ are bounded. A critical point of function f is defined as a point u belonging to set X such that the absolute value of the derivative of f evaluated at u is equal to zero. Given that the function $u \to |df|(u)$ is lower semicontinuous, it follows that every accumulation point of a (PS) sequence is unequivocally a crucial point of f.

Given that we are seeking a positive solution for system (1.1), we examine the functional J defined on the closed positive cone P of H,

$$P = \{u | u \in H, u(x) \geq 0, \text{ a.e. } x \in \Omega\}.$$

In this context, P refers to a metric space that is considered complete, whereas J is a continuous functional defined on P.

The following lemma is true if $|dJ|(u) < +\infty$.

Lemma 3.2. *If $|dJ|(u) < +\infty$, then for any $v \in P$, the following is true:*

$$\begin{aligned} \lambda \int_\Omega f(x,u)(v-u)dx \leq & \int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} (v-u)dx - \int_\Omega \phi_u |u|^{2_s^*-3} u(v-u)dx \\ & - \int_\Omega |u|^{2_s^*-2} u(v-u)dx + |dJ|(u)\|v-u\|. \end{aligned} \tag{3.5}$$

If u is a crucial point of J in particular. The issue (2.5) then has a weak solution in u.

Proof. The proof provided is analogous to the demonstration of Lemma 2.2 in reference [23]. Assume $\delta < \frac{1}{2}\|v-u\|$, $v \neq u$, $v \in P$ and $|dJ|(u) < c$. The mapping $\sigma : U \times [0, \delta] \to P$ is defined by the function

$$\sigma(z,t) = z + t\frac{v-z}{\|v-z\|},$$

where U represents a neighborhood of u. Then $\|\sigma(z,t) - z\| = t$. According to equation (3.4), there is a pair $(z,t) \in U \times [0, \delta]$ such that

$$J(\sigma(z,t)) > J(z) - ct.$$

Hence, we suppose that sequences $\{u_n\} \subset P$ and $\{t_n\} \subset [0, +\infty)$ such that $u_n \to u$, as $t_n \to 0^+$, and

$$J(u_n + t_n \frac{v - u_n}{\|v - u_n\|}) \geq J(u_n) - ct_n.$$

Namely,

$$J(u_n + s_n(v - u_n)) \geq J(u_n) - cs_n\|v - u_n\|, \tag{3.6}$$

where $s_n = \frac{t_n}{\|v-u_n\|} \to 0^+$ as $n \to \infty$. When dividing (3.6) by s_n, the following equation is obtained:

$$\begin{aligned}
&\lambda \int_\Omega \frac{F(x, u_n + s_n(v - u_n)) - F(x, u_n)}{s_n} dx \\
\leq& \frac{1}{2} \frac{\|u_n + s_n(v - u_n)\|^2 - \|u_n\|^2}{s_n} \\
&- \frac{1}{2(2_s^* - 1)} \int_\Omega \frac{\phi_{u_n + s_n(v-u_n)} |u_n + s_n(v - u_n)|^{2_s^*-1} - \phi_{u_n} |u_n|^{2_s^*-1}}{s_n} dx \\
&- \frac{1}{2_s^*} \int_\Omega \frac{|u_n + s_n(v - u_n)|^{2_s^*} - |u_n|^{2_s^*}}{s_n} dx + c\|v - u_n\|.
\end{aligned} \tag{3.7}$$

Suppose that

$$I_{1,n} = \int_\Omega \frac{F(x, u_n + s_n(v - u_n)) - F(x, (1 - s_n)u_n)}{s_n} dx$$

and

$$I_{2,n} = \int_\Omega \frac{F(x, (1 - s_n)u_n) - F(x, u_n)}{s_n} dx.$$

Notice that

$$I_{1,n} = \int_\Omega \frac{f(x, \xi_n)s_n v}{s_n} dx = \int_\Omega f(x, \xi_n)v dx$$

where $\xi_n \in (u_n - s_n u_n, u_n + s_n(v - u_n))$, since $\xi_n \to u,\ u_n \to u$, as $s_n \to 0^+$. Given that $F(x, t)$ is growing in t, $I_{1,n} \geq 0$ for every n. By employing Fatou's Lemma to $I_{1,n}$, we derive

$$\liminf_{n\to\infty} I_{1,n} \geq \int_\Omega f(x, u)v dx$$

for $v \in P$. By applying the differential mean value theorem to $I_{2,n}$, we obtain

$$\lim_{n\to\infty} I_{2,n} = \int_\Omega f(x, u)u dx.$$

Based on the above facts, one has

$$\lambda \int_\Omega f(x, u)(v - u)dx \leq \liminf_{n\to\infty}(I_{1,n} + I_{2,n}). \tag{3.8}$$

For the second term of (3.7), since $s_n \to 0^+$ as $n \to \infty$, we have

$$\begin{aligned}\frac{1}{2s_n}\Big(\|u_n + s_n(v-u_n)\|^2 - \|u_n\|^2\Big) =&\frac{1}{2s_n}\Big(\langle u_n + s_n(v-u_n), u_n + s_n(v-u_n)\rangle \\ &- \langle u_n, u_n\rangle\Big) \\ =&\frac{1}{2s_n}\Big[(\langle u_n, u_n\rangle + 2s_n\langle u_n, v-u_n\rangle \\ &+ s_n^2\langle v-u_n, v-u_n\rangle) - \langle u_n, u_n\rangle\Big] \\ =&\langle u_n, v-u_n\rangle + \frac{s_n}{2}\|v-u_n\|^2 \\ =&\langle u, v-u\rangle. \end{aligned} \tag{3.9}$$

On the fourth item, we need proof

$$\lim_{n\to\infty}\int_\Omega \frac{|u_n + s_n(v-u_n)|^{2_s^*} - |u_n|^{2_s^*}}{2_s^* s_n}dx = \int_\Omega |u_n|^{2_s^*-1}(v-u)dx. \tag{3.10}$$

According to the differential mean value theorem, we obtain

$$\int_\Omega \frac{|u_n + s_n(v-u_n)|^{2_s^*} - |u_n|^{2_s^*}}{2_s^* s_n}dx = \int_\Omega \frac{2_s^*\xi_{n_1}^{2_s^*-1}s_n(v-u_n)}{2_s^* s_n}dx = \int_\Omega \xi_{n_1}^{2_s^*-1}(v-u_n)dx,$$

where $\xi_{n_1} \in (u_n, u_n + s_n(v-u_n))$, since $\xi_{n_1} \to u,\ u_n \to u$ as $s_n \to 0^+$, and so

$$\lim_{n\to\infty}\int_\Omega \frac{|u_n + s_n(v-u_n)|^{2_s^*} - |u_n|^{2_s^*}}{2_s^* s_n}dx = \int_\Omega u^{2_s^*-1}(v-u)dx.$$

Therefore, combining (3.8), (3.9), (3.10) and Lemma 2.1, we get

$$\begin{aligned}\lambda\int_\Omega f(x,u)(v-u)dx \leq& \liminf_{n\to\infty}(I_{1,n} + I_{2,n}) \\ \leq& \int_\Omega (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}(v-u)dx - \int_\Omega \phi_u|u|^{2_s^*-3}u(v-u)dx \\ &- \int_\Omega |u|^{2_s^*-2}u(v-u)dx + c\|v-u\|\end{aligned}$$

for every $v \in P$. Given that $|dJ|(u) < c$ is arbitrarily chosen, (3.5) is valid. Assume that u is a critical point of the functional J. For $\varphi \in H$, $\omega > 0$, taking $v = (u + \omega\varphi)^+ \in P$, as test function in the aforementioned inequality, we may conclude that

$$
\begin{aligned}
0 \leq &\int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} [(u+\omega\varphi)^+ - u] dx - \int_\Omega \phi_u |u|^{2_s^*-3} u [(u+\omega\varphi)^+ - u] dx \\
&- \int_\Omega |u|^{2_s^*-2} [(u+\omega\varphi)^+ - u] dx - \lambda \int_\Omega f(x,u)[(u+\omega\varphi)^+ - u] dx \\
=&\omega\Big[\int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx - \int_\Omega \phi_u |u|^{2_s^*-3} u\varphi dx - \int_\Omega |u|^{2_s^*-2} u\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx\Big] \\
&- \int_{\{u+\omega\varphi<0\}} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} (u+\omega\varphi) dx + \int_{\{u+\omega\varphi<0\}} \phi_u |u|^{2_s^*-3} u(u+\omega\varphi) dx \\
&+ \int_{\{u+\omega\varphi<0\}} |u|^{2_s^*-2} u(u+\omega\varphi) dx + \lambda \int_{\{u+\omega\varphi<0\}} f(x,u)(u+\omega\varphi) dx \\
\leq&\omega\Big[\int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx - \int_\Omega \phi_u |u|^{2_s^*-3} u\varphi dx - \int_\Omega |u|^{2_s^*-2} u\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx\Big] \\
&- \omega \int_{\{u+\omega\varphi<0\}} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx.
\end{aligned}
$$

Given that $(-\Delta)^{\frac{s}{2}} u(x) = 0$ for a.e. $x \in \Omega$ with $u(x) = 0$ and $\{x \in \Omega | u(x) + \omega\varphi(x) < 0, u(x) > 0\} \to 0$ as $\omega \to 0$. We possess

$$
\int_{\{u+\omega\varphi<0\}} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx = \int_{\{u+\omega\varphi<0, u>0\}} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx \to 0 \text{ as } \omega \to 0.
$$

Therefore

$$
\begin{aligned}
0 \leq &\omega\Big[\int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx - \int_\Omega \phi_u |u|^{2_s^*-3} u\varphi dx - \int_\Omega |u|^{2_s^*-2} u\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx\Big] \\
&+ o(\omega)
\end{aligned}
$$

as $\omega \to 0$, we obtain

$$
\int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx - \int_\Omega \phi_u |u|^{2_s^*-3} u\varphi dx - \int_\Omega |u|^{2_s^*-2} u\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx \geq 0.
$$

From the arbitrary nature of the sign of φ, we can infer that

$$
\int_\Omega (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx - \int_\Omega \phi_u |u|^{2_s^*-3} u\varphi dx - \int_\Omega |u|^{2_s^*-2} u\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx = 0
$$

for every $\varphi \in H$. As so, u is a system (1.1) weak solution.

We then show, that $u \geq \omega_\lambda$ a.e. in Ω. Choosing in (2.6) $\varphi = (u - \omega_\lambda)^-$, one has

$$
\begin{aligned}
\int_{\{u<\omega_\lambda\}} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} (u-\omega_\lambda) dx = &\int_{\{u<\omega_\lambda\}} \phi_u |u|^{2_s^*-3} u(u-\omega_\lambda) dx \\
&+ \int_{\{u<\omega_\lambda\}} (|u|^{2_s^*-2} u + \frac{\lambda}{\omega_\lambda})(u-\omega_\lambda) dx. \qquad (3.11)
\end{aligned}
$$

Notice that

$$\int_{\{u<\omega_\lambda\}} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} (u-\omega_\lambda) dx = \lambda \int_{\{u<\omega_\lambda\}} \frac{u-\omega_\lambda}{\omega_\lambda} dx. \tag{3.12}$$

Therefore, it may be deduced from equations (3.11) and (3.12) that

$$\begin{aligned}\int_{\{u<\omega_\lambda\}} |(-\Delta)^{\frac{s}{2}} (u-\omega_\lambda)|^2 dx =& \int_{\{u<\omega_\lambda\}} \phi_u |u|^{2_s^*-3} u(u-\omega_\lambda) dx \\ &+ \int_{\{u<\omega_\lambda\}} |u|^{2_s^*-2} u(u-\omega_\lambda) dx \\ \leq & 0.\end{aligned}$$

It follows that $\|(u-\omega_\lambda)^-\| = 0$, that is, $u(x) \geq \omega_\lambda(x)$ a.e. in Ω. Therefore, it follows that $u \in P$. Summarizing, u is a positive solution of system (1.1) if it is a critical point of J. □

Lemma 3.3. *There are positive values $\lambda_* > 0$, r, and $\rho > 0$ such that the functional J fulfills the following statements:*
(i) $J(u)|_{u \in S_\rho} \geq r$ and $\inf_{u \in \Omega_\rho} J(u) < 0$ for $\lambda \in (0, \lambda_*)$;
(ii) There exists $e \in H$ such that $\|e\| > \rho$ and $J(e) < 0$.

Proof. (i) Based on the definitions of S, Hölder inequality, Sobolev inequality, and Lemma 2.1, equations (3.2) and (3.3), it can be shown that

$$\begin{aligned}J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)} \int_\Omega \phi_u |u|^{2_s^*-1} dx - \frac{1}{2_s^*} \int_\Omega |u|^{2_s^*} dx - \lambda \int_\Omega F(x,u) dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)} S^{-1} \|u\|^{2(2_s^*-1)} - \frac{1}{2_s^*} \|u\|^{2_s^*} S^{-\frac{2_s^*}{2}} - C\lambda.\end{aligned}$$

It is observed that there are certain constants $r,\ \rho,\ \lambda_* > 0$ such that $J(u)|_{S_\rho} \geq r$ for every λ in the interval $(0, \lambda_*)$. Furthermore, if $u \in H$, then the following statement is true:

$$\begin{aligned}\lim_{t\to 0^+} \frac{J(tu)}{t} &= -\lambda \lim_{t \to 0^+} \int_\Omega \frac{\int_0^{tu} f(x,s) ds}{t} dx \\ &= -\lambda \int_\Omega f(x,0) u dx \\ &= -\lambda \int_\Omega \frac{u}{\omega_\lambda} dx < 0.\end{aligned}$$

Therefore, we can deduce that $J(tu) < 0$ for all $t \neq 0$, provided that t is sufficiently small. Thus, given sufficiently small values of $\|u\|$, the following inequality holds

$$d \triangleq \inf_{u \in \Omega_\rho} J(u) < 0. \tag{3.13}$$

(ii) For every $u \in H$, $u \neq 0$, one has $J(tu) \to -\infty$ as $t \to +\infty$. Consequently, $J(e) < 0$ and $\|e\| > \rho$ for every $e \in H$. The proof is finished. □

Lemma 3.4. *The functional J fulfills the $(PS)_c$ condition if and only if $c \in (0, c^*)$, where*

$$c^* = \Big(\frac{\sqrt{5}-1}{2}\Big)^{\frac{2}{2_s^*-2}} \frac{(2_s^*-2)(2\cdot 2_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*} S^{\frac{N}{2s}} - \theta\lambda,\ \theta = \theta\Big(N,\ S,\ |\Omega|,\ \int_\Omega \frac{1}{\sqrt{\phi_1}}dx\Big).$$

Proof. Let $\{u_n\} \subset P \subset H$ be such that $|dJ|(u_n) \to 0,\ J(u_n) \to c$ as $n \to \infty$. According to Lemma 3.2, we obtain

$$\lambda \int_\Omega f(x,u_n)(v-u_n)dx \le \int_\Omega (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}}(v-u_n)dx - \int_\Omega \phi_{u_n}|u_n|^{2_s^*-3}u_n(v-u_n)dx$$
$$- \int_\Omega |u_n|^{2_s^*-2}u_n(v-u_n)dx + |dJ|(u_n)\|v-u_n\|. \tag{3.14}$$

By substituting $v = 2u_n$ into equation (3.14), we obtain

$$\lambda \int_\Omega f(x,u_n)u_n dx \le \int_\Omega (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} u_n dx - \int_\Omega \phi_{u_n}|u_n|^{2_s^*-1}dx - \int_\Omega |u_n|^{2_s^*}dx$$
$$+ |dJ|(u_n)\|u_n\|.$$

By (3.2) and (3.3),

$$\begin{aligned}
1 + c + o(1)\|u_n\| \ge& J(u_n) - \frac{1}{2_s^*}|dJ|(u_n)\|u_n\| \\
=& \frac{1}{2}\|u_n\|^2 - \frac{1}{2(2_s^*-1)}\int_\Omega \phi_{u_n}|u_n|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_\Omega |u_n|^{2_s^*}dx \\
&- \lambda\int_\Omega F(x,u_n)dx - \frac{1}{2_s^*}\|u_n\|^2 + \frac{1}{2_s^*}\int_\Omega \phi_{u_n}|u_n|^{2_s^*-1}dx + \frac{1}{2_s^*}\int_\Omega |u_n|^{2_s^*}dx \\
&+ \frac{\lambda}{2_s^*}\int_\Omega f(x,u_n)u_n dx \\
\ge& \frac{2_s^*-2}{2\cdot 2_s^*}\|u_n\|^2 + \frac{2_s^*-2}{2(2_s^*-1)2_s^*}\int_\Omega \phi_{u_n}|u_n|^{2_s^*-1}dx - \lambda\int_\Omega F(x,u_n)dx \\
&+ \frac{\lambda}{2_s^*}\int_\Omega f(x,u_n)u_n dx \\
\ge& \frac{2_s^*-2}{2\cdot 2_s^*}\|u_n\|^2 - \lambda\int_\Omega F(x,u_n)dx \\
\ge& \frac{s}{N}\|u_n\|^2 - c\|u_n\|,
\end{aligned}$$

for some positive constant c. Thus, the sequence $\{u_n\}$ is bounded in H. Therefore, there is a subsequence $\{u_n\} \subset P \subset H$ (which is still denoted by itself) and $v \in P$ such that

$$\begin{cases} u_n \rightharpoonup v \ \text{ in } \ H, \\ u_n \to v \ \text{ in } \ L^q(\Omega),\ 1 \le q \le 2_s^*, \\ u_n(x) \to v(x) \ \text{ a.e. in } \Omega. \end{cases}$$

Given that $f(x, u_n)v \leq 1$ for every n, by the dominated convergence theorem, we can get

$$\int_\Omega f(x, u_n)u_n dx \to \int_\Omega f(x, v)v dx. \tag{3.15}$$

Let $\omega_n = u_n - v$. The conclusion is true if $\|\omega_n\| \to 0$. Alternatively, there is a subsequence (which is still referred to as itself) such that $\lim_{n\to\infty}\|\omega_n\|^2 = l > 0$. Thus by $\lim_{n\to\infty}\langle J'(u_n), v\rangle = 0$ and (3.15) we know

$$\|v\|^2 - \int_\Omega \phi_v |v|^{2_s^*-1} dx - \int_\Omega |v|^{2_s^*} dx - \lambda \int_\Omega f(x, v)v dx = 0. \tag{3.16}$$

As stated by the Brézis-Lieb Lemma, the following holds true:

$$\begin{cases} \|u_n\|^2 = \|\omega_n\|^2 + \|v\|^2 + o_n(1), \\ \int_\Omega |u_n|^{2_s^*} dx = \int_\Omega |\omega_n|^{2_s^*} dx + \int_\Omega |v|^{2_s^*} dx + o_n(1). \end{cases} \tag{3.17}$$

According to Lemma 2.1, (3.15) and (3.17), it follows that

$$\begin{aligned} o_n(1) =& \|v\|^2 + \|\omega_n\|^2 - \Big(\int_\Omega \phi_v |v|^{2_s^*-1} dx + \int_\Omega \phi_{\omega_n} |\omega_n|^{2_s^*-1} dx\Big) \\ & - \Big(\int_\Omega |v|^{2_s^*} dx + \int_\Omega |\omega_n|^{2_s^*} dx\Big) - \lambda \int_\Omega f(x, v)v dx. \end{aligned}$$

Consequently, we infer that

$$\|\omega_n\|^2 = \int_\Omega \phi_{\omega_n} |\omega_n|^{2_s^*-1} dx + \int_\Omega |\omega_n|^{2_s^*} dx + o_n(1), \tag{3.18}$$

For the sake of simplicity, we can suppose that

$$a_n = \int_\Omega \phi_{\omega_n} |\omega_n|^{2_s^*-1} dx \to a \quad \text{and} \quad b_n = \int_\Omega |\omega_n|^{2_s^*} dx \to b. \tag{3.19}$$

Notice that

$$\begin{aligned} \int_\Omega |\omega_n|^{2_s^*} dx =& \int_\Omega (-\Delta)^{\frac{s}{2}} \phi_{\omega_n} (-\Delta)^{\frac{s}{2}} \omega_n dx \\ \leq& \frac{\varepsilon^2}{2} \int_\Omega |(-\Delta)^{\frac{s}{2}} |\omega_n||^2 dx + \frac{1}{2\varepsilon^2} \int_\Omega |(-\Delta)^{\frac{s}{2}} \phi_{\omega_n}|^2 dx \\ \leq& \frac{\varepsilon^2}{2} \int_\Omega |(-\Delta)^{\frac{s}{2}} \omega_n|^2 dx + \frac{1}{2\varepsilon^2} \int_\Omega \phi_{\omega_n} |\omega_n|^{2_s^*-1} dx. \end{aligned}$$

Thus, taking the limit when $n \to \infty$, we deduce that

$$b < \frac{1}{2\varepsilon^2} a + \frac{\varepsilon^2}{2} l.$$

We conclude that $a \geq \frac{3-\sqrt{5}}{2}l$ by choosing $\varepsilon^2 = \frac{\sqrt{5}-1}{2}$ and combining (3.18) and (3.19). It can be deduced from equations (3.18) and (3.19) that

$$\begin{aligned}&\frac{1}{2}\|\omega_n\|^2 - \frac{1}{2(2_s^*-1)}\int_\Omega \phi_{\omega_n}|\omega_n|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_\Omega |\omega_n|^{2_s^*}dx\\ =&\frac{1}{2}l - \frac{1}{2(2_s^*-1)}a - \frac{1}{2_s^*}b + o_n(1)\\ \geq&\frac{(2_s^*-2)(2\cdot 2_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}l. \end{aligned} \tag{3.20}$$

In addition, the estimations (2.1) and (3.18) result in

$$l \leq S^{-2_s^*}l^{2_s^*-1} + S^{-\frac{2_s^*}{2}}l^{\frac{2_s^*}{2}}.$$

We thus get $l \geq (\frac{\sqrt{5}-1}{2})^{\frac{2}{2_s^*-2}}S^{\frac{N}{2s}}$. Taking it into (3.20), we conclude that

$$\begin{aligned}&\frac{1}{2}\|\omega_n\|^2 - \frac{1}{2(2_s^*-1)}\int_\Omega \phi_{\omega_n}|\omega_n|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_\Omega |\omega_n|^{2_s^*}dx\\ \geq&\Big(\frac{\sqrt{5}-1}{2}\Big)^{\frac{2}{2_s^*-2}}\frac{(2_s^*-2)(2\cdot 2_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}S^{\frac{N}{2s}}. \end{aligned} \tag{3.21}$$

On one hand, according to the Young, Hölder, and Sobolev inequalities, the following statement is true

$$\begin{aligned}J(v) &= \frac{1}{2}\|v\|^2 - \frac{1}{2(2_s^*-1)}\int_\Omega \phi_v|v|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_\Omega |v|^{2_s^*}dx - \lambda\int_\Omega F(x,v)dx\\ &= \frac{2_s^*-2}{2\cdot 2_s^*}\|v\|^2 + \frac{2_s^*-2}{2(2_s^*-1)2_s^*}\int_\Omega \phi_v|v|^{2_s^*-1}dx - \lambda\int_\Omega F(x,v)dx + \frac{\lambda}{2_s^*}\int_\Omega f(x,v)vdx\\ &\geq \frac{2_s^*-2}{2\cdot 2_s^*}\|v\|^2 - C\lambda\\ &= \frac{s}{N}\|v\|^2 - C\lambda\\ &\geq -\theta\lambda, \end{aligned}$$

where $\theta = \theta(N,\ S,\ |\Omega|,\ \int_\Omega \frac{1}{\sqrt{\phi_1}}dx)$. On the other hand, according to (3.15), (3.17), (3.21) and Lemma 2.1, we obtain

$$\begin{aligned}J(v) =&J(u_n) - \Big[\frac{1}{2}\|\omega_n\|^2 - \frac{1}{2(2_s^*-1)}\int_\Omega \phi_{\omega_n}|\omega_n|^{2_s^*-1}dx - \frac{1}{2_s^*}\int_\Omega |\omega_n|^{2_s^*}dx\Big] + o_n(1)\\ \leq&c - \Big(\frac{\sqrt{5}-1}{2}\Big)^{\frac{2}{2_s^*-2}}\frac{(2_s^*-2)(2\cdot 2_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}S^{\frac{N}{2s}}\\ <&\Big[\Big(\frac{\sqrt{5}-1}{2}\Big)^{\frac{2}{2_s^*-2}}\frac{(2_s^*-2)(2\cdot 2_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}S^{\frac{N}{2s}} - \theta\lambda\Big]\\ &- \Big(\frac{\sqrt{5}-1}{2}\Big)^{\frac{2}{2_s^*-2}}\frac{(2_s^*-2)(2\cdot 2_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}S^{\frac{N}{2s}}\\ =&-\theta\lambda. \end{aligned}$$

It's a contradiction. As thus, $l = 0$ suggests that $u_n \to u$ in H. The proof of Lemma 3.5 is therefore finished. $\square$

Theorem 3.5. *For $0 < \lambda < \lambda_0$, system (1.1) has a positive solution (u_0, ϕ_{u_0}) with $J(u_0) = d < 0$.*

To demonstrate the existence of a positive solution for system (1.1) using mountain pass theorem, as described in [25]. We state that $v_\varepsilon(x) = \psi(x)U_\varepsilon(x)$, $x \in \Omega$, where $U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} u^*(\frac{x}{\varepsilon})$, $u^*(x) = \frac{\tilde{u}(x \backslash S^{\frac{1}{2s}})}{\|\tilde{u}\|_{2_s^*}}$, and $\tilde{u}(x) = k(\mu_0^2 + |x|^2)^{-\frac{N-2s}{2}}$ with $k > 0$ and $\mu_0 > 0$ being fixed constants; the cut-off function $\psi \in C_0^\infty(\Omega_\delta(x_0), [0,1])$ such that $\psi(x) = 1$ near $x = x_0$, where $\Omega_\delta(x_0) \subset \Omega$. Let $v_\varepsilon(x) = \psi(x)U_\varepsilon(x)$, our conclusion is as follows.

By multiplying the second equation of system (1.1) by $|u|$ and integrating, we obtain

$$\begin{aligned}\int_\Omega |v_\varepsilon|^{2_s^*} dx &= \int_\Omega (-\Delta)^{\frac{s}{2}} \phi_{v_\varepsilon} (-\Delta)^{\frac{s}{2}} |v_\varepsilon| dx \\ &\le \frac{1}{2} \int_\Omega |(-\Delta)^{\frac{s}{2}} |v_\varepsilon||^2 dx + \frac{1}{2} \int_\Omega |(-\Delta)^{\frac{s}{2}} \phi_{v_\varepsilon}|^2 dx \\ &\le \frac{1}{2} \int_\Omega |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx + \frac{1}{2} \int_\Omega \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx. \end{aligned} \tag{3.22}$$

Consequently, if we define a new functional $G : H \to \mathbb{R}$ defined as follows

$$\begin{aligned} G(u) &\triangleq \frac{N}{N+2s} \|u\|^2 - \lambda \int_\Omega F(x,u) dx - \frac{1}{2_s^* - 1} \int_\Omega |u|^{2_s^*} dx \\ &= \frac{2N}{N+2s} \Big(\frac{1}{2} \|u\|^2 - \frac{\lambda(N+2s)}{2N} \int_\Omega F(x,u) dx - \frac{1}{2_s^*} \int_\Omega |u|^{2_s^*} dx \Big) \\ &\triangleq \frac{2N}{N+2s} I(u) \end{aligned}$$

where

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda(N+2s)}{2N} \int_\Omega F(x,u) dx - \frac{1}{2_s^*} \int_\Omega |u|^{2_s^*} dx.$$

By (3.22) we have

$$J(u) \le G(u) = \frac{2N}{N+2s} I(u) \tag{3.23}$$

for each $u \in H$. Now, let us consider the following problem:

$$\begin{cases} (-\Delta)^s u = \dfrac{\lambda(N+2s)}{2N} f(x,u) + |u|^{2_s^*-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.24}$$

The critical point of the functional I corresponds to a weak solution of problem (3.24). The energy functional corresponding to problem (3.24) is as follows

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda(N+2s)}{2N} \int_\Omega F(x,u) dx - \frac{1}{2_s^*} \int_\Omega |u|^{2_s^*} dx.$$

where $F(x,u) = \int_0^u f(x,t) dt$.

Lemma 3.6. (see[24]) *Every positive weak solution of problem* (3.24) *belongs to* $L^\infty(\Omega)\cap C^\infty(\Omega)$.

Remark 3.7. By employing a similar approach as the proof of Lemma 3.3, we can demonstrate that the functional I also fulfills mountain pass theorem. Thus, problem (3.24) has a positive solution u_0 with $I(u_0) < 0$.

Lemma 3.8. *Assume* $0 < \lambda < \lambda_1$, *for each* $s \in (0,1)$ *it holds*

$$\sup_{t\geq 0} I(u_0 + tv_\varepsilon) \leq c^* - \theta\lambda$$

where $\lambda_1 = min\{1, (\frac{C_2}{C_3+C_1+\theta})^2\}$. *In particular*

$$\sup_{t\geq 0} J(u_0 + tv_\varepsilon) \leq \frac{2N}{N+2s}c^* - \theta\lambda \tag{3.25}$$

Proof. From [25], we have

$$\int_\Omega |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx \leq S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}), \quad \int_\Omega |v_\varepsilon|^{2_s^*} dx = S^{\frac{N}{2s}} + O(\varepsilon^N) \tag{3.26}$$

and

$$|v_\varepsilon|_p^p = \begin{cases} O(\varepsilon^{\frac{N(2-p)+2sp}{2}}) & \text{if } p > \frac{N}{N-2s}, \\ O(\varepsilon^{\frac{N(2-p)+2sp}{2}}|log\varepsilon|) & \text{if } p = \frac{N}{N-2s}, \\ O(\varepsilon^{\frac{(N-2s)p}{2}}) & \text{if } p < \frac{N}{N-2s}. \end{cases} \tag{3.27}$$

Then via (3.22), for $\varepsilon > 0$ small enough, we derive

$$\begin{aligned} \int_\Omega \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx \geq & 2\int_\Omega |v_\varepsilon|^{2_s^*} dx - \int_\Omega |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx \\ = & S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}). \end{aligned} \tag{3.28}$$

By (3.23), we have

$$\int_\Omega |v_\varepsilon|^{2_s^*-1} dx = C\varepsilon^{\frac{N-2s}{2}}. \tag{3.29}$$

Since $I(u_0 + tv_\varepsilon) \to -\infty$ as $t \to \infty$, $I(u_0) < 0$, according to Lemma 3.3 (i), we can suppose that there exist $t_1,\ t_2 > 0$ such that $\sup_{t\geq 0} I(u_0 + tv_\varepsilon) = \sup_{t\in[t_1,t_2]} I(u_0 + tv_\varepsilon)$.

Since u_0 is positive solution of problem (3.24), we have

$$\begin{cases} I(u_0) < 0, \\ \int_\Omega (-\Delta)^{\frac{s}{2}} u_0 (-\Delta)^{\frac{s}{2}} v_\varepsilon dx = \int_\Omega |u_0|^{2_s^*-2} u_0 v_\varepsilon dx + \frac{2N}{N+2s}\lambda \int_\Omega \frac{v_\varepsilon}{u_0} dx, \\ \frac{v_\varepsilon}{u_0} \in L^1(\Omega), \text{ that is } \int_\Omega \frac{v_\varepsilon}{u_0} dx \leq C \text{ for some } C > 0. \end{cases} \tag{3.30}$$

Moreover, according to $u_0 > \omega_\lambda$, we deduce that

$$\begin{aligned} F(x, u_0) = & \int_0^{\omega_\lambda} \frac{1}{\omega_\lambda} dt + \int_{\omega_\lambda}^{u_0} \frac{1}{t} dt \\ = & 1 + ln u_0 - ln\omega_\lambda. \end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_\Omega [F(x,u_0+tv_\varepsilon)-F(x,u_0)]dx =& \int_\Omega [1+ln(u_0+tv_\varepsilon)-ln\omega_\lambda]dx \\ &-\int_\Omega (1+lnu_0-ln\omega_\lambda)dx \\ =&\int_\Omega ln(1+\frac{tv_\varepsilon}{u_0})dx \\ \geq& 0 \end{aligned} \tag{3.31}$$

for all $t \geq 0$. Thus, based on equation (3.30), equation (3.31), and the following inequality

$$(a+b)^{2_s^*} \geq a^{2_s^*} + 2_s^* a^{2_s^*-1}b + 2_s^* ab^{2_s^*-1} + b^{2_s^*}, \text{ for } a,\ b \geq 0,$$

we have

$$\begin{aligned} I(u_0+tv_\varepsilon) =& \frac{1}{2}\|u_0+tv_\varepsilon\|^2 - \frac{1}{2_s^*}\int_\Omega |u_0+tv_\varepsilon|^{2_s^*}dx - \frac{N+2s}{2N}\lambda\int_\Omega F(x,u_0+tv_\varepsilon)dx \\ =& \frac{1}{2}\|u_0\|^2 + \frac{t^2}{2}\|v_\varepsilon\|^2 + t\int_\Omega (-\Delta)^{\frac{s}{2}}u_0\cdot(-\Delta)^{\frac{s}{2}}v_\varepsilon dx \\ &- \frac{1}{2_s^*}\int_\Omega |u_0+tv_\varepsilon|^{2_s^*}dx - \frac{N+2s}{2N}\lambda\int_\Omega F(x,u_0+tv_\varepsilon)dx \\ =& I(u_0) + \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{1}{2_s^*}\int_\Omega \Big(|u_0+tv_\varepsilon|^{2_s^*} - |u_0|^{2_s^*} - 2_s^*|u_0|^{2_s^*-1}tv_\varepsilon\Big)dx \\ &- \frac{N+2s}{2N}\lambda\int_\Omega \Big(F(x,u_0+tv_\varepsilon)-F(x,u_0)\Big)dx + \frac{2N}{N+2s}\lambda\int_\Omega \frac{tv_\varepsilon}{u_0}dx \\ \leq& \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |v_\varepsilon|^{2_s^*}dx - t^{2_s^*-1}\int_\Omega |u_0||v_\varepsilon|^{2_s^*-1}dx + \frac{2N}{N+2s}\lambda\int_\Omega \frac{tv_\varepsilon}{u_0}dx. \end{aligned}$$

For $t \geq 0$, let

$$g(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |v_\varepsilon|^{2_s^*}dx - t^{2_s^*-1}\int_\Omega |u_0||v_\varepsilon|^{2_s^*-1}dx + \frac{2N}{N+2s}\lambda\int_\Omega \frac{tv_\varepsilon}{u_0}dx.$$

Since $\lim_{t\to\infty} g(t) = -\infty$. It is demonstrated by the [26] that there exist positive constants t_1, t_2 (independent of ε, λ) such that

$$0 < t_1 \leq t_\varepsilon \leq t_2 < +\infty.$$

There exists a positive constant C such that $u_0 < C$ and one has $u_0 \in C^1(\Omega, \mathbb{R}^+)$ via a standard regularity argument. Therefore, it can be deduced from equations (3.21) and (3.26)-(3.30) that

$$\begin{aligned}\sup_{t\in[t_1,t_2]} I(u_0+tv_\varepsilon) \leq& \Big(\frac{\sqrt{5}-1}{2}\Big)^{\frac{2}{2_s^*-2}} \frac{(2_s^*-2)(2\cdot 2_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*} S^{\frac{N}{2s}} + C_1\varepsilon^{N-2s} - C_2\varepsilon^{\frac{N-2s}{2}} \\ &+ C_3\lambda. \end{aligned}$$

where $C_1, C_2, C_3 > 0$(independent of ε, λ). Let $\varepsilon = \lambda^{\frac{1}{N-2s}}$, $\lambda_1 = \min\{1, (\frac{C_2}{C_3+C_1+\theta})^2\}$ we have

$$\begin{aligned} C_1\varepsilon^{N-2s} - C_2\varepsilon^{\frac{N-2s}{2}} + C_3\lambda =& (C_1 + C_3)\lambda - C_2\lambda^{\frac{1}{2}} \\ =& \lambda(C_1 + C_3 - C_2\lambda^{-\frac{1}{2}}) \\ \leq& -\theta\lambda \end{aligned}$$

for all $\lambda \in (0, \lambda_1)$. We may deduce from (3.23) and the discussion above that (3.25) is likewise true for $\lambda \in (0, \lambda_1)$. Thus it completes the proof of Lemma 3.8. □

Proof of Theorem 1.1: Let $0 < \lambda < \lambda_* = \min\{\lambda_0, \lambda_1\}$. Lemma 3.4 and Lemma 3.6 imply that J fulfills the Palais-Smale criterion at the level c. There is a Palais-Smale sequence $\{u_n\}$ such that, as $n \to \infty$, $|dJ|(u_n) \to 0$, $J(u_n) \to c$, by Lemma 3.3. Up to a subsequence, $u_n \to v$ in H, and $J(v) = \lim_{n\to\infty} J(v) = c > 0$, $|dJ|(u_n) \to 0$. By utilizing mountain pass lemma and Lemma 3.2, it can be concluded that (v, ϕ_v) is a positive solution of system (1.1). By combining with Theorem 3.5, it follows that system (1.1) has two positive solutions. The proof of Theorem 1.1 is now finished.

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