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EXISTENCE AND HYERS-ULAM STABILITY OF IMPULSIVE DELAY INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract In recent years, impulsive ordinary differential equations with delay terms have garnered significant attention due to their wide applications in various fields, including mechanics, population dynamics, and nuclear reactor physics. The primary objective of this paper is to establish the Hyers-Ulam stability and Hyers-Ulam-Rassias stability for impulsive delay ordinary differential equations by employing a novel generalized Gronwall inequality alongside fixed-point methods and Picard's operator technique. An example is provided to illustrate the stability of impulsive differential equation which is based on a new deep learning framework, and the integral operators are learned using neural networks.

Keywords Hyers-Ulam stability, Hyers-Ulam-Rassias stability, delay integro-differential equation, impulses.

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1. Introduction

In the early 20th century, the works of V. Volterra on population dynamics have motivated the theory of impulsive differential equation (IDE), which stemmed from the necessity of modeling systems [1,2]. By the mid-20th century, with the development of computer technology and numerical calculation, IDE became an essential mathematical tool in various scientific and engineering fields [3,4]. Nowadays, IDEs are used in a wide range of scientific and engineering fields, such as mechanics, chemical reactions, communication systems, population dynamics, medical models, optimal control models, nuclear reactor physics, economics, deep learning, pharmacokinetics and frequency modulation systems (see [5]- [10], and the bibliography therein). The naturally evolutionary behavior in many real-world problems can be characterized by impulse effects [11].

In 1940s, Ulam proposed a problem regarding the stability of the Cauchy equation, to which Hyers provided a partial solution. In 1978, Rassias provided a more

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extensive elaboration on the idea of Hyers, where the bound for the norm of the Cauchy difference was determined in a more general form. This concept of stability is known as Ulam's type stability(including Hyers-Ulam stability and Hyers-Ulam-Rassias stability). Subsequently, numerous researchers have delved into the investigation of Hyers-Ulam stability in ordinary differential equations, partial differential equations, and fractional differential equations (see [5–8, 10, 12, 15, 16, 18–24, 27] and references therein). The Hyers-Ulam stability problem stands as a fundamental issue in classical physical systems, encompassing plasma physics, electrical circuits, aerodynamics and various other fields.

Among these results, Ulam's type stability of IDEs draw many researchers' attention, the first result on IDEs was obtained by Wang et al. [12] in 2012. They gave the Ulam's type stability for the first-order nonlinear IDEs on closed bounded interval with finite impulses. In addition, various generalizations of Hyers-Ulam stability have been extensively studied and many elegant results have been obtained by using different approaches, for examples, see [13]- [27]. C. Tunç [18] investigated the stability of zero solution and boundedness of all solutions of the nonlinear Volterra integro-differential equation with delay by defining new suitable Lyapunov functions. In 2016, using abstract Gronwall lemma and Gronwall integral inequality, A. Zada et al. [19] considered the Hyers-Ulam stability and Hyers-Ulam-Rassias stability for first-order IDEs with delay of the form:

$$z'(t) = F(t, z(t), z(h(t))), I = [t_0, T], t \in I' \triangleq I \setminus \{t_1, t_2, \cdots, t_m\},$$
$$z(t) = \alpha(t), t \in [t_0 - \tau, t_0],$$
$$\triangle z(t_k) = z(t_k^+) - z(t_k^-) = Y_k(z(t_k^-)), k = 1, 2, \cdots, m,$$

where $\tau > 0, T > t_0 \geq 0$ are fixed points, and $F : [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}, Y_k : \mathbb{R} \to \mathbb{R}$ and $\alpha : [t_0 - \tau, t_0] \to \mathbb{R}$ are continuous functions. $z(t_k^+) = \lim_{s\to 0^+} z(t_k + s)$ and $z(t_k^-) = \lim_{s\to 0^+} z(t_k - s)$ are the right and left side limits of $z(t)$ at t_k, $k = 1, 2, \cdots, m$, where t_k satisfy $t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T < +\infty$. In 2019, using Gronwall integral inequality, A. Zada et al. [20] investigated the existence and uniqueness theorem for the solutions of a class of nonlinear impulsive integral equations with a bounded variable delay. Moreover, the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the integral equations were obtained with the help of open mapping theorem approach.

A. R. Aruldass et al. [21] proposed a new method for investigating the Ulam stability of linear differential equations of the form $u'(t) + \mu u(t) = 0$ and the non-homogeneous linear differential equation $u'(t) + \mu u(t) = r(t)$ by applying Kamal transform method in 2021. In the same year, using fixed point method in the sense of Cadariu and Radu, R. Murali et al. [22] proved the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the $n-$order differential equation. D. A. Refaai et al. [23] discussed the Hyers-Ulam stability of fractional impulsive Volterra delay integro-differential equations of the form:

$$\eta_1(t) = I^{\alpha}_{t_0,t} f\left(t, \eta_1(t), \eta_1(h(t)), \int_{t_0}^{t} g(t, \tau, \eta_1(\tau), \eta_1(h(\tau)))d\tau\right), t \in I',$$
$$\triangle \eta_1(t_k) = \eta_1(t_k^+) - \eta_1(t_k^-) = \beta_k \int_{t_k - \tau_k}^{t_k - \theta_k} U(\eta_1(s))ds, k = 1, 2, \cdots, m,$$
$$\eta_1(t) = \phi(t), t \in [t_0 - \tau, t_0],$$

where $\tau > 0, \beta_k \geq 0, 0 \leq \theta_k \leq \tau_k \leq t_k - t_{k-1}$ for $k = 1, 2, \cdots, m, T > t_0 \geq 0, f : [t_0, T] \times \mathbb{R}^3 \to \mathbb{R}$ and $g : [t_0, T] \times [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $\phi : [t_0 - \tau, t_0] \to \mathbb{R}$ is a delay function, and $I^{\alpha}_{t_0,t} f$ is the Riemann-Liouville fractional integral of order α. Their analysis was based on Pachpatte's inequality and the fixed point approach represented by the Picard operators.

Thereafter, using the fixed point theory and the generalized metric, E. El-hady et al. [24] obtained Hyers-Ulam-Rassias stability for the following impulsive Volterra integral equation of second kind

$$u(t) = \int_0^t f(s, u(s))ds + \sum_{0<t_k<t} U(u(t_k^-)),$$

where $U : \mathbb{C} \to \mathbb{C}$, $u(t_k^-)$ represents the left limit of $u(t)$ at $t = t_k$, $k = 1, 2, \cdots, m$ and f is a continuous function.

Motivated by the above mentioned papers, in this paper, using a novel generalized Gronwall inequality and the fixed-point method, we investigate the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of impulsive delay integro-differential equation of the form:

$$\begin{aligned} u'(t) &= F(t, u(t), u(h(t))) + \int_{t_0}^t G(t, s, u(s), u(h(s)))ds, t \in I', \\ u(t) &= \alpha(t), t \in [t_0 - \tau, t_0], \\ \triangle u(t_k) &= u(t_k^+) - u(t_k^-) = \phi_k(u(t_k^-)), k = 1, 2, \cdots, m, \end{aligned} \tag{1.1}$$

where $\tau > 0, T > t_0 \geq 0, \alpha(t) : [t_0 - \tau, t_0] \to \mathbb{R}$ is a continuous function, $F : [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}$, $G : [t_0, T] \times [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}$, and $\phi_k : \mathbb{R} \to \mathbb{R}$. Moreover, we assume that $h : [t_0, T] \to [t_0 - \tau, T]$ is a continuous delay function such that $h(t) \leq t$. $u(t_k^+) = \lim_{\triangle t \to 0^+} u(t_k + \triangle t)$ and $u(t_k^-) = \lim_{\triangle t \to 0^-} u(t_k + \triangle t)$ are the right and left side limits of $u(t)$ at t_k, where t_k satisfy $t_0 < t_1 < \cdots < t_{m+1} = T < +\infty$.

The rest of this paper is structured as follows: Section 2 introduces important notations, recalls some concepts and preliminary results, and proposes the new Gronwall inequality. Section 3 establishes the existence, uniqueness, and Hyers-Ulam stability for equation (1.1) by utilizing the fixed point theorem and a new generalized Gronwall inequality. In Section 4, the Hyers-Ulam-Rassias stability for equation (1.1) is demonstrated using a generalized Gronwall inequality. An illustrative example based on the neural integro-differential equation is provided in Section 5 to show the application of our main results.

2. Preliminaries

In this section, we provide preliminaries including some important notations, definitions and lemmas.

Let $C(J)$ be the Banach space of all continuous real valued functions defined on J with norm $\|z\|_c = \sup\{|z(t)| : t \in J\}$, where J is a compact interval. Let $PC([t_0 - \tau, T])$ be the collection of piecewise continuous functions z: $[t_0 - \tau, T] \to \mathbb{R}$ with discontinuous points t_k satisfying $t_0 < t_1 < t_2 < \cdots < t_m < T \triangleq t_{m+1}$ and $z(t_k^+), z(t_k^-)$ exist and are finite for $k = 1, 2, \cdots, m$. With norm $\|z\|_{PC} = \sup\{|z(t)| : t \in [t_0 - \tau, T]\}$, it is easy to see that $PC([t_0 - \tau, T])$ is a Banach space.

Similarly, we define the Banach space

$$PC^1([t_0-\tau,T]) = \{z|z \in PC\left([t_0-\tau,T]\right) \text{ and } z' \in PC([t_0-\tau,T])\}$$

with norm

$$\|z\|_{PC^1} = \max\left\{\|z'\|_{PC}, \|z\|_{PC}\right\}.$$

Now, for $\varepsilon > 0$, and a nonnegative, increasing function $\varphi(t) \in PC([t_0-\tau,T])$ with $\varphi'(t) > 0$ and $\chi > 0$ with $\varphi(t^*) = \chi > 0$ for some $t^* \in [t_0-\tau,T]$, we consider the following inequalities:

$$\begin{aligned} &\left|u' - F(t,u(t),u(h(t))) - \int_{t_0}^t G(t,s,u(s),u(h(s)))ds\right| \le \varepsilon, t \in I', \\ &\left|\triangle u(t_k) - \phi_k(u(t_k^-))\right| \le \varepsilon, k = 1,2,\cdots,m, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\left|u' - F(t,u(t),u(h(t))) - \int_{t_0}^t G(t,s,u(s),u(h(s)))ds\right| \le \varphi(t), t \in I', \\ &\left|\triangle u(t_k) - \phi_k(u(t_k^-))\right| \le \chi, k = 1,2,\cdots,m. \end{aligned} \tag{2.2}$$

Definition 2.1. Equation (1.1) is Hyers-Ulam stable on $[t_0-\tau,T]$ if for each $u \in PC([t_0-\tau,T]) \cap PC^1([t_0,T])$ satisfying (2.1), there exists a solution $u_0 \in PC([t_0-\tau,T]) \cap PC^1([t_0,T])$ of (1.1) with $|u_0(t)-u(t)| \le K\varepsilon$ for all $t \in [t_0-\tau,T]$, where $K > 0$ is a constant.

Definition 2.2. Equation (1.1) is Hyers-Ulam-Rassias stable on $[t_0-\tau,T]$ with respect to (φ,χ) if for each $u \in PC([t_0-\tau,T]) \cap PC^1([t_0,T])$ satisfying (2.2), there exists a solution $u_0 \in PC([t_0-\tau,T]) \cap PC^1([t_0,T])$ of (1.1) with $|u_0(t)-u(t)| < M\varphi(t)$ for all $t \in [t_0-\tau,T]$, where $M > 0$ is a constant.

Definition 2.3. (Picard operator [27]) Let $(Z;d)$ be a metric space. An operator $T: Z \to Z$ is said to be a Picard operator if there exists $z^* \in Z$ such that :

(i) $F_T = \{z^*\}$, where $F_T = \{z \in Z : T(z) = z\}$ is the fixed point set of T;

(ii) the sequence $\{T^n(z)\}_{n\in N}$ converges to z^* for all $z \in Z$.

Lemma 2.1. *(Generalized Gronwall Lemma) If for $t \ge t_0 \ge 0$, we have*

$$z(t) \le a(t) + \int_{t_0}^t b(s)\left(z(s) + \int_{t_0}^s c(\tau)z(\tau)d\tau\right)ds + \sum_{t_0<t_k<t} z(t_k^-)\xi_k, \tag{2.3}$$

where $z, a, b, c \in PC([t_0,\infty))$ are nonnegative functions, $a(t)$ is nondecreasing and $\xi_k > 0$ for $k = 1,2,\cdots,m$. Then for $t \ge t_0$, the following inequality holds

$$z(t) \le a_k(t)H(t_k,t), \quad t \in (t_k,t_{k+1}] \tag{2.4}$$

for $k = 0,1,2,\cdots,m$ with $a_0(t) = a(t)$, where

$$a_k(t) = a(t)\prod_{i=1}^k \left[1 + H(t_{i-1},t_i)\left(A(t_{i-1},t_i) + \xi_i\right)\right], \tag{2.5}$$

$$A(t_{i-1},t) = \int_{t_{i-1}}^t b(s)\left(1 + \int_{t_0}^s c(\tau)d\tau\right)ds, \tag{2.6}$$

$$H(t_k,t) = 1 + A(t_k,t)\left(1 + \int_{t_k}^t b(s)\exp\left\{\int_{t_k}^s (b(\tau)+c(\tau))d\tau\right\}ds\right). \tag{2.7}$$

Proof: Since z, a, b and c are piecewise continuous nonnegative functions, $a(t)$ is nondecreasing and $\xi_k > 0$, for $t_0 \le t \le t_1$, we have

$$z(t) \le a(t) + \int_{t_0}^{t} b(s)z(s)ds + \int_{t_0}^{t} b(s)\int_{t_0}^{s} c(\tau)z(\tau)d\tau ds.$$

Obviously

$$z(t) \le a(t) + w(t), \tag{2.8}$$

where $w(t) = \int_{t_0}^{t} b(s)z(s)ds + \int_{t_0}^{t} b(s)\int_{t_0}^{s} c(\tau)z(\tau)d\tau ds$ and $w(t_0) = 0$. It is easy to see that

$$\begin{aligned} w(t) &\le \int_{t_0}^{t} b(s)\left(a(s) + w(s) + \int_{t_0}^{s} c(\tau)(a(\tau) + w(\tau))d\tau\right) ds \\ &= \int_{t_0}^{t} b(s)a(s)ds + \int_{t_0}^{t} b(s)\int_{t_0}^{s} c(\tau)a(\tau)d\tau ds \\ &\quad + \int_{t_0}^{t} b(s)\left(w(s) + \int_{t_0}^{s} c(\tau)w(\tau)d\tau\right) ds. \end{aligned} \tag{2.9}$$

Let $J(t) = \int_{t_0}^{t} b(s)a(s)ds + \int_{t_0}^{t} b(s)\int_{t_0}^{s} c(\tau)a(\tau)d\tau ds$, we get $J(t_0) = 0$. Since $a(t)$ is nondecreasing, we get $J(t) \le a(t)A(t_0, t)$, where $A(t_0, t)$ is defined by (2.6).

For $t > t_0$, dividing both sides of (2.9) by $J(t)$, we have

$$\begin{aligned} \frac{w(t)}{J(t)} &\le 1 + \frac{1}{J(t)}\int_{t_0}^{t} b(s)\left(w(s) + \int_{t_0}^{s} c(\tau)w(\tau)d\tau\right) ds \\ &\le 1 + \int_{t_0}^{t} b(s)\left(\frac{w(s)}{J(s)} + \int_{t_0}^{s} c(\tau)\frac{w(\tau)}{J(\tau)}d\tau\right) ds. \end{aligned}$$

Let

$$Y(t) = \frac{w(t)}{J(t)}, \tag{2.10}$$

we get

$$Y(t) \le 1 + \int_{t_0}^{t} b(s)\left(Y(s) + \int_{t_0}^{s} c(\tau)Y(\tau)d\tau\right) ds.$$

Using nonlinear Pachpatte's integral inequalities [28], we get

$$Y(t) \le 1 + \int_{t_0}^{t} b(s)\exp\left\{\int_{t_0}^{s} [b(\tau) + c(\tau)]d\tau\right\} ds.$$

Then (2.10) implies that

$$\begin{aligned} w(t) &\le J(t)\left(1 + \int_{t_0}^{t} b(s)\exp\left\{\int_{t_0}^{s} [b(\tau) + c(\tau)]d\tau\right\} ds\right) \\ &\le a(t)A(t_0, t)\left(1 + \int_{t_0}^{t} b(s)\exp\left\{\int_{t_0}^{s} [b(\tau) + c(\tau)]d\tau\right\} ds\right), \end{aligned} \tag{2.11}$$

thus, for $t \in [t_0, t_1]$, using (2.8) and (2.11), we get

$$z(t) \le a(t)H(t_0, t). \tag{2.12}$$

Particularly, we have

$$z(t_1^-) = z(t_1) \le a(t_1)H(t_0, t_1). \tag{2.13}$$

For $t \in (t_1, t_2]$, using (2.12) and (2.13), we get

$$\begin{aligned} z(t) &\le a(t) + \xi_1 z(t_1^-) + \int_{t_0}^{t} b(s)\left(z(s) + \int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds \\ &= B_1(t) + \int_{t_1}^{t} b(s)\left(z(s) + \int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds, \end{aligned}$$

where

$$\begin{aligned} B_1(t) &= a(t) + \xi_1 z(t_1^-) + \int_{t_0}^{t_1} b(s)\left(z(s) + \int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds \\ &\le a(t) + z(t_1^-)(A(t_0, t_1) + \xi_1) \\ &\le a(t)\left[1 + H(t_0, t_1)(A(t_0, t_1) + \xi_1)\right] = a_1(t). \end{aligned}$$

So we get

$$z(t) \le a_1(t) + \int_{t_1}^{t} b(s)\left(z(s) + \int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds.$$

Let

$$Y_1(t) = \frac{z(t)}{a_1(t)}, \tag{2.14}$$

we get

$$Y_1(t) \le 1 + \int_{t_1}^{t} b(s)\left(Y_1(s) + \int_{t_0}^{s} c(\tau)Y_1(\tau)d\tau\right)ds.$$

Then we obtain

$$Y_1(t) \le 1 + k(t), \tag{2.15}$$

where

$$k(t) = \int_{t_1}^{t} b(s)\left(Y_1(s) + \int_{t_0}^{s} c(\tau)Y_1(\tau)d\tau\right)ds,$$

and $k(t_1) = 0$. So we have

$$\begin{aligned} k'(t) &= b(t)\left(Y_1(s) + \int_{t_0}^{t} c(s)Y_1(s)ds\right) \\ &\le b(t)\left(1 + \int_{t_0}^{t} c(s)ds\right) + b(t)\left(k(t) + \int_{t_0}^{t_1} c(s)k(s)ds + \int_{t_1}^{t} c(s)k(s)ds\right) \\ &\le b(t)\left(1 + \int_{t_0}^{t} c(s)ds\right) + b(t)\left(k(t) + \int_{t_1}^{t} c(s)k(s)ds\right). \end{aligned} \tag{2.16}$$

Integrating on both sides of (2.16) from t_1 to t, we have

$$k(t) \le A(t_1, t) + \int_{t_1}^{t} b(s)\left(k(s) + \int_{t_1}^{s} c(\tau)k(\tau)d\tau\right)ds.$$

Since $A(t_1,t)>0$ for $t>t_1$, we get

$$\frac{k(t)}{A(t_1,t)} \leq 1+\int_{t_1}^{t} b(s)\left(\frac{k(s)}{A(t_1,s)}+\int_{t_1}^{s} c(\tau)\frac{k(\tau)}{A(t_1,\tau)}d\tau\right)ds.$$

Using nonlinear Pachpatte's integral inequalities, we obtain

$$k(t) \leq A(t_1,t)\left(1+\int_{t_1}^{t} b(s)\exp\left\{\int_{t_1}^{s}(b(\tau)+c(\tau))d\tau\right\}ds\right). \tag{2.17}$$

By (2.15) and (2.17), it is easy to see that

$$Y_1(t) \leq 1+M(t)\left(1+\int_{t_1}^{t} b(s)\exp\left\{\int_{t_1}^{s}(b(\tau)+c(\tau))d\tau\right\}ds\right). \tag{2.18}$$

By (2.14) and (2.18), we get

$$z(t) \leq a_1(t)Y_1(t) \leq a_1(t)(1+k(t)) = a_1(t)H(t_1,t),$$

and as a consequence, we get

$$z(t_2^-) = z(t_2) \leq a_1(t_2)H(t_1,t_2). \tag{2.19}$$

Suppose for $t\in(t_{k-1},t_k]$, one has $z(t)\leq a_{k-1}(t)H(t_{k-1},t)$ and $z(t_k^-)=z(t_k)\leq a_{k-1}(t_k)H(t_{k-1},t_k)$, then for $t\in(t_k,t_{k+1}]$, we get

$$\begin{aligned} z(t) \leq & a(t)+\int_{t_0}^{t} b(s)\left(z(s)+\int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds+\sum_{t_0<t_k<t} z(t_k^-)\xi_k \\ = & B_k(t)+\int_{t_k}^{t} b(s)\left(z(s)+\int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} B_k(t) = & a(t)+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} b(s)\left(z(s)+\int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds+\sum_{i=1}^{k} z(t_i^-)\xi_i \\ \leq & a(t)+\sum_{i=1}^{k}\left[A(t_{i-1},t_i)+\xi_i\right]z(t_i^-) \\ \leq & a(t)+\sum_{i=1}^{k}\left[A(t_{i-1},t_i)+\xi_i\right]a_{i-1}(t_i)H(t_{i-1},t_i) \\ \leq & a(t)+\sum_{i=1}^{k}\left[A(t_{i-1},t_i)+\xi_i\right]a_{i-1}(t)H(t_{i-1},t_i) \\ \leq & a_1(t)+\sum_{i=2}^{k}\left[A(t_{i-1},t_i)+\xi_i\right]a_{i-1}(t)H(t_{i-1},t_i) \\ \leq & a_2(t)+\sum_{i=3}^{k}\left[A(t_{i-1},t_i)+\xi_i\right]a_{i-1}(t)H(t_{i-1},t_i) \\ \leq & \cdots \leq a_k(t). \end{aligned}$$

So we get by (2.20) that

$$z(t) \le a_k(t) + \int_{t_k}^{t} b(s)\left(z(s) + \int_{t_0}^{s} c(\tau)z(\tau)d\tau\right)ds.$$

Similar method as above can deduce that $z(t) \le a_k(t)H(t_k, t)$. By the method of mathematical induction, we prove that (2.4) holds for the whole interval I'. This completes the proof of Lemma 2.1.

Lemma 2.2. *(Abstract Gronwall Lemma [19]) Let $(Z, d, \le)$ be an ordered metric space and let $T : Z \to Z$ be an increasing Picard operator with fixed point z^*. Then for any $z \in Z$, $z \le T(z)$ implies $z \le z^*$ and $z \ge T(z)$ implies $z \ge z^*$, where z^* is the fixed point of T in Z.*

Lemma 2.3. *(c.f. [12]) A function $u \in PC^1([t_0, T])$ satisfies (2.1) if and only if there is a function $f \in PC([t_0 - \tau, T])$ and a sequence $\{f_k\}$ (which depends on f) such that $|f(t)| \le \varepsilon$ for all $t \in [t_0 - \tau, T]$, $|f_k| \le \varepsilon$ for all $k = 1, 2, \cdots, m$ and*

$$\begin{aligned} &u'(t) = F(t, u(t), u(h(t))) + \int_{t_0}^{t} G(t, s, u(h(s)))ds + f(t), t \in I', \\ &\triangle u(t_k) = \phi_k(u(t_k^-)) + f_k, k = 1, 2, 3, \cdots, m. \end{aligned} \tag{2.21}$$

Remark 2.1. A function $u \in PC^1([t_0, T])$ satisfies (2.2) if and only if there is a function $f \in PC([t_0 - \tau, T])$ and a sequence $\{f_k\}$ (which depends on f) such that $|f(t)| \le \varphi(t)$ for all $t \in [t_0 - \tau, T]$, $|f_k| \le \chi$ for all $k = 1, 2, 3, \cdots, m$ and (2.21) holds.

Lemma 2.4. *Each solution $u \in PC^1([t_0, T])$ of (2.1) satisfies the following integral inequality*

$$\begin{aligned} &\Bigg| u(t) - u(t_0) - \sum_{j=1}^{k} \phi_j(u(t_j^-)) - \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ &- \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx \Bigg| \le (t - t_0 + m)\varepsilon \end{aligned}$$

for all $t \in (t_k, t_{k+1}] \subset [t_0, T]$, $k = 0, 1, 2, \cdots, m$.

Proof: If $u \in PC^1([t_0, T])$ satisfies (2.1), then by Lemma 2.3, we have

$$\begin{aligned} &u'(t) = F(t, u(t), u(h(t))) + \int_{t_0}^{t} G(t, s, u(s), u(h(s)))ds + f(t), t \in I', \\ &\triangle u(t_k) = \phi_k(u(t_k^-)) + f_k, k = 1, 2, \cdots, m. \end{aligned} \tag{2.22}$$

For $t \in [t_0, t_1]$, integrating (2.22) from t_0 to t implies that

$$\int_{t_0}^{t} u'(x)dx = \int_{t_0}^{t}\left[F(x, u(x), u(h(x))) + \int_{t_0}^{x} G(x, s, u(s), u(h(s)))ds + f(x)\right]dx,$$

then we have

$$u(t) = u(t_0) + \int_{t_0}^{t}\left[F(x, u(x), u(h(x))) + \int_{t_0}^{x} G(x, s, u(s), u(h(s)))ds + f(x)\right]dx,$$

and

$$u(t_1^-) = u(t_1) = u(t_0) + \int_{t_0}^{t_1} F(x, u(x), u(h(x)))dx + \int_{t_0}^{t_1} \int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_0}^{t_1} f(x)dx.$$

For $t \in (t_1, t_2]$, integrating (2.22) from t_1 to t implies that

$$\int_{t_1}^{t} u'(x)dx = \int_{t_1}^{t} \left[F(x, u(x), u(h(x))) + \int_{t_0}^{x} G(x, s, u(s), u(h(s)))ds + f(x) \right] dx.$$

Then we get

$$\begin{aligned} u(t) =& u(t_1^+) + \int_{t_1}^{t} F(x, u(x), u(h(x)))dx + \int_{t_1}^{t} \int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx \\ &+ \int_{t_1}^{t} f(x)dx \\ =& \phi_1(u(t_1^-)) + u(t_1^-) + f_1 + \int_{t_1}^{t} F(x, u(x), u(h(x)))dx \\ &+ \int_{t_1}^{t} \int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_1}^{t} f(x)dx \\ =& \phi_1(u(t_1^-)) + u(t_0) + f_1 + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ &+ \int_{t_0}^{t} \int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_0}^{t} f(x)dx. \end{aligned} \tag{2.23}$$

So we have

$$\begin{aligned} u(t_2^-) =& \phi_1(u(t_1^-)) + u(t_0) + f_1 + \int_{t_0}^{t_2} F(x, u(x), u(h(x)))dx \\ &+ \int_{t_0}^{t_2} \int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_0}^{t_2} f(x)dx. \end{aligned}$$

Now, suppose that for $t \in (t_{k-1}, t_k]$, we have

$$\begin{aligned} u(t) =& u(t_0) + \sum_{i=1}^{k-1} f_i + \sum_{j=1}^{k-1} \phi_j(u(t_j^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ &+ \int_{t_0}^{t} \int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_0}^{t} f(x)dx, \end{aligned}$$

and

$$\begin{aligned} u(t_k^-) =& u(t_0) + \sum_{i=1}^{k-1} f_i + \sum_{j=1}^{k-1} \phi_j(u(t_j^-)) + \int_{t_0}^{t_k} F(x, u(x), u(h(x)))dx \\ &+ \int_{t_0}^{t_k} \int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_0}^{t_k} f(x)dx. \end{aligned}$$

Then for $t \in (t_k, t_{k+1}]$, integrating (2.22) from t_k to t implies that

$$\begin{aligned}
u(t) =& u(t_k^+) + \int_{t_k}^{t} F(x, u(x), u(h(x)))dx \\
&+ \int_{t_k}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_k}^{t} f(x)dx \\
=& \phi_k(u(t_k^-)) + u(t_k^-) + f_k + \int_{t_k}^{t} F(x, u(x), u(h(x)))dx \\
&+ \int_{t_k}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx + \int_{t_k}^{t} f(x)dx \\
=& u(t_0) + \sum_{i=1}^{k} f_i + \sum_{j=1}^{k} \phi_j(u(t_j^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\
&+ \int_{t_0}^{t} f(x)dx + \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx.
\end{aligned}$$

Hence, by the method of mathematical induction, we have

$$\begin{aligned}
u(t) = u(t_0) &+ \sum_{i=1}^{k} f_i + \sum_{j=1}^{k} \phi_j(u(t_j^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\
&+ \int_{t_0}^{t} f(x)dx + \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_k, t_{k+1}].
\end{aligned} \tag{2.24}$$

It follows by (2.24) that

$$\begin{aligned}
&\Bigg| u(t) - u(t_0) - \sum_{j=1}^{k} \phi_j(u(t_j^-)) - \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\
&- \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx \Bigg| \\
&= \left| \int_{t_0}^{t} f(x)dx + \sum_{i=1}^{k} f_i \right| \le \int_{t_0}^{t} |f(x)|dx + \sum_{i=1}^{k} |f_i| \\
&\le (t - t_0 + k)\varepsilon \le (t - t_0 + m)\varepsilon, t \in (t_k, t_{k+1}].
\end{aligned}$$

This completes the proof of Lemma 2.4.

Remark 2.2. Each solution $u \in PC^1([t_0, T])$ of (2.2) satisfies the following integral inequality

$$\begin{aligned}
&\Bigg| u(t) - u(t_0) - \sum_{j=1}^{k} \phi_j(u(t_j^-)) - \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\
&- \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx \Bigg| \le \rho\varphi(t) + m\chi, \ \ for\ t \in (t_k, t_{k+1}] \subset [t_0, T].
\end{aligned}$$

3. Hyers-Ulam Stability

In this section, the Hyers-Ulam stability for the impulsive delay integro-differential equation is studied by using Definition 2.1, Lemma 2.1 and Lemma 2.2.

Theorem 3.1. *Suppose that the following hypotheses hold:*

(A_1) $F : [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}$, $G : [t_0, T] \times [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous, and* F, G *are Lipschitz continuous with respect to the last two variables,*

$$|F(x, \eta_1, \eta_2) - F(x, \xi_1, \xi_2)| \le \sum_{i=1}^{2} L_1 |\eta_i - \xi_i|, \tag{3.1}$$

$$|G(x, s, \eta_1, \eta_2) - G(x, s, \xi_1, \xi_2)| \le \sum_{i=1}^{2} L_1 L_2 |\eta_i - \xi_i|, \tag{3.2}$$

where L_1, $L_2 > 0$, *for all* $x, s \in I'$;

(A_2) $\phi_j : \mathbb{R} \to \mathbb{R}$ *is such that* $|\phi_j(\eta_1) - \phi_j(\eta_2)| \le M_j |\eta_1 - \eta_2|$, $M_j > 0$ *for all* $j \in \{1, 2, \cdots, m\}$ *and* $\eta_1, \eta_2 \in \mathbb{R}$;

(A_3) $\sum_{j=1}^{m} M_j + 2L_1(T - t_0) + (T - t_0)^2 L_1 L_2 < 1.$

Then there exists a unique solution of problem (1.1) *in* $PC([t_0 - \tau, T]) \cap PC^1([t_0, T])$ *and equation* (1.1) *is Hyers-Ulam stable on* $[t_0 - \tau, T]$.

Proof: (1) We define an operator $T : PC([t_0 - \tau, T]) \to PC([t_0 - \tau, T])$ as

$$(Tu)(t) = \begin{cases} \alpha(t), t \in [t_0 - \tau, t_0], \\ \alpha(t_0) + \displaystyle\int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ \qquad + \displaystyle\int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_0, t_1], \\ \alpha(t_0) + \phi_1(u(t_1^-)) + \displaystyle\int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ \qquad + \displaystyle\int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_1, t_2], \\ \alpha(t_0) + \displaystyle\sum_{j=1}^{2} \phi_j(u(t_j^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ \qquad + \displaystyle\int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_2, t_3], \\ \qquad \vdots \\ \alpha(t_0) + \displaystyle\sum_{j=1}^{m} \phi_j(u(t_j^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ \qquad + \displaystyle\int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_m, t_{m+1}]. \end{cases} \tag{3.3}$$

For any u_1, $u_2 \in PC([t_0-\tau,T])$, and for all $t\in[t_0-\tau,t_0]$, we have

$$|(Tu_1)(t)-(Tu_2)(t)|=0.$$

For $t\in(t_k,t_{k+1}]$, we deduce that

$$\begin{aligned}
&|(Tu_1)(t)-(Tu_2)(t)|\\
\le&\sum_{j=1}^{k}\left|\phi_j(u_1(t_j^-))-\phi_j(u_2(t_j^-))\right|\\
&+\int_{t_0}^{t}|F(x,u_1(x),u_1(h(x)))-F(x,u_2(x),u_2(h(x)))|\,dx\\
&+\int_{t_0}^{t}\int_{t_0}^{x}|G(x,s,u_1(s),u_1(h(s)))-G(x,s,u_2(s),u_2(h(s)))|\,dsdx\\
\le&\sum_{j=1}^{k}M_j\left|u_1(t_j^-)-u_2(t_j^-)\right|+L_1\int_{t_0}^{t}|u_1(x)-u_2(x)|\,dx\\
&+L_1\int_{t_0}^{t}|u_1(h(x))-u_2(h(x))|\,dx+L_1L_2\int_{t_0}^{t}\int_{t_0}^{x}|u_1(s)-u_2(s)|\,dsdx\\
&+L_1L_2\int_{t_0}^{t}\int_{t_0}^{x}|u_1(h(s))-u_2(h(s))|\,dsdx\\
\le&\left(\sum_{j=1}^{k}M_j+2L_1(T-t_0)+(T-t_0)^2L_1L_2\right)\cdot\sup_{t\in[t_0-\tau,T]}|u_1(x)-u_2(x)|\\
\le&\left(\sum_{j=1}^{m}M_j+2L_1(T-t_0)+(T-t_0)^2L_1L_2\right)\|u_1-u_2\|\,.
\end{aligned}$$

By (A_3), the operator T is strictly contractive on $(t_k,t_{k+1}],k=0,1,2,\cdots,m$, and hence it is a Picard operator on $PC([t_0-\tau,T])$. By (3.1) and (3.2), the unique fixed point of this operator is in fact the unique solution of (1.1) in $PC([t_0-\tau,T])\cap PC^1([t_0,T])$.

Next, let $y\in PC([t_0-\tau,T])\cap PC^1([t_0,T])$ be a solution of (2.1). The unique solution $u\in PC([t_0-\tau,T])\cap PC^1([t_0,T])$ of the following initial value problem

$$\begin{aligned}
u'(t)&=F(t,u(t),u(h(t)))+\int_{t_0}^{t}G(t,s,u(s),u(h(s)))ds,t\in I',\\
u(t)&=y(t),t\in[t_0-\tau,t_0]\,,\\
\triangle u(t_k)&=\phi_k(u(t_k^-)),k=1,2,\cdots,m
\end{aligned}\tag{3.4}$$

is given by

$$u(t) = \begin{cases} y(t), t \in [t_0 - \tau, t_0], \\ y(t_0) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx + \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_0, t_1], \\ y(t_0) + \phi_1(u(t_1^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ \qquad + \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_1, t_2], \\ y(t_0) + \sum_{j=1}^{2} \phi_j(u(t_j^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ \qquad + \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_2, t_3], \\ \qquad \vdots \\ y(t_0) + \sum_{j=1}^{m} \phi_j(u(t_j^-)) + \int_{t_0}^{t} F(x, u(x), u(h(x)))dx \\ \qquad + \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, u(s), u(h(s)))dsdx, t \in (t_m, t_{m+1}]. \end{cases} \tag{3.5}$$

We observe that for $t \in [t_0 - \tau, t_0]$, $|y(t) - u(t)| = 0$.

For $t \in (t_k, t_{k+1}]$, using Lemma 2.4, let

$$\begin{aligned} B(t) = y(t) - y(t_0) - \sum_{j=1}^{k} \phi_j(y(t_j^-)) - \int_{t_0}^{t} F(x, y(x), y(h(x)))dx \\ - \int_{t_0}^{t}\int_{t_0}^{x} G(x, s, y(s), y(h(s)))dsdx, \end{aligned}$$

we get

$$\begin{aligned} |y(t) - u(t)| \leq & |B(t)| + \int_{t_0}^{t} |F(x, y(x), y(h(x))) - F(x, u(x), u(h(x)))| \, dx \\ & + \int_{t_0}^{t}\int_{t_0}^{x} |G(x, s, y(s), y(h(s))) - G(x, s, u(s), u(h(s)))| \, dsdx \\ & + \sum_{j=1}^{k} \left|\phi_j(y(t_j^-)) - \phi_j(u(t_j^-))\right| \\ \leq & (m + t - t_0)\varepsilon + L_1 \int_{t_0}^{t} |y(x) - u(x)| \, dx + L_1 \int_{t_0}^{t} |y(h(x)) - u(h(x))| \, dx \\ & + L_1 L_2 \int_{t_0}^{t}\int_{t_0}^{x} |y(s) - u(s)| \, dsdx + L_1 L_2 \int_{t_0}^{t}\int_{t_0}^{x} |y(h(s)) - u(h(s))| \, dsdx \\ & + \sum_{j=1}^{k} M_j \left|y(t_j^-) - u(t_j^-)\right|. \end{aligned}$$

Next, we show that the operator $\Lambda : PC([t_0-\tau,T]) \to PC([t_0-\tau,T])$ given below is an increasing Picard operator on $PC([t_0-\tau,T])$

$$(\Lambda v)(t) = \begin{cases} 0, \qquad t \in [t_0-\tau, t_0], \\ (t-t_0)\varepsilon + L_1 \int_{t_0}^{t} v(x)dx + L_1 \int_{t_0}^{t} v(h(x))dx \\ \ + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(s)dsdx + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(h(s))dsdx, t \in (t_0, t_1], \\ (1+t-t_0)\varepsilon + M_1 v(t_1^-) + L_1 \int_{t_0}^{t} v(x)dx + L_1 \int_{t_0}^{t} v(h(x))dx \\ \ + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(s)dsdx + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(h(s))dsdx, t \in (t_1, t_2], \\ (2+t-t_0)\varepsilon + \Sigma_{j=1}^{2} M_j v(t_j^-) + L_1 \int_{t_0}^{t} v(x)dx + L_1 \int_{t_0}^{t} v(h(x))dx \\ \ + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(s)dsdx + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(h(s))dsdx, t \in (t_2, t_3], \\ \vdots \\ (m+t-t_0)\varepsilon + \sum_{j=1}^{m} M_j v(t_j^-) + L_1 \int_{t_0}^{t} v(x)dx + L_1 \int_{t_0}^{t} v(h(x))dx \\ \ + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(s)dsdx + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v(h(s))dsdx, t \in (t_m, t_{m+1}]. \end{cases} \tag{3.6}$$

For any $v_1,\ v_2 \in PC([t_0-\tau,T])$, $|(\Lambda v_1)(t) - (\Lambda v_2)(t)| = 0$ for $t \in [t_0-\tau, t_0]$. For $t \in (t_k, t_{k+1}]$, we see that

$$\begin{aligned} |(\Lambda v_1)(t) - (\Lambda v_2)(t)| \le & \sum_{j=1}^{k} M_j \left| v_1(t_j^-) - v_2(t_j^-) \right| + L_1 \int_{t_0}^{t} |v_1(x) - v_2(x)|\, dx \\ & + L_1 \int_{t_0}^{t} |v_1(h(x)) - v_2(h(x))|\, dx \\ & + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} |v_1(s) - v_2(s)|\, dsdx \\ & + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} |v_1(h(s)) - v_2(h(s))|\, dsdx \\ \le & \sum_{j=1}^{k} M_j \sup_{t\in[t_0-\tau,T]} |v_1(t) - v_2(t)| + L_1(t-t_0) \sup_{t\in[t_0-\tau,T]} |v_1(t) - v_2(t)| \\ & + L_1(t-t_0) \sup_{t\in[t_0-\tau,T]} |v_1(h(t)) - v_2(h(t))| \\ & + L_1L_2 \int_{t_0}^{t} (x-t_0)dx \sup_{t\in[t_0-\tau,T]} |v_1(t) - v_2(t)| \\ & + L_1L_2 \int_{t_0}^{t} (x-t_0)dx \sup_{t\in[t_0-\tau,T]} |v_1(h(t)) - v_2(h(t))| . \end{aligned}$$

So we get

$$|(\Lambda v_1)(t)-(\Lambda v_2)(t)| \leq \left(\sum_{j=1}^{m} M_j + 2L_1(T-t_0) + (T-t_0)^2 L_1 L_2\right) \|v_1 - v_2\|.$$

According to (A_3), the operator Λ is contractive on $PC([t_0-\tau, T])$ in each interval $(t_k, t_{k+1}]$, where $k = 0, 1, 2, \cdots, m$. Applying Banach contraction principle, we get Λ is a Picard operator and hence it has a unique fixed point, that is $v^* \in PC([t_0-\tau, T])$, and

$$\begin{aligned} v^*(t) =& (m+t-t_0)\varepsilon + \sum_{j=1}^{k} M_j v^*(t_j^-) + L_1 \int_{t_0}^{t} v^*(x)dx + L_1 \int_{t_0}^{t} v^*(h(x))dx \\ &+ L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v^*(s)dsdx + L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v^*(h(s))dsdx, t \in (t_k, t_{k+1}]. \end{aligned}$$

v^* is increasing, so $v^*(h(t)) \leq v^*(t)$ and hence we can get

$$v^*(t) \leq (m+T-t_0)\varepsilon + \sum_{j=1}^{k} M_j v^*(t_j^-) + 2L_1 \int_{t_0}^{t} v^*(x)dx + 2L_1L_2 \int_{t_0}^{t}\int_{t_0}^{x} v^*(s)dsdx. \tag{3.7}$$

Using Lemma 2.1, we get

$$v^*(t) \leq (m+T-t_0)\varepsilon \prod_{i=1}^{k} \left[1 + H(t_{i-1}, t_i)\left(A(t_{i-1}, t_i) + M_i\right)\right] H(t_k, t).$$

where

$$A(t_{i-1}, t_i) = 2L_1 \left[1 - L_2 t_0 + \frac{L_2}{2}(t_i + t_{i-1})\right](t_i - t_{i-1}) \tag{3.8}$$

$$\begin{aligned} H(t_{i-1}, t_i) = 1 + &\frac{2L_1}{2L_1+L_2}(t_i - t_{i-1})\left[1 - L_2 t_0 + \frac{L_2}{2}(t_i + t_{i-1})\right] \\ &\cdot \left[L_2 + 2L_1 \exp\{(2L_1+L_2)(t_i - t_{i-1})\}\right] \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} H(t_k, t) = 1 + &\frac{2L_1}{2L_1+L_2}(t - t_k)\left[1 - L_2 t_0 + \frac{L_2}{2}(t_k + t)\right] \\ &\cdot \left[L_2 + 2L_1 \exp\{(2L_1+L_2)(t - t_k)\}\right] \end{aligned} \tag{3.10}$$

Set $v(t) = |y(t) - u(t)|$, by (3.6), $v(t) \leq (\Lambda v)(t)$, then by using abstract Gronwall Lemma 2.2, we get $v(t) \leq v^*$. Thus

$$\begin{aligned} |y(t)-u(t)| \leq v^*(t) &\leq (m+T-t_0)\varepsilon \prod_{i=1}^{k} \left[1 + H(t_{i-1}, t_i)\left(A(t_{i-1}, t_i) + M_i\right)\right] H(t_k, t) \\ &\leq k\varepsilon \end{aligned}$$

for all $t \in [t_0-\tau, T]$, where

$$k = (m+T-t_0)\prod_{i=1}^{k} \left[1 + H(t_{i-1}, t_i)\left(A(t_{i-1}, t_i) + M_i\right)\right] H(t_k, T).$$

Consequently, equation (1.1) is Hyers-Ulam stable and the proof is completed.

Remark 3.1. Equation (1.1) has the following two special cases.

$$\begin{aligned} &u'(t) = F(t, u(t), u(h(t))) + \int_{t_0}^{t} G(t, s, u(s), u(h(s)))ds, t \in I, \\ &u(t) = \alpha(t), t \in [t_0 - \tau, t_0], \tau \geq 0. \end{aligned} \tag{3.11}$$

$$\begin{aligned} &u'(t) = F(t, u(t), u(h(t))) + \int_{t_0}^{t} G(t, s, u(s), u(h(s)))ds, t \in I', \\ &u(t_0) = \alpha(t_0), \triangle u(t_k) = u(t_k^+) - u(t_k^-) = \phi_k(u(t_k^-)), k = 1, 2, \cdots, m. \end{aligned} \tag{3.12}$$

For these two special cases, the following corollaries can be obtained by applying Theorem 3.1.

Corollary 3.1. *Suppose that condition* (A_1) *is satisfied and* $2L_1(T - t_0) + (T - t_0)^2 L_1 L_2 < 1$*, then there exists a unique solution of problem* (3.11) *in* $C([t_0 - \tau, T]) \cap C^1([t_0, T])$ *and equation* (3.11) *is Hyers-Ulam stable on* $[t_0 - \tau, T]$.

Corollary 3.2. *If conditions* $(A_1) - (A_3)$ *are satisfied, then there exists a unique solution of problem* (3.12) *in* $PC([t_0, T])$ *and equation* (3.12) *is Hyers-Ulam stable on* $[t_0, T]$.

4. Hyers-Ulam-Rassias Stability

In this section, we will prove the Hyers-Ulam-Rassias stability of the impulsive delay integro-differential equations by using Definition 2.2, Lemma 2.1, Remark 2.1 and Lemma 2.2 on $[t_0 - \tau, T]$.

Theorem 4.1. *Suppose that the following hypotheses hold:*

(A_1') $F : [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}$, $G : [t_0, T] \times [t_0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous with the Lipschitz condition:*

$$|F(x, \eta_1, \eta_2) - F(x, \xi_1, \xi_2)| \leq \sum_{i=1}^{2} L_1 |\eta_i - \xi_i| ; \tag{4.1}$$

$$|G(x, s, \eta_1, \eta_2) - G(x, s, \xi_1, \xi_2)| \leq \sum_{i=1}^{2} L_1 L_2 |\eta_i - \xi_i| ; \tag{4.2}$$

where L_1, $L_2 > 0$ *for all* x, $s \in I'$;

(A_2') $\phi_j : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous, i.e.,* $|\phi_j(\eta_1) - \phi_j(\eta_2)| \leq M_j |\eta_1 - \eta_2|$ *for some constant* $M_j > 0$*, and for all* $j \in \{1, 2, \cdots, m\}$ *and* η_1, $\eta_2 \in \mathbb{R}$;

(A_3') $\sum_{j=1}^{m} M_j + 2L_1(T - t_0) + (T - t_0)^2 L_1 L_2 < 1$;

(A_4') $\varphi(t) : [t_0 - \tau, T] \to \mathbb{R}^+$ *is an increasing function, and* $\int_{t_0}^{t} \varphi(r)dr \leq \rho\varphi(t)$ *for some constant* $\rho > 0$.

Then there exists a unique solution of problem (1.1) *in* $PC([t_0-\tau,T])\cap PC^1([t_0,T])$ *and equation* (1.1) *is Hyers-Ulam-Rassias stable on* $[t_0-\tau,T]$.

Proof: For given $\varepsilon>0$, $\varphi(t)\in PC([t_0-\tau,T])$, where $\varphi(t)$ is an increasing and nonnegative functions, $\varphi(t_1)=\chi>0$ for some $t_1\in[t_0-\tau,T]$. Following the same proof steps as Theorem 3.1, we have $|T(u_1)(t)-T(u_2)(t)|\leq(\sum_{j=1}^m M_j+2L_1(T-t_0)+(T-t_0)^2L_1L_2)\cdot\|u_1-u_2\|$, where the operator T is defined by (3.3), $t\in(t_k,t_k+1]$, $k=0,1,2,\cdots,m$.

Using(A_3'), the operator T is strictly contractive on $(t_k,t_k+1]$, $k=1,2,\cdots,m$, and T is a Picard operator on $PC([t_0-\tau,T])$. Thus, the unique fixed point of this operator is in fact the unique solution of (1.1) in $PC([t_0-\tau,T])\cap PC^1([t_0,T])$.

Let $y\in PC([t_0-\tau,T])\cap PC^1([t_0,T])$ be a solution to (2.2). The unique solution $u\in PC([t_0-\tau,T])\cap PC^1([t_0,T])$ of the differential equation (3.4) is given by (3.5). Following the proof Theorem 3.1, we have

$$\begin{aligned}|y(t)-u(t)|\leq&\int_{t_0}^t\varphi(x)dx+k\chi+L_1\int_{t_0}^t|y(x)-u(x)|\,dx\\&+L_1\int_{t_0}^t|y(h(x))-u(h(x))|\,dx+L_1L_2\int_{t_0}^t\int_{t_0}^x|y(s)-u(s)|\,dsdx\\&+L_1L_2\int_{t_0}^t\int_{t_0}^x|y(h(s))-u(h(s))|\,dsdx+\sum_{j=1}^k M_j|y(t_j^-)-u(t_j^-)|.\end{aligned}\tag{4.3}$$

Next we show that operator $\Lambda_1:PC([t_0-\tau,T])\to PC([t_0-\tau,T])$ given below is an increasing Picard operator on $PC([t_0-\tau,T])$:

$$(\Lambda_1v)(t)=\begin{cases}0,\qquad t\in[t_0-\tau,t_0];\\ \rho\varphi(t)+L_1\int_{t_0}^t v(x)dx+L_1\int_{t_0}^t v(h(x))dx+L_1L_2\int_{t_0}^t\int_{t_0}^x v(s)dsdx\\ \qquad+L_1L_2\int_{t_0}^t\int_{t_0}^x v(h(s))dsdx,\qquad t\in(t_0,t_1];\\ \rho\varphi(t)+\chi+M_1v(t_1^-)+L_1\int_{t_0}^t v(x)dx+L_1\int_{t_0}^t v(h(x))dx\\ \qquad+L_1L_2\int_{t_0}^t\int_{t_0}^x v(s)dsdx+L_1L_2\int_{t_0}^t\int_{t_0}^x v(h(s))dsdx,t\in(t_1,t_2];\\ \rho\varphi(t)+2\chi+\Sigma_{j=1}^2M_jv(t_j^-)+L_1\int_{t_0}^t v(x)dx+L_1\int_{t_0}^t v(h(x))dx\\ \qquad+L_1L_2\int_{t_0}^t\int_{t_0}^x v(s)dsdx+L_1L_2\int_{t_0}^t\int_{t_0}^x v(h(s))dsdx,t\in(t_2,t_3];\\ \quad\vdots\\ \rho\varphi(t)+m\chi+\sum_{j=1}^m M_jv(t_j^-)+L_1\int_{t_0}^t v(x)dx+L_1\int_{t_0}^t v(h(x))dx\\ \qquad+L_1L_2\int_{t_0}^t\int_{t_0}^x v(s)dsdx+L_1L_2\int_{t_0}^t\int_{t_0}^x v(h(s))dsdx,t\in(t_m,t_{m+1}].\end{cases}$$

Using the same proof as Theorem 3.1, we obtain that the operator Λ_1 is contractive on $PC([t_0-\tau,T])$ for $t\in(t_k,t_k+1]$ where $k=0,1,2,\cdots,m$. Applying Banach contraction principle, we get Λ_1 is a Picard operator with unique fixed $v^*\in PC([t_0-\tau,T])$, that is

$$\begin{aligned}v^*(t)=&\rho\varphi(t)+k\chi+\sum_{j=1}^{k}M_jv^*(t_j^-)+L_1\int_{t_0}^{t}v^*(x)dx+L_1\int_{t_0}^{t}v^*(h(x))dx\\&+L_1L_2\int_{t_0}^{t}\int_{t_0}^{x}v^*(s)dsdx+L_1L_2\int_{t_0}^{t}\int_{t_0}^{x}v^*(h(s))dsdx,t\in(t_k,t_{k+1}].\end{aligned}\tag{4.4}$$

Since v^* is increasing, $v^*(h(t))\le v^*(t)$ and hence we get by (4.4) that

$$v^*(t)\le\rho\varphi(t)+k\chi+\sum_{j=1}^{k}M_jv^*(t_j^-)+2L_1\int_{t_0}^{t}v^*(x)dx+2L_1L_2\int_{t_0}^{t}\int_{t_0}^{x}v^*(s)dsdx.$$

Using Lemma 2.1, we get

$$v^*(t)\le(k\chi+\rho\varphi(t))\varepsilon\prod_{i=1}^{k}\left[1+H(t_{i-1},t_i)\left(A(t_{i-1},t_i)+M_i\right)\right]H(t_k,t).$$

where $A(t_{i-1},t_i)$ is defined by (2.6), and $H(t_k,t)$ is defined by (2.7). If we set $v(t)=|y(t)-u(t)|$, then by (3.6), $v(t)\le(\Lambda_1v)(t)$ and using the abstract Gronwall lemma, it follows that $v(t)\le v^*$. Thus

$$|y(t)-u(t)|\le(k\chi+\rho\varphi(t))\varepsilon\prod_{i=1}^{k}\left[1+H(t_{i-1},t_i)\left(A(t_{i-1},t_i)+M_i\right)\right]H(t_k,T).$$

Consequently, equation (1.1) is Hyers-Ulam-Rassias stable, and the proof is completed.

Corollary 4.1. *If conditions (A_1') and (A_4') are satisfied and $2L_1(T-t_0)+(T-t_0)^2L_1L_2<1$, then there exists a unique solution of problem (3.11) in $C([t_0-\tau,T])\cap C^1([t_0,T])$ and equation (3.11) is Hyers-Ulam-Rassias stable on $[t_0-\tau,T]$.*

Corollary 4.2. *If conditions $(A_1')-(A_4')$ are satisfied, then there exists a unique solution of problem (3.12) in $PC^1([t_0,T])$ and equation (3.12) is Hyers-Ulam-Rassias stable on $[t_0,T]$.*

5. Example

Integro-differential equations determined by non-local operators describe many dynamical systems, such as brain dynamics, population dynamics, infectious disease spreading, and learning dynamics through neural networks [4]- [7]. In the 1950s, two sensational results appeared in the United Kingdom and the United States, namely, the Hodgkin-Huxley equation, a mathematical model describing the conduction of nerve impulses, and the Hartland-Ratliff equation, which described the side inhibition of the visual system, both of which were complex nonlinear equations, and aroused the interest of mathematicians and biologists.

The application of differential equation solvers to learn the dynamical behaviors using neural networks was initially introduced in [25, 26], aiming to fulfill the requirement for continuous deep learning models. In 2023, E. Zappala et al. [7] introduce the new neural integro-differential equation, a novel deep learning framework is based on the theory of integro-differential equations where integral operators are learned using neural networks. The general form of neural integro-differential equation is given as

$$u'(t) = F(t, u(t)) + \int_{\alpha(t)}^{\beta(t)} G(t, s, u(s))ds, \tag{5.1}$$

where $u : \mathbb{R} \to \mathbb{R}^n$ is a vector function of independent time variable t, $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $G : \mathbb{R}^{n+2} \to \mathbb{R}^n$. The theory of integro-differential equations is better understood in the setting where the function G is a product of type $k(t, s)f(u(s))$ for some function $f : \mathbb{R}^n \to \mathbb{R}^n$, and a matrix valued function $k : \mathbb{R}^2 \to M(\mathbb{R}, n)$ is called a kernel function, where $M(\mathbb{R}, n)$ indicates the space of square matrices with real coefficients. So we obtain the special case of (5.1) as

$$u'(t) = F(t, u(t)) + \int_{\alpha(t)}^{\beta(t)} k(t, s)f(u(s))ds,$$

where both k and f are neural networks that will be learned during training.

Example 5.1. We consider the system of neural integro-differential equation

$$\begin{cases} u_1'(t) = -\dfrac{|u_1(t)|}{3(1 + |u_1(t)|)} + \displaystyle\int_0^t \dfrac{\sin 2\pi t}{7} \left(\sin |u_1(s)| + \dfrac{|u_2(s)|}{1 + |u_2(s)|}\right) ds, \\ u_2'(t) = -\dfrac{|u_2(t)|}{3(1 + |u_2(t)|)} + \displaystyle\int_0^t \dfrac{\cos 2\pi t}{7} \left(\sin |u_1(s)| + \dfrac{|u_2(s)|}{1 + |u_2(s)|}\right) ds, \end{cases} \tag{5.2}$$

where $t \in [0, 100] \setminus \{25, 50\}$, $t_0 = 0$, $t_1 = 25$, $t_2 = 50$, and $t_3 = 100$. For vector $u = (u_1, u_2)^T \in \mathbb{R}^2$ and matrix $A = (a_{ij})_{2\times 2}$, we define $\|U\| = \sum_{i=1}^2 |u_i|$, $\|A\| = \max_{1\le j\le 2} \sum_{i=1}^2 |a_{ij}|$. Then equation (5.2) can be written in vector form

$$U'(t) = F(t, U(t)) + \int_0^t G(t, s, U(s))ds,$$

where $F(t, U) = (F_1(t, U), F_2(t, U))^T = \left(-\frac{|u_1|}{3(1+|u_1|)}, -\frac{|u_2|}{3(1+|u_2|)}\right)^T$, and $G(t, s, U) = K(t, s)f(U)$,

$$K(t, s) = \begin{pmatrix} \frac{\sin 2\pi t}{7} & \frac{\sin 2\pi t}{7} \\ \frac{\cos 2\pi t}{7} & \frac{\cos 2\pi t}{7} \end{pmatrix},$$

$f(U) = (f_1(u_1), f_2(u_2))^T = \left(\sin |u_1|, \frac{|u_2|}{1+|u_2|}\right)^T$. Let $M_1 = \frac{1}{10}$, $M_2 = \frac{1}{100}$, $L_1 = \frac{1}{300}$, $L_2 = \frac{1}{1000}$. For $U = (u_1, u_2)^T$, $\widetilde{U} = (\widetilde{u_1}, \widetilde{u_2})^T \in (PC[0, 100]\backslash\{25, 50\})^2$, we get

$$|F_i(t, U) - F_i(t, \widetilde{U})| = \left|\frac{1}{3}\left(\frac{|\widetilde{u}_i|}{1 + |\widetilde{u}_i|} - \frac{|u_i|}{1 + |u_i|}\right)\right| \le \frac{1}{3}L_1\|u_i - \widetilde{u}_i\|$$

for $i = 1, 2$, so we get

$$\|F(t,U) - F(t,\widetilde{U})\| \leq \frac{1}{3}L_1\|U - \widetilde{U}\| = \frac{1}{900}\|U - \widetilde{U}\|.$$

$$|f_1(u_1) - f_1(\widetilde{u}_1)| = |\sin|u_1| - \sin|\widetilde{u}_1|| \leq \|u_1 - \widetilde{u}_1\|,$$

$$|f_2(u_2) - f_2(\widetilde{u}_2)| = \left|\frac{|u_2|}{1+|u_2|} - \frac{|\widetilde{u}_2|}{1+|\widetilde{u}_2|}\right| \leq \|u_2(t) - \widetilde{u}_2(t)\|.$$

Then we have

$$|f(U) - f(\widetilde{U})| \leq \|U - \widetilde{U}\|.$$

We can show that

$$\| K(t,s) \| = \frac{|\sin 2\pi t||\cos 2\pi t|}{7},$$

$$|G(t,s,U) - G(t,s,\widetilde{U})| \leq \| K(t,s) \| \cdot \|U - \widetilde{U}\| \leq \frac{1}{7}\|U - \widetilde{U}\|.$$

Then

$$M_1 + M_2 + 2L_1(T - t_0) + L_1L_2(T - t_0)^2 = \frac{243}{300} < 1.$$

Finally, by Theorem 3.1, the neural integro-differential equation has a unique solution in $PC^1[0, 100]$ and also all the conditions in Theorem 3.1 hold. Therefore, the equation (5.2) is Hyers-Ulam stable on $[0, 100]$.

6. Conclusions

Differential equations and differential-integral equations are used to solve continuous deep learning models through neural network learning dynamics, and are applied to a new generation of information technology. In this paper, the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of impulsive delay ordinary differential equations is obtained by using a novel generalized Gronwall inequality, fixed-point method and Picard's operator technique. For third-order and higher order differential equations, fractional differential equations,the results on Ulam type stability are very little, which can be researched in future. Moreover, the application of differential equations and differential-integral equations in artificial intelligence and deep learning has just begun, and there are many novel and interesting problems to be solved.

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