Convergence analysis of the Uzawa iterative method for the thermally coupled stationary incompressible magnetohydrodynamics flow

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Abstract

In this paper, we designed an Uzawa iterative method for solving the thermally coupled stationary incompressible magnetohydrodynamics system, where a decoupled discrete system is obtained and no saddle point problem is required to deal with at each iterative step except the initial guess. Then, the convergence analysis of the presented method is provided. Finally, the effectiveness of the proposed method is illustrated with some numerical examples.

Keywords: Uzawa iterative method; Thermally coupled magnetohydrodynamics; Relaxation parameter; Convergence analysis

1. Introduction

This paper is concerned with numerical methods for solving the thermally coupled incompressible magnetohydrodynamics (MHD) flow for applications like the design of electromagnetic pumps, nuclear reactor cooling, steel casting, and crystal growth. The thermally coupled MHD model can describe buoyancy effects due to temperature differences in the MHD flow. The governing equations of this model are the MHD equations coupled with the heat equation by the Boussinesq approximation. The strong coupling between the saddle-point subproblems, the nonlinearity, and the extra temperature field needed for the MHD flow make accurate simulation of the MHD flow challenging.

Although multi-physical field coupling of the stationary thermally coupled incompressible magnetohydrodynamics model makes its numerical simulation challenging, the research is of great significance due to the wide applications of the model. There have been many literatures on numerical investigations for the considered equations in recent years. For example, the thermally coupled MHD problem is studied initially by Meir [18, 19], where the existence and uniqueness of the solutions to the considered equations are addressed, and the

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finite element approximation for the problem is studied. Further, Meir and Schmidt [21] have proposed a general approach to stationary, electromagnetically, and thermally driven liquid-metal flows. Some existence results of weak solutions to two stationary MHD systems of equations including Joule heating have been given in [4]. Ravindran [26] has investigated and proposed an efficient partitioned time-stepping scheme for solving the MHD system with temperature-dependent coefficients. Badia et al. [3] have extended block preconditioning techniques to the thermally coupled incompressible inductionless MHD problem. Additionally, a stabilized finite element method is designed to solve thermally coupled MHD flows [6]. Sheikholeslami [27] has conducted an investigation into the importance of the generated magnetic field in enhancing Brownian diffusion and its profound influence on the hydrothermal analysis of the base liquid. This stabilized method is based on splitting the unknown into a finite element component and a subscale and on giving an approximation for the latter. Recently, Yang and Zhang [29] have proposed three iterative finite element methods for the thermally coupled stationary incompressible MHD equations. Zhang et al. [31] have designed the two-level finite element iterative methods for the stationary thermally coupled incompressible MHD equations. Keram et al. [16] have designed an iterative method based on linearization approach for the thermally coupled stationary incompressible MHD equations at high physical parameters. A decoupled Crank-Nicolson time-stepping scheme is designed in [25], and the unconditional stability and optimal order error estimates of the scheme are proved. In addition, for the non-stationary equations, Ding et al. [9] have given the Crank-Nicolson extrapolated fully discrete scheme based on the finite element method and obtained some optimal error estimates for the velocity, magnetic induction, and temperature under a weak regularity hypothesis. Moreover, Qiu et al. [24] have studied a fully discrete Euler semi-implicit scheme for the nonstationary electromagnetically and thermally driven flow, which is describing the motion of a nonisothermal incompressible magnetohydrodynamics fluid subject to generalized Boussinesq problem with temperature dependent parameters. Ma et al. [20] have proposed a fully discrete decoupled finite element method for the thermally coupled incompressible magnetohydrodynamic problem. In addition, Liu et al. [17] have presented a grad-div stabilization with the Jacobi iteration to the thermally coupled incompressible MHD system. Yang et al. [29] have proved existence and uniqueness of weak solution to a Voigt regularization of the three-dimensional thermally coupled MHD equations and proposed a fully discrete scheme that has unconditional stability and is convergent. Although the previous works obtain many efficient numerical results, relatively little attention is given to the development of efficient numerical methods to deal with the strong coupling between the saddle-point subproblems.

As is known, the Uzawa iterative method [2] is first proposed to solve the constrained optimization problems, where the saddle point problem naturally arises. Since it is simple, efficient, and has minimal computer memory requirements, it has been widely used in computational science and engineering. For the nonlinear partial differential equations, Chen et al. [7] have constructed some Uzawa-type iterative methods for solving the steady incompressible Navier-Stokes equations and have proved that the methods converge geometrically with a contraction number. Further, the steady-state MHD equations are solved by applying some Uzawa-type iterative algorithms [30]. The lines of arguments in the presented paper follow closely those used in [30]. Recently, Hong et al. [15] have presented an augmented Lagrangian Uzawa-type method for qquasi-static multiple-network poroelasticity equations and the robust uniform linear convergence of its parameters is proved. Moreover, Çıbık et al. [8] have constructed the Ramshaw-Mesina iteration to solve some saddle-point problems. Ouertani et al. [23] have presented two algorithms for the resolution of the time-dependent S-tokes problem with nonstandard boundary conditions by the domain-decomposition spectral element method.

In this paper, to conquer the numerical difficulties mentioned earlier and find an efficient and accurate approximation of the thermally coupled stationary incompressible MHD problem, we are going to devise an Uzawa iterative method for the considered problem based on a mixed finite element method, where a decoupled discrete system is solved and no saddle-point system is required to solve at each iterative step except the initial guess. The remainder of the paper is organized as follows. In Section 2, we describe the problem to be solved, some notations, and the basic facts to be used throughout the paper. The method is proposed and fully analyzed for the considered problem in Section 3. Numerical examples are presented in the final section.

2. Problem statement

Let Ω be a bounded, simple-connected domain in $\mathbb{R}^2$, which is convex or has a $C^{1,1}$ boundary $\partial\Omega$. In this paper, we consider the following thermally coupled stationary incompressible MHD equations [18, 19], i.e., the stationary incompressible Navier-Stokes equations and Maxwell's equations coupled to the heat equation by the Boussinesq approximation

$$\begin{aligned} -Re^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + s\mathbf{H}\times \mathrm{curl}\mathbf{H} =& \mathbf{f} + \beta T\mathbf{j}, \quad \text{in } \Omega,\\ sRm^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{H}) - s\mathrm{curl}(\mathbf{u}\times\mathbf{H}) =& \mathbf{g}, \quad \text{in } \Omega,\\ -\kappa\Delta T + \mathbf{u}\cdot\nabla T =& \gamma, \quad \text{in } \Omega,\\ \mathrm{div}\mathbf{u} = 0, \ \ \mathrm{div}\mathbf{H} =& 0, \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\mathbf{u} = (u_1(x), u_2(x), 0)$, $\mathbf{H} = (H_1(x), H_2(x), 0)$, p and T are the velocity field, magnetic field, pressure and temperature, respectively. Several coefficients appeared in (1) are the hydrodynamic Reynolds number Re, the magnetic Reynolds number Rm, the thermal expansion coefficient β, the thermal conductivity κ and the coupling number s. In addition, $\mathbf{g} = (g_1(x), g_2(x), 0)$ represents the known applied current with $\mathrm{div}\mathbf{g} = 0$, $\mathbf{f} = (f_1(x), f_2(x), 0)$, is a force term for the magnetic induction, γ is a given heat source, and $\mathbf{j}$ denotes a unit vector in the direction opposite to the direction of gravity for $\mathbf{u}$.

Furthermore, the system (1) is considered in conjunction with the following boundary conditions [18, 14, 5, 9]:

$$\begin{cases} \mathbf{u}|_{\partial\Omega} = 0, \quad \text{(no-slip condition)},\\ \mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{n}\times\mathrm{curl}\mathbf{H}|_{\partial\Omega} = 0, \quad \text{(perfectly conducting wall)},\\ T|_{\Gamma_D} = 0, \quad \nabla T\cdot\mathbf{n}|_{\Gamma_N} = 0, \quad \text{(insulated wall)}, \end{cases} \tag{2}$$

where $\mathbf{n}$ the outer unit normal vector to $\partial\Omega$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ is a regular open subset of $\partial\Omega$.

For $1 \le q \le \infty$ and $m \in \mathbb{N}^+$, we define the usual Sobolev space $W^{m,q}(\Omega)$ norm and Lebegue space $L^q(\Omega)$ norm by$\|\cdot\|_{W^{m,q}(\Omega)}$ and $\|\cdot\|_{L^q(\Omega)}$, respectively. Particularly, $L^2(\Omega)$ norm and its inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$. In addition, we write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ represents the norm in $H^m(\Omega)$. Next, to write the variational form of the system (1)-(2), we introduce the following necessary function spaces:

$$\mathbf{X} = \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v}|_{\partial\Omega} = 0\}, \quad \mathbf{W} = \{\mathbf{B} \in H^1(\Omega)^2 : \mathbf{B}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$
$$Q = \{S \in H^1(\Omega) : S|_{\Gamma_D} = 0\}, \quad M = \{q \in L^2(\Omega) : (q,1) = 0\}.$$

Now, we introduce the product space $\mathbf{D} = \mathbf{X} \times \mathbf{W}$, which is equipped with the norm for all $(\mathbf{w}, \mathbf{\Phi}) \in \mathbf{D}$, $\|\nabla(\mathbf{w},\mathbf{\Phi})\|_0^2 = \|\nabla\mathbf{w}\|_0^2 + \|\nabla\mathbf{\Phi}\|_0^2$.

Moreover, we define three continuous bilinear forms $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ on $Q\times Q$, $\mathbf{X}\times\mathbf{X}$ and $\mathbf{W}\times\mathbf{W}$, respectively, by

$$a_0(T,S) =\kappa(\nabla T, \nabla S), \quad \forall T,S \in Q, \qquad a_1(\mathbf{u},\mathbf{v}) = Re^{-1}(\nabla\mathbf{u},\nabla\mathbf{v}), \quad \forall\,\mathbf{u},\mathbf{v}\in\mathbf{X},$$
$$a_2(\mathbf{H},\mathbf{B}) =sRm^{-1}\big((\mathrm{curl}\mathbf{H},\mathrm{curl}\mathbf{B}) + (\mathrm{div}\mathbf{H},\mathrm{div}\mathbf{B})\big), \quad \forall\,\mathbf{H},\mathbf{B}\in\mathbf{W},$$

and three trilinear forms $b_0(\cdot,\cdot,\cdot)$, $b_1(\cdot,\cdot,\cdot)$ and $b_2(\cdot,\cdot,\cdot)$ on $\mathbf{X}\times Q\times Q$, $\mathbf{X}\times\mathbf{X}\times\mathbf{X}$ and $\mathbf{W}\times\mathbf{W}\times\mathbf{X}$, by

$$b_0(\mathbf{u},T,S) =(\mathbf{u}\cdot\nabla T,S) + \frac{1}{2}((\mathrm{div}\mathbf{u})T,S) = \frac{1}{2}(\mathbf{u}\cdot\nabla T,S) - \frac{1}{2}(\mathbf{u}\cdot\nabla S,T), \quad \forall\,\mathbf{u}\in\mathbf{X},\ T,S\in Q,$$
$$b_1(\mathbf{u},\mathbf{w},\mathbf{v}) =((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}) + \frac{1}{2}((\mathrm{div}\mathbf{u})\mathbf{w},\mathbf{v}) = \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}) - \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}), \quad \forall\,\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X},$$
$$b_2(\mathbf{H},\mathbf{B},\mathbf{v}) =s(\mathbf{H}\times\mathrm{curl}\mathbf{B},\mathbf{v}), \quad \forall\,\mathbf{v}\in\mathbf{X},\ \mathbf{H},\mathbf{B}\in\mathbf{W}.$$

These trilinear forms satisfy the following properties [14, 12, 11]:

$$|b_0(\mathbf{u},T,S)| \le N_0\|\nabla\mathbf{u}\|_0\|\nabla T\|_0\|\nabla S\|_0, \quad |b_1(\mathbf{u},\mathbf{w},\mathbf{v})| \le N_1\|\nabla\mathbf{u}\|_0\|\nabla\mathbf{w}\|_0\|\nabla\mathbf{v}\|_0, \tag{3}$$
$$|b_2(\mathbf{H},\mathbf{B},\mathbf{v})| \le sN_2\|\nabla\mathbf{H}\|_0\|\nabla\mathbf{B}\|_0\|\nabla\mathbf{v}\|_0, \tag{4}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$ and $\mathbf{H},\mathbf{B}\in\mathbf{W}$, where $N_i > 0$, $i = 0,1,2$ are constants depending on Ω. In addition, from Gerbeau et al. [11], Girault and Raviart [12], Nochetto and Pyo [22] and Gunzburger et al. [13], it is seen that the following inequalities hold: for all $\mathbf{B}\in\mathbf{W}$,

$$\|\nabla\mathbf{B}\|_0^2 \le c_1(\|\mathrm{curl}\mathbf{B}\|_0^2 + \|\mathrm{div}\mathbf{B}\|_0^2), \tag{5}$$
$$\|\mathrm{curl}\mathbf{B}\|_0 \le \sqrt{2}\|\nabla\mathbf{B}\|_0, \tag{6}$$

where c_1 is positive constant and only dependent on Ω.

Then, the thermally coupled stationary incompressible MHD problem (1)-(2) can be rewritten as: For all $((\mathbf{v},\mathbf{B}),S,q)\in\mathbf{D}\times Q\times M$, search for $((\mathbf{u},\mathbf{H}),T,p)\in\mathbf{D}\times Q\times M$ such that

$$\begin{aligned} A_0((\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{B})) + A_1((\mathbf{u},\mathbf{H}),(\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{B})) - d((\mathbf{v},\mathbf{B}),p) &=(\mathbf{F},(\mathbf{v},\mathbf{B})) + G(T,(\mathbf{v},\mathbf{B})),\\ d((\mathbf{u},\mathbf{H}),q) &=0,\\ a_0(T,S) + b_0(\mathbf{u},T,S) &=(\gamma,S), \end{aligned} \tag{7}$$

where $A_1((\mathbf{u},\mathbf{H}),(\mathbf{w},\mathbf{\Phi}),(\mathbf{v},\mathbf{B})) = b_1(\mathbf{u},\mathbf{w},\mathbf{v})+b_2(\mathbf{H},\mathbf{\Phi},\mathbf{v})-b_2(\mathbf{H},\mathbf{B},\mathbf{w})$, $A_0((\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{B})) = a_1(\mathbf{u},\mathbf{v})+a_2(\mathbf{H},\mathbf{B})$, $(\mathbf{F},(\mathbf{v},\mathbf{B})) = (\mathbf{f},\mathbf{v})+(\mathbf{g},\mathbf{B})$, $G(T,(\mathbf{v},\mathbf{B})) = \beta(T\mathbf{j},\mathbf{v})$ and $d((\mathbf{v},\mathbf{B}),p) = (\mathrm{div}\mathbf{v},p)$.

To discuss the well-posedness of the above mixed variational formulation, we recall the coercive and continuous properties of $A_0((\cdot,\cdot),(\cdot,\cdot))$ and the continuous property of $A_1((\cdot,\cdot),(\cdot,\cdot),(\cdot,\cdot))$.

Lemma 2.1. *[10, 30] For all* $(\mathbf{u},\mathbf{H}),(\mathbf{w},\mathbf{\Phi}),(\mathbf{v},\mathbf{B}) \in \mathbf{D}$, *there hold*

$$\begin{aligned} A_0((\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{B})) &\le c_A\|\nabla(\mathbf{u},\mathbf{H})\|_0\|\nabla(\mathbf{v},\mathbf{B})\|_0,\\ A_0((\mathbf{u},\mathbf{H}),(\mathbf{u},\mathbf{H})) &\ge \nu_A\|\nabla(\mathbf{u},\mathbf{H})\|_0^2,\\ A_1((\mathbf{u},\mathbf{H}),(\mathbf{w},\mathbf{\Phi}),(\mathbf{v},\mathbf{B})) &\le N\|\nabla(\mathbf{u},\mathbf{H})\|_0\|\nabla(\mathbf{w},\mathbf{\Phi})\|_0\|\nabla(\mathbf{v},\mathbf{B})\|_0, \end{aligned}$$

where $c_A = \max\{Re^{-1}, 4sRm^{-1}\}$, $\nu_A = \min\{Re^{-1}, sRm^{-1}c_1^{-1}\}$ *and* $N = \sqrt{2}\max\{N_1, sN_2\}$.

Now, we recall the following existence and uniqueness results for the problem (1)-(2) [18, 19, 29].

Theorem 2.1. *Let* $\gamma \in Q'$, $\mathbf{F} \in \mathbf{D}'$ *and* ν_A *satisfy the following uniqueness condition:*

$$0 < \delta < 1,$$

where $\delta = \delta_1+\delta_2$ *with* $\delta_1 := \nu_A^{-2}N(\|\mathbf{F}\|_{-1}+\kappa^{-1}\beta\|\gamma\|_{-1})$ *and* $\delta_2 := \nu_A^{-1}\kappa^{-2}\beta N_0\|\gamma\|_{-1}$, $\|\gamma\|_{-1} = \sup\limits_{T\in Q, T\neq 0}\frac{(\gamma,T)}{\|\nabla T\|_0}$, $\|\mathbf{F}\|_{-1} = \sup\limits_{(\mathbf{u},\mathbf{H})\in\mathbf{D},(\mathbf{u},\mathbf{H})\neq(\mathbf{0},\mathbf{0})}\frac{(\mathbf{F},(\mathbf{u},\mathbf{H}))}{\|\nabla(\mathbf{u},\mathbf{H})\|_0}$, *and* C_F *is the Poincaré constant. Then the problem* (1) *and* (2) *admits a unique solution* $((\mathbf{u},\mathbf{H}),p,T) \in \mathbf{D}\times M\times Q$ *such that*

$$\nu_A\|\nabla(\mathbf{u},\mathbf{H})\|_0 \le (\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}), \qquad \kappa\|\nabla T\|_0 \le \|\gamma\|_{-1}.$$

Noting that in the above theorem, one applies the fact that $\|v\|_{-1} \le C_F^2\|\nabla v\|_0$, for all $v \in H_0^1(\Omega)$.

3. An Uzawa iterative method

From now on, let h be a real positive parameter. The conforming finite element subspaces $(\mathbf{X}_h,\mathbf{W}_h,Q_h,M_h)$ of $(\mathbf{X},\mathbf{W},Q,M)$ is characterized by $K_h = K_h(\Omega)$, a partitioning of $\overline{\Omega}$ into triangles K, assumed to be uniformly regular as $h \to 0$. Next, we define the product space $\mathbf{D}_h = \mathbf{X}_h\times\mathbf{W}_h$. Further, we assume that the couple $\mathbf{X}_h\times M_h$ admits the following discrete inf-sup condition: for each $q_h \in M_h$, there exists $\mathbf{v}_h \in \mathbf{X}_h$, such that [12]

$$\sup_{0\neq\mathbf{v}_h\in\mathbf{X}_h}\frac{\int_\Omega q_h\,\mathrm{div}\mathbf{v}_h\,dx}{\|\nabla\mathbf{v}_h\|_0} \ge \widetilde{\beta}\|q_h\|_0, \tag{8}$$

where $\widetilde{\beta} > 0$ is a constant depending on Ω.

Then, according to the above finite element subspaces, the finite element approximation for (7) is to seek $((\mathbf{u}_h, \mathbf{H}_h), T_h, p_h) \in \mathbf{D}_h \times Q_h \times M_h$ such that for all $((\mathbf{v}, \mathbf{B}), S, q) \in \mathbf{D}_h \times Q_h \times M_h$

$$
\begin{aligned}
&a_0(T_h, S) + b_0(\mathbf{u}_h, T_h, S) = (\gamma, S), \\
&A_0((\mathbf{u}_h, \mathbf{H}_h), (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{u}_h, \mathbf{H}_h), (\mathbf{u}_h, \mathbf{H}_h), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), p_h) = (\mathbf{F}, (\mathbf{v}, \mathbf{B})) \\
&\quad + G(T_h, (\mathbf{v}, \mathbf{B})), \\
&d((\mathbf{u}_h, \mathbf{H}_h), q) = 0.
\end{aligned} \tag{9}
$$

The following results can be found in [19, 29], which describe the stability of the numerical solutions obtained by (9).

Theorem 3.1. *Let $((\mathbf{u}_h, \mathbf{H}_h), T_h, p_h) \in \mathbf{D}_h \times Q_h \times M_h$ be a solution of the finite element discretization (9). Then, under the assumptions of Theorem 2.1, there hold*

$$
\nu_A \|\nabla(\mathbf{u}_h, \mathbf{H}_h)\|_0 \leq (\|\mathbf{F}\|_{-1} + C_F^2 \beta \kappa^{-1} \|\gamma\|_{-1}), \qquad \kappa \|\nabla T_h\|_0 \leq \|\gamma\|_{-1}.
$$

Now, we present an Uzawa iterative method for the finite element scheme (9) of the considered problem, and then analyze its convergence based on a positive number ρ called relaxation parameter.

Algorithm 3.1. *(Uzawa algorithm). Given an initial guess $((\mathbf{u}_h^0, \mathbf{H}_h^0), T_h^0, p_h^0) \in \mathbf{D}_h \times Q_h \times M_h$, search for $((\mathbf{u}_h^{n+1}, \mathbf{H}_h^{n+1}), T_h^{n+1}, p_h^{n+1}) \in \mathbf{D}_h \times Q_h \times M_h$ such that*

$$
\begin{cases}
a_0(T_h^{n+1}, S) + b_0(\mathbf{u}_h^n, T_h^{n+1}, S) = (\gamma, S), \\
A_0((\mathbf{u}_h^{n+1}, \mathbf{H}_h^{n+1}), (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{u}_h^n, \mathbf{H}_h^n), (\mathbf{u}_h^{n+1}, \mathbf{H}_h^{n+1}), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), p_h^n) \\
\quad = (\mathbf{F}, (\mathbf{v}, \mathbf{B})) + G(T_h^{n+1}, (\mathbf{v}, \mathbf{B})), \\
(p_h^{n+1}, q) = (p_h^n, q) - \rho d((\mathbf{u}_h^{n+1}, \mathbf{H}_h^{n+1}), q),
\end{cases} \tag{10}
$$

for all $(\mathbf{v}, \mathbf{B}) \in \mathbf{D}_h$, $S \in Q_h$ and $q \in M_h$, where $\rho > 0$ is the relaxation parameter and n denotes iterative step.

Note that the nonlinear terms in (9) are linearized by allowing the nonlinearities to lag one time step behind. Then, the velocity, magnetic, pressure, and temperature are decoupled, a decoupled discrete system is obtained and no saddle point problem is required to solve at each iterative step except the initial guess. Furthermore, for Algorithm 3.1, the initial guess $((\mathbf{u}_h^0, \mathbf{B}_h^0), T_h^0, p_h^0) \in \mathbf{D}_h \times Q_h \times M_h$ is defined by solving the following equations:

$$
\begin{cases}
a_0(T_h^0, S) = (\gamma, S), \quad \forall S \in Q_h, \\
A_0((\mathbf{u}_h^0, \mathbf{H}_h^0), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), p_h^0) = (\mathbf{F}, (\mathbf{v}, \mathbf{B})) + G(T_h^0, (\mathbf{v}, \mathbf{B})), \quad \forall (\mathbf{v}, \mathbf{B}) \in \mathbf{D}_h, \\
d((\mathbf{u}_h^0, \mathbf{H}_h^0), q) = 0, \quad \forall q \in M_h.
\end{cases} \tag{11}
$$

Although (11) is a saddle point problem, it is also a linear problem. Compared with the nonlinear saddle point problem (9), it is much simpler to solve. In the numerical tests, we apply the Crout solver to obtain the initial guess.

Then, we will expect to prove the following iterative error bounds between the finite element solutions of (9) and the Uzawa iterative solutions of (10). For convenience, we set $\mathbf{e}_h^n = \mathbf{u}_h - \mathbf{u}_h^n$, $\boldsymbol{\xi}_h^n = \mathbf{H}_h - \mathbf{H}_h^n$, $\eta_h^n = p_h - p_h^n$ and $\theta_h^n = T_h - T_h^n$, $n \geq 0$. Firstly, we need to derive the iterative error bounds between the equations (9) and (11).

Lemma 3.1. *Let* $((\mathbf{u}_h^0, \mathbf{H}_h^0), p_h^0, T_h^0) \in \mathbf{D}_h \times Q_h \times M_h$ *be the solution of (11). Then, under the assumptions of Theorem 3.1, we have the following results*

$$\begin{aligned} \|\nabla\theta_h^0\|_0 &\leq N_0\kappa^{-2}\nu_A^{-1}\|\gamma\|_{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}), \\ \|\nabla(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)\|_0 &\leq \nu_A^{-1}\delta(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}), \\ \|\eta_h^0\|_0 &\leq \widetilde{\beta}^{-1}\delta(c_A\nu_A^{-1} + 1)(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}). \end{aligned}$$

Proof. By subtracting the first equation of (9) from the first equation of (11), we have

$$a_0(\theta_h^0, S) + b_0(\mathbf{u}_h, T_h, S) = 0. \tag{12}$$

Taking $S = \theta_h^0$ in (12) and using (3) yield

$$\|\nabla\theta_h^0\|_0 \leq \kappa^{-1}N_0\|\nabla\mathbf{u}_h\|_0\|\nabla T_h\|_0 \leq \kappa^{-1}N_0\|\nabla(\mathbf{u}_h, \mathbf{H}_h)\|_0\|\nabla T_h\|_0.$$

In view of Theorem 3.1, we deduce that

$$\|\nabla\theta_h^0\|_0 \leq N_0\kappa^{-2}\nu_A^{-1}\|\gamma\|_{-1}(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}). \tag{13}$$

Next, subtract the second equation of (9) from the second equation of (11) to get

$$A_0((\mathbf{e}_h^0, \boldsymbol{\xi}_h^0), (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{u}_h, \mathbf{H}_h), (\mathbf{u}_h, \mathbf{H}_h), (\mathbf{v}, \mathbf{B})) - d((\mathbf{v}, \mathbf{B}), \eta_h^0) = G(\theta_h^0, (\mathbf{v}, \mathbf{B})). \tag{14}$$

Choosing $(\mathbf{v}, \mathbf{B}) = (\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)$ in (14), we obtain

$$A_0((\mathbf{e}_h^0, \boldsymbol{\xi}_h^0), (\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)) = -A_1((\mathbf{u}_h, \mathbf{H}_h), (\mathbf{u}_h, \mathbf{H}_h), (\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)) + G(\theta_h^0, (\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)),$$

where we have used the third equation of (9) and the third equation of (11). Hence, utilizing Lemma 2.1, we get

$$\nu_A\|\nabla(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)\|_0 \leq N\|\nabla(\mathbf{u}_h, \mathbf{H}_h)\|_0^2 + \beta\|\theta_h^0\|_{-1},$$

which combines with (13) and Theorem 3.1 to give

$$\|\nabla(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)\|_0 \leq \nu_A^{-1}\delta(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}). \tag{15}$$

Finally, use the discrete inf-sup condition (8) and (14) to bound

$$\begin{aligned}\|\eta_h^0\|_0 &\le \widetilde{\beta}^{-1}(c_A\|\nabla(\mathbf{e}_h^0,\boldsymbol{\xi}_h^0)\|_0 + N\|\nabla(\mathbf{u}_h,\mathbf{H}_h)\|_0^2 + \beta\|\theta_h^0\|_{-1}) \\ &\le \widetilde{\beta}^{-1}\delta(c_A\nu_A^{-1}+1)(\|\mathbf{F}\|_{-1} + C_F^2\beta\kappa^{-1}\|\gamma\|_{-1}),\end{aligned}$$

where we have applied (13) and (15). □

Next, we will consider the convergence of the Uzawa iterative method for the thermally coupled stationary incompressible MHD problem. First, we show that the function sequence generated by this iterative algorithm is bounded, and then we will develop the corresponding convergence rate analysis based on the relaxation parameter.

Theorem 3.2. *Let $\{(\mathbf{u}_h^n,\mathbf{H}_h^n),p_h^n,T_h^n\}$ be the function sequence of Algorithm 3.1. Then, under the assumptions of Theorem 3.1, if the relaxation parameter $\rho\in(0,2\nu_A(1-\delta))$, then we have*

$$\begin{aligned}D_1\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\eta_h^{n+1}\|_0^2 &\le D_1\|\nabla(\mathbf{e}_h^0,\boldsymbol{\xi}_h^0)\|_0^2 + \|\eta_h^0\|_0^2, \\ D_1\|\nabla\theta_h^{n+1}\|_0^2 &\le \kappa^{-4}N_0^2\|\gamma\|_{-1}^2(D_1\|\nabla(\mathbf{e}_h^0,\boldsymbol{\xi}_h^0)\|_0^2 + \|\eta_h^0\|_0^2),\end{aligned}$$

where $D_1=\frac{\rho}{2}((2\nu_A-\rho)+\sqrt{\Delta})$ and $\Delta=(2\nu_A-\rho)^2-4\nu_A^2\delta^2$.

Proof. Subtracting the first equation of (9) from the first equation of (10), we have

$$a_0(\theta_h^{n+1},S) + b_0(\mathbf{e}_h^n,T_h,S) + b_0(\mathbf{u}_h^n,\theta_h^{n+1},S) = 0. \tag{16}$$

Taking $S=\theta_h^{n+1}$ in (16) and combining (3) with the fact that $b_0(\mathbf{u}_h^n,\theta_h^{n+1},\theta_h^{n+1})=0$, we obtain

$$\kappa\|\nabla\theta_h^{n+1}\|_0 \le N_0\|\nabla\mathbf{e}_h^n\|_0\|\nabla T_h\|_0,$$

which combines Theorem 3.1 to give

$$\|\nabla\theta_h^{n+1}\|_0 \le \kappa^{-2}N_0\|\gamma\|_{-1}\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0. \tag{17}$$

Then, subtracting the second equation of (9) from the second equation of (10), we have

$$\begin{aligned}&A_0((\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),(\mathbf{v},\mathbf{B})) + A_1((\mathbf{e}_h^n,\boldsymbol{\xi}_h^n),(\mathbf{u}_h,\mathbf{H}_h),(\mathbf{v},\mathbf{B})) - d((\mathbf{v},\mathbf{B}),\eta_h^n) \\ &\quad + A_1((\mathbf{u}_h^n,\mathbf{H}_h^n),(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),(\mathbf{v},\mathbf{B})) = G(\theta_h^{n+1},(\mathbf{v},\mathbf{B})).\end{aligned} \tag{18}$$

Selecting $(\mathbf{v},\mathbf{B})=(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})$ in (18) yields

$$\begin{aligned}&\nu_A\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2 - d((\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),\eta_h^n) \\ &\le -A_1((\mathbf{e}_h^n,\boldsymbol{\xi}_h^n),(\mathbf{u}_h,\mathbf{H}_h),(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})) + G(\theta_h^{n+1},(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})).\end{aligned} \tag{19}$$

Note that $A_1((\mathbf{u}_h^n,\mathbf{H}_h^n),(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}))=0$.

Moreover, according to the Polarization identity $2(a,b)=\|a+b\|_0^2-\|a\|_0^2-\|b\|_0^2$, combining the third equation of (9) and the third equation of (10), we arrive at

$$\begin{aligned}&-d((\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),\eta_h^n)=-d((\mathbf{u}_h,\mathbf{H}_h),\eta_h^n)+d((\mathbf{u}_h^{n+1},\mathbf{H}_h^{n+1}),\eta_h^n)=\rho^{-1}(p_h^n-p_h^{n+1},\eta_h^n)\\&=\rho^{-1}(\eta_h^{n+1}-\eta_h^n,\eta_h^n)=(2\rho)^{-1}(\|\eta_h^{n+1}\|_0^2-\|\eta_h^n\|_0^2-\|\eta_h^{n+1}-\eta_h^n\|_0^2).\end{aligned}\tag{20}$$

Plugging (20) into (19) leads to

$$\begin{aligned}&2\rho\nu_A\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2+\|\eta_h^{n+1}\|_0^2\le\|\eta_h^n\|_0^2+\|\eta_h^{n+1}-\eta_h^n\|_0^2+2\rho G(\theta_h^{n+1},(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}))\\&\quad-2\rho A_1((\mathbf{e}_h^n,\boldsymbol{\xi}_h^n),(\mathbf{u}_h,\mathbf{H}_h),(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})).\end{aligned}\tag{21}$$

Next, applying the third equation of (9) and third equation of (10) again to estimate the term $\|\eta_h^{n+1}-\eta_h^n\|_0$, we get

$$(\eta_h^{n+1}-\eta_h^n,q)=(p_h^n-p_h^{n+1},q)=\rho d((\mathbf{u}_h^{n+1},\mathbf{H}_h^{n+1}),q)=-\rho d((\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1}),q).\tag{22}$$

Setting $q=\eta_h^{n+1}-\eta_h^n$ in (22) and noticing the fact that $\|\mathrm{div}\mathbf{v}\|_0\le\|\nabla\mathbf{v}\|_0$ proved in [22] yield

$$\|\eta_h^{n+1}-\eta_h^n\|_0\le\rho\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0.\tag{23}$$

Hence, making use of (23) and Lemma 2.1, we rewrite (21) as

$$\begin{aligned}&\rho(2\nu_A-\rho)\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2+\|\eta_h^{n+1}\|_0^2\le\|\eta_h^n\|_0^2+2\rho\beta\|\theta_h^{n+1}\|_{-1}\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0\\&\quad+2\rho N\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0\|\nabla(\mathbf{u}_h,\mathbf{H}_h)\|_0\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0.\end{aligned}$$

Then, using (17) and the Young inequality, we derive that

$$\rho(\nu_A(2-\delta\varsigma)-\rho)\|\nabla(\mathbf{e}_h^{n+1},\boldsymbol{\xi}_h^{n+1})\|_0^2+\|\eta_h^{n+1}\|_0^2\le\|\eta_h^n\|_0^2+\rho\nu_A\delta\varsigma^{-1}\|\nabla(\mathbf{e}_h^n,\boldsymbol{\xi}_h^n)\|_0^2,\tag{24}$$

where $\varsigma>0$ is a parameter to be determined later on.

Furthermore, we solve a quadratic algebraic equation

$$\nu_A\delta\varsigma^2-(2\nu_A-\rho)\varsigma+\nu_A\delta=0,$$

to get a positive root $\varsigma=\varsigma^\star$ which makes $\nu_A(2-\delta\varsigma)-\rho=\nu_A\delta\varsigma^{-1}$ hold. It is easy to obtain

$$\varsigma^\star=\frac{(2\nu_A-\rho)-\sqrt{\Delta}}{2\nu_A\delta},$$

where $\Delta = (2\nu_A - \rho)^2 - 4\nu_A^2\delta^2$. Note that the condition $\rho \in (0, 2\nu_A(1-\delta))$. Hence, $\varsigma^\star$ is a positive root, due to the fact that $\Delta = (2\nu_A(1+\delta) - \rho)(2\nu_A(1-\delta) - \rho)$.

Finally, let $D_1 = \rho(\nu_A(2 - \delta\varsigma^\star) - \rho) = \rho\nu_A\delta(\varsigma^\star)^{-1}$. Then, $D_1 = \frac{\rho}{2}((2\nu_A - \rho) + \sqrt{\Delta})$ and (24) is rewritten as

$$D_1\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\eta_h^{n+1}\|_0^2 \leq D_1\|\nabla(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n)\|_0^2 + \|\eta_h^n\|_0^2. \tag{25}$$

In view of (17), it follows from (25) that

$$\begin{aligned} D_1\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\eta_h^{n+1}\|_0^2 \leq D_1\|\nabla(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)\|_0^2 + \|\eta_h^0\|_0^2, \\ D_1\|\nabla\theta_h^{n+1}\|_0^2 \leq \kappa^{-4}N_0^2\|\gamma\|_{-1}^2(D_1\|\nabla(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)\|_0^2 + \|\eta_h^0\|_0^2). \end{aligned}$$

□

Now, we are going to develop our convergence rate analysis for the Uzawa iterative algorithm.

Theorem 3.3. *Under assumptions of Theorem 3.2, the following estimates hold:*

$$\begin{aligned} D\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\eta_h^{n+1}\|_0^2 \leq & H^{n+1}(D\|\nabla(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)\|_0^2 + \|\eta_h^0\|_0^2), \\ D\|\nabla\theta_h^{n+1}\|_0^2 \leq & \kappa^{-4}N_0^2\|\gamma\|_{-1}^2 H^n(D\|\nabla(\mathbf{e}_h^0, \boldsymbol{\xi}_h^0)\|_0^2 + \|\eta_h^0\|_0^2), \end{aligned}$$

where $D \in \big(0, \frac{1}{2}\nu_A^2\big)$ and $H \in \Big(1 - \frac{1}{4}\big(\frac{\nu_A}{c_A}\big)^2, 1\Big)$ are two parameters independent of n and h.

Proof. In fact, according to Lemma 3.1 and Theorem 3.2, 3.1, there exists a positive constant D_2 independent of n and h such that

$$\|\nabla(\mathbf{u}_h^n, \mathbf{H}_h^n)\|_0 \leq D_2.$$

Then, rewrite (18) to give

$$\begin{aligned} d((\mathbf{v}, \mathbf{B}), \eta_h^n) = & A_0((\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}), (\mathbf{v}, \mathbf{B})) - G(\theta_h^{n+1}, (\mathbf{v}, \mathbf{B})) + A_1((\mathbf{e}_h^n, \boldsymbol{\xi}_h^n), (\mathbf{u}_h, \mathbf{H}_h), (\mathbf{v}, \mathbf{B})) \\ & + A_1((\mathbf{u}_h^n, \mathbf{H}_h^n), (\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1}), (\mathbf{v}, \mathbf{B})). \end{aligned}$$

Applying the inf-sup condition (8) to the above equation, we obtain

$$\widetilde{\beta}\|\eta_h^n\|_0 \leq (c_A + ND_2)\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0 + \nu_A\delta\|\nabla(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n)\|_0,$$

where we have used Theorem 3.1 and (17). That is

$$\widetilde{\beta}^2\|\eta_h^n\|_0^2 \leq 2\big((c_A + ND_2)^2\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \nu_A^2\delta^2\|\nabla(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n)\|_0^2\big).$$

Hence, one gets

$$\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 \geq D_3\|\eta_h^n\|_0^2 - D_4\|\nabla(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n)\|_0^2, \tag{26}$$

where $D_3 = \frac{\widetilde{\beta}^2}{2(c_A+ND_2)^2}$ and $D_4 = \frac{\nu_A^2\delta^2}{(c_A+ND_2)^2}$.

Next, denote $c_{\rho,\varsigma} = \rho(\nu_A(2-\delta\varsigma)-\rho)$. Then (24) becomes

$$\sigma\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + (c_{\rho,\varsigma}-\sigma)\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\eta_h^{n+1}\|_0^2 \leq \rho\nu_A\delta\varsigma^{-1}\|\nabla(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n)\|_0^2 + \|\eta_h^n\|_0^2,$$

where $\sigma \in (0, c_{\rho,\varsigma})$ is a parameter to be determined. Substituting (26) into the above inequality, we obtain

$$\begin{aligned}&(c_{\rho,\varsigma}-\sigma)\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\eta_h^{n+1}\|_0^2\\ \leq\ &(\varsigma^{-1}\rho\nu_A\delta + \sigma D_4)\|\nabla(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n)\|_0^2 + (1-\sigma D_3)\|\eta_h^n\|_0^2.\end{aligned} \tag{27}$$

Further, we will choose parameters ς and σ such that

$$\frac{c_{\rho,\varsigma}-\sigma}{1} = \frac{\varsigma^{-1}\rho\nu_A\delta + \sigma D_4}{1-\sigma D_3}, \tag{28}$$

and $1-\sigma D_3 > 0$, which leads to

$$D_3\sigma^2 - (1 + c_{\rho,\varsigma}D_3 + D_4)\sigma + c_{\rho,\varsigma} - \delta\rho\nu_A\varsigma^{-1} = 0. \tag{29}$$

Because

$$c_{\rho,\varsigma} - \delta\rho\nu_A\varsigma^{-1} = (1 + c_{\rho,\varsigma}D_3 + D_4)\sigma - D_3\sigma^2 > c_{\rho,\varsigma}D_3\sigma - D_3\sigma^2 > 0,$$

we get $c_{\rho,\varsigma} - \delta\rho\nu_A\varsigma^{-1} > 0$, which combines the definition of $c_{\rho,\varsigma}$ to yield

$$\delta\nu_A\varsigma^2 - (2\nu_A-\rho)\varsigma + \nu_A\delta < 0.$$

Next, we solve this quadratic inequality to get

$$\frac{(2\nu_A-\rho)-\sqrt{\Delta}}{2\nu_A\delta} < \varsigma < \frac{(2\nu_A-\rho)+\sqrt{\Delta}}{2\nu_A\delta}.$$

Noticing that $\rho \in (0, 2\nu_A(1-\delta))$, we arrive at $\Delta = ((2\nu_A-\rho)+2\nu_A\delta)((2\nu_A-\rho)-2\nu_A\delta) > 0$. Here, we select

$$\varsigma = \varsigma^\dagger = \frac{2\nu_A-\rho}{2\nu_A\delta}.$$

Substituting the parameter $\varsigma^\dagger$ into (29), we get $c_{\rho,\varsigma^\dagger} = \rho\nu_A - \frac{\rho^2}{2}$, and a quadratic algebraic equation

$$a\sigma^2 - b\sigma + c = 0, \tag{30}$$

where $a = D_3$, $b = 1+D_4+c_{\rho,\varsigma^\dagger}D_3$ and $c = c_{\rho,\varsigma^\dagger} - \frac{\rho^2\nu_A^2\delta^2}{c_{\rho,\varsigma^\dagger}}$. It is easy to verify that $b > 1+c_{\rho,\varsigma^\dagger}a$ and $c < c_{\rho,\varsigma^\dagger}$, which lead to

$$b^2 - 4ac > (1 + c_{\rho,\varsigma^\dagger}a)^2 - 4ac_{\rho,\varsigma^\dagger} \geq 0.$$

Hence, if we select $\varsigma = \varsigma^\dagger$, then (29) has a real root $\sigma = \sigma^\dagger = \frac{b-\sqrt{b^2-4ac}}{2a}$.

Next, we choose the parameters ς and σ as $\varsigma^\dagger$ and $\sigma^\dagger$, so the estimate (27) can be expressed as

$$D\|\nabla(\mathbf{e}_h^{n+1}, \boldsymbol{\xi}_h^{n+1})\|_0^2 + \|\eta_h^{n+1}\|_0^2 \leq H(D\|\nabla(\mathbf{e}_h^n, \boldsymbol{\xi}_h^n)\|_0^2 + \|\eta_h^n\|_0^2), \tag{31}$$

where $H = 1 - \sigma^\dagger D_3$ and $D = c_{\rho,\varsigma^\dagger} - \sigma^\dagger$. Note that $D > 0$ and $H > 0$. Next, we will prove them.

For the quadratic algebraic equation (30), we consider its quadratic function $f(x) = ax^2 - bx + c$. Because $a > 0$, $c_{\rho,\varsigma^\dagger} > 0$, $b > 1+c_{\rho,\varsigma^\dagger}a$ and $c < c_{\rho,\varsigma^\dagger}$, we obtain $\lim_{x\to-\infty} f(x) = -\infty$ and

$$f(c_{\rho,\varsigma^\dagger}) < ac_{\rho,\varsigma^\dagger}^2 - (1 + ac_{\rho,\varsigma^\dagger})c_{\rho,\varsigma^\dagger} + c_{\rho,\varsigma^\dagger} = 0.$$

In fact, the chosen smaller root $\sigma^\dagger = \frac{b-\sqrt{b^2-4ac}}{2a}$ of (30) must belong to $(-\infty, c_{\rho,\varsigma^\dagger}) \cap (0, +\infty)$, so $D > 0$, which combines with (28) to get

$$DH = \frac{2\nu_A^2\delta^2\rho}{2\nu_A - \rho} + \sigma^* D_4 > 0.$$

Hence, $H > 0$.

Finally, note that $0 < D < c_{\rho,\varsigma^\dagger} = \rho\nu_A - \frac{\rho^2}{2} = \frac{1}{2}\rho(2\nu_A - \rho) \leq \frac{1}{2}\nu_A^2$. In light of the definition of D_3 and the fact that $\widetilde{\beta} \leq 1$ proved in [22], we get $D_3 < \frac{1}{2c_A^2}$. Noticing that $\sigma^\dagger < c_{\rho,\varsigma^\dagger} < \frac{\nu_A^2}{2}$, we arrive at $1 > H = 1 - \sigma^\dagger D_3 > 1 - \frac{\nu_A^2}{4c_A^2}$.

Combining (31) with (17), we finish the proof. $\square$

4. Numerical experiments

In this section, we give some numerical experiments to test the accuracy and performance of the proposed algorithm for the thermally coupled stationary incompressible MHD flow.

4.1. Experiment 1

In the first experiment, we access to the numerical performance of the presented Uzawa iterative algorithm for the thermally coupled incompressible MHD equations. Here, let the domain $\Omega = [0,1] \times [0,1]$ and the right-hand sides $\mathbf{f}, \mathbf{g}$ and γ are selected such that the exact solutions are given by

$$\begin{aligned} u_1(x,y) &= x^2(x-1)^2 y(y-1)(2y-1), \quad u_2(x,y) = -y^2(y-1)^2 x(x-1)(2x-1), \\ H_1(x,y) &= \sin(\pi x)\cos(\pi y), \quad H_2(x,y) = -\cos(\pi x)\sin(\pi y), \\ p(x,y) &= (2x-1)(2y-1), \quad T(x,y) = u_1(x,y) + u_2(x,y). \end{aligned}$$

We employ the MINI element [1] for approximating the velocity and pressure, and the continuous linear finite element for discretizing the temperature and magnetic field.

Then, we set the parameters $s = Re = Rm = \beta = \kappa = 1$. Additionally, the stopping criterion of the iteration is set to be

$$\sqrt{\|\mathbf{u}_h^{n-1} - \mathbf{u}_h^n\|_0^2 + \|\mathbf{H}_h^{n-1} - \mathbf{H}_h^n\|_0^2} < 1.0e - 6.$$

In Figure 1, we plot the log errors of the numerical solutions in H^1-seminorms of the velocity, magnetic, temperature, and L^2-norm of the pressure at different iterative step n. Here, we set the relaxation parameter $\rho = 1.5$ and pick five different mesh sizes h. From Figure 1, we can find that the Uzawa iterative algorithm works well and the iterative error decreases when iteration step increases. Moreover, we can see that it converges faster when the mesh size is smaller.

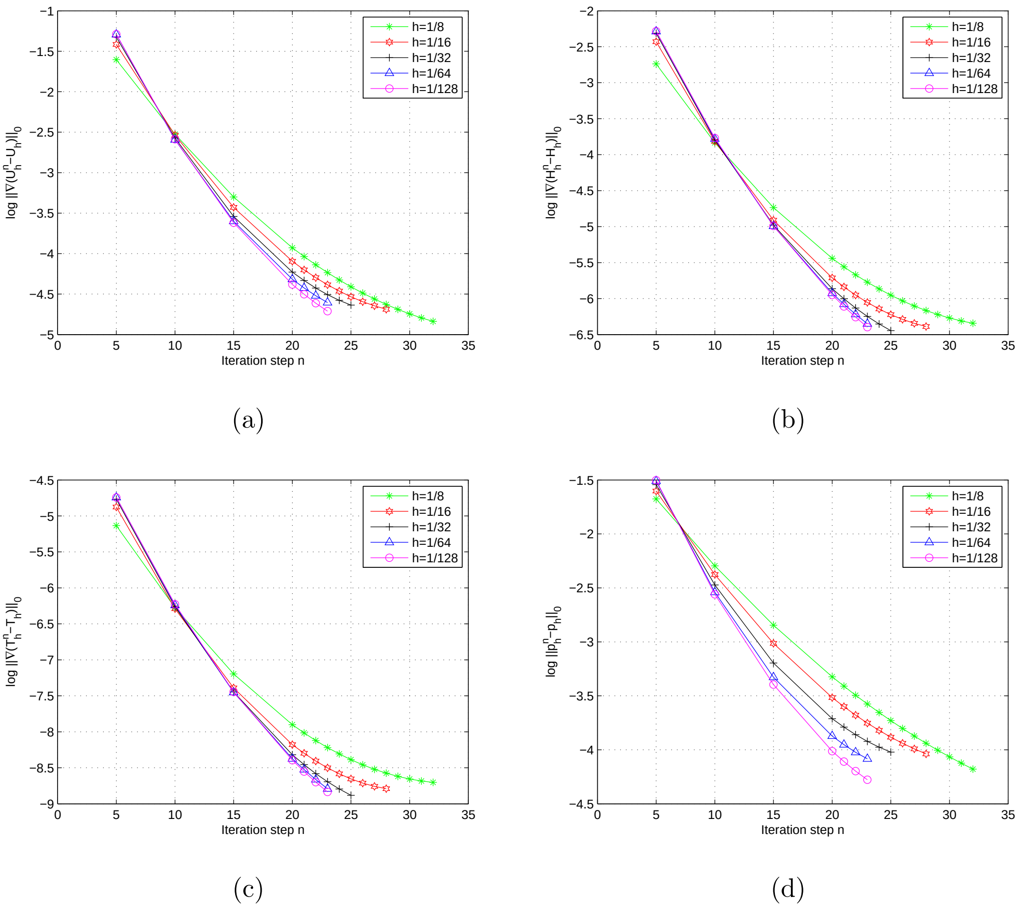
In the above test, we choose a fixed relaxation parameter and pick the different mesh sizes. Now, we consider a fixed mesh size $h = \frac{1}{64}$ and test the Uzawa iterative algorithm with the different relaxation parameters. Figure 2 shows the log errors at the different iterative step for the different relaxation parameter. From Figure 2, we can observe that the Uzawa iterative algorithm converges faster when ρ becomes larger. However, it becomes slow when ρ is too large (e.g. $\rho = 1.9$), which is not surprising. Because from Theorem 3.2 and 3.3 the relaxation parameter ρ has its limited interval.

Finally, to find the relaxation parameter that makes the Uzawa iterative algorithm converge fast, we consider the relation between n and ρ with $h = \frac{1}{64}$. In Table 1, we list the iterative steps n used for reaching the numerical solution in terms of the stopping rule. Obviously, from this table, the Uzawa iterative algorithm converges faster when we choose larger ρ, and the Uzawa iterative algorithm with $\rho = 1.5$ has the least iterative step $n = 23$. However, if $\rho > 1.5$, then it needs more iterative step to converge or it may not converge.

Table 1: The iterative step n with the relaxation parameter ρ.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
n	247	139	98	77	63	54	47	42	37	34	31	29	27	25	23	24	31	47	93	—

"—" means that the iterative step is larger than 600.



(a) (b) (c) (d)

Figure 1: The log errors of the velocity (a), magnetic (b), temperature (c) and pressure (d) at the different iterative steps.

4.2. Experiment 2

In this experiment, we will test the presented Uzawa iterative method by using the thermal driven cavity problem, which is investigated in [29]. The computational domain consists of a square cavity with differentially heated vertical walls where left and right walls are kept at $T = 1$ and $T = 0$, respectively. The remaining walls are insulated and there is no heat transfer through them. No-slip boundary conditions are imposed for the velocity at all walls. For the magnetic field, we set $H_1 = 1$, $\frac{\partial H_2}{\partial \mathbf{n}} = 0$ at the horizontal walls and $H_2 = 0$, $\frac{\partial H_1}{\partial \mathbf{n}} = 0$ at the vertical walls.

In the numerical example, the computations are obtained on the uniform grid 30×30. Here, we set the model parameters $s = Re = \kappa = 1$, $R_m = 0.1$ and take $\mathbf{f} = \mathbf{0}, \mathbf{g} = \mathbf{0}$, $\gamma = 0$. The performances of the presented Uzawa iterative method with $\rho = 1.5$ are compared with Newton iterative and Oseen iterative method in [29]. Note that the selection of the parameters is the same as that in [29]. In Table 2, we show that the maximum velocity at $y = 0.5$ with different thermal expansion coefficient β. From this table, we can see that the presented Uzawa iterative method spends the least CPU times to get almost the

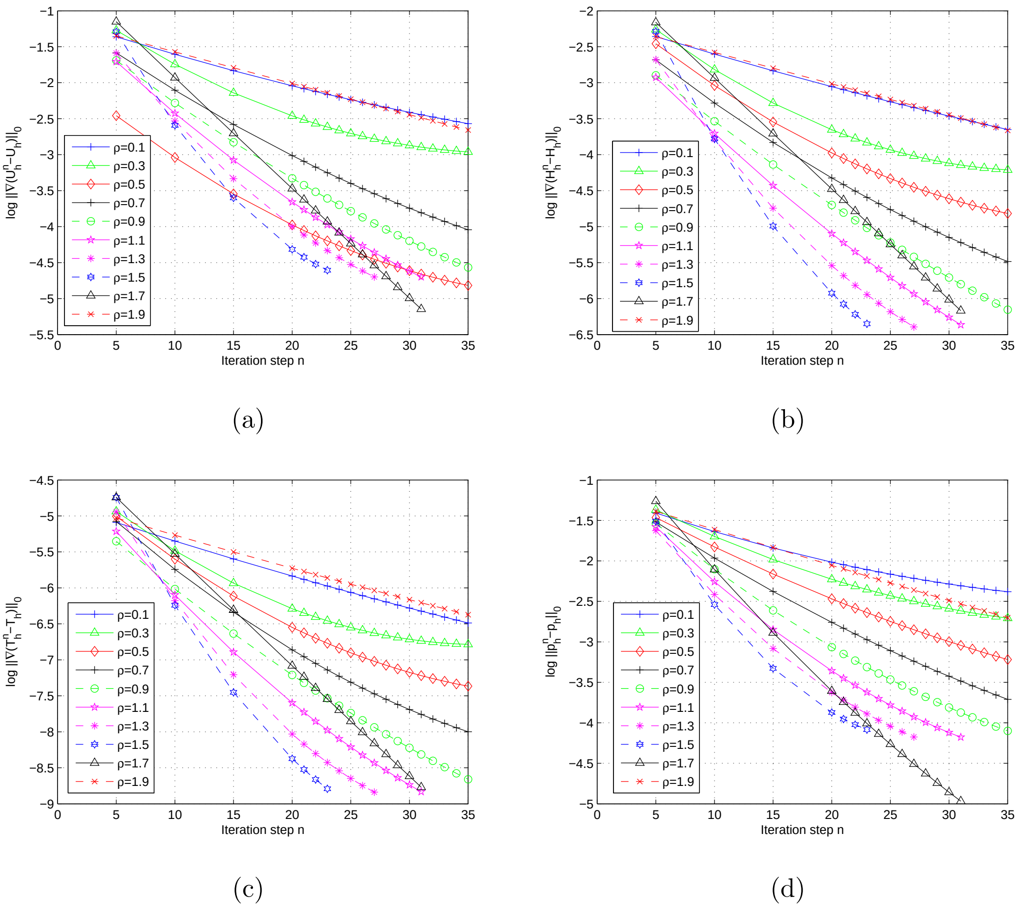


Figure 2: The log errors of the velocity (a), magnetic (b), temperature (c) and pressure (d) at the different iterative steps.

same maximum velocity values obtained by the other iterative methods. Furthermore, in Figure 3, we give the numerical velocity streamlines, magnetic and isotherms of the thermally coupled incompressible MHD problem by the Uzawa iterative method with different thermal expansion coefficient. From this figure, the Uzawa iterative method runs well and can capture this model well.

Table 2: Comparisons of the maximum velocity values obtained by the different iterative methods.

	$\beta = 1$	$\beta = 10$	$\beta = 100$	CPU time
Uzawa iterative method	0.189	0.224	0.570	18.578
Newton iterative method [29]	0.188	0.223	0.576	32.829
Oseen iterative method [29]	0.190	0.224	0.578	31.859

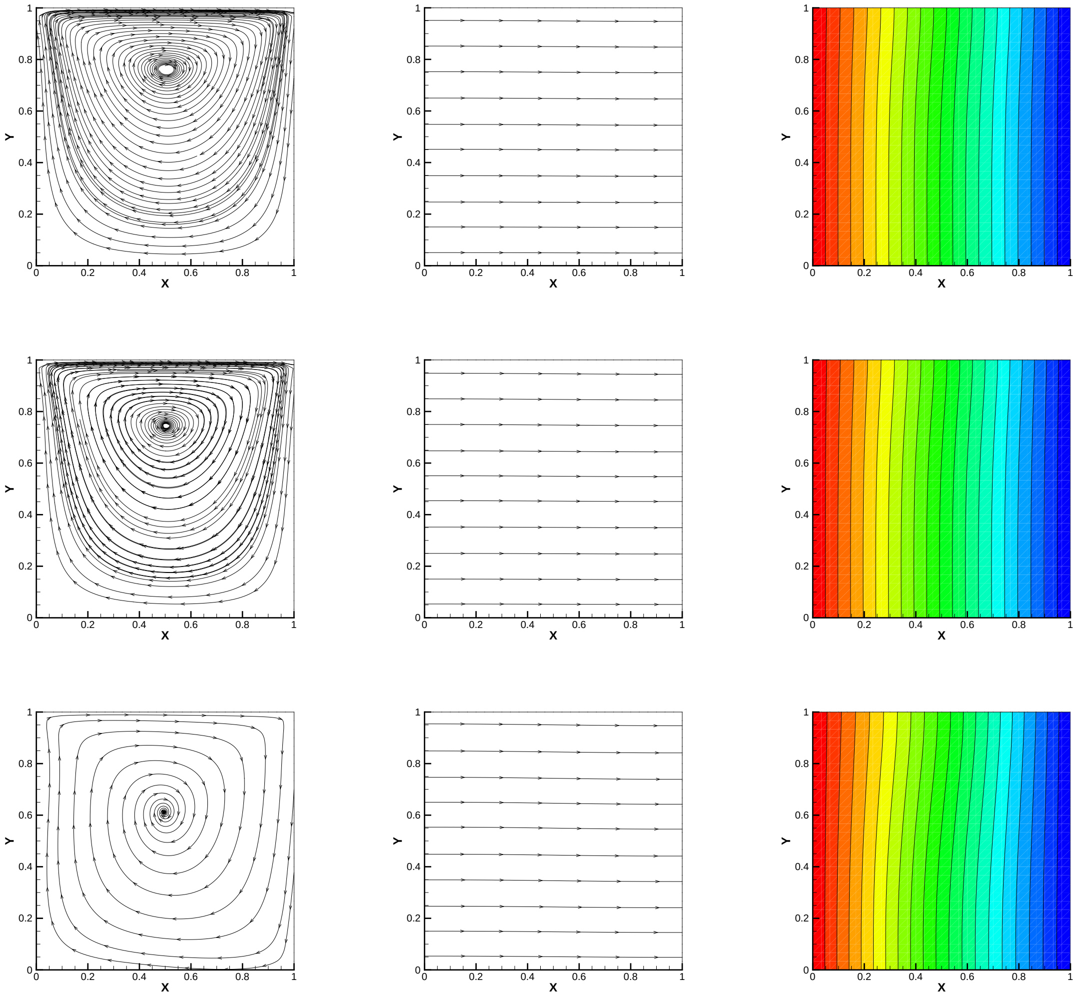


Figure 3: Numerical velocity streamlines, magnetic and isotherms with different thermal expansion coefficient $\beta = 1$ (the first line), $\beta = 10$ (the second line) and $\beta = 100$ (the third line).

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