Existence of Solutions and Ulam Stability of Hilfer-Hadamard Sequential Fractional Differential Equations with Multi-Point Fractional Integral Boundary Value Problem

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Abstract

In this paper, we study the existence and uniqueness of solutions for the boundary value problem of Hilfer-Hadamard sequential fractional differential equations via fixed point theorems. The existence of a solution is proved by the Krasnoselskii fixed point theorem, the Leray-Schauder alternative, and the Leray-Schauder nonlinear alternative. Moreover, we prove the uniqueness of the solution using the Banach contraction principle. We also discuss the Ulam-Hyers, Ulam-Hyers-Rassias, generalized Ulam-Hyers and generalized Ulam-Hyers-Rassias stability for the problem. Illustrative examples are also provided.

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1 Introduction

Fractional calculus has its origins in the question of extending the meaning of derivatives and integrals to real and complex numbers. The concept was first introduced by Gottfried Wilhelm Leibniz in the late 17th century, and it was later developed by various mathematicians, including Niels Henrik Abel, Joseph Liouville, and Oliver Heaviside. It has numerous applications across various fields. This is because fractional differential equations are a class of mathematical models that have gained significant attention in recent years due to their ability to describe complex phenomena in various fields, including physics, chemistry, biology, engineering, economics, signal and image processing, control theory, and so on; see the monographs [2,7,12,13,16,19,22,25,37]. These equations extend the traditional concept of differential equations by incorporating fractional derivatives, which are a generalization of the classical derivatives used in traditional differential equations and can model complex phenomena more accurately.

Various types of fractional derivatives have been introduced, among which the Riemann-Liouville fractional derivative and the Caputo fractional derivative are the most widely used. The Hilfer-Hadamard fractional derivative is new fractional derivative introduced in 2012 by M.D. Qasim [23]. It is an interpolation between the well-known Hadamard fractional derivative and the Caputo fractional derivative. It was introduced as a means to bridge the gap between these two types of fractional derivatives, offering a more flexible framework for modeling dynamic systems. The Hilfer-Hadamard fractional derivative is particularly useful in fields where systems exhibit memory and hereditary properties, which are not adequately captured by integer-order derivatives. It has been applied in areas such as control theory, signal processing, viscoelasticity, and anomalous diffusion.

The existence of solutions and stability to fractional differential equations are a topic of significant interest in the field of fractional calculus. The existence and uniqueness of solutions are studied using classical fixed point theorems; see the monographs [3,4,27,34,36]. Stability is a crucial concept in the analysis of fractional differential equations (FDEs), just as it is for ordinary differential equations (ODEs). The stability of a solution ensures that the system's behavior remains predictable and consistent under small perturbations. There are several types of stability relevant to FDEs, often

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extending classical stability concepts to accommodate the fractional nature of these equations such as Lyapunov stability, Ulam stability, practical stability, input-to-state stability and Mittag-Leffler stability. In this paper, we focus on Ulam stability. Ulam stability examines the sensitivity of the solutions of differential equations to perturbations in their initial conditions or equation itself. If a small change in the input leads to a small change in the output, the system is considered stable. This foundational idea has been expanded by Hyers and later by Rassias, who introduced their own concepts of stability. Ulam-Hyers stability addresses the existence of a solution that remains close to the original solution under slight perturbations, while Ulam-Hyers-Rassias stability further generalizes this concept by allowing for greater flexibility in the types of perturbations considered. These stability continues to be a vital area of research in mathematics. It provides essential insights into the stability and robustness of solutions to differential equations, enabling the modeling and analysis of a wide range of real-world phenomena. Researchers have investigated the Ulam-Hyers and Ulam-Hyers-Rassias stability of various types of fractional differential equations, including linear and nonlinear equations, and have developed several methods for establishing stability results, see the monographs [5, 8, 10, 15, 17, 18, 20, 21, 30–33, 35].

Fractional differential equations involving the Hilfer-Hadamard fractional derivative have been studied extensively in recent years. Researchers have investigated topics such as the existence and uniqueness of solutions, stability, and numerical methods. The research results cover various aspects of Hilfer-Hadamard fractional differential equations, including their applications, properties, and solution techniques. Many mathematicians have conducted research on fractional differential equations with the Hilfer-Hadamard fractional derivative. Abbas *et al.* [1] in 2017, studied the existence and Ulam-Hyers-Rassias stability results for a class of fractional differential equations involving the Hilfer-Hadamard fractional derivative,

$$\begin{cases} {}_H D_1^{\alpha,\beta} u(t) = f(t, u(t)), \quad t \in J = [1, T], \\ I_1^{1-\gamma} u(t)|_{t=1} = \phi, \end{cases}$$

where $\alpha \in (0,1), \beta \in [0,1], \gamma = \alpha + \beta - \alpha\beta, T > 1, \phi \in \mathbb{R}, f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, ${}_H I_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$ and ${}_H D_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β. Schauder fixed point theorem is used to show the existence result and then the solution is proved to be generalized Ulam-Hyers-Rassias stable.

In 2018, Vivek *et al.* [29] researched the existence, uniqueness and Ulam stabilities of solutions for Hilfer-Hadamard fractional differential equations with boundary conditions,

$$\begin{cases} {}_H D_{1^+}^{\alpha,\beta} x(t) = f(t, x(t)), \quad t \in J = [1, T], \\ I_{1^+}^{1-\gamma} x(1) = a, \quad I_{1^+}^{1-\gamma} x(T) = b, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

where ${}_H D_{1^+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $1 < \alpha < 2$ and type $\beta \in [0,1]$, $f : J \times X \to X$ is given continuous function and X is a Banach space. The existence results is shown by Schaefer's fixed point theorem while Banach's fixed point theorem is used to obtain uniqueness. Then the solution is shown to be generalized Ulam-Hyers stable.

In 2020, Ahmad and Pawar [24] studied the existence and uniqueness for Hilfer-Hadamard fractional differential equations,

$$ {}_H D^{\alpha,\beta} x(t) + f(t, x(t)) = 0, \quad t \in J = (1, e],$$

with boundary value condition,

$$x(1 + \epsilon) = 0, \quad {}_H D^{1,1} x(e) = v_H D^{1,1} x(\zeta),$$

where ${}_H D^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $1 < \alpha \le 2$ and type $\beta \in [0,1]$, $0 \le v < 1, \zeta \in (1, e), 0 < \epsilon < 1$, ${}_H D^{1,1} = t\frac{d}{dt}$ and $f : J \to \mathbb{R}_+$, $(R_+ := [0, \infty))$. This paper uses Leray-Schauder alternative and Banach's fixed point theorem to show the existence and uniqueness of the solution.

Recently, in [28], the authors studied the existence and uniqueness of solutions for boundary value problems for sequential Hilfer-Hadamard fractional differential equations with three-point boundary conditions,

$$({}_HD_{1+}^{\alpha,\beta} + k\,{}_HD_{1+}^{\alpha-1,\beta})u(t) = f(t, u(t)), \quad t \in [1, e],$$
$$u(1) = 0,\ u(e) = \lambda u(\theta), \quad \theta \in (1, e),$$

where ${}_HD_{1+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha \in (1, 2]$ and type $\beta \in [0, 1]$, $k \in \mathbb{R}_+ := [0, \infty)$, $\lambda \in \mathbb{R} \setminus \{\frac{1}{(\log\ \theta)^{\gamma-1}}\}$ and $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. However, it has been observed that the literature on Hilfer-Hadamard sequential fractional differential equations of order in $(1, 2]$ is scarce and needs to be developed further.

Motivated by the ongoing research in Hilfer-Hadamard fractional differential equations, this paper investigates the existence and uniqueness of solutions for sequential Hilfer-Hadamard fractional differential equation

$$({}_HD_{1+}^{\alpha,\beta} + k\,{}_HD_{1+}^{\alpha-1,\beta})u(t) = f(t, u(t)), \quad t \in [1, e], \tag{1}$$

with multi-points integral boundary conditions,

$$u(1) = 0,\ u(e) = \sum_{i=1}^{m} \lambda_i I^{\delta_i} u(\theta_i) = \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} u(s) ds, \tag{2}$$

where ${}_HD_{1+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha \in (1, 2]$ and type $\beta \in [0, 1]$, $k \in \mathbb{R}_+ := [0, \infty)$, $\lambda_i \in \mathbb{R}$, $\theta_i \in (1, e), i = 1, 2, \ldots, m$ and $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. $I^{\delta_i}, i = 1, 2, \ldots, m$ are the Riemann-Liouville fractional integral of positive order.

Existence and uniqueness of solutions are established via classical fixed point theorems, such as Banach, Krasnoselskii and Schaefer fixed point theorems, and the Leray-Schauder nonlinear alternative. The Ulam-Hyers, Ulam-Hyers-Rassias, generalized Ulam-Hyers and generalized Ulam-Hyers-Rassias stability are also discussed for the Hilfer-Hadamard boundary value problem (1)-(2). Illustrative examples are also provided.

This paper is structured as follows: In Section 2, we recall some definitions, notations, and theorems needed for our proof. The main results regarding existence and uniqueness are proved in Section 3. The stability results in the sense of Ulam-Hyers, Ulam-Hyers-Rassias, generalized Ulam-Hyers, and generalized Ulam-Hyers-Rassias are discussed in Section 4, while examples illustrating the main results are provided in Section 5.

2 Preliminaries

In this section, some basic definitions and theorems are presented. Let $L^1[a, b]$ be the Banach space of an Lebesgue integrable function. We consider $AC[a, b]$, the space of absolutely continuous function on the interval $[a, b]$, and $AC_\delta^n[a, b]$, the space of n-times $\delta-$differentiable absolutely continuous functions on the interval $[a, b]$, as follows

$$AC[a, b] = \left\{ f : f(t) = c + \int_a^t \varphi(\tau) d\tau,\ c \in \mathbb{R},\ \varphi \in L^1[a, b] \right\},$$
$$AC_\delta^n[a, b] = \left\{ f : [a, b] \to \mathbb{R} : \delta^{(n-1)} f(t) \in AC[a, b] \right\},$$

where δ is the Euler operator $t\frac{d}{dt}$.

Definition 2.1 (The Riemann-Liouville fractional integral [13]). *The Riemann-Liouville integral of order $\alpha > 0$ of a function $f : [a, \infty) \to \mathbb{R}$ is defined by*

$$I_{a+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > a.$$

Definition 2.2 (Hadamard fractional integral [13]). *The Hadamard fractional integral of order* $\alpha > 0$ *for a function* $f : [a, \infty) \to \mathbb{R}$ *is defined as*

$$ {}_H I^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha - 1} \frac{f(\tau)}{\tau} d\tau, \quad t > a $$

provided the integral exists, where $\log(.) = \log_e(.)$.

Definition 2.3 (Hadamard fractional derivative [13]). *The Hadamard fractional derivative of order* $\alpha > 0$, *applied to the function* $f : [a, \infty) \to \mathbb{R}$ *is defined as*

$$ {}_H D^{\alpha}_{a+} f(t) = \delta^n ({}_H I^{n-\alpha}_{a+} f(t)), \quad n - 1 < \alpha < n, \quad n = [\alpha] + 1, $$

where $\delta^n = (t\frac{d}{dt})^n$ *and* $[\alpha]$ *denotes the integer part of the real number* α.

Definition 2.4 (Hilfer-Hadamard fractional derivative [11,24]). *Let* $n - 1 < \alpha < n$ *and* $0 \leq \beta \leq 1$, $f \in L^1[a, b]$. *The Hilfer-Hadamard fractional derivative of order* α *and type* β *of* f *is defined as*

$$ \begin{aligned} ({}_H D^{\alpha,\beta}_{a+} f)(t) &= ({}_H I^{\beta(n-\alpha)}_{a+} \delta^n \ {}_H I^{(n-\alpha)(1-\beta)}_{a+} f)(t), \\ &= ({}_H I^{\beta(n-\alpha)}_{a+} \delta^n \ {}_H I^{n-\gamma}_{a+} f)(t) \\ &= ({}_H I^{\beta(n-\alpha)}_{a+} \ {}_H D^{\gamma}_{a+} f)(t), \end{aligned} $$

where $\gamma = \alpha + n\beta - \alpha\beta$, ${}_H I^{(.)}_{a+}$ *and* ${}_H D^{(.)}_{a+}$ *are the Hadamard fractional integral and derivative defined by Definitions 2.2 and 2.3, respectively.*

The Hilfer-Hadamard fractional derivative can be viewed as an interpolation between the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Specifically, when $\beta = 0$, this derivative reduces to the Hadamard fractional derivative, and when $\beta = 1$, it corresponds to the Caputo-Hadamard fractional derivative.

We recommend some theorems of the Hadamard fractional integral and Hilfer-Hadamard fractional derivative by Kilbas *et al.* [13].

Theorem 2.5. *([13]) Let* $\alpha > 0$, $n = [\alpha] + 1$ *and* $0 < a < b < \infty$. *If* $f \in L^1[a, b]$ *and* $({}_H I^{n-\alpha}_{a+} f)(t) \in AC^n_\delta[a, b]$, *then*

$$ ({}_H I^{\alpha}_{a+} \ {}_H D^{\alpha}_{a+} f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_H I^{n-\alpha}_{a+} f))(a)}{\Gamma(\alpha - j)} \left(\log \frac{t}{a} \right)^{\alpha - j - 1}. $$

Theorem 2.6. *([24]) Let* $\alpha > 0$, $0 \leq \beta \leq 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n - 1 < \gamma \leq n$, $n = [\alpha] + 1$, *and* $0 < a < b < \infty$. *If* $f \in L^1[a, b]$ *and* $({}_H I^{n-\gamma}_{a+} f)(t) \in AC^n_\delta[a, b]$, *then*

$$ {}_H I^{\alpha}_{a+} \ ({}_H D^{\alpha,\beta}_{a+} f)(t) = \ {}_H I^{\gamma}_{a+} \ ({}_H D^{\gamma}_{a+} f)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_H I^{n-\gamma}_{a+} f))(a)}{\Gamma(\gamma - j)} \left(\log \frac{t}{a} \right)^{\gamma - j - 1}. $$

From this theorem, we notice that if $\beta = 0$ the formula reduces to the formula in the Theorem 2.5.

We will use the following well-known classical fixed point theorems in Banach spaces to prove the existence and uniqueness of solution of the Hilfer-Hadamard fractional differential problem.

Theorem 2.7. (Krasnoselskii's fixed point theorem [14]). *Let* Y *be a bounded, closed, convex and nonempty subset of a Banach space* X. *Let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be the operators satisfying the conditions: (i)* $\mathcal{F}_1 y_1 + \mathcal{F}_2 y_2 \in Y$ *whenever* $y_1, y_2 \in Y$; *(ii)* $\mathcal{F}_1$ *is compact and continuous; (iii)* $\mathcal{F}_2$ *is a contraction mapping. Then there exists* $y \in Y$ *such that* $y = \mathcal{F}_1 y + \mathcal{F}_2 y$.

Theorem 2.8. (Schaefer fixed point theorem [26]). *Let $\mathcal{F} : E \to E$ be a completely continuous operator (i.e., a continuous map $\mathcal{F}$ restricted to any bounded set in E is compact). Let $\varepsilon(\mathcal{F})=\{x \in E : x = \lambda\mathcal{F}(x), 0 \leqslant \lambda \leqslant 1\}$. Then, either the set $\varepsilon(\mathcal{F})$ is unbounded, or $\mathcal{F}$ has at least one fixed point.*

Theorem 2.9. (Leray-Schauder nonlinear alternative for single valued maps [9]). *Let X be a Banach space, C a closed, convex subset of X, U an open subset of C and $0 \in U$. Suppose that $\mathcal{F} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{F}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) $\mathcal{F}$ has a fixed point in $\bar{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $x = \lambda\mathcal{F}(x)$.

Theorem 2.10. (Banach fixed point theorem [6]). *Let X be a Banach space, $D \subset X$ nonempty closed subset, and $\mathcal{F} : D \to D$ a strict contraction, i.e., there exists $k \in (0, 1)$ such that $\|\mathcal{F}x - \mathcal{F}y\| \leq k\|x - y\|$ for all $x, y \in D$. Then, $\mathcal{F}$ has a fixed point in D.*

In this paper, we are interested in the stability in the sense that the solution of the problem (1)-(2) remains continuous under changes to the equation while preserving the structure of the boundary condition. We present and discuss four types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the fractional differential problems (1)-(2).

Let ϵ be a positive real number, $\alpha \in (1, 2]$, $\beta \in [0, 1]$, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ be the continuous function and $\varphi : [1, e] \to \mathbb{R}_+$. We consider the fractional differential problem (1)-(2) and the following fractional differential inequalities

$$|({}_HD^{\alpha,\beta}_{1+} + k\,{}_HD^{\alpha-1,\beta}_{1+})v(t) - f(t, v(t))| \leq \epsilon, \quad t \in [1, e], \tag{3}$$

$$|({}_HD^{\alpha,\beta}_{1+} + k\,{}_HD^{\alpha-1,\beta}_{1+})v(t) - f(t, v(t))| \leq \varphi(t), \quad t \in [1, e], \tag{4}$$

$$|({}_HD^{\alpha,\beta}_{1+} + k\,{}_HD^{\alpha-1,\beta}_{1+})v(t) - f(t, v(t))| \leq \epsilon\varphi(t), \quad t \in [1, e]. \tag{5}$$

with the integral boundary condition

$$v(1) = 0, \;\; v(e) = \sum_{i=1}^{m} \lambda_i I^{\delta_i} v(\theta_i) = \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} v(s) ds, \tag{6}$$

where $\lambda_i \in \mathbb{R}$ and $\theta_i \in (1, e)$, for $i = 1, 2, \ldots, m$.

Definition 2.11 (Ulam-Hyers stable [32]). *Problem (1)-(2) is Ulam-Hyers stable, if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C^1([1, e], \mathbb{R})$ of the inequality (3) with (6), there exists a solution $u \in C^1([1, e], \mathbb{R})$ of problem (1)-(2) satisfying*

$$|v(t) - u(t)| \leq c_f\epsilon, \quad t \in [1, e].$$

Definition 2.12 (Generalized Ulam-Hyers stable [32]). *Problem (1)-(2) is generalized Ulam-Hyers stable, if there exists a continuous function $\theta_f : R_+ \to R_+$ with $\theta_f(0) = 0$ such that, for each solution $v \in C^1([1, e], \mathbb{R})$ of the inequality (3) with (6), there exists a solution $u \in C^1([1, e], \mathbb{R})$ of problem (1)-(2) satisfying*

$$|v(t) - u(t)| \leq \theta_f(\epsilon), \quad t \in [1, e].$$

Definition 2.13 (Ulam-Hyers-Rassias stable [32]). *Problem (1)-(2) is Ulam-Hyers-Rassias stable with respect to φ, if there exists a constant $c_{f,\varphi} > 0$ such that, for each $\epsilon > 0$ and for each solution $v \in C^1([1, e], \mathbb{R})$ of the inequality (5) with (6), there exists a solution $u \in C^1([1, e], \mathbb{R})$ of problem (1)-(2) satisfying*

$$|v(t) - u(t)| \leq c_{f,\varphi}\epsilon\varphi(t), \quad t \in [1, e].$$

Definition 2.14 (Generalized Ulam-Hyers-Rassias stable [32])**.** *Problem (1)-(2) is generalized Ulam-Hyers-Rassias stable with respect to* φ*, if there exists constant* $c_{f,\varphi} > 0$ *such that, for each solution* $v \in C^1([1,e],\mathbb{R})$ *of the inequality (4) with (6), there exists a solution* $u \in C^1([1,e],\mathbb{R})$ *of problem (1)-(2) satisfying*

$$|v(t) - u(t)| \leq c_{f,\varphi}\varphi(t), \quad t \in [1,e].$$

Remark 2.15. *It is clear that (i) Definition 2.11* $\implies$ *Definition 2.12; (ii) Definition 2.13* $\implies$ *Definition 2.14; (iii) Definition 2.13* $\implies$ *Definition 2.11.*

Remark 2.16. *A function* $v \in C^1([1,e],\mathbb{R})$ *is a solution of the inequality (3) if and only if there exists a function* $g \in C^1([1,e],\mathbb{R})$ *such that* $|g(t)| \leq \epsilon, \; t \in [1,e]$ *and*

$$({}_HD_{1+}^{\alpha,\beta} + k\,{}_HD_{1+}^{\alpha-1,\beta})v(t) = f(t,v(t)) + g(t), \quad t \in [1,e].$$

One can make similar observations as Remark 2.16 for the inequalities (4) and (5).

3 Existence and Uniqueness Results

In this section, we prove existence and uniqueness of solutions for Hilfer-Hadamard sequential fractional integral boundary value problem (1)-(2).

3.1 An Auxiliary Lemma

We start by proving a basic lemma concerning a linear variant of the boundary value problem (1)-(2), which will be used to transform the boundary value problem (1)-(2) into an equivalent integral equation.

Lemma 3.1. *Let* $h \in C([1,e],\mathbb{R})$ *and*

$$\Delta = 1 - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} (\log\ s)^{\gamma-1} ds \neq 0.$$

Then, $u \in C([1,e],\mathbb{R})$ *is a solution of the Hilfer-Hadamard sequential fractional differential equation*

$$({}_HD_{1+}^{\alpha,\beta} + k\,{}_HD_{1+}^{\alpha-1,\beta})u(t) = h(t), \quad 1 < \alpha \leq 2, \; 1 \leq \beta \leq 2, \; t \in [1,e] \tag{7}$$

supplemented with the boundary conditions (2) if and only if

$$\begin{aligned} u(t) =& \frac{(\log\ t)^{\gamma-1}}{\Delta}\Bigg\{k\int_1^e \frac{u(s)}{s}ds - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds \\ &+ \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(-k\int_1^s \frac{u(r)}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{h(r)}{r}dr\Bigg)ds\Bigg\} \\ &- k\int_1^t \frac{u(s)}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds, \end{aligned} \tag{8}$$

where $\gamma = \alpha + 2\beta - \alpha\beta$*.*

Proof. Taking the Hadamard fractional integral of order α to both sides of (7), we get

$${}_HI_{1+}^{\alpha}({}_HD_{1+}^{\alpha,\beta})u(t) + k\,{}_HI_{1+}^{\alpha}({}_HD_{1+}^{\alpha-1,\beta})u(t) = {}_HI_{1+}^{\alpha}h(t).$$

By Theorem 2.6, one has

$$u(t) - \sum_{j=0}^{1}\frac{\left(\delta^{(2-j-1)}({}_HI_{1+}^{2-\gamma}u)\right)(1)}{\Gamma(\gamma - j)}(\log t)^{\gamma-j-1} + k\,{}_HI_{1+}^{\alpha}({}_HD_{1+}^{\alpha-1,\beta})u(t) = {}_HI_{1+}^{\alpha}h(t), \tag{9}$$

where $\gamma \in (1,2]$. From equation (9), by Definition 2.4, we obtain

$$u(t) - \frac{(\delta_H I_{1^+}^{2-\gamma} u)(1)}{\Gamma(\gamma)}(\log t)^{\gamma-1} - \frac{({}_H I_{1^+}^{2-\gamma} u)(1)}{\Gamma(\gamma-1)}(\log t)^{\gamma-2} + k_H I_{1^+}^1 ({}_H I_{1^+}^{\gamma-1}{}_H D_{1^+}^{\gamma-1}) u(t) = {}_H I_{1^+}^{\alpha} h(t).$$

Then, by Theorem 2.5, one has

$$\begin{aligned} u(t) &- \frac{(\delta_H I_{1^+}^{2-\gamma} u)(1)}{\Gamma(\gamma)}(\log t)^{\gamma-1} - \frac{({}_H I_{1^+}^{2-\gamma} u)(1)}{\Gamma(\gamma-1)}(\log t)^{\gamma-2} \\ &+ k_H I_{1^+}^1 \left(u(t) - \frac{({}_H I_{1^+}^{2-\gamma} u)(1)}{\Gamma(\gamma-1)}(\log t)^{\gamma-2} \right) = {}_H I_{1^+}^{\alpha} h(t). \end{aligned} \tag{10}$$

The equation (10) can be written as follows

$$\begin{aligned} u(t) = \ & c_0(\log t)^{\gamma-1} + c_1\left((\log\ t)^{\gamma-2} + k\int_1^t \frac{(\log\ s)^{\gamma-2}}{s} ds \right) - k\int_1^t \frac{u(s)}{s} ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t \frac{h(s)}{s}\left(\log\ \frac{t}{s}\right)^{\alpha-1} ds, \end{aligned} \tag{11}$$

where c_0 and c_1 are arbitrary constants. Now, the first boundary condition $u(1) = 0$ together with (11) yield $c_1 = 0$. The equation (11) can be written as follows

$$u(t) = c_0(\log\ t)^{\gamma-1} - k\int_1^t \frac{u(s)}{s} ds + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds. \tag{12}$$

Next, the second boundary condition of (2) together with (12) yields

$$\begin{aligned} & c_0 - k\int_1^e \frac{u(s)}{s} ds + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\ &= \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i} (\theta_i - s)^{\delta_i-1}\left(c_0(\log\ s)^{\gamma-1} - k\int_1^s \frac{u(r)}{r} dr + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1} \frac{h(r)}{r} dr \right) ds. \end{aligned}$$

Rearranging the above equation, we get

$$\begin{aligned} c_0 = & \frac{1}{\Delta}\left\{ k\int_1^e \frac{u(s)}{s} ds - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \right. \\ & \left. + \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i} (\theta_i - s)^{\delta_i-1}\left(-k\int_1^s \frac{u(r)}{r} dr + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1} \frac{h(r)}{r} dr \right) ds \right\}. \end{aligned}$$

Substituting the value of c_0 into (12), we obtain the integral equation (8). The converse follows by direct computation. Thus, the proof is completed. □

Let us introduce the Banach space $X = C\big([1,e],\mathbb{R}\big)$ endowed with the norm defined by $\|u\| := \max_{t\in[1,e]}|u(t)|$. In view of Lemma 3.1, we define an operator $\mathcal{F} : X \to X$, by

$$\begin{aligned} (\mathcal{F}u)(t) = & \frac{(\log\ t)^{\gamma-1}}{\Delta}\left\{ k\int_1^e \frac{u(s)}{s} ds - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds \right. \\ & \left. + \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i} (\theta_i - s)^{\delta_i-1}\left(-k\int_1^s \frac{u(r)}{r} dr + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1} \frac{f(r,u(r))}{r} dr \right) ds \right\} \\ & - k\int_1^t \frac{u(s)}{s} ds + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s} ds. \end{aligned} \tag{13}$$

We use the following notations in the proofs for computational convenience:

$$\omega = \sum_{i=1}^{m} |\lambda_i| \frac{(\theta_i - 1)^{\delta_i}}{\Gamma(\delta_i + 1)}, \quad M = \frac{1}{|\Delta|}[1 + \omega + |\Delta|].$$

We need the following hypotheses in the sequel:

(H_1) There exists a continuous nonnegative function $\phi : [1, e] \to \mathbb{R}_+$, $(\mathbb{R}_+ := [0, \infty))$ such that

$$|f(t, u(t))| \leq \phi(t), \quad \text{for each} \quad (t, u(t)) \in [1, e] \times \mathbb{R}.$$

(H_2) There exists a constant $l > 0$ such that, for all $t \in [1, e]$ and $u_i \in \mathbb{R}$, $i = 1, 2$,

$$|f(t, u_1) - f(t, u_2)| \ \leq \ l|u_1 - u_2|.$$

(H_3) There exist a real constant $N > 0$ such that, for all $t \in [1, e]$, $u \in \mathbb{R}$,

$$|f(t, u)| \leq N.$$

(H_4) There exists a continuous function $p : [1, e] \to \mathbb{R}_+$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, u)| \leq p(t)\psi(|u|), \quad \text{for each} \quad (t, u) \in [1, e] \times \mathbb{R}.$$

(H_5) There exists a constant $C > 0$ such that

$$\frac{\Gamma(\alpha + 1)(1 - kM)C}{M\|p\|\psi(C)} > 1.$$

(H_6) Let $\varphi : [1, e] \to \mathbb{R}_+$ be an increasing continuous function. There exists $\lambda_\varphi > 0$ such that

$${}_H I^{\alpha}_{1^+} \varphi(t) \leq \lambda_\varphi \varphi(t), \quad t \in [1, e].$$

3.2 Existence Result via Krasnoselskii's Fixed Point Theorem

In this subsection, we prove an existence result based on Krasnoselskii's fixed point theorem.

Theorem 3.2. *Assume that* (H_1) *holds. Then the problem (1)-(2) has at least one solution on* $[1, e]$, *provided that* $kM < 1$.

Proof. By assumption (H_1), we can fix

$$R \geq \frac{M\|\phi\|}{\Gamma(\alpha + 1)(1 - kM)},$$

where $\|\phi\| = \sup_{t \in [1,e]} |\phi(t)|$,and we consider $B_R = \{u \in X : \|u\| \leq R\}$. We split the operator $\mathcal{F} : X \to X$ defined by (13) as $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by

$$\begin{aligned}(\mathcal{F}_1 u)(t) \ &= \ \frac{(\log\ t)^{\gamma-1} k}{\Delta} \left\{ \int_1^e \frac{u(s)}{s} ds - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\delta_i)} \int_1^{\theta_i} (\theta_i - s)^{\delta_i - 1} \left(\int_1^s \frac{u(r)}{r} dr \right) ds \right\} \\ &\quad -k \int_1^t \frac{u(s)}{s} ds, \quad t \in [1, e],\end{aligned}$$

and

$$\begin{aligned}(\mathcal{F}_2u)(t) &= \frac{(\log\ t)^{\gamma-1}}{\Delta}\Bigg\{-\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{f(s,u(s))}{s}ds\\
&+\sum_{i=1}^m\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{f(r,u(r))}{r}dr\bigg)ds\Bigg\}\\
&+\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{f(s,u(s))}{s}ds,\quad t\in[1,e].\end{aligned}$$

Step I : We will show that $\mathcal{F}_1u+\mathcal{F}_2v\in B_R$, whenever $u,v\in B_R$. Let $u,v\in B_R$, we have

$$\begin{aligned}&|(\mathcal{F}_1u)(t)+(\mathcal{F}_2v)(t)|\\
&\le\frac{1}{|\Delta|}\Bigg\{k\int_1^e\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{|f(s,v(s))|}{s}ds\\
&+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(k\int_1^s\frac{|u(r)|}{r}dr+\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{|f(r,v(r))|}{r}dr\bigg)ds\Bigg\}\\
&+k\int_1^t\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{|f(s,v(s))|}{s}ds\\
&\le\frac{1}{|\Delta|}\Bigg\{k\|u\|+\frac{\|\phi\|}{\Gamma(\alpha+1)}+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(k\|u\|\log s+\frac{\|\phi\|(\log s)^\alpha}{\Gamma(\alpha+1)}\bigg)ds\Bigg\}\\
&+k\|u\|\log t+\frac{\|\phi\|(\log t)^\alpha}{\Gamma(\alpha+1)}\\
&\le\frac{1}{|\Delta|}\Bigg\{k\|u\|+\frac{\|\phi\|}{\Gamma(\alpha+1)}+\bigg(k\|u\|+\frac{\|\phi\|}{\Gamma(\alpha+1)}\bigg)\omega\Bigg\}+k\|u\|+\frac{\|\phi\|}{\Gamma(\alpha+1)}\\
&=\frac{1}{|\Delta|}\bigg(k\|u\|+\frac{\|\phi\|}{\Gamma(\alpha+1)}\bigg)[1+\omega+|\Delta|]=\bigg(k\|u\|+\frac{\|\phi\|}{\Gamma(\alpha+1)}\bigg)M\end{aligned}$$

which, upon taking the norm for $t\in[1,e]$, yields

$$\|\mathcal{F}_1u+\mathcal{F}_2v\|\le R(kM)+\frac{M\|\phi\|}{\Gamma(\alpha+1)}\le R.$$

Hence, $\mathcal{F}_1u+\mathcal{F}_2v\in B_R$.

Step II : Next, we will show that the operator $\mathcal{F}_1$ is a contraction. Let $u_1,u_2\in X$. Then, for any $t\in[1,e]$, we have

$$\begin{aligned}&|(\mathcal{F}_1u_2)(t)-(\mathcal{F}_1u_1)(t)|\\
&\le\frac{k}{|\Delta|}\Bigg\{\int_1^e\frac{|u_2(s)-u_1(s)|}{s}ds+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\bigg(\int_1^s\frac{|u_2(r)-u_1(r)|}{r}dr\bigg)ds\Bigg\}\\
&+k\int_1^t\frac{|u_2(s)-u_1(s)|}{s}ds\\
&\le\frac{k}{|\Delta|}\Bigg\{\|u_2-u_1\|+\|u_2-u_1\|\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\ ds\Bigg\}+k\|u_2-u_1\|\\
&=\frac{k}{|\Delta|}\Bigg\{\|u_2-u_1\|+\|u_2-u_1\|\sum_{i=1}^m|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\Bigg\}+k\|u_2-u_1\|\\
&=\|u_2-u_1\|\frac{k}{|\Delta|}[1+\omega+|\Delta|]=kM\|u_2-u_1\|,\end{aligned}$$

which on taking the norm for $t \in [1,e]$, yields

$$\|\mathcal{F}_1 u_2 - \mathcal{F}_1 u_1\| \le kM\|u_2 - u_1\|.$$

By $kM < 1$, the operator $\mathcal{F}_1$ is a contraction.

Step III : Finally, we will show that the operator $\mathcal{F}_2$ is continuous and compact. First, we show that the operator $\mathcal{F}_2$ is continuous. Let $\{u_n\}$ be a sequence such that $\{u_n\} \to u$ in X for any $t \in [1,e]$. Then, we have

$$\begin{aligned}
&|(\mathcal{F}_2 u_n)(t) - (\mathcal{F}_2 u)(t)| \\
&\quad \le \frac{1}{|\Delta|}\left\{\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u_n(s)) - f(s,u(s))|}{s}ds\right.\\
&\qquad \left. + \sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\left(\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u_n(r)) - f(r,u(r))|}{r}dr\right)ds\right\}\\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u_n(s)) - f(s,u(s))|}{s}ds.
\end{aligned}$$

Since f is continuous, we get

$$\|\mathcal{F}_2 u_n - \mathcal{F}_2 u\| \to 0 \quad as \quad \{u_n\} \to u.$$

Hence, the operator $\mathcal{F}_2$ is continuous.

Next, we will show that $\mathcal{F}_2$ is compact by using Arzelá-Ascoli theorem. First, $\mathcal{F}_2$ is uniformly bounded since

$$\|\mathcal{F}_2 u\| \le \frac{M\|\phi\|}{\Gamma(\alpha+1)}.$$

Finally, we show that $\mathcal{F}_2$ is equicontinuous. We define $\sup\limits_{(t,u)\in[1,e]\times B_R}|f(t,u(t))| = \bar{f}$ and take $t_1, t_2 \in [1,e]$ which $t_1 < t_2$. Then, we have

$$\begin{aligned}
&|(\mathcal{F}_2 u)(t_2) - (\mathcal{F}_2 u)(t_1)| \\
&\le \frac{\left[(\log\ t_2)^{\gamma-1} - (\log\ t_1)^{\gamma-1}\right]}{|\Delta|}\left\{\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right.\\
&\quad \left. + \sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\left(\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r))|}{r}dr\right)ds\right\}\\
&\quad + \frac{1}{\Gamma(\alpha)}\left[\int_1^{t_1}\left(\left(\log\ \frac{t_2}{s}\right)^{\alpha-1} - \left(\log\ \frac{t_1}{s}\right)^{\alpha-1}\right)\frac{|f(s,u(s))|}{s}ds + \int_{t_1}^{t_2}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right]\\
&\le \left[(\log\ t_2)^{\gamma-1} - (\log\ t_1)^{\gamma-1}\right]\frac{1}{|\Delta|}\left\{\frac{\bar{f}}{\Gamma(\alpha+1)}(1+\omega)\right\} + \frac{\bar{f}}{\Gamma(\alpha+1)}\left[(\log t_2)^{\alpha} - (\log t_1)^{\alpha}\right].
\end{aligned}$$

Taking $t_2 \to t_1$ from the above inequality, we have $|(\mathcal{F}_2 u)(t_2) - (\mathcal{F}_2 u)(t_1)| \to 0$. Thus, $\mathcal{F}_2$ is equicontinuous. By Arzelá-Ascoli theorem, we conclude that the operator $\mathcal{F}_2$ is compact on B_R.

Hence all the conditions of Krasnoselskii's fixed point theorem (2.7) are satisfied, and therefore the boundary value problem (1)-(2) has at least one solution on $[1,e]$. □

3.3 Existence Result via Schaefer Fixed Point Theorem

We will show the existence result based on Schaefer fixed point theorem.

Theorem 3.3. *Assume that* (H_3) *holds. Then, the boundary value problem (1)-(2) has at least one solution on* $[1,e]$*, provided that* $kM < 1$.

Proof. We divide the proof into two steps.
Step I : We show that the operator $\mathcal{F}: X \to X$ defined by (13), is completely continuous.
Step I.1 : First we show that $\mathcal{F}$ is continuous. Let $\{u_n\}$ be a sequence such that $\{u_n\} \to u$ in X. Then, for each $t \in [1, e]$, we have

$$
\begin{aligned}
&|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)| \\
&\quad \leq \frac{1}{|\Delta|}\Bigg\{\Bigg|k\int_1^e \frac{u_n(s)-u(s)}{s}ds\Bigg| + \Bigg|\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{f(s,u_n(s))-f(s,u(s))}{s}ds\Bigg| \\
&\qquad + \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(\Bigg|k\int_1^s \frac{u_n(r)-u(r)}{r}dr\Bigg| \\
&\qquad + \Bigg|\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{f(r,u_n(r))-f(r,u(r))}{r}dr\Bigg|\Bigg)ds\Bigg\}\Big|(\log\ t)^{\gamma-1}\Big| \\
&\qquad + \Bigg|k\int_1^t \frac{u_n(s)-u(s)}{s}ds\Bigg| + \Bigg|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{f(s,u_n(s))-f(s,u(s))}{s}ds\Bigg| \\
&\quad \leq \frac{1}{|\Delta|}\Bigg\{k\int_1^e \frac{|u_n(s)-u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u_n(s))-f(s,u(s))|}{s}ds \\
&\qquad + \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k\int_1^s \frac{|u_n(r)-u(r)|}{r}dr \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u_n(r))-f(r,u(r))|}{r}dr\Bigg)ds\Bigg\} + k\int_1^t \frac{|u_n(s)-u(s)|}{s}ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u_n(s))-f(s,u(s))|}{s}ds.
\end{aligned}
$$

Since f is continuous, $\|\mathcal{F}u_n - \mathcal{F}u\| \to 0$ as $\{u_n\} \to u$. Thus, $\mathcal{F}$ is continuous.

Step I.2 : Now, we show that $\mathcal{F}$ is compact. Let $u \in B_R := \{u \in X : \|u\| \leq R\}$. Then, we have

$$
\begin{aligned}
|(\mathcal{F}u)(t)| &\leq \frac{1}{|\Delta|}\Bigg\{k\int_1^e \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
&\quad + \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k\int_1^s \frac{|u(r)|}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r))|}{r}dr\Bigg)ds\Bigg\} \\
&\quad + k\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
&\leq \frac{1}{|\Delta|}\Bigg\{k\|u\|\int_1^e \frac{ds}{s} + \frac{N}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\quad + \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\Bigg(k\|u\|\int_1^s \frac{dr}{r} + \frac{N}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{dr}{r}\Bigg)ds\Bigg\} \\
&\quad + k\|u\|\int_1^t \frac{ds}{s} + \frac{N}{\Gamma(\alpha)}\int_1^t \left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\leq \frac{1}{|\Delta|}\Bigg\{k\|u\| + \frac{N}{\Gamma(\alpha+1)} + \left(k\|u\| + \frac{N}{\Gamma(\alpha+1)}\right)\sum_{i=1}^m |\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\Bigg\} + k\|u\| + \frac{N}{\Gamma(\alpha+1)} \\
&= \left(k\|u\| + \frac{N}{\Gamma(\alpha+1)}\right)M,
\end{aligned}
$$

which on taking the norm for $t \in [1, e]$, yields

$$
\|\mathcal{F}u\| \leq kMR + \frac{NM}{\Gamma(\alpha+1)}.
$$

Hence $\mathcal{F}$ is uniformly bounded.

Finally, we show that $\mathcal{F}$ is equicontinuous. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $u \in B_R$. Then we have

$$\begin{aligned}
&|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)| \\
\leq& \frac{\left[(\log\ t_2)^{\gamma-1} - (\log\ t_1)^{\gamma-1}\right]}{|\Delta|}\Bigg\{k\int_1^e \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
&+ \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\left(k\int_1^s \frac{|u(r)|}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s \left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r))|}{r}dr\right)ds\Bigg\} \\
&+ k\int_{t_1}^{t_2}\frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\Bigg[\int_1^{t_1}\left(\left(\log\ \frac{t_2}{s}\right)^{\alpha-1} - \left(\log\ \frac{t_1}{s}\right)^{\alpha-1}\right)\frac{|f(s,u(s))|}{s}ds \\
&+ \int_{t_1}^{t_2}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\Bigg] \\
\leq& \frac{\left[(\log\ t_2)^{\gamma-1} - (\log\ t_1)^{\gamma-1}\right]}{|\Delta|}\Bigg\{k\|u\|\int_1^e \frac{ds}{s} + \frac{N}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&+ \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\left(k\|u\|\int_1^s \frac{dr}{r} + \frac{N}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{dr}{r}\right)ds\Bigg\} \\
&+ k\|u\|\int_{t_1}^{t_2}\frac{ds}{s} + \frac{N}{\Gamma(\alpha)}\Bigg[\int_1^{t_1}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s} - \int_1^{t_1}\left(\log\ \frac{t_1}{s}\right)^{\alpha-1}\frac{ds}{s} + \int_{t_1}^{t_2}\left(\log\ \frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}\Bigg] \\
\leq& \left[(\log\ t_2)^{\gamma-1} - (\log\ t_1)^{\gamma-1}\right]\frac{1}{|\Delta|}\left\{k\|u\| + \frac{N}{\Gamma(\alpha+1)} + \left(k\|u\| + \frac{N}{\Gamma(\alpha+1)}\right)\sum_{i=1}^m |\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\right\} \\
&+ k\|u\|(\log t_2 - \log t_1) + \frac{N}{\Gamma(\alpha+1)}\left[(\log t_2)^\alpha - (\log t_1)^\alpha\right] \\
=& \frac{\left[(\log\ t_2)^{\gamma-1} - (\log\ t_1)^{\gamma-1}\right]}{|\Delta|}\left\{\left(k\|u\| + \frac{N}{\Gamma(\alpha+1)}\right)[1+\omega]\right\} + k\|u\|(\log t_2 - \log t_1) \\
&+ \frac{N}{\Gamma(\alpha+1)}\left[(\log t_2)^\alpha - (\log t_1)^\alpha\right].
\end{aligned}$$

Taking $t_2 \to t_1$ on the above equation, we have $|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)| \to 0$. Thus, $\mathcal{F}$ is equicontinuous. By Arzelá-Ascoli theorem, we get that $\mathcal{F}(\Omega)$ is compact, that is $\mathcal{F}$ is compact on Ω. Therefore $\mathcal{F}$ is completely continuous.

Step II : We show that the set $\mathcal{E} = \{u \in X : u = \eta(\mathcal{F}u), 0 \leq \eta \leq 1\}$ is bounded. Let $u \in \mathcal{E}$, then $u = \eta(\mathcal{F}u)$. For any $t \in [1, e]$, we have $u(t) = \eta(\mathcal{F}u)(t)$. Then, in view of the hypothesis (H_3), we obtain

$$\begin{aligned}
|u(t)| \leq& |(\mathcal{F}u)(t)| \\
\leq& \frac{1}{|\Delta|}\Bigg\{k\int_1^e \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
&+ \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\left(k\int_1^s \frac{|u(r)|}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r))|}{r}dr\right)ds\Bigg\} \\
&+ k\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
\leq& \frac{1}{|\Delta|}\Bigg\{k\|u\|\int_1^e \frac{ds}{s} + \frac{N}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&+ \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\left(k\|u\|\int_1^s \frac{dr}{r} + \frac{N}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{dr}{r}\right)ds\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+k\|u\|\int_1^t \frac{ds}{s}+\frac{N}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
\leq&\frac{1}{|\Delta|}\left\{k\|u\|+\frac{N}{\Gamma(\alpha+1)}+\left(k\|u\|+\frac{N}{\Gamma(\alpha+1)}\right)\sum_{i=1}^m|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\right\}+k\|u\|+\frac{N}{\Gamma(\alpha+1)}\\
=&kM\|u\|+\frac{N}{\Gamma(\alpha+1)}M.
\end{aligned}
$$

Taking maximum for $t\in[1,e]$, yields

$$
\|u\|\leq\frac{MN}{\Gamma(\alpha+1)(1-kM)},
$$

which shows that the set $\mathcal{E}$ is bounded.

By Theorem 2.8, we get that the operator $\mathcal{F}$ has at least one fixed point. Therefore, the boundary value problem (1)-(2) has at least one solution on $[1,e]$. This completes the proof. □

3.4 Existence Result via Leray-Schauder Nonlinear Alternative.

Our final existence result is proved via Leray-Schauder nonlinear alternative.

Theorem 3.4. *Assume that* (H_4) *and* (H_5) *hold. Then, the boundary value problem (1)-(2) has at least one solution on* $[1,e]$, *if* $kM<1$.

Proof. As shown in Theorem 3.3, the operator $\mathcal{F}$ is completely continuous.

We will prove that there exists an open set $U\subseteq X$ with for all $u\in\partial U, u\neq\mu(\mathcal{F}u)$ for $\mu\in(0,1)$. Let $u\in X$ such that $u=\mu(\mathcal{F}u)$, for some $0<\mu<1$. Then, for each $t\in[1,e]$, we have

$$
\begin{aligned}
|u(t)|=&\mu|(\mathcal{F}u)(t)|\leq|(\mathcal{F}u)(t)|\\
\leq&\frac{1}{|\Delta|}\left\{k\int_1^e\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right.\\
&\left.+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\left(k\int_1^s\frac{|u(r)|}{r}dr+\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r))|}{r}dr\right)ds\right\}\\
&+k\int_1^t\frac{|u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\\
\leq&\frac{1}{|\Delta|}\left\{k\|u\|\int_1^e\frac{ds}{s}+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\right.\\
&\left.+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\left(k\|u\|\int_1^s\frac{dr}{r}+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{1}{r}dr\right)ds\right\}\\
&+k\|u\|\int_1^t\frac{ds}{s}+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
\leq&\frac{1}{|\Delta|}\left\{k\|u\|+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}+\left(k\|u\|+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}\right)\sum_{i=1}^m|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\right\}+k\|u\|+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}\\
=&kM\|u\|+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}M,
\end{aligned}
$$

which, upon taking maximum for $t\in[1,e]$, yields

$$
\|u\|\leq kM\|u\|+\frac{\|p\|\psi(\|u\|)}{\Gamma(\alpha+1)}M\quad\text{or}\quad\frac{\Gamma(\alpha+1)(1-kM)\|u\|}{M\|p\|\psi(\|u\|)}\leq 1.
$$

In view of (H_5), there is no solution u such that $\|u\| \neq C$. Let us set

$$U = \{u \in X : \|u\| < C\}.$$

The operator $\mathcal{F} : \overline{U} \to X$ is completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \mu(\mathcal{F}u)$, for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder Theorem 2.9, we deduce that $\mathcal{F}$ has a fixed point $u \in \overline{U}$ which is a solution of the boundary value problem (1)-(2). □

3.5 Existence and Uniqueness Result via Banach's Fixed Point Theorem

Next, we prove an existence and uniqueness result based on Banach's contraction principle.

Theorem 3.5. *Assume that (H_2) holds. Then the boundary value problem (1)-(2) has a unique solution on $[1,e]$, provided that*

$$\Xi := kM + \frac{lM}{\Gamma(\alpha+1)} < 1. \tag{14}$$

Proof. We will use the Banach contraction principle to prove that $\mathcal{F}$, defined by (13), has a unique fixed point. Fixing $M_0 = \max_{t\in[1,e]} |f(t,0)| < \infty$ and using the assumption (H_2), we obtain

$$|f(t,u(t))| \leq |f(t,u(t)) - f(t,0)| + |f(t,0)| \leq l\|u\| + M_0. \tag{15}$$

We choose

$$R \geq \frac{\dfrac{M_0 M}{\Gamma(\alpha+1)}}{1 - \left[kM + \dfrac{lM}{\Gamma(\alpha+1)}\right]}.$$

We divide the proof into two steps:
Step I : First, we show that $\mathcal{F}(B_R) \subset B_R$, where $B_R = \{u \in X : \|u\| \leq R\}$. Let $u \in B_R$. Then, using (15), we obtain

$$\begin{aligned}
|(\mathcal{F}u)(t)| \leq & \frac{1}{|\Delta|}\left\{k\int_1^e \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right. \\
& \left. + \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\left(k\int_1^s \frac{|u(r)|}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r))|}{r}dr\right)ds\right\} \\
& + k\int_1^t \frac{|u(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds \\
\leq & \frac{1}{|\Delta|}\left\{k\|u\|\int_1^e \frac{ds}{s} + \frac{(l\|u\|+M_0)}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{ds}{s}\right. \\
& \left. + \sum_{i=1}^m \frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\left(k\|u\|\int_1^s\frac{dr}{r} + \frac{(l\|u\|+M_0)}{\Gamma(\alpha)}\int_1^s\left(\log\frac{s}{r}\right)^{\alpha-1}\frac{dr}{r}\right)ds\right\} \\
& + k\|u\|\int_1^t\frac{ds}{s} + \frac{(l\|u\|+N)}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\
\leq & \frac{1}{|\Delta|}\left\{k\|u\| + \frac{(lR+M_0)}{\Gamma(\alpha+1)} + \left(k\|u\| + \frac{(lR+M_0)}{\Gamma(\alpha+1)}\right)\sum_{i=1}^m|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\right\} + k\|u\| + \frac{(lR+M_0)}{\Gamma(\alpha+1)} \\
= & \left(k\|u\| + \frac{(lR+M_0)}{\Gamma(\alpha+1)}\right)\frac{1}{|\Delta|}[1+\omega+|\Delta|] = kM\|u\| + \frac{(lR+M_0)}{\Gamma(\alpha+1)}M,
\end{aligned}$$

which, upon taking the norm for $t \in [1,e]$, yields

$$\|\mathcal{F}(u)\| \le kM\|u\| + \frac{(lR+M_0)}{\Gamma(\alpha+1)}M.$$

Hence

$$\|\mathcal{F}(u)\| \le R\Big[kM + \frac{lM}{\Gamma(\alpha+1)}\Big] + \frac{M_0 M}{\Gamma(\alpha+1)} \le R.$$

Thus $\|\mathcal{F}u\| \le R$, that is, $\mathcal{F}u \in B_R$. Hence $\mathcal{F}(B_R) \subset B_R$.

Step II : We show that the operator $\mathcal{F}$ is a contraction. Let $u, v \in X$. Then, for any $t \in [1,e]$, we have

$$\begin{aligned}
&|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\
&\le \frac{1}{|\Delta|}\Bigg\{k\int_1^e \frac{|u(s)-v(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^e \Big(\log\,\frac{e}{s}\Big)^{\alpha-1}\frac{|f(s,u(s))-f(s,v(s))|}{s}ds \\
&+ \sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k\int_1^s\frac{|u(r)-v(r)|}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\,\frac{s}{r}\Big)^{\alpha-1}\frac{|f(r,u(r))-f(r,v(r))|}{r}dr\Big)ds\Bigg\} \\
&+ k\int_1^t \frac{|u(s)-v(s)|}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\,\frac{t}{s}\Big)^{\alpha-1}\frac{|f(s,u(s))-f(s,v(s))|}{s}ds \\
&\le \frac{1}{|\Delta|}\Bigg\{k\|u-v\| + \frac{l\|u-v\|}{\Gamma(\alpha+1)} + \sum_{i=1}^{m}|\lambda_i|\frac{1}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(k\|u-v\|\log s + \frac{l\|u-v\|}{\Gamma(\alpha+1)}(\log s)^{\alpha}\Big)ds\Bigg\} \\
&+ k\|u-v\|\log t + \frac{l\|u-v\|}{\Gamma(\alpha+1)}(\log t)^{\alpha} \\
&\le \frac{1}{|\Delta|}\Bigg\{k\|u-v\| + \frac{l\|u-v\|}{\Gamma(\alpha+1)} + \Big(k\|u-v\| + \frac{l\|u-v\|}{\Gamma(\alpha+1)}\Big)\sum_{i=1}^{m}|\lambda_i|\frac{(\theta_i-1)^{\delta_i}}{\Gamma(\delta_i+1)}\Bigg\} + k\|u-v\| + \frac{l\|u-v\|}{\Gamma(\alpha+1)} \\
&= \Big(kM + \frac{lM}{\Gamma(\alpha+1)}\Big)\|u-v\|,
\end{aligned}$$

which, upon taking the norm for $t \in [1,e]$, yields

$$\|\mathcal{F}u - \mathcal{F}v\| \le \Xi\|u-v\|. \tag{16}$$

In view of (14), the operator $\mathcal{F}$ is a contraction mapping. Therefore by Theorem 2.10, operator $\mathcal{F}$ has a unique fixed point. Therefore the boundary value problem (1)-(2) has a unique solution on $[1,e]$. □

4 Ulam Stability Results

Lastly, we study the Ulam-Hyers and Ulam-Hyers-Rassias stability of the Hilfer-Hadamard fractional differential equation (1) with boundary condition (2).

Theorem 4.1. *If assumption (H_2) and condition (14) are satisfied, then the boundary value problem (1)-(2) is Ulam-Hyers stable, and hence generalized Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and v be a solution of the inequality (3) with the boundary (6). Then by Remark 2.16, there exists a function $g \in C^1([1,e],\mathbb{R})$ such that $|g(t)| \le \epsilon,\ t \in [1,e]$, and

$$\begin{cases} ({}_H D_{1+}^{\alpha,\beta} + k\,{}_H D_{1+}^{\alpha-1,\beta})v(t) = f(t,v(t)) + g(t), \quad t \in [1,e], \\ v(1) = 0,\ v(e) = \displaystyle\sum_{i=1}^{m}\lambda_i I^{\delta_i}v(\theta_i) = \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}v(s)ds, \quad \theta_i \in (1,e). \end{cases} \tag{17}$$

By Lemma 3.1, the solution of (17) can be written as

$$\begin{aligned}v(t)=&\frac{(\log\ t)^{\gamma-1}}{\Delta}\Big\{k\int_1^e\frac{v(s)}{s}ds-\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{f(s,v(s))+g(s)}{s}ds\\&+\sum_{i=1}^m\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(-k\int_1^s\frac{v(r)}{r}dr\\&+\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{f(r,v(r))+g(r)}{r}dr\Big)ds\Big\}-k\int_1^t\frac{v(s)}{s}ds\\&+\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{f(s,v(s))+g(s)}{s}ds,\end{aligned}$$

which can be rearranged as

$$\begin{aligned}v(t)-&\frac{(\log\ t)^{\gamma-1}}{\Delta}\Big\{k\int_1^e\frac{v(s)}{s}ds-\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{f(s,v(s))}{s}ds\\&+\sum_{i=1}^m\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(-k\int_1^s\frac{v(r)}{r}dr\\&+\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{f(r,v(r))}{r}dr\Big)ds\Big\}+k\int_1^t\frac{v(s)}{s}ds\\&-\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{f(s,v(s))}{s}ds\\=&\frac{(\log\ t)^{\gamma-1}}{\Delta}\Big\{-\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{g(s)}{s}ds\\&+\sum_{i=1}^m\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{g(r)}{r}dr\Big)ds\Big\}\\&+\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{g(s)}{s}ds.\end{aligned}$$

Using $|g(t)|\le\epsilon$, we obtain

$$\begin{aligned}&\Big|v(t)-\frac{1}{\Delta}\Big\{k\int_1^e\frac{v(s)}{s}ds-\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{f(s,v(s))}{s}ds+\sum_{i=1}^m\frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\\&\quad\Big(-k\int_1^s\frac{v(r)}{r}dr+\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{f(r,v(r))}{r}dr\Big)ds\Big\}(\log\ t)^{\gamma-1}+k\int_1^t\frac{v(s)}{s}ds\\&\quad-\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{f(s,v(s))}{s}ds\Big|\\&\le\frac{1}{|\Delta|}\Big\{\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{|g(s)|}{s}ds\\&\quad+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{|g(r)|}{r}dr\Big)ds\Big\}(\log\ t)^{\gamma-1}\\&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\ \frac{t}{s}\Big)^{\alpha-1}\frac{|g(s)|}{s}ds\\&\le\epsilon\Bigg(\frac{1}{|\Delta|}\Big\{\frac{1}{\Gamma(\alpha)}\int_1^e\Big(\log\ \frac{e}{s}\Big)^{\alpha-1}\frac{ds}{s}\\&\quad+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Big(\frac{1}{\Gamma(\alpha)}\int_1^s\Big(\log\ \frac{s}{r}\Big)^{\alpha-1}\frac{dr}{r}\Big)ds\Big\}(\log\ t)^{\gamma-1}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\Bigg)\\
=&\epsilon\Bigg(\frac{1}{|\Delta|}\Bigg\{\frac{1}{\Gamma(\alpha+1)}+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\frac{(\log\ s)^\alpha}{\Gamma(\alpha+1)}ds\Bigg\}(\log\ t)^{\gamma-1}+\frac{(\log\ t)^\alpha}{\Gamma(\alpha+1)}\Bigg)\\
\le&\epsilon\Bigg(\frac{1}{|\Delta|}\Bigg\{\frac{1}{\Gamma(\alpha+1)}+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\frac{1}{\Gamma(\alpha+1)}ds\Bigg\}+\frac{1}{\Gamma(\alpha+1)}\Bigg)\\
=&\frac{\epsilon}{\Gamma(\alpha+1)}\Bigg(\frac{1}{|\Delta|}\Bigg\{1+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}ds\Bigg\}+1\Bigg)\\
=&\frac{\epsilon}{\Gamma(\alpha+1)}\Bigg(\frac{1}{|\Delta|}[1+\omega+|\Delta|]\Bigg)=\frac{M\epsilon}{\Gamma(\alpha+1)}.
\end{aligned}
$$

By virtue of Theorem 3.5, we denote by u the unique solution of the problem (1)-(2). Notice that we take the left boundary, $u(1)=v(1)=0$, and the right boundary $u(e)$ is arbitrary, satisfying $u(e)=\sum_{i=1}^m\lambda_i I^{\delta_i}u(\theta_i)$. Then, we have $u(t)=(\mathcal{F}u)(t)$, where $\mathcal{F}$ defined by (13). From above inequality, it follows

$$
\begin{aligned}
&|v(t)-u(t)|=|v(t)-(\mathcal{F}u)(t)|\\
&\le\frac{M\epsilon}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\Bigg\{k\int_1^e\frac{|v(s)-u(s)|}{s}ds+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{|f(s,v(s))-f(s,u(s))|}{s}ds\\
&+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Bigg(k\int_1^s\frac{|v(r)-u(r)|}{r}dr\\
&+\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{|f(r,v(r))-f(r,u(r))|}{r}dr\Bigg)ds\Bigg\}(\log\ t)^{\gamma-1}+k\int_1^t\frac{|v(s)-u(s)|}{s}ds\\
&+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{|f(s,v(s))-f(s,u(s))|}{s}ds.
\end{aligned}
$$

It follows by assumption (H_2) that

$$
\begin{aligned}
\|v-u\|\le&\frac{M\epsilon}{\Gamma(\alpha+1)}+\|v-u\|\Bigg[\frac{1}{|\Delta|}\Bigg\{k\int_1^e\frac{ds}{s}+\frac{l}{\Gamma(\alpha)}\int_1^e\left(\log\ \frac{e}{s}\right)^{\alpha-1}\frac{ds}{s}\\
&+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Bigg(k\int_1^s\frac{dr}{r}+\frac{l}{\Gamma(\alpha)}\int_1^s\left(\log\ \frac{s}{r}\right)^{\alpha-1}\frac{dr}{r}\Bigg)ds\Bigg\}(\log\ t)^{\gamma-1}\\
&+k\int_1^t\frac{ds}{s}+\frac{l}{\Gamma(\alpha)}\int_1^t\left(\log\ \frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\Bigg]\\
\le&\frac{M\epsilon}{\Gamma(\alpha+1)}+\|v-u\|\Bigg[\frac{1}{|\Delta|}\Bigg\{k+\frac{l}{\Gamma(\alpha+1)}\\
&+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}\Bigg(k\log\ s+\frac{l(\log\ s)^\alpha}{\Gamma(\alpha+1)}\Bigg)ds\Bigg\}(\log\ t)^{\gamma-1}+k\log\ t+\frac{l(\log\ t)^\alpha}{\Gamma(\alpha+1)}\Bigg]\\
\le&\frac{M\epsilon}{\Gamma(\alpha+1)}+\|v-u\|\left(k+\frac{l}{\Gamma(\alpha+1)}\right)\Bigg(\frac{1}{|\Delta|}\Bigg\{1+\sum_{i=1}^m\frac{|\lambda_i|}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i-s)^{\delta_i-1}ds\Bigg\}+1\Bigg)\\
=&\frac{M\epsilon}{\Gamma(\alpha+1)}+\|v-u\|\left(k+\frac{l}{\Gamma(\alpha+1)}\right)M.
\end{aligned}
$$

Therefore,

$$
\|v-u\|\le c_f\epsilon,\quad\text{where}\quad c_f=\frac{M}{(1-\Xi)\Gamma(\alpha+1)}>0.
$$

Hence the problem (1)-(2) is Ulam-Hyers stable. Moreover, it is generalized Ulam-Hyers stable as $\|v-u\| \leq \theta_f(\epsilon)$, with $\theta_f(\epsilon) = c\epsilon, \theta_f(0) = 0$. □

Theorem 4.2. *Assume that assumption (H_2) and condition (14) hold, and that there exists a function φ satisfying assumption (H_6). Then the problem (1)-(2) is Ulam-Hyers-Rassias stable, and hence generalized Ulam-Hyers-Rassias with respect to φ.*

Proof. Let $\epsilon > 0$ and v satisfies the differential inequality (5) with the boundary condition (6). By integration of (5) and using (H_6), for any $t \in [1, e]$ one has

$$\bigg| v(t) - c_0(\log t)^{\gamma-1} - c_1\bigg((\log\ t)^{\gamma-2} + k\int_1^t \frac{(\log\ s)^{\gamma-2}}{s}ds\bigg) + k\int_1^t \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha)}\int_1^t \frac{f(s,v(s))}{s}\bigg(\log\ \frac{t}{s}\bigg)^{\alpha-1}ds\bigg| \leq \epsilon\, {}_HI_{1^+}^{\alpha}\varphi(t) \leq \epsilon\lambda_\varphi\varphi(t),$$

for all $c_0 = \frac{(\delta_H I_{1^+}^{2-\gamma}v)(1)}{\Gamma(\gamma)}$, $c_1 = \frac{({}_HI_{1^+}^{2-\gamma}v)(1)}{\Gamma(\gamma-1)} \in \mathbb{R}$. By virtue of the proof of Lemma 3.1, we will choose c_0 and c_1 such that v in the above inequality satisfies the boundary condition (6), as follows

$$\begin{aligned} c_0 =& \frac{1}{\Delta}\bigg\{k\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha)}\int_1^e \bigg(\log\ \frac{e}{s}\bigg)^{\alpha-1}\frac{f(s,v(s))}{s}ds \\ &+ \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\bigg(-k\int_1^s \frac{v(r)}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s \bigg(\log\ \frac{s}{r}\bigg)^{\alpha-1}\frac{f(r,v(r))}{r}dr\bigg)ds\bigg\} \end{aligned}$$

and set $c_1 = 0$. Then we have the inequality

$$\begin{aligned} \bigg| v(t) &- \frac{(\log\ t)^{\gamma-1}}{\Delta}\bigg\{k\int_1^e \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha)}\int_1^e \bigg(\log\ \frac{e}{s}\bigg)^{\alpha-1}\frac{f(s,v(s))}{s}ds \\ &+ \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\delta_i)}\int_1^{\theta_i}(\theta_i - s)^{\delta_i-1}\bigg(-k\int_1^s \frac{v(r)}{r}dr + \frac{1}{\Gamma(\alpha)}\int_1^s \bigg(\log\ \frac{s}{r}\bigg)^{\alpha-1}\frac{f(r,v(r))}{r}dr\bigg)ds\bigg\} \\ &+ k\int_1^t \frac{v(s)}{s}ds - \frac{1}{\Gamma(\alpha)}\int_1^t \bigg(\log\ \frac{t}{s}\bigg)^{\alpha-1}\frac{f(s,v(s))}{s}ds\bigg| \leq \epsilon\lambda_\varphi\varphi(t), \quad t \in [1, e]. \end{aligned}$$

Now, by virtue of Theorem 3.5, we let u be the unique solution of the problem (1)-(2). That is defined as $u(t) = (\mathcal{F}u)(t)$, where $\mathcal{F}$ is defined by (13). From above inequality, the same method as in the proof of Theorem 4.1, it follows that

$$|v(t) - u(t)| = |v(t) - (\mathcal{F}u)(t)| \leq \epsilon\lambda_\varphi\varphi(t) + \|v-u\|\bigg(k + \frac{l}{\Gamma(\alpha+1)}\bigg)M.$$

Therefore,

$$\|v-u\| \leq c_{f,\varphi}\epsilon\varphi(t), \quad \text{where} \quad c_{f,\varphi} = \frac{\lambda_\varphi}{1-\Xi} > 0.$$

Hence, the problem (1)-(2) is Ulam-Hyers-Rassias stable with respect to φ. Moreover, it is generalized Ulam-Hyers-Rassias stable with respect to φ, if we take $\epsilon = 1$ then $\|v-u\| \leq c_{f,\varphi}\varphi(t)$. This completes the proof. □

5 Examples

In this section, we give four examples to illustrate our main results.

Example 5.1. *Consider the following boundary value problem*

$$\begin{cases} \left({}_H D_{1^+}^{\frac{3}{2},\frac{1}{6}} + \dfrac{1}{5} {}_H D_{1^+}^{\frac{1}{2},\frac{1}{6}}\right) u(t) = e^t \sin u(t), \quad t \in [1,e], \\ u(1) = 0, \ u(e) = 4 I^{\frac{5}{2}} u\left(\dfrac{3}{2}\right) + \dfrac{1}{4} I^{\frac{3}{2}} u(2). \end{cases} \tag{18}$$

Here $\alpha = \frac{3}{2}$, $\beta = \frac{1}{6}$, $k = \frac{1}{5}$, $\lambda_1 = 4$, $\lambda_2 = \frac{1}{4}$, $\delta_1 = \frac{5}{2}$, $\delta_2 = \frac{3}{2}$, $\gamma = \frac{19}{12}$, $\theta_1 = \frac{3}{2}$, *and* $\theta_2 = 2$. *For each* $u \in \mathbb{R}$, *we have* $|f(t,u)| \le e^t$, *and thus* (H_1) *is satisfied. Using the given data, we find that* $\omega = 0.4008$, $\Delta = 0.6854$, $M = 3.0438$, *and*

$$kM = \frac{k}{|\Delta|}[1 + \omega + |\Delta|] \approx 0.6088 < 1.$$

Hence, all the conditions of Theorem 3.2 are satisfied. Therefore, the boundary value problem (18) has at least one solution on $[1,e]$.

Example 5.2. *Consider the following boundary value problem*

$$\begin{cases} \left({}_H D_{1^+}^{2,\frac{1}{4}} + \dfrac{1}{3} {}_H D_{1^+}^{1,\frac{1}{4}}\right) u(t) = \dfrac{\sin u(t)}{(\log t + 10)} + \dfrac{1}{(t+1)^3}, \quad t \in [1,e], \\ u(1) = 0, \ u(e) = -\dfrac{1}{20} I^{\frac{3}{2}} u(2) + \dfrac{1}{6} I^{\frac{7}{4}} u\left(\dfrac{9}{5}\right) - 4 I^{\frac{5}{3}} u\left(\dfrac{4}{3}\right). \end{cases} \tag{19}$$

Here $\alpha = 2$, $\beta = \frac{1}{4}$, $k = \frac{1}{3}$, $\lambda_1 = -\frac{1}{20}$, $\lambda_2 = \frac{1}{6}$, $\lambda_3 = -4$, $\delta_1 = \frac{3}{2}$, $\delta_2 = \frac{7}{4}$, $\delta_3 = \frac{5}{3}$, $\gamma = \frac{9}{4}$, $\theta_1 = 2$, $\theta_2 = \frac{9}{5}$, *and* $\theta_3 = \frac{4}{3}$. *For each* $u \in \mathbb{R}$, *we have*

$$|f(t,u)| \le \left|\frac{1}{(\log t + 10)}\right| + \left|\frac{1}{(t+1)^3}\right| \le \frac{1}{10} + \frac{1}{8} = \frac{9}{40},$$

thus (H_3) *is satisfied. Using the given data, we find that* $\omega = 0.5338$, $\Delta = 1.0272$, $M = 2.4931$, *and*

$$kM = \frac{k}{|\Delta|}[1 + \omega + |\Delta|] \approx 0.83103 < 1.$$

Hence, all the conditions of Theorem 3.3 are satisfied. Therefore, the boundary value problem (19) has at least one solution on $[1,e]$.

Example 5.3. *Consider the following boundary value problem*

$$\begin{cases} \left({}_H D_{1^+}^{\frac{3}{2},\frac{1}{4}} + \frac{1}{5} {}_H D_{1^+}^{\frac{1}{2},\frac{1}{4}}\right) u(t) = [u(t)]^2 \cos t, \quad t \in [1,e], \\ u(1) = 0, \ u(e) = \dfrac{1}{20} I^{\frac{10}{7}} u(2) + \dfrac{1}{6} I^{\frac{9}{2}} u\left(\dfrac{9}{5}\right), \end{cases} \tag{20}$$

Here $\alpha = \frac{3}{2}$, $\beta = \frac{1}{4}$, $k = \frac{1}{5}$, $\lambda_1 = \frac{1}{20}$, $\lambda_2 = \frac{1}{6}$, $\delta_1 = \frac{10}{7}$, $\delta_2 = \frac{9}{2}$, $\gamma = \frac{13}{8}$, $\theta_1 = 2$, *and* $\theta_2 = \frac{9}{5}$. *For each* $u \in \mathbb{R}$, *there exists a constant function* $p(t) = 1$ *and continuous nondecreasing function* $\psi(x) = x^2$, *for all* $x \in \mathbb{R}_+$ *such that* $|f(t,u)| \le p(t)\psi(|u|) = |u|^2$, *and then* (H_4) *is satisfied. Using the given data, we find that* $\omega = 0.0388$, $\Delta = 0.9822$, $M = 2.0576$, *and*

$$kM = \frac{k}{|\Delta|}[1 + \omega + |\Delta|] \approx 0.41152 < 1.$$

There exists a constant $C = 0.3801$ such that

$$\frac{\Gamma(\alpha+1)(1-kM)C}{M\|p\|\psi(C)} = \frac{\Gamma(\alpha+1)(1-kM)C}{MC^2} \approx 1.0003 > 1.$$

Hence, (H_5) is satisfied. We set $U = \{u \in X : \|u\| < 0.3801\}$. Therefore, all the conditions of Theorem 3.4 are satisfied. Thus, the boundary value problem (20) has at least one solution on $[1, e]$.

Example 5.4. *Consider the following boundary value problem*

$$\begin{cases} \left({}_H D_{1+}^{\frac{4}{3},\frac{1}{2}} + \frac{1}{10}\,{}_H D_{1+}^{\frac{1}{3},\frac{1}{2}}\right)u(t) = \dfrac{u(t)}{\sqrt{99+t^2}(5+u(t))} + \dfrac{1}{10+t^3}, \quad t \in [1,e], \\ u(1) = 0, \;\; u(e) = -10 I^{\frac{7}{2}} u\left(\dfrac{5}{3}\right) + \dfrac{13}{5} I^{\frac{5}{2}} u(2), \end{cases} \tag{21}$$

Here $\alpha = \frac{4}{3}$, $\beta = \frac{1}{2}$, $k = \frac{1}{10}$, $\lambda_1 = -10$, $\lambda_2 = \frac{13}{5}$, $\delta_1 = \frac{7}{2}$, $\delta_2 = \frac{5}{2}$, $\gamma = \frac{5}{3}$, $\theta_1 = \frac{5}{3}$, and $\theta_2 = 2$. Notice (H_2) is satisfied with $l = \frac{1}{50}$, because

$$\begin{aligned} |f(t,u_1) - f(t,u_2)| &= \left| \frac{u_1(t)}{\sqrt{99+t^2}(5+u_1(t))} - \frac{u_2(t)}{\sqrt{99+t^2}(5+u_2(t))} \right| \\ &\le \frac{1}{10}\left| \frac{u_1(t)}{(5+u_1(t))} - \frac{u_2(t)}{(5+u_2(t))} \right| \le \frac{1}{50}|u_1 - u_2|. \end{aligned}$$

Using the given data, we find that $\omega = 0.9903$, $\Delta = 0.7675$, $M = 3.5933$, and

$$\Xi := \left[kM + \frac{lM}{\Gamma(\alpha+1)} \right] \approx 0.4197 < 1.$$

Hence, all the conditions of Theorem 3.5 are satisfied. Therefore, the boundary value problem (21) has a unique solution on $[1, e]$. Moreover, the problem (21) is Ulam-Hyers stable and generalized Ulam-Hyers stable according to Theorem 4.1. In addition, by virtue of Theorem 4.2, if there exists a function $\varphi : [1, e] \to \mathbb{R}_+$ satisfying the assumption (H_6), then the problem (21) is Ulam-Hyers-Rassias stable, and generalized Ulam-Hyers-Rassias stable on $[1, e]$ with respect to φ.

6 Conclusion

This paper presents existence and uniqueness results for Hilfer-Hadamard sequential fractional differential equations (1) with multi-point Riemann-Liouville fractional integral boundary conditions (2). Firstly, by considering a linear variant of the given problem, we converted the nonlinear problem into a fixed point problem. Once the fixed point operator was established, the existence results were derived using the Krasnoselskii's fixed point theorem, the Schaefer fixed point theorem, and the Leray-Schauder nonlinear alternative. The Banach contraction principle was then applied to achieve the existence and uniqueness result.

Moreover, the stability of the problem in the sense of Ulam-Hyers and Ulam-Hyers-Rassias were determined. We found that if the problem has a unique solution according to the assumptions of Theorem 3.5, it is also Ulam-Hyers stable and generalized Ulam-Hyers stable on $[1, e]$. Furthermore, by adding one more condition as (H_6), we obtained Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability results. Additionally, we provide examples that illustrate the obtained results.

In summary, we established results regarding existence, uniqueness, and stability for the Hilfer-Hadamard sequential fractional differential equations with multi-point fractional integral boundary conditions, thereby extending their applicability to a wider range of mathematical models.

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