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TIMOSHENKO SYSTEM WITH INTERNAL DISSIPATION OF FRACTIONAL DERIVATIVE TYPE

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Abstract This manuscript deals with the well-posedness and asymptotic behavior of the Timoshenko system with internal dissipation of fractional derivative type. We use semigroup theory. The existence and uniqueness of the solution are obtained by applying the Lumer-Phillips Theorem. We present two results for the asymptotic behavior: strong stability of the C_0-semigroup associated with the system using the Arendt-Batty and Lyubich-Vũ's general criterion and the polynomial stability applying the Borichev-Tomilov's theorem. This results expand the understanding of the asymptotic behavior of Timoshenko systems with fractional internal dissipation, providing clear criteria for both strong and polynomial stability.

Keywords Timoshenko system, well-posedness, polynomial stability, fractional derivative type damping.

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1. Introduction

In 1921, Timoshenko [35] introduced the pioneer system of beams given by

$$\begin{cases} \rho_1\phi_{tt} - k(\phi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) = 0, \end{cases} \tag{1.1}$$

where $(x, t) \in (0, L) \times (0, +\infty)$. The functions $\phi = \phi(x, t)$ is the transverse displacement, and $\psi = \psi(x, t)$ is the rotation of the neutral axis due to bending. The coefficients are positive, being $\rho_1 = \rho S$, $\rho_2 = \rho I$, $b = EI$, $k = \kappa GS$, where S and I are the cross-sectional area and the second moment of the cross-sectional area, respectively; E, G, and κ are Young's modulus, the modulus of rigidity, and the transverse shear factor, respectively.

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About this pioneering system, we have a wide literature; see, for instance, [1,6, 29,32] and references therein. It is well known that when the system (1.1) is full damped, like

$$\begin{cases} \rho_1\phi_{tt} - k(\phi_x + \psi)_x + \phi_t = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) + \psi_t = 0, \end{cases}$$

the exponential stability holds, see [29]. However, when partially damped,

$$\begin{cases} \rho_1\phi_{tt} - k(\phi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) + b(x)\psi_t = 0, \end{cases}$$

with $b(x) \in C^0([0,\, L]),\ 0 < b_0 \le b(x)$ the exponential stability holds if and only if the wave velocities from the system equations are the same, see [32], that is

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}. \tag{1.2}$$

Condition (1.2) is a consequence of a second non-physical frequency spectrum in the Timoshenko beam. The discovery of the second spectrum, which acts in opposition to the dissipative properties of the system (1.1), is credited to Manevich and Kolakowski [24] and Nesterenko [25]. Elishakoff [15] presented a model that eliminates (1.2). For a historical review of Timoshenko's theory, including essential phases of his life and recent arguments about the Timoshenko-Ehrenfest partnership, see, for instance, [16–18].
We are interested in internal damping of fractional order. In this direction, we consider the following model:

$$\begin{cases} \rho_1\phi_{tt} - k(\phi_x + \psi)_x + a\partial_t^{\alpha,\,\eta}\phi = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) + c\partial_t^{\beta,\,\zeta}\psi = 0, \\ \phi(0,\, t) = \phi(L,\, t) = 0 \quad \text{and} \quad \psi(0,\, t) = \psi(L,\, t) = 0, \\ \phi(x,\, 0) = \phi_0(x) \quad \text{and} \quad \psi(x,\, 0) = \psi_0(x), \\ \phi_t(x,\, 0) = \phi_1(x) \quad \text{and} \quad \psi_t(x,\, 0) = \psi_1(x), \end{cases} \tag{1.3}$$

where $(x,\, t) \in (0,\, L) \times (0,\, +\infty)$, $\eta,\, \zeta \ge 0$, $\alpha,\, \beta \in (0,\, 1)$ and L, ρ_1, ρ_2, k, a, b, c positive real constants.

For the reader's taste, we briefly review fractional calculus. There are many definitions for fractional derivatives [14], among which Riemann-Liouville's and Caputo's are the most widely used. A fractional derivative with a non-singular kernel involving exponential and trigonometric functions was proposed in [3]. The suggested fractional operator includes the Caputo-Fabrizio fractional derivative as a particular case. In this paper, the fractional derivative damping force is regarded as a control force to study the properties of free-damped vibration of the system, so the Caputo definition [10–12] is used here.

Let $0 < \omega < 1$. The Caputo fractional integral operator of order ω is defined by

$$I^\omega f(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} f(s)ds, \tag{1.4}$$

where Γ is the well-known gamma function, and $f \in L^1([0,\, +\infty))$.

The Caputo fractional derivative operator of order α is defined by

$$\partial_t^{\omega} f(t) = I^{1-\omega} f'(t) := \frac{1}{\Gamma(1-\omega)} \int_0^t (t-s)^{-\omega} f'(s) ds, \tag{1.5}$$

with $f \in W^{1,1}([0, +\infty))$.

Besides, we note that the Caputo definition of the fractional derivative does possess a straightforward but interesting interpretation: if the function $f(t)$ represents the strain history within a viscoelastic material whose relaxation function is $[\Gamma(1-\alpha)t^{\alpha}]^{-1}$ then the material will experience at any time t total stress given the expression $\partial_t^{\alpha} f(t)$. Also, it easy to show that ∂_t^{α} is a left inverse of I^{α}, but in general it is not a right inverse. More precisely, we have

$$\partial_t^{\omega} I^{\omega} f = f, \qquad I^{\omega} \partial_t^{\omega} f(t) = f(t) - f(0).$$

For the proof of above equalities and more properties of fractional calculus see [31].

This work considers slightly different versions of (1.4) and (1.5). In [13], Choi and MacCamy defined fractional integro-differential operators with exponential weight. Let $0 < \omega < 1$, $\delta \geq 0$, the exponential fractional integral of order ω and weight δ is defined by

$$I^{\omega,\delta} f(t) = \frac{1}{\Gamma(\omega)} \int_0^t e^{-\delta(t-s)} (t-s)^{\omega-1} f(s) ds, \tag{1.6}$$

with $f \in L^1([0, +\infty))$.

The exponential fractional derivative operator of order ω and weight δ is defined by

$$\partial_t^{\omega,\delta} f(t) = \frac{1}{\Gamma(1-\omega)} \int_0^t e^{-\delta(t-s)} (t-s)^{-\alpha} f'(s) ds, \tag{1.7}$$

with $f \in W^{1,1}([0, +\infty))$.

Note that

$$\partial_t^{\omega,\delta} f(t) = I^{1-\omega,\delta} f'(t). \tag{1.8}$$

An essential advantage of fractional differential equations in applications is the non-local property, making fractional calculus more attractive. In [9], the stabilization of a wave equation with general internal control of the diffusive type was analyzed with Caputo's fractional derivative damping for a particular kernel. Fractional calculus has been increasingly applied in different fields of science, for example: applications in bioengineering [23], dynamics of particles, fields, and media [34], Bats–Hosts–Reservoir–People transmission fractional-order COVID-19 model for simulating the potential transmission with individual response and control measures [30], electrical circuits [3], and science and engineering [27, 33, 36]. Recently, Ammari et al. [4] have given unified methods for stabilizing some fractional evolution systems; they consider the stabilization for some abstract evolution equations with fractional damping and validate the abstract results with concrete examples. They also study the stabilization of fractional evolution systems with memory. In general, fractional order derivative is used as boundary damping or as delays of fractional order [21]. In both situations, the Lax-Milgran Theorem can be applied naturally to obtain a well-posedness of the extended problem. In this context, Benaissa-Benazzouz [6] studied the following Timoshenko's system with boundary dissipation

$$\begin{cases} \phi(0,\,t) = 0 \quad \text{and} \quad \psi(0,\,t) = 0, \\ m_1\phi_{tt}(L,\,t) + k(\phi_x + \psi)(L,\,t) = -\gamma_1 \partial_t^{\alpha,\,\eta}\phi(L,\,t), \\ m_2\psi_{tt}(L,\,t) + b\psi_{xx}(L,\,t) = -\gamma_2 \partial_t^{\alpha,\,\eta}\psi(L,\,t), \end{cases} \tag{1.9}$$

and provides a global solution by applying the Hille–Yosida Theorem. From a stability point of view, in (1.9), the action of two dynamic control boundary conditions of the fractional derivative type gives the polynomial stability.

In [2], a one-dimensional Timoshenko system's indirect boundary stability and exact controllability are studied. The authors show that the system is strongly stable but not uniformly stable. They proved that the energy decay rate depends on the coefficients appearing in the system and on the order of the fractional damping. Moreover, under the equal speed propagation condition, the optimal polynomial energy decay rate was obtained.

Adnane *et al.* [1] considered the Timoshenko system with a delay in fractional order given by

$$\begin{cases} \rho_1\phi_{tt}(x,\,t) - k(\phi_x + \psi)(x,\,t) + a_1\partial_t^{\alpha,\,\eta}(x,\,t-\tau_1) + a_2\psi_t(x,\,t) = 0, \\ \rho_2\psi_{tt}(x,\,t) - b(\psi_{xx}(x,\,t) + k(\phi_z + \psi)(x,\,t) + \tilde{a_1}\partial_t^{\beta,\,\zeta}\psi(x,\,t-\tau_2) + \tilde{a_2}\psi_t(x,\,t) = 0. \end{cases} \tag{1.10}$$

Under a condition on the fractional delay, using a classical semigroup theory gives the existence and uniqueness of the solution by Lumer-Phillips Theorem. From a stability point of view, in (1.10), the two fractional time delays associated with internal frictional dampings lead to an exponential stability result.

Our result differs from works [2, 6] in that it presents stability analysis using fractional dampings in the domain. We appoint the differs from the work [1] by two relevant aspects. The two fractional smoothings are not in the same order, and stability is achieved without delays. Thus, this is the first contribution to the literature regarding the Timoshenko beam with fractional damping on the domain.

The remainder of this paper is organized as follows. Section 2 is concerned with reformulating the model (1.3) into an augmented system. Section 3, the existence of a solution is given by applying the Lumer-Phillips Theorem. Section 4 presents the strong stability of the C_0-semigroup associated with the system using Arendt-Batty and Lyubich-Vũ's general criterion. In Section 5, the polynomial stability is proved by applying Borichev-Tomilov's Theorem.

2. Augmented model and Preliminary results

This section concerns reformulating the model (1.3) into an augmented system. The following proposition is fundamental to building the augmented model.

Proposition 2.1. *[See [4]] Let p be the function*

$$p(y) = |y|^{\frac{2\omega-1}{2}}, \quad y \in \mathbb{R}, \quad 0 < \omega < 1. \tag{2.1}$$

Then the relation between the Input $\mathcal{U}$ and the Output $\mathcal{O}$ if the following system

$$\begin{cases} \varphi_t(t,y) + (|y|^2 + \delta)\varphi(t,y) = p(y)\mathcal{U}(t), \\ \varphi(0,\, y) = 0, \\ \mathcal{O}(t) = \gamma \displaystyle\int_{\mathbb{R}} p(y)\varphi(t,y)dy, \end{cases} \tag{2.2}$$

where $\mathcal{U} \in C([0,\, +\infty))$, and $\gamma = \dfrac{\sin\omega\pi}{\pi} = \dfrac{1}{\Gamma(\omega)\Gamma(1-\omega)}$, is given by

$$\mathcal{O}(t) = I^{1-\omega,\,\delta}\mathcal{U}(t). \tag{2.3}$$

Taking $\mathcal{U} = a\phi_t$ in Proposition 2.1 and applying a expression (2.3), we obtain

$$\gamma_1 \int_{\mathbb{R}} p(y)\varphi_1(x,\, t,\, y)dy = a\partial_t^{\alpha,\,\eta}\phi(x,\, t),$$

where $\gamma_1 = \dfrac{a\sin\alpha\pi}{\pi} = \dfrac{a}{\Gamma(\alpha)\Gamma(1-\alpha)}$ and $p(y) = |y|^{\frac{2\alpha-1}{2}}$.

Similahirly, applying Proposition 2.1 with $\mathcal{U} = c\psi_t$, we get

$$\gamma_2 \int_{\mathbb{R}} q(y)\varphi_2(x,\, t,\, y)dy = c\partial_t^{\beta,\,\zeta}\psi(x,\, t),$$

where $\gamma_2 = \dfrac{c\sin\beta\pi}{\pi} = \dfrac{c}{\Gamma(\beta)\Gamma(1-\beta)}$ and $q(y) = |y|^{\frac{2\beta-1}{2}}$.

Thus, the problem (1.3) is equivalent to the following augmented model

$$\begin{cases} \rho_1\phi_{tt}(x,\, t) - k(\phi_x(x,\, t) + \psi(x,\, t))_x + \gamma_1 \displaystyle\int_{\mathbb{R}} p(y)\varphi_1(x,\, t,\, y)dy = 0, \\ \rho_2\psi_{tt}(x,\, t) - b\psi_{xx}(x,\, t) + k(\phi_x(x,\, t) + \psi(x,\, t)) + \gamma_2 \displaystyle\int_{\mathbb{R}} q(y)\varphi_2(x,\, t,\, y)dy = 0, \\ (\varphi_1)_t(x,\, t,\, y) + (|y|^2 + \eta)\varphi_1(x,\, t,\, y) - p(y)\phi_t(x,\, t) = 0, \\ (\varphi_2)_t(x,\, t,\, y) + \left(|y|^2 + \zeta\right)\varphi_2(x,\, t,\, y) - q(y)\psi_t(x,\, t) = 0, \\ \phi(0,\, t) = \phi(L,\, t) = 0 \quad \text{and} \quad \psi(0,\, t) = \psi(L,\, t) = 0, \\ \varphi_1(0,\, 0,\, y) = \varphi_1(L,\, 0,\, y) = 0 \quad \text{and} \quad \varphi_2(0,\, 0,\, y) = \varphi_2(L,\, 0,\, y) = 0, \\ \phi(x,\, 0) = \phi_0(x), \quad \psi(x,\, 0) = \psi_0(x), \quad \phi_t(x,\, 0) = \phi_1(x), \quad \psi_t(x,\, 0) = \psi_1(x), \\ \varphi_1(x,\, 0,\, y) = 0, \quad \varphi_2(x,\, 0,\, y) = 0. \end{cases} \tag{2.4}$$

Now, consider the following technical lemmas. The Lemma 2.1 will be used for well-posedness, the Lemma 2.3 will be used for the proof of strong stability, and the Lemma 2.2 will be used for the proof of polynomial stability.

Lemma 2.1. *If $0 < \omega < 1$ and $\delta \geq 0$, then*

$$C(\omega,\, \delta) := \int_{\mathbb{R}} \frac{|y|^{2\omega-1}dy}{|y|^2 + \delta + 1} < +\infty \text{ and } D(\omega,\, \delta) := \int_{\mathbb{R}} \frac{|y|^{2\omega-1}dy}{(|y|^2 + \delta + 1)^2} < +\infty.$$

Proof. Note that

$$C(\omega,\,\delta):=\int_{\mathbb{R}}\frac{|y|^{2\omega-1}dy}{|y|^2+\delta+1}=\frac{2}{(1+\delta)}\int_0^{+\infty}\frac{|y|^{2\omega-1}dy}{1+\dfrac{|y|^2}{(1+\delta)}}.$$

As $0<\omega<1$, by making a change of variable, we have that

$$0<C(\omega,\,\delta):=\frac{1}{(1+\delta)^{1-\omega}}\int_1^{+\infty}\frac{d\sigma}{\sigma(\sigma-1)^{1-\omega}}<+\infty.$$

To do this, it is sufficient to show that $\frac{1}{\sigma(\sigma-1)^{1-\omega}}\le\frac{1}{\sigma^{1+\omega}}$ para σ sufficiently large. Indeed

$$\begin{aligned}\int_1^{+\infty}\frac{d\sigma}{\sigma(\sigma-1)^{1-\omega}}&\le\int_1^{N}\frac{d\sigma}{\sigma(\sigma-1)^{1-\omega}}+\int_N^{+\infty}\frac{d\sigma}{\sigma^{1+\omega}}\\&=K+\lim_{t\to+\infty}\int_N^{t}\frac{d\sigma}{\sigma^{1+\omega}}=K+\frac{1}{\omega N^{-\omega}}.\end{aligned}$$

Let us now prove the statement. Multiplying both sides of the inequality by $\sigma\sigma^{1+\omega}=\sigma^{2+\omega}$, we have

$$\frac{\sigma\sigma^{1+\omega}}{\sigma(\sigma-1)^{1-\omega}}\le\frac{\sigma^{2+\omega}}{\sigma^{1+\omega}}=\sigma.$$

Thus

$$\frac{1}{\sigma(\sigma-1)^{1-\omega}}\le\frac{1}{\sigma^{1+\omega}}\iff\frac{\sigma^{\omega}}{(\sigma-1)^{1-\omega}}\le 1.$$

Finally, taking σ sufficiently large such that $\sigma^2>3\sigma-1$, has $\sigma-1>\dfrac{\sigma}{\sigma-1}>1$. As $0<\omega<1$, we have $\left(\dfrac{\sigma}{\sigma-1}\right)^{\omega}<\dfrac{\sigma}{\sigma-1}<\sigma-1$. Logo $\sigma^{\omega}<(\sigma-1)^{1-\omega}$. Moreover, note that

$$D(\omega,\,\delta):=\int_{\mathbb{R}}\frac{|y|^{2\omega-1}dy}{(|y|^2+\delta+1)^2}\le\int_{\mathbb{R}}\frac{|y|^{2\omega-1}dy}{|y|^2+\delta+1}=C(\omega,\,\delta)<+\infty.$$

□

Lemma 2.2. *If $0<\omega<1$, $\lambda\ge 0$ and $\delta>0$, then*

$$J(\lambda,\omega,\delta):=\int_{\mathbb{R}}\frac{|y|^{2\omega-1}dy}{|y|^2+\delta+\lambda}<+\infty\ \text{and}\ L(\lambda,\omega,\,\delta):=\int_{\mathbb{R}}\frac{|y|^{2\omega-1}dy}{(|y|^2+\delta+\lambda)^2}<+\infty.$$

Proof. *Analogous to the proof of Lemma 2.1.* □

Lemma 2.3. *Let $0<\omega<1$. If $\delta>0$ and $\lambda\in\mathbb{R}$, or if $\delta=0$ and $\lambda>0$, then*

$$E(\lambda,\,\omega,\,\delta):=\int_{\mathbb{R}}\frac{|y|^{2\omega-1}dy}{|y|^2+\delta+\lambda i}<\infty.$$

Furthermore, for $j=1,\,2$, we have that

$$H_j(x,\,\lambda,\,\omega,\,\delta):=\int_{\mathbb{R}}\frac{|y|^{\frac{2\omega-1}{2}}h_j(x,\,y)dy}{|y|^2+\delta+\lambda i}\in L^2(0,\,L).$$

Proof. Note that $E(\lambda, \omega, \delta) = F(\lambda, \omega, \delta) + \lambda i G(\lambda, \omega, \delta)$, where

$$F(\lambda, \omega, \delta) := \int_{\mathbb{R}} \frac{(|y|^2+\delta)|y|^{2\omega-1}dy}{\lambda^2+(|y|^2+\delta)^2} \quad \text{and} \quad G(\lambda, \omega, \delta) := \int_{\mathbb{R}} \frac{|y|^{2\omega-1}dy}{\lambda^2+(|y|^2+\delta)^2}.$$

We use that

$$G(\lambda, \omega, \delta) = 2\int_0^1 \frac{|y|^{2\omega-1}dy}{\lambda^2+(|y|^2+\delta)^2} + 2\int_1^{+\infty} \frac{|y|^{2\omega-1}dy}{\lambda^2+(|y|^2+\omega)^2}.$$

Since in both cases, ($\delta > 0$ and $\lambda \in \mathbb{R}$) or ($\delta = 0$ and $\lambda > 0$), we obtain

$$\frac{|y|^{2\omega-1}}{\lambda^2+(|y|^2+\delta)^2} \sim \frac{|y|^{2\omega-1}}{\lambda^2+\delta^2} \quad \text{as} \quad |y| \to 0$$

and

$$\frac{|y|^{2\omega-1}}{\lambda^2+(|y|^2+\delta)^2} \sim \frac{1}{|y|^{5-2\omega}} \quad \text{as} \quad |y| \to +\infty,$$

it follows that $G(\lambda, \delta) < \infty$. Similarly,

$$F(\lambda, \omega, \delta) = 2\int_0^1 \frac{(|y|^2+\delta)|y|^{2\omega-1}dy}{\lambda^2+(|y|^2+\delta)^2} + 2\int_1^{+\infty} \frac{(|y|^2+\delta)|y|^{2\omega-1}dy}{\lambda^2+(|y|^2+\omega)^2},$$

and, if ($\delta > 0$ and $\lambda \in \mathbb{R}$) or ($\delta = 0$ and $\lambda > 0$), we obtain

$$\frac{(|y|^2+\delta)|y|^{2\omega-1}}{\lambda^2+(|y|^2+\delta)^2} \sim \frac{(|y|^2+\delta)|y|^{2\omega-1}}{\lambda^2+\delta^2} \quad \text{for} \quad |y| \to 0$$

and

$$\frac{(|y|^2+\delta)|y|^{2\omega-1}}{\lambda^2+(|y|^2+\delta)^2} \sim \frac{1}{|y|^{3-2\omega}} \quad \text{for} \quad |y| \to +\infty.$$

Thus, $F(\lambda, \omega, \delta) < \infty$, and consequently, it follows that $E(\lambda, \omega, \delta) < \infty$. Moreover, from the Cauchy-Schwarz inequality and the fact that $h_j \in L^2(\mathbb{R}; L^2(0, L))$, it follows that

$$\int_0^L |H(x, \lambda, \omega, \delta)|^2 \, dx = \left(\int_{\mathbb{R}} \frac{|y|^{2\omega-1}dy}{\lambda^2+(|y|^2+\delta)^2}\right) \int_0^L \int_{\mathbb{R}} |h_j(x, y)|^2 dy dx < +\infty.$$

□

Theorem 2.1 (Lumer-Phillips, [26]). *Let $\mathcal{A}$ be a linear operator with domain $\mathcal{D}(\mathcal{A})$ dense in a Hilbert space $\mathcal{H}$ satisfying*

(i) $Re\langle \mathcal{A}u, u\rangle \leq 0;\ \forall u \in \mathcal{H}$ (dissipatividade).

(ii) There exists $\lambda > 0$ such that $(\lambda I - \mathcal{A})(\mathcal{H}) = \mathcal{H}$ (maximalidade).

Then $\mathcal{A}$ is the infinitesimal generator of a contraction C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{H}$.

Theorem 2.2 (Arendt-Batty [5], Lyubich-Vũ [22]). *Let $\mathcal{A}$ be the generator of a C_0 contraction semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ in a reflexive Banach space X If the following conditions are satisfied:*

(i) $\mathcal{A}$ has no purely imaginary eigenvalues;

(ii) $\sigma(\mathcal{A}) \cap i\mathbb{R}$ *is countable.*

Then, $(e^{t\mathcal{A}})_{t\geq 0}$ *is strongly stable. That is*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}x\| = 0;\ \forall x \in X.$$

Theorem 2.3 (Borichev-Tomilov [8])**.** *Let* $(e^{t\mathcal{A}})_{t\geq 0}$ *be a* C_0*-semigroup of contractions on a Hilbert space* $\mathcal{H}$ *such that* $i\mathbb{R} \subset \rho(\mathcal{A})$*. Then* $(e^{t\mathcal{A}})_{t\geq 0}$ *is polynomial stable, that is, for every* $x \in \mathcal{D}(\mathcal{A})$*,*

$$\|e^{t\mathcal{A}}x\| \leq \frac{C}{t^{\omega}}\|x\|_{\mathcal{D}(\mathcal{A})};\ \forall t \geq 0, \tag{2.5}$$

for some $C > 0$ *and for* $\omega > 0$*, if and only if*

$$\limsup_{|\lambda|\to\infty} \frac{1}{|\lambda|^{1/\omega}} \left\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 2.4 (Gearhart-Prüss-Huang [19,20,28])**.** *Let* $(e^{t\mathcal{A}})_{t\geq 0}$ *be a* C_0*-semigroup of contractions defined on a Hilbert space* $\mathcal{H}$ *and generated by* $\mathcal{A}$*. Then,* $(e^{t\mathcal{A}})_{t\geq 0}$ *is exponentially stable, that is,*

$$\|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{-wt}$$

for some $C > 0$ *and for* $\omega > 0$*, if and only if*

$$\rho(\mathcal{A}) \supset i\mathbb{R} \quad \textit{and} \quad \limsup_{|\lambda|\to+\infty} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} < C, \quad \forall\ \beta \in \mathbb{R}.$$

We close this section with an important functional analysis result that will be of utmost importance in this article.

Theorem 2.5 (Fredholm alternative [7])**.** *Let* X *be a Banach space. If* $\mathcal{L} : X \to X$ *is a a compact linear operator on* X*, then*

(i) $\ker(\mathcal{I} - \mathcal{L})$ *is finite dimension.*

(ii) $(\mathcal{I} - \mathcal{L})(X)$ *is closed.*

(iii) $\ker(\mathcal{I} - \mathcal{L}) = \{0\} \Leftrightarrow (\mathcal{I} - \mathcal{L})(X) = X$*.*

3. Energy and well-posedness of the ofaugmented system

In this section, we use results of the semigroup theory of linear operators, see [26], to obtain an existence theorem of solutions of system (2.4). The Lumer-Phillips Theorem will be applied.

Proposition 3.1. *The energy associated with to previous system* (2.4) *is given by*

$$\begin{aligned}E(t) =& \frac{k}{2}\|(\phi_x + \psi)(t)\|^2_{L^2(0,\,L)} + \frac{\rho_1}{2}\|\phi_t(t)\|^2_{L^2(0,\,L)} + \frac{\rho_2}{2}|\psi_t(t)\|^2_{L^2(0,\,L)}\\ &+ \frac{b}{2}\|\psi_x(t)\|^2_{L^2(0,\,L)} + \frac{\gamma_1}{2}\int_{\mathbb{R}}\|\varphi_1(t,\,y)\|^2_{L^2(0,\,L)}dy + \frac{\gamma_2}{2}\int_{\mathbb{R}}\|\varphi_2(t,\,y)\|^2_{L^2(0,\,L)}dy,\end{aligned}$$

and verifies that

$$\frac{d}{dt}E(t) = -\gamma_1 \int_{\mathbb{R}} (|y|^2 + \eta)\|\varphi_1(t,\, y)\|^2_{L^2(0,\, L)} dy - \gamma_2 \int_{\mathbb{R}} (|y|^2 + \zeta)\|\varphi_2(t,\, y)\|^2_{L^2(0,\, L)} dy. \tag{3.1}$$

Proof. Multiplying the first equation in (2.4) by ϕ_t, integrating over x, and using the boundary conditions, we obtain

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\phi_t(x,\, t)|^2 dx + k\int_0^L (\phi_x(x,\, t) + \psi(x,\, t))\phi_{xt}(x,\, t) dx \\ &+ \gamma_1 \int_0^L \phi_t(x,\, t) \int_{\mathbb{R}} p(y)\varphi_1(x,\, t,\, y) dy dx = 0. \end{aligned} \tag{3.2}$$

Similarly, by multiplying the second equation in (2.4) by ψ_t, integrating over x, and using the boundary conditions, we get

$$\begin{aligned} &\frac{\rho_2}{2}\frac{d}{dt}\int_0^L |\psi_t|^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^L |\psi_x|^2 dx + k\int_0^L (\phi_x + \psi)\psi_t dx \\ &+ \gamma_2 \int_0^L \psi_t(x,\, t) \int_{\mathbb{R}} q(y)\varphi_2(x,\, t,\, y) dy dx = 0. \end{aligned} \tag{3.3}$$

Now, by summing the equations (3.2) and (3.3), we have

$$\begin{aligned} &\frac{k}{2}\frac{d}{dt}\int_0^L |(\phi_x + \psi)(x,\, t)|^2 dx + \frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\phi_t(x,\, t)|^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^L |\psi_t(x,\, t)|^2 dx \\ &+ \frac{b}{2}\frac{d}{dt}\int_0^L |\psi_x(x,\, t)|^2 dx + \gamma_1 \int_0^L \phi_t(x,\, t) \int_{\mathbb{R}} p(y)\varphi_1(x,\, t,\, y) dy dx \\ &+ \gamma_2 \int_0^L \psi_t(x,\, t) \int_{\mathbb{R}} q(y)\varphi_2(x,\, t,\, y) dy dx = 0. \end{aligned} \tag{3.4}$$

On the other hand, by multiplying the last two equations in (2.4) by $\gamma_1\varphi_1(x,\, t,\, y)$ and $\gamma_2\varphi_2(x,\, t,\, y)$ respectively, and then integrating over the variable y, we get

$$\begin{aligned} &\frac{\gamma_1}{2}\frac{d}{dt}\int_{\mathbb{R}} |\varphi_1(x,\, t,\, y)|^2 dy + \gamma_1 \int_{\mathbb{R}} \left(|y|^2 + \eta\right) |\varphi_1(x,\, t,\, y)|^2 dy \\ &= \gamma_1 \phi_t(x,\, t) \int_{\mathbb{R}} p(y)\varphi_1(x,\, t,\, y) dy, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\frac{\gamma_2}{2}\frac{d}{dt}\int_{\mathbb{R}} |\varphi_2(x,\, t,\, y)|^2 dy + \gamma_2 \int_{\mathbb{R}} \left(|y|^2 + \zeta\right) |\varphi_2(x,\, t,\, y)|^2 dy \\ &= \gamma_2 \psi_t(x,\, t) \int_{\mathbb{R}} q(y)\varphi_2(x,\, t,\, y) dy. \end{aligned} \tag{3.6}$$

Substituting the expressions (3.5) and (3.6) into (3.4), we obtain

$$\frac{d}{dt}E(t) = -\gamma_1 \int_{\mathbb{R}} (|y|^2 + \eta)\|\varphi_1(t,\, y)\|^2_{L^2(0,\, L)} dy - \gamma_2 \int_{\mathbb{R}} (|y|^2 + \zeta)\|\varphi_2(t,\, y)\|^2_{L^2(0,\, L)} dy.$$

which establishes (3.1). □

To achieve the goal, consider the phase space

$$\mathcal{H} = [H_0^1(0,\, L)]^2 \times [L^2(0,\, L)]^2 \times [L^2(\mathbb{R};\, L^2(0,\, L))]^2,$$

equipped with the following inner product

$$\begin{aligned}\langle U,\, \widetilde{U}\rangle_{\mathcal{H}} = {} & k\langle \phi_x + \psi,\, \tilde{\phi}_x + \tilde{\psi}\rangle_{L^2(0,\,L)} + b\langle \psi,\, \tilde{\psi}\rangle_{H_0^1(0,\,L)} + \rho_1\langle u,\, \tilde{u}\rangle_{L^2(0,\,L)} \\ & + \rho_2\langle v,\, \tilde{v}\rangle_{L^2(0,\,L)} + \gamma_1\langle \varphi_1,\, \tilde{\varphi_1}\rangle_{L^2(\mathbb{R};\, L^2(0,\,L))} + \gamma_2\langle \varphi_2,\, \tilde{\varphi_2}\rangle_{L^2(\mathbb{R};\, L^2(0,\,L))},\end{aligned}$$

where $U = (\phi,\, \psi,\, u,\, v, \varphi_1,\, \varphi_2)^T$ and $\widetilde{U} = (\tilde{\phi},\, \tilde{\psi},\, \tilde{u},\, \tilde{v},\, \tilde{\varphi_1},\, \tilde{\varphi_2})^T$.

Note that, by setting $u = \phi_t$ and $v = \psi_t$ in the matrix $U = (\phi,\, \psi,\, u,\, v, \varphi_1,\, \varphi_2)^T$, the system (2.4) is equivalent to the following Cauchy problem

$$\begin{cases} U_t = \mathcal{A}U, \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{3.7}$$

where $U_0 = (\phi_0,\, \psi_0,\, \phi_1,\, \psi_1,\, 0,\, 0)^T$ and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} u \\ v \\ \dfrac{1}{\rho_1}\left[k(\phi_x + \psi)_x - \gamma_1 \displaystyle\int_{\mathbb{R}} p(y)\varphi_1(y)dy\right] \\ \dfrac{1}{\rho_2}\left[b\psi_{xx} - k(\phi_x + \psi) - \gamma_2 \displaystyle\int_{\mathbb{R}} q(y)\varphi_2(y)dy\right] \\ -(|y|^2 + \eta)\varphi_1(y) + p(y)u \\ -(|y|^2 + \zeta)\varphi_2(y) + q(y)v \end{pmatrix}. \tag{3.8}$$

The domain $\mathcal{D}(\mathcal{A})$ is definide by

$$D(\mathcal{A}) = \left\{ (\phi,\, \psi,\, u,\, v, \varphi_1,\, \varphi_2)^T \in \mathcal{V} \;\middle|\; \begin{array}{l} |y|\varphi_1,\ |y|\varphi_2 \in L^2(\mathbb{R};\, L^2(0,\, L)), \\ -(|y|^2 + \eta)\varphi_1 + p(y)u \in L^2(\mathbb{R};\, L^2(0,\, L)), \\ -(|y|^2 + \zeta)\varphi_2 + q(y)v \in L^2(\mathbb{R};\, L^2(0,\, L)). \end{array} \right\}$$

where $\mathcal{V} = [H_0^1(0,\, L) \cap H^2(0,\, L)]^2 \times [H_0^1(0,\, L)]^2 \times [L^2(\mathbb{R};\, L^2(0,\, L))]^2$ is dense in $\mathcal{H}$.

Theorem 3.1. *If $U_0 \in \mathcal{H}$, then the Cauchy problem* (3.7) *exists and admits a unique weak solution*

$$U \in C^0\left([0,\,+\infty);\,\mathcal{H}\right),$$

given by $U(t) = e^{t\mathcal{A}}U_0$. If $U_0 \in \mathcal{D}(\mathcal{A})$, then the obtained solution is a strong solution with the following regularity

$$U \in C^0\left([0,\,+\infty);\,\mathcal{D}\left(\mathcal{A}\right)\right) \cap C^1\left([0,\,+\infty);\,\mathcal{H}\right).$$

Proof. Consider $U = (\phi,\,\psi,\,u,\,v,\,\varphi_1,\,\varphi_2)^T \in \mathcal{D}\left(\mathcal{A}\right)$. Employing integration by parts, using boundary conditions, and exploiting properties of complex conjugation, we have

$$\begin{aligned}
\langle \mathcal{A}U,\,U\rangle_{\mathcal{H}} &= k\int_0^L (u_x+v)(x)(\overline{\phi_x+\psi})(x)dx + b\int_0^L v_x(x)\overline{\psi}_x(x)dx \\
&\quad + k\int_0^L (\phi_x+\psi)_x(x)\overline{u}(x)dx - \gamma_1\int_0^L\int_{\mathbb{R}} p(y)\varphi_1(x,\,y)\overline{u}(x)dydx \\
&\quad + b\int_0^L \psi_{xx}(x)\overline{v}(x)dx - k\int_0^L(\phi_x+\psi)(x)\overline{v}(x)dx \\
&\quad - \gamma_2\int_0^L\int_{\mathbb{R}} q(y)\varphi_2(x,\,y)\overline{v}(x)dydx - \gamma_1\int_0^L\int_{\mathbb{R}}(|y|^2+\eta)|\varphi_1(x,\,y)|^2dydx \\
&\quad + \gamma_1\int_0^L\int_{\mathbb{R}} p(y)u(x)\overline{\varphi}_1(x,\,y)dydx - \gamma_2\int_0^L\int_{\mathbb{R}}(|y|^2+\zeta)|\varphi_2(x,\,y)|^2dydx \\
&\quad + \gamma_2\int_0^L\int_{\mathbb{R}} q(y)v(x)\overline{\varphi}_2(x,\,y)dydx \\
&= k\int_0^L (u_x+v)(x)(\overline{\phi_x+\psi})(x)dx + b\int_0^L v_x(x)\overline{\psi}_x(x)dx \\
&\quad - k\int_0^L(\phi_x+\psi)(x)\overline{u}_x(x)dx - b\int_0^L \psi_x(x)\overline{v}_x(x)dx \\
&\quad - k\int_0^L(\phi_x+\psi)(x)\overline{v}(x) - \gamma_2\int_{\mathbb{R}}(|y|^2+\zeta)\|\varphi_2(y)\|^2_{L^2(0,\,L)}dy \\
&\quad + \gamma_1\int_0^L\int_{\mathbb{R}} p(y)\left[u(x)\overline{\varphi}_1(x,\,y) - \overline{u(x)\overline{\varphi}_1(x,\,y)}\right]dydx \\
&\quad + \gamma_2\int_0^L\int_{\mathbb{R}} q(y)\left[v(x)\overline{\varphi}_2(x,\,y) - \overline{v(x)\overline{\varphi}_2(x,\,y)}\right]dydx \\
&\quad - \gamma_1\int_{\mathbb{R}}(|y|^2+\eta)\|\varphi_1(y)\|^2_{L^2(0,\,L)}dy \\
&= 2ik\int_0^L Im\left[(u_x+v)(x)(\overline{\phi_x+\psi})(x)\right]dx + 2ib\int_0^L Im\left[v_x(x)\overline{\psi}_x(x)\right]dx \\
&\quad + 2i\gamma_1\int_0^L\int_{\mathbb{R}} p(y)Im\left[u(x)\overline{\varphi}_1(x,\,y)\right]dydx \\
&\quad + 2i\gamma_2\int_0^L\int_{\mathbb{R}} q(y)Im\left[v(x)\overline{\varphi}_2(x,\,y)\right]dydx \\
&\quad - \gamma_1\int_{\mathbb{R}}(|y|^2+\eta)\|\varphi_1(y)\|^2_{L^2(0,\,L)}dy - \gamma_2\int_{\mathbb{R}}(|y|^2+\zeta)\|\varphi_2(y)\|^2_{L^2(0,\,L)}dy.
\end{aligned}$$

Therefore

$$\begin{aligned} Re\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} =& -\gamma_1\int_{\mathbb{R}}(|y|^2+\eta)\|\varphi_1(y)\|^2_{L^2(0,\,L)}dy - \gamma_2\int_{\mathbb{R}}(|y|^2+\zeta)\|\varphi_2(y)\|^2_{L^2(0,\,L)}dy \\ \leq& \, 0. \end{aligned} \tag{3.9}$$

Hence, the linear operator $\mathcal{A}$ defined in (3.8) is a dissipative operator.

As $\mathcal{A}$ is a dissipative linear operator with $D(\mathcal{A})$ dense in $\mathcal{H}$, to use the Lumer-Phillips Theorem, it is sufficient to show that $(\mathcal{I}-\mathcal{A})U = W$. Thus, given $W = (f_1,\, f_2,\, g_1,\, g_2,\, h_1,\, h_2)^T \in \mathcal{H}$, we want to show that there is some vector $U = (\phi,\, \psi,\, u,\, v,\, \varphi_1,\, \varphi_2)^T \in \mathcal{D}(\mathcal{A})$ such that $(\mathcal{I}-\mathcal{A})U = W$. This is,

$$\begin{cases} \phi - u = f_1, \\ \psi - v = f_2, \\ \rho_1 u - k\left(\phi_x+\psi\right)_x + \gamma_1\displaystyle\int_{\mathbb{R}} p(y)\varphi_1(y)dy = \rho_1 g_1, \\ \rho_2 v - b\psi_{xx} + k(\phi_x+\psi) + \gamma_2\displaystyle\int_{\mathbb{R}} q(y)\varphi_2(y)dy = \rho_2 g_2, \\ \varphi_1(y) + (|y|^2+\eta)\varphi_1(y) - p(y)u = h_1(y), \quad \forall\, y\in\mathbb{R}, \\ \varphi_2(y) + (|y|^2+\zeta)\varphi_2(y) - q(y)v = h_2(y), \quad \forall\, y\in\mathbb{R}. \end{cases} \tag{3.10}$$

From the first two equations in (3.10), we get

$$u = \phi - f_1 \quad \text{and} \quad v = \psi - f_2. \tag{3.11}$$

On the other hand, from the last two equations in (3.10), we obtain

$$\varphi_1(y) = \frac{h_1(y)}{|y|^2+\eta+1} - \frac{p(y)f_1}{|y|^2+\eta+1} + \frac{p(y)\phi}{|y|^2+\eta+1}, \tag{3.12}$$

$$\varphi_2(y) = \frac{h_2(y)}{|y|^2+\zeta+1} - \frac{q(y)f_2}{|y|^2+\zeta+1} + \frac{q(y)\phi}{|y|^2+\zeta+1}. \tag{3.13}$$

Applying Lemma 2.1 to the expressions above, we get

$$\gamma_1\int_{\mathbb{R}} p(y)\varphi_1(y)dy = \gamma_1\left[\int_{\mathbb{R}}\frac{p(y)h_1(y)dy}{|y|^2+\eta+1} + C(\alpha,\,\eta)(\phi - f_1)\right] \tag{3.14}$$

and

$$\gamma_2\int_{\mathbb{R}} q(y)\varphi_2(y)dy = \gamma_2\left[\int_{\mathbb{R}}\frac{q(y)h_2(y)dy}{|y|^2+\zeta+1} + C(\beta,\,\zeta)(\psi - f_2)\right]. \tag{3.15}$$

Finally, applying expressions (3.11), (3.14), and (3.15) to the third and fourth equations of the system (3.10), we have

$$\rho_1\phi - \rho_1 f_1 - k\left(\phi_x+\psi\right)_x + \gamma_1 C(\alpha,\,\eta)(\phi-f_1) + \gamma_1\int_{\mathbb{R}}\frac{p(y)h_1(y)dy}{|y|^2+\eta+1} = \rho_1 g_1$$

and

$$\rho_2\psi - \rho_2 f_2 - b\psi_{xx} + k(\phi_x+\psi) + \gamma_2 C(\beta,\,\zeta)(\psi-f_2) + \gamma_2\int_{\mathbb{R}}\frac{q(y)h_2(y)dy}{|y|^2+\zeta+1} = \rho_2 g_2.$$

Then

$$\rho_1\phi - k(\phi_x+\psi)_x + \gamma_1 C(\alpha,\,\eta)\phi = \rho_1(f_1+g_1) + \gamma_1 C(\alpha,\,\eta) f_1 - \gamma_1 \int_{\mathbb{R}} \frac{p(y)h_1(y)dy}{|y|^2+\eta+1} \tag{3.16}$$

and

$$\rho_2\psi - b\psi_{xx} + k(\phi_x+\psi) + \gamma_2 C(\beta,\,\zeta)\psi = \rho_2(f_2+g_2) + \gamma_2 C(\beta,\,\zeta) f_2 - \gamma_2 \int_{\mathbb{R}} \frac{q(y)h_2(y)dy}{|y|^2+\zeta+1} \tag{3.17}$$

Multiplying equations (3.16) and (3.17) by $\tilde{\phi} \in H_0^1(0,\,L)$ and $\tilde{\psi} \in H_0^1(0,\,L)$ respectively, integrating over x and applying integration by parts, we obtain the following equivalent system

$$\begin{cases} C_1 \displaystyle\int_0^L \phi\tilde{\phi}dx + k\int_0^L (\phi_x+\psi)\,\tilde{\phi}_x dx = \int_0^L F_1\tilde{\phi}dx - \gamma_1 \int_0^L \tilde{\phi} \int_{\mathbb{R}} \frac{p(y)h_1(y)dydx}{|y|^2+\eta+1}, \\ C_2 \displaystyle\int_0^L \psi\tilde{\psi}dx + \int_0^L b\psi_x\tilde{\psi}_x dx + k\int_0^L (\phi_x+\psi)\tilde{\psi}dx = \int_0^L F_2\tilde{\psi}dx \\ \quad -\gamma_2 \displaystyle\int_0^L \tilde{\psi} \int_{\mathbb{R}} \frac{q(y)h_2(y)dydx}{|y|^2+\zeta+1}, \end{cases} \tag{3.18}$$

where $C_1 = \rho_1 + \gamma_1 C(\alpha,\,\eta)$, $C_2 = \rho_2 + \gamma_2 C(\beta,\,\zeta)$, $F_1 = \rho_1(f_1+g_1) + \gamma_1 C(\alpha,\,\eta)$ and $F_2 = \rho_2(f_2+g_2) + \gamma_2 C(\beta,\,\zeta)$.

Note that the system (3.18) is equivalent to the problem of finding a vector $(\phi,\,\psi) \in \left[H_0^1(0,\,L)\right]^2$ such that

$$\mathcal{B}((\phi,\,\psi),\,(\tilde{\phi},\,\tilde{\psi})) = \mathcal{L}(\tilde{\phi},\,\tilde{\psi}), \tag{3.19}$$

where $\mathcal{B} : [H_0^1(0,\,L)]^2 \times [H_0^1(0,\,L)]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\mathcal{B}((\phi,\,\psi),\,(\tilde{\phi},\,\tilde{\psi})) = \\ C_1\int_0^L \phi\tilde{\phi}dx + k\int_0^L (\phi_x+\psi)(\tilde{\phi}_x+\tilde{\psi})dx + C_2\int_0^L \psi\tilde{\psi}dx + b\int_0^L \psi_x\tilde{\psi}_x dx,$$

and $\mathcal{L} : [H_0^1(0,\,L)]^2 \longrightarrow \mathbb{R}$ is the linear form defined by

$$\mathcal{L}(\tilde{\phi},\tilde{\psi}) = \\ \int_0^L F_1\tilde{\phi}dx + \int_0^L F_2\tilde{\psi}dx - \gamma_1\int_0^L \tilde{\phi}\int_{\mathbb{R}} \frac{p(y)h_1(y)dydx}{|y|^2+\eta+1} - \gamma_2\int_0^L \tilde{\psi}\int_{\mathbb{R}} \frac{q(y)h_2(y)dydx}{|y|^2+\zeta+1}.$$

It is easy to observe that $\mathcal{B}$ is a continuous and coercive bilinear form. On the other hand,

$$\left|\gamma_1\int_0^L \tilde{\phi}\int_{\mathbb{R}} \frac{p(y)h_1(y)dydx}{|y|^2+\eta+1}\right| \le L\gamma_1\sqrt{D(\alpha,\,\eta)}\|\tilde{\phi}\|_{H_0^1(0,\,L)}\|h_1\|_{L^2(\mathbb{R};\,L^2(0,\,L))}$$

and

$$\left|\gamma_2\int_0^L \tilde{\psi}\int_{\mathbb{R}} \frac{q(y)h_2(y)dydx}{|y|^2+\zeta+1}\right| \le L\gamma_2\sqrt{D(\beta,\,\zeta)}\|\tilde{\psi}\|_{H_0^1(0,\,L)}\|h_2\|_{L^2(\mathbb{R};\,L^2(0,\,L))}.$$

It follows that $\mathcal{L}$ is a bounded linear functional. Then, from Lax-Milgram theorem, we deduce the existence of a unique solution $(\phi,\, \psi) \in H_1^0(0,\, L) \times H_0^1(0,\, L)$ to the variational problem (3.18). By elliptic regularity, it follows that $\phi,\, \psi \in H_0^1(0,\, L) \cap H^2(0,\, L)$. Now, define u and v from (3.11), then we have $u,\, v \in H^1(0,\, L)$. Finally, as $h_1,\, h_2 \in L^2(\mathbb{R};\, L^2(0,\, L))$, defining $\varphi_1(y)$ and $\varphi_2(y)$ using the respective expressions given in (3.12), (3.13) and (3.14), it is evident that $|y|\varphi_1 \in L^2(\mathbb{R};\, L^2(0,\, L))$, $|y|\varphi_2 \in L^2(\mathbb{R};\, L^2(0,\, L))$, $-\left(|y|^2+\eta\right)\varphi_1 + p(y)u \in L^2(\mathbb{R};\, L^2(0,\, L))$ and $-\left(|y|^2+\zeta\right)\varphi_2 + q(y)v \in L^2(\mathbb{R};\, L^2(0,\, L))$. □

4. Strong stability

In this section, we will use the Arendt-Batty Theorem (Theorem 2.2) to prove that the semigroup associated with our problem is strongly stable. In other words, our solution decays to zero pointwise as t tends to infinity.

Proposition 4.1. *If $\lambda \in \mathbb{R}$, then $\lambda i\mathcal{I} - \mathcal{A}$ is injective.*

Proof. Let $\lambda \in \mathbb{R}$ such that λi is an eigenvalue of the operator $\mathcal{A}$, and let $U = (\phi,\, \psi,\, u,\, v,\, \varphi_1,\, \varphi_2) \in \mathcal{D}(\mathcal{A})$ be the associated eigenvector. Then $\mathcal{A}U = \lambda iU$. Equivalently

$$\begin{cases} u = \lambda i\phi, \\ v = \lambda i\psi, \\ k(\phi_x+\psi)_x - \gamma_1 \displaystyle\int_{\mathbb{R}} p(y)\varphi_1(y)dy = i\lambda\rho_1 u, \\ b\psi_{xx} - k(\phi_x+\psi) - \gamma_2 \displaystyle\int_{\mathbb{R}} q(y)\varphi_2(y)dy = i\lambda\rho_2 v, \\ (|y|^2+\eta+\lambda i)\varphi_1(y) = p(y)u, \quad \forall\, y \in \mathbb{R}, \\ (|y|^2+\zeta+\lambda i)\varphi_2(y) = q(y)v, \quad \forall\, y \in \mathbb{R}. \end{cases} \tag{4.1}$$

Note that

$$\begin{aligned} 0 =& Re\langle \lambda iU,\, U\rangle_{\mathcal{H}} \\ =& -\gamma_1 \int_{\mathbb{R}} (|y|^2+\eta)\|\varphi_1(y)\|^2_{L^2(0,\,L)}dy - \gamma_2 \int_{\mathbb{R}} (|y|^2+\zeta)\|\varphi_2(y)\|^2_{L^2(0,\,L)}dy. \end{aligned}$$

Therefore

$$\varphi_1(x,\, y) = 0 \quad \text{and} \quad \varphi_2(x,\, y) = 0 \quad \text{a.e. in} \quad (x,\, y) \in (0,\, L)\times\mathbb{R}. \tag{4.2}$$

Applying (4.2) to the last two equations of the system (4.1), we obtain

$$u(x) = 0 \quad \text{and} \quad v(x) = 0 \quad \text{a.e.} \quad \text{in} \quad x \in (0,\, L). \tag{4.3}$$

Now, applying (4.3) to the first two equations of the system (4.1), we have

$$\lambda i\phi(x) = 0 \quad \text{and} \quad \lambda i\psi(x) = 0 \quad \text{a.e. in} \quad x \in (0,\, L). \tag{4.4}$$

If $\lambda \neq 0$, then $\phi = 0$ and $\psi = 0$ almost everywhere on $(0,\, L)$. Otherwise, from third and fourth equations of the system (4.1), along with the boundary conditions of the problem, we obtain the following system

$$\begin{cases} k\phi_{xx}(x,\,t) + k\psi(x,\,t)_x = 0, \quad 0 < x < L, \quad t > 0, \\ b\psi_{xx}(x,\,t) - k\phi_x(x,\,t) - k\psi(x,\,t) = 0, \quad 0 < x < L, \quad t > 0, \\ \phi_t(x,\,t) = \psi_t(x,\,t) = 0, \quad 0 < x < L, \quad t > 0, \\ \phi(0,\,t) = \phi(L,\,t) = 0 \quad \text{and} \quad \psi(0,\,t) = \psi(L,\,t) = 0, \quad t > 0. \end{cases} \tag{4.5}$$

Applying the operator method to the system (4.5), we obtain that $\phi \equiv 0$ and $\psi \equiv 0$. Therefore, in any case, $\ker(\lambda i\mathcal{I} - \mathcal{A}) = \{0\}$. □

Corollary 4.1. *If $\lambda \in \mathbb{R}$, then λi is not an eigenvalue of $\mathcal{A}$.*

Proposition 4.2. *If $\eta = 0$ or $\zeta = 0$ then the operator $\mathcal{A}$ is not invertible and consequently* $0 \in \sigma(\mathcal{A})$.

Proof. If $\eta = 0$, let $W_0 = (\sin(\pi x/L),\, 0,\, 0,\, 0,\, 0, 0) \in \mathcal{H}$, and assume there exists

$$U = (\phi,\, \psi,\, u,\, v,\, \varphi_1\, \varphi_2) \in \mathcal{D}(\mathcal{A}) \text{ such that } \mathcal{A}U = W_0.$$

In this case, $\varphi_1(y) = |y|^{\frac{2\alpha-5}{2}} \sin(\pi x/L)$, and however $\varphi_1 \notin L^2(\mathbb{R};\, L^2(0,\, L))$ for $0 < \alpha < 1$.

The case where $\zeta = 0$ is similar. Just choose a vector $U \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = W_1$, where $W_1 = (0,\, \sin(\pi x/L),\, 0,\, 0,\, 0, 0)$. □

Proposition 4.3. *(a) If $\eta = 0$ or $\zeta = 0$, then $\lambda i\mathcal{I} - \mathcal{A}$ is surjective, for any $\lambda \neq 0$.*

(b) If $\eta, \zeta > 0$ and $\lambda \in \mathbb{R}$, then $\lambda i\mathcal{I} - \mathcal{A}$ is surjective.

Proof. Given $W = (f_1,\, f_2,\, g_1,\, g_2,\, h_1,\, h_2)^T \in \mathcal{H}$, we aim to show that there exists a vector $U = (\phi,\, \psi,\, u,\, v, \varphi_1,\, \varphi_2)^T \in \mathcal{D}(\mathcal{A})$ such that $(\lambda i\mathcal{I} - \mathcal{A})U = W$. That is,

$$\begin{cases} \lambda i\phi - u = f_1, \\ \lambda i\psi - v = f_2, \\ i\lambda\rho_1 u - k\left(\phi_x + \psi\right)_x + \gamma_1 \displaystyle\int_{\mathbb{R}} p(y)\varphi_1(y)dy = \rho_1 g_1, \\ i\lambda\rho_2 v - b\psi_{xx} + k\left(\phi_x + \psi\right) + \gamma_2 \displaystyle\int_{\mathbb{R}} q(y)\varphi_2(y)dy = \rho_2 g_2, \\ (\lambda i + |y|^2 + \eta)\varphi_1(y) - p(y)u = h_1(y); \quad \forall\, y \in \mathbb{R}, \\ (\lambda i + |y|^2 + \zeta)\varphi_2(y) - q(y)v = h_2(y); \quad \forall\, y \in \mathbb{R}. \end{cases} \tag{4.6}$$

From (4.6), we get

$$u = \lambda i\phi - f_1 \quad \text{and} \quad v = \lambda i\psi - f_2, \tag{4.7}$$

$$\varphi_1(y) = \frac{h_1(y)}{|y|^2 + \eta + \lambda i} - \frac{p(y)f_1}{|y|^2 + \eta + \lambda i} + \frac{\lambda i p(y)\phi}{|y|^2 + \eta + \lambda i}, \tag{4.8}$$

$$\varphi_2(y) = \frac{h_2(y)}{|y|^2 + \zeta + \lambda i} - \frac{q(y)f_2}{|y|^2 + \zeta + \lambda i} + \frac{\lambda i q(y)\psi}{|y|^2 + \zeta + \lambda i}. \tag{4.9}$$

Applying Lemma 2.3 to the expressions (4.8) and (4.9), it follows that

$$\gamma_1 \int_{\mathbb{R}} p(y)\varphi_1(y)dy = \gamma_1 \left[\int_{\mathbb{R}} \frac{p(y)h_1(y)dy}{|y|^2 + \eta + \lambda i} + E(\lambda,\, \alpha,\, \eta)(\lambda i\phi - f_1)\right] \tag{4.10}$$

and

$$\gamma_2 \int_{\mathbb{R}} q(y)\varphi_2(y)dy = \gamma_2 \left[\int_{\mathbb{R}} \frac{q(y)h_2(y)dy}{|y|^2+\zeta+\lambda i} + E(\lambda,\, \beta,\, \zeta)(\lambda i\psi - f_2) \right]. \tag{4.11}$$

Thus, applying expressions (4.7), (4.10) and (4.11) in $(4.6)_{3,4}$ respectively, we have

$$\begin{aligned} &-\lambda^2\rho_1\phi - i\lambda\rho_1 f_1 - k\left(\phi_x+\psi\right)_x + \gamma_1 E(\lambda,\, \alpha,\, \eta)(\lambda i\phi - f_1) \\ &+\gamma_1 \int_{\mathbb{R}} \frac{p(y)h_1(y)dy}{|y|^2+\eta+\lambda i} = \rho_1 g_1. \end{aligned}$$

and

$$\begin{aligned} &-\lambda^2\rho_2\psi - i\lambda\rho_2 f_2 - b\psi_{xx} + k\left(\phi_x+\psi\right) + \gamma_2 E(\lambda,\, \beta,\, \zeta)(\lambda i\psi - f_2) \\ &+\gamma_2 \int_{\mathbb{R}} \frac{q(y)h_2(y)dy}{|y|^2+\zeta+\lambda i} = \rho_2 g_2. \end{aligned}$$

Then

$$\begin{aligned} &-\lambda^2\rho_1\phi - k(\phi_x+\psi)_x + \gamma_1\lambda i E(\lambda,\, \alpha,\, \eta)\phi = \rho_1(i\lambda f_1 + g_1) + \gamma_1 E(\lambda,\, \alpha,\, \eta) f_1 \\ &-\gamma_1 \int_{\mathbb{R}} \frac{p(y)h_1(y)dy}{|y|^2+\eta+\lambda i} \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &-\lambda^2\rho_2\psi - b\psi_{xx} + k(\phi_x+\psi) + \gamma_2\lambda i E(\lambda,\, \beta,\, \zeta)\psi = \rho_2(i\lambda f_2 + g_2) + \gamma_2 E(\lambda,\, \beta,\, \zeta) f_2 \\ &-\gamma_2 \int_{\mathbb{R}} \frac{q(y)h_2(y)dy}{|y|^2+\zeta+\lambda i}. \end{aligned} \tag{4.13}$$

If $\lambda = 0$, by hypothesis, we have $\eta, \zeta > 0$. In that case, we have

$$-k(\phi_x+\psi)_x = \rho_1 g_1 + \gamma_1 E(0, \alpha,\, \eta) f_1 - \gamma_1 H_1(x, 0, \alpha, \eta), \tag{4.14}$$

$$-b\psi_{xx} + k(\phi_x+\psi) = \rho_2 g_2 + \gamma_2 E(0, \beta,\, \zeta) f_2 - \gamma_2 H_2(x, 0, \beta, \zeta). \tag{4.15}$$

Multiplying equations (4.14) and (4.15) by $\widetilde{\phi} \in H_0^1(0,\, L)$ and $\widetilde{\psi} \in H_0^1(0,\, L)$ respectively, and proceeding in a manner similar to the approach used in the proof of Theorem 3.1, we get the problem of finding a vector $(\phi,\, \psi) \in \left[H_0^1(0,\, L)\right]^2$ such that

$$\mathcal{B}((\phi,\, \psi),\, (\widetilde{\phi},\, \widetilde{\psi})) = \mathcal{L}(\widetilde{\phi},\, \widetilde{\psi}), \tag{4.16}$$

where $\mathcal{B} : [H_0^1(0,\, L)]^2 \times [H_0^1(0,\, L)]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\mathcal{B}((\phi,\, \psi),\, (\widetilde{\phi},\, \widetilde{\psi})) = k\int_0^L (\phi_x+\psi)(\tilde{\phi}_x + \tilde{\psi})dx + b\int_0^L \psi_x \widetilde{\psi}_x dx \tag{4.17}$$

and $\mathcal{L} : [H_0^1(0,\, L)]^2 \longrightarrow \mathbb{R}$ is the linear form defined by

$$\mathcal{L}(\widetilde{\phi},\, \widetilde{\psi}) = \int_0^L F\widetilde{\phi}dx + \int_0^L G\widetilde{\psi}dx, \tag{4.18}$$

where $F = \rho_1 g_1 + \gamma_1 E(0, \alpha,\, \eta) f_1 - \gamma_1 H_1(x, 0, \alpha, \eta)$ and $G = \rho_2 g_2 + \gamma_2 E(0, \beta,\, \zeta) f_2 - \gamma_2 H_2(x, 0, \beta, \zeta)$.

Now, it suffices to utilize the Lax-Milgram Theorem.

Finally, suppose that $\lambda \neq 0$. Define the linear unbounded operator

$$\mathcal{M} : [H_0^1(0, L)]^2 \to \left([H_0^1(0, L)]^2\right)^*$$

given by

$$\mathcal{M}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix} k(\phi_x + \psi)_x - I_1(\lambda, \alpha, \eta)\phi \\ b\psi_{xx} - k(\phi_x + \psi) - I_2(\lambda, \beta, \zeta)\psi \end{pmatrix},$$

where $I_j(\lambda, \omega, \delta) = \gamma_j \lambda i E(\lambda, \omega, \delta)$ $(j = 1, 2)$.

From Lax-Milgram Theorem, it follows that it is an isomorphism. Thus, we have that the system (4.12)-(4.13) is equivalent to

$$\left(-\lambda^2 \begin{pmatrix}\rho_1 & \\ & \rho_2\end{pmatrix} \mathcal{M}^{-1} - \mathcal{I}\right)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \mathcal{M}^{-1}\begin{pmatrix}\widetilde{F}\\ \widetilde{G}\end{pmatrix}, \tag{4.19}$$

$\widetilde{F} = [\rho_1 \lambda i + \gamma_1 E(\lambda, \alpha, \eta)] f_1 + \rho_1 g_1 - \gamma_1 H_1(x, \lambda, \alpha, \eta)$ and $\widetilde{G} = [\rho_2 \lambda i + \gamma_2 E(\lambda, \beta, \zeta)] f_2 + \rho_2 g_2 - \gamma_2 H_2(x, \lambda, \beta, \zeta)$.

Since the operator $\mathcal{M}^{-1}$ is isomorphism and $\mathcal{I}$ is a compact operator from $[H_0^1(0, L)]^2$ to $\left([H_0^1(0, L)]^2\right)^*$. Then $\mathcal{M}^{-1}$ is compact operator from $[H_0^1(0, L)]^2$ to $[H_0^1(0, L)]^2$. Consequently, by Fredholm alternative (Theorem 2.5), proving the existence of $(\phi, \psi) \in [H_0^1(0, L)]^2$ solution of (4.19) reduces to proving

$$\ker\left(-\lambda^2 \begin{pmatrix}\rho_1 & \\ & \rho_2\end{pmatrix} \mathcal{M}^{-1} - \mathcal{I}\right) = \left\{\begin{pmatrix}0\\ 0\end{pmatrix}\right\}.$$

Indeed, if $(\widetilde{\phi}, \widetilde{\psi})^T \in \ker\left(-\lambda^2 \begin{pmatrix}\rho_1 & \\ & \rho_2\end{pmatrix} \mathcal{M}^{-1} - \mathcal{I}\right)$, then

$$\left(-\lambda^2 \begin{pmatrix}\rho_1 & \\ & \rho_2\end{pmatrix} \mathcal{I} - \mathcal{M}\right)\begin{pmatrix}\widetilde{\phi}\\ \widetilde{\psi}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

That is,

$$\begin{cases} (I_1(\lambda, \alpha,\, \eta) - \rho_1 \lambda^2)\widetilde{\phi} - k(\widetilde{\phi}_x + \widetilde{\psi})_x = 0, \\ (I_2(\lambda, \beta,\, \zeta) - \rho_2 \lambda^2)\widetilde{\psi} - b\widetilde{\psi}_{xx} + k(\widetilde{\phi}_x + \widetilde{\psi}) = 0. \end{cases} \tag{4.20}$$

Multiplying (4.20), by $\overline{\widetilde{\phi}}$ and $\overline{\widetilde{\psi}}$ respectively, integrating by parts and using the boundary conditions, it follows that $(\widetilde{\phi}, \widetilde{\psi})^T = (0,0)$. So, it follows that from Fredholm alternative (Theorem 2.5) there is a unique solution $(\phi, \psi) \in [H_0^1(0,L)]^2$ for (4.19). By elliptic regularity, it follows that $\phi, \psi \in H_0^1(0, L) \cap H^2(0, L)$. Now just take u, v, as given in (4.7) and $\varphi_1(y)$, $\varphi_2(y)$ as given in (4.8)-(4.9). Evidently, $U = (\phi, \psi, u, v, \varphi_1, \varphi_2) \in \mathcal{D}(\mathcal{A})$ and $(\lambda i\mathcal{I} - \mathcal{A})U = W$. ☐

Corollary 4.2.

(a) If $\eta = 0$ or $\zeta = 0$, then $i\lambda \notin \sigma(\mathcal{A})$, for any $\lambda \neq 0$,

(b) If η, $\zeta > 0$ and $\lambda \in \mathbb{R}$, then $i\lambda \notin \sigma(\mathcal{A})$.

Theorem 4.1. *The C_0 contraction semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable in $\mathcal{H}$, that is,*

$$\lim_{t\to+\infty} \left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} = 0, \quad \forall\, U_0 \in \mathcal{H}.$$

Proof. From the Corollary 4.1, it follows that the operator $\mathcal{A}$ does not have purely imaginary eigenvalues. However, if $\eta = 0$ or $\zeta = 0$, Proposition 4.2 and item (a) of the Corollary 4.2, imply that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$. In the case of $\eta, \zeta > 0$, using item (b) of the corollary 4.2, we conclude that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Therefore, in both cases, we can apply Arendt and Batty's Theorem, leading to the desired result. ☐

5. Polynomial stability

In this section, we show the main result of the manuscript. Initially note that, as shown in Proposition 4.2, for $\eta = 0$ or $\zeta = 0$, we have $0 \in \sigma(\mathcal{A})$, and thus, $i\mathbb{R} \not\subset \varrho(\mathcal{A})$. Therefore, according to the Gearhart-Prüss-Huang Theorem, it follows that the semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ generated by the operator $\mathcal{A}$ is not exponentially stable, and therefore, the solution $U(t) = \mathcal{S}(t)U_0$ the of problem (3.7) does not decay exponentially. On the other hand, taking $\eta, \zeta > 0$, the item (b) of the Corollary 4.2 we guarantee that $i\mathbb{R} \subset \varrho(\mathcal{A})$. For this case we will use the result below to prove that the system decays polynomially.

Theorem 5.1. *For $U_0 \in \mathcal{D}(\mathcal{A})$ and $\eta, \zeta > 0$, the C_0-semigroup $S(t) = e^{\mathcal{A}t}$ is polynomially stable if*

$$||e^{t\mathcal{A}}U_0|| \leq \frac{C}{t^\omega}\|U_0\|_{\mathcal{D}(\mathcal{A})},\ t > 0 \text{ and } \omega > 0.$$

Proof. The resolvent equation $(i\lambda\mathcal{I} - \mathcal{A})U = W$, for $U = (\phi, \psi, u, v, \varphi_n, \varphi_2) \in \mathcal{D}(\mathcal{A})$, $W = (f_1, f_2, g_1, g_2, h_1, h_2) \in \mathcal{H}$ e $\lambda \in \mathbb{R}$, is equivalent to

$$\begin{cases} i\lambda\phi - u = f_1, \\ i\lambda\psi - v = f_2, \\ i\lambda\rho_1 u - k(\phi_x + \psi)_x + \gamma_1 \displaystyle\int_{\mathbb{R}} p(y)\varphi_1(y)dy = \rho_1 g_1, \\ i\lambda\rho_2 v - b\psi_{xx} + k(\phi_x + \psi) + \gamma_2 \displaystyle\int_{\mathbb{R}} q(y)\varphi_2(y)dy = \rho_2 g_2, \\ (i\lambda + |y|^2 + \eta)\varphi_1(y) - p(y)u = h_1(y), \quad \forall\, y \in \mathbb{R}, \\ (i\lambda + |y|^2 + \zeta)\varphi_2(y) - q(y)v = h_2(y), \quad \forall\, y \in \mathbb{R}. \end{cases} \tag{5.1}$$

Taking the inner product of $(i\lambda\mathcal{I}-\mathcal{A})U$ with U in $\mathcal{H}$, we obtain $i\lambda\|U\|^2_{\mathcal{H}}-\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=\langle W,U\rangle$. Then, from the Cauchy-Schwarz inequality, it follows that

$$Re(-\langle\mathcal{A}U,U\rangle_{\mathcal{H}})\leq|\langle W,U\rangle|\leq\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}}. \tag{5.2}$$

By using (3.9) we get

$$\gamma_1\int_{\mathbb{R}}(|y|^2+\eta)\|\varphi_1(t,\,y)\|^2_{L^2(0,\,L)}dy+\gamma_2\int_{\mathbb{R}}(|y|^2+\zeta)\|\varphi_2(t,\,y)\|^2_{L^2(0,\,L)}dy\leq\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}}. \tag{5.3}$$

On the other hand, from the fifth equation of the system (5.1), it follows that

$$p(y)|u(x)|\leq(|\lambda|+|y|^2+\eta)|\varphi_1(x,y)|+|h_1(x,y)|.$$

Multiplying the last equation by $\left(|\lambda|+|y|^2+\eta\right)^{-1}p(y)$, we get

$$\left(|\lambda|+|y|^2+\eta\right)^{-1}[p(y)]^2|u(x)|\leq p(y)|\varphi_1(x,y)|+\left(|\lambda|+|y|^2+\eta\right)^{-1}p(y)|h_1(x,y)|.$$

Integrating into the variable y and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}J(|\lambda|,\alpha,\eta)|u(x)|\leq J(0,\alpha,\eta)\left(\int_{\mathbb{R}}(|y|^2+\eta)|\varphi_1(x,y)|^2dy\right)^{1/2}\\+L(|\lambda|,\alpha,\eta)\left(\int_{\mathbb{R}}|h_1(x,y)|^2dy\right)^{1/2},\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}J(|\lambda|,\alpha,\eta)&=\int_{\mathbb{R}}\frac{|y|^{2\alpha-1}dy}{|y|^2+\eta+|\lambda|},\\J(0,\alpha,\eta)&=\int_{\mathbb{R}}\frac{|y|^{2\alpha-1}dy}{|y|^2+\eta},\\L(|\lambda|,\alpha,\eta)&=\int_{\mathbb{R}}\frac{|y|^{2\alpha-1}dy}{(|y|^2+\eta+|\lambda|)^2},\end{aligned}$$

are constants given by Lemma 2.2.
Applying Young's inequality in (5.4), we have

$$\begin{aligned}[J(|\lambda|,\alpha,\eta)]^2|u(x)|^2\leq 2[J(0,\alpha,\eta)]^2\left(\int_{\mathbb{R}}(|y|^2+\eta)|\varphi_1(x,y)|^2dy\right)\\+2[L(|\lambda|,\alpha,\eta)]^2\left(\int_{\mathbb{R}}|h_1(x,y)|^2dy\right).\end{aligned} \tag{5.5}$$

Note that

$$[J(|\lambda|,\alpha,\eta)]^2=\left(\int_{\mathbb{R}}\frac{|y|^{2\alpha-1}dy}{|y|^2+\eta+|\lambda|}\right)^2=C_0(\eta+|\lambda|)^{2\alpha-2}\leq C|\lambda|^{2\alpha-2}, \tag{5.6}$$

$$[L(|\lambda|,\alpha,\eta)]^2=\int_{\mathbb{R}}\frac{|y|^{2\alpha-1}dy}{(y^2+\eta+|\lambda|)^2}=C_1(\eta+|\lambda|)^{\alpha-2}\leq C|\lambda|^{2\alpha-2},\ \forall|\lambda|>1, \tag{5.7}$$

where C is a constant.
Using (5.6) and (5.7) in (5.5) we obtain

$$C|\lambda|^{2\alpha-2}|u(x)|^2\leq C\left(\int_{\mathbb{R}}(|y|^2+\eta)|\varphi_1(x,y)|^2dy\right)+C|\lambda|^{2\alpha-2}\left(\int_{\mathbb{R}}|h_1(x,y)|^2dy\right),$$

that is,

$$|u(x)|^2 \leq C|\lambda|^{2-2\alpha}\left(\int_{\mathbb{R}}(y^2+\eta)|\varphi_1(x,y)|^2dy\right)+C\left(\int_{\mathbb{R}}|h_1(x,y)|^2dy\right).$$

Integrating the expression above over the variable $x \in (0,L)$, we have

$$\int_0^L |u(x)|^2dx \leq C|\lambda|^{2-2\alpha}\int_0^L\int_{\mathbb{R}}(y^2+\eta)|\varphi_1(x,y)|^2dydx + C\int_0^L\int_{\mathbb{R}}|h_1(x,y)|^2dydx.$$

Now using the inequality (5.3), we have

$$\int_0^L |u(x)|^2dx \leq C|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}}+C\|W\|^2_{\mathcal{H}} \tag{5.8}$$

In an entirely analogous way, we have

$$\int_0^L |v(x)|^2dx \leq C|\lambda|^{2-2\beta}\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}}+C\|W\|^2_{\mathcal{H}}. \tag{5.9}$$

Multiplying the third equation in (5.1) by $\overline{\phi}$ and integrating in $x \in (0,L)$, we have

$$i\lambda\rho_1\int_0^L u\overline{\phi}dx - k\int_0^L \phi_{xx}\overline{\phi}dx - k\int_0^L \psi_x\overline{\phi} + \gamma_1\int_0^L\overline{\phi}\int_{\mathbb{R}}p(y)\varphi_1(y)dydx = \rho_1\int_0^L g_1\overline{\phi}dx.$$

On the other hand, from the first equation it follows in (5.1) that$-i\lambda\overline{\phi} = \overline{f_1}+\overline{u}$. Then

$$\begin{aligned}&-k\int_0^L \phi_{xx}\overline{\phi}dx - k\int_0^L\psi_x\overline{\phi} = \rho_1\int_0^L|u|^2dx - \gamma_1\int_0^L\overline{\phi}\int_{\mathbb{R}}p(y)\varphi_1(y)dydx\\&+\rho_1\int_0^L u\overline{f_1}dx + \rho_1\int_0^L g_1\overline{\phi}dx.\end{aligned}$$

Integrating by parts and using boundary conditions we have

$$\begin{aligned}&k\int_0^L |\phi_x|^2dx + k\int_0^L\psi\overline{\phi}_xdx = \rho_1\int_0^L|u|^2dx - \gamma_1\int_0^L\overline{\phi}\int_{\mathbb{R}}p(y)\varphi_1(y)dydx\\&+\rho_1\int_0^L u\overline{f_1}dx + \rho_1\int_0^L g_1\overline{\phi}dx.\end{aligned} \tag{5.10}$$

Similarly, multiplying the fourth equation in (5.1) by $\overline{\psi}$, using the second equation in (5.1), integrating by parts in $x \in (0,L)$ and using the boundary conditions, we obtain

$$\begin{aligned}&b\int_0^L|\psi_x|^2dx + k\int_0^L\phi_x\overline{\psi}dx + k\int_0^L|\psi|^2dx = \rho_2\int_0^L|v|^2dx\\&-\gamma_2\int_0^L\overline{\psi}\int_{\mathbb{R}}q(y)\varphi_2(y)dydx + \rho_2\int_0^L v\overline{f_2}dx + \rho_2\int_0^L g_2\overline{\psi}dx.\end{aligned} \tag{5.11}$$

As $|\phi_x+\psi|^2 = |\phi_x|^2+\phi_x\overline{\psi}+\psi\overline{\phi}_x+|\psi|^2$, adding (5.10) and (5.11), we obtain

$$k\int_0^L|\phi_x+\psi|^2dx + b\int_0^L|\psi_x|^2dx \leq \rho_1\int_0^L|u|^2dx + \rho_2\int_0^L|v|^2dx$$

$$
- \gamma_1 \left| \int_0^L \overline{\phi} \int_{\mathbb{R}} p(y)\varphi_1(y)dydx \right| - \gamma_2 \left| \int_0^L \overline{\psi} \int_{\mathbb{R}} q(y)\varphi_2(y)dydx \right|
+ \rho_1 \int_0^L |u\overline{f_1} + g_1\overline{\phi}|dx + \rho_2 \int_0^L |v\overline{f_2} + g_2\overline{\psi}|dx. \tag{5.12}
$$

From the Cauchy-Schwarz, Young, and Poincaré inequalities, we obtain

$$
\begin{aligned}
&\left| \int_0^L \overline{\phi} \int_{\mathbb{R}} p(y)\varphi_1(y)dydx \right| \\
&\leq \|\phi\|_{L^2(0,L)} \left(\int_{\mathbb{R}} \frac{[p(y)]^2 dy}{y^2+\eta} \right)^{1/2} \left(\int_0^L \int_{\mathbb{R}} (y^2+\eta)|\varphi_1(y)|^2 dydx \right)^{1/2} \\
&\leq \frac{1}{2} \left(\int_{\mathbb{R}} \frac{[p(y)]^2 dy}{y^2+\eta} \right) \|\phi\|_{L^2(0,L)} + \frac{1}{2} \int_0^L \int_{\mathbb{R}} (y^2+\eta)|\varphi_1(y)|^2 dydx \\
&\leq \frac{C_2}{2} \left(\int_{\mathbb{R}} \frac{[p(y)]^2 dy}{y^2+\eta} \right) \|\phi_x\|_{L^2(0,L)} + \frac{1}{2} \int_0^L \int_{\mathbb{R}} (y^2+\eta)|\varphi_1(y)|^2 dydx.
\end{aligned} \tag{5.13}
$$

In a similar way we obtain the following estimate

$$
\left| \int_0^L \overline{\psi} \int_{\mathbb{R}} q(y)\varphi_2(y)dydx \right| \leq \frac{C_2}{2} \left(\int_{\mathbb{R}} \frac{[q(y)]^2 dy}{y^2+\zeta} \right) \|\psi_x\|_{L^2(0,L)}
+ \frac{1}{2} \int_0^L \int_{\mathbb{R}} (y^2+\zeta)|\varphi_2(y)|^2 dydx. \tag{5.14}
$$

Using the estimates (5.13) and (5.14) in (5.12), we have

$$
\begin{aligned}
&k \int_0^L |\phi_x+\psi|^2 dx + b \int_0^L |\psi_x|^2 dx \leq \rho_1 \int_0^L |u|^2 dx + \rho_2 \int_0^L |v|^2 dx \\
&+ \frac{\gamma_1 C_p}{2} \left(\int_{\mathbb{R}} \frac{[p(y)]^2 dy}{y^2+\eta} \right) \|\phi_x\|_{L^2(0,L)} + \frac{\gamma_1}{2} \int_0^L \int_{\mathbb{R}} (y^2+\eta)|\varphi_1(y)|^2 dydx \\
&\frac{\gamma_2 C_p}{2} \left(\int_{\mathbb{R}} \frac{[q(y)]^2 dy}{y^2+\zeta} \right) \|\psi_x\|_{L^2(0,L)} + \frac{\gamma_2}{2} \int_0^L \int_{\mathbb{R}} (y^2+\zeta)|\varphi_2(y)|^2 dydx \\
&+ \rho_1 \int_0^L |u\overline{f_1} + g_1\overline{\phi}|dx + \rho_2 \int_0^L |v\overline{f_2} + g_2\overline{\psi}|dx.
\end{aligned} \tag{5.15}
$$

From (5.3), (5.8), (5.9) and (5.15), follow that

$$
\begin{aligned}
&k \int_0^L |\phi_x+\psi|^2 dx + b \int_0^L |\psi_x|^2 dx \\
&\leq C|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}} + C|\lambda|^{2-2\beta}\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}} + C\|W\|_{\mathcal{H}}^2,
\end{aligned} \tag{5.16}
$$

where C is a constant.
Since

$$
\|U\|_{\mathcal{H}}^2 = k\|\phi_x+\psi\|_{L^2(0,L)}^2 + b\|\psi\|_{H_0^1(0,L)}^2 + \rho_1\|u\|_{L^2(0,L)}^2 + \rho_2\|v\|_{L^2(0,L)}^2
$$

$$+ \gamma_1 \|\varphi_1\|^2_{L^2(\mathbb{R};L^2(0,L))} + \gamma_2 \|\varphi_2\|^2_{L^2(\mathbb{R};L^2(0,L))},$$

from (5.3), (5.8), (5.9) and (5.16), follows that

$$\|U\|^2_{\mathcal{H}} \leq C\left(|\lambda|^{4-2\min\{\alpha,\beta\}} + 1\right)\|U\|_{\mathcal{H}}\|W\|_{\mathcal{H}} + C\|W\|^2_{\mathcal{H}}. \tag{5.17}$$

Applying the Young's inequality for $\lambda \neq 0$, we have

$$\|U\|^2_{\mathcal{H}} \leq C|\lambda|^{8-4\min\{\alpha,\beta\}}\|W\|^2_{\mathcal{H}}.$$

That is, $\|U\|_{\mathcal{H}} \leq C|\lambda|^{4-2\min\{\alpha,\beta\}}\|W\|_{\mathcal{H}};\ \forall\ U \in \mathcal{D}(\mathcal{A})$. But this is equivalent to

$$\frac{\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}W\|_{\mathcal{H}}}{\|W\|_{\mathcal{H}}} \leq C|\lambda|^{4-2\min\{\alpha,\beta\}}.$$

Therefore

$$\frac{1}{|\lambda|^{4-2\min\{\alpha,\beta\}}}\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C.$$

Finally, taking $1/\omega = 4 - 2\min\{\alpha,\beta\} > 0$ and letting $|\lambda| \to \infty$ rom Theorem 2.3, it follows that $||e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \leq \dfrac{C}{t^{\frac{1}{4-2\min\{\alpha,\beta\}}}} = \dfrac{C}{t^{\omega}}$. □

Conclusions

This work investigates the well-posedness, strong stability, and polynomial stability of a Timoshenko system with fractional derivative internal dissipation. Our study differs from the existing literature by considering fractional damping applied directly to the domain without delays or the need for equal acceleration speed conditions, as in previous works.

Our results expand the understanding of the asymptotic behavior of Timoshenko systems with fractional internal dissipation, providing clear criteria for strong and polynomial stability. These results are important for the specialized audience in complex dynamical systems, where fractional dissipation is essential for modeling viscoelastic materials.

A natural continuation of the research developed in this paper is the study of the stability conditions of the Timoshenko beam system with only fractional internal damping. In this context, this work will open new fronts of investigation, encouraging the exploration of other models of dissipative systems and their applications in engineering and applied sciences.

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References

[1] A. Adnane, A. Benaissa and K. Benomar, *Uniform stabilization for a Timoshenko beam system with delays in fractional order internal dampings*, SEMA., 2023, 80, 283–302.

[2] M. Akil, Y. Chitour, M. Ghader and A. wehbe, *Stability and exact controllability of a Timoshenko system with only one fractional damping on the boundary*, Asymptot. Anal., 2020, 119(3-4), 221–280.

[3] A. Alshabanat, M. Jleli, S. Kumar and B. Samet, *Generalization of Caputo-Fabrizio fractional derivative and applications to electrical circuits*, Front. Phys., 2020, 8, 1–10.

[4] K. Ammari, F. Hassine and L. Robbiano, *Stabilization for Some Fractional-Evolution Systems*, SpringerBriefs in Mathematics, Switzerland, 2022.

[5] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Am. Math. Soc., 1988, 306(2), 837–852.

[6] A. Benaissa and S. Benazzouz, *Well-posedness and asymptotic behavior of Timoshenko beam system with dynamic boundary feedback of fractional derivative type*, Z. Angew. Math. Phys., 2017, 68, 1–38.

[7] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, New York, 1999.

[8] A. Borichev and Y. Tomilov, *Optimal polynomial decay of function and operator semigroups*, Math. Ann., 2010, 347, 455–478.

[9] A. Boudaoud and A. Benaissa, *Stabilization of a wave equation with a general internal control of diffusive type*, Discontinuity, Nonlinearity, and Complexity, (2023), 12(4), 879–891.

[10] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent - II*, Geophys. J. Int., 1967, 13(5), 529–539.

[11] M. Caputo, *Elasticitá e Dissipazione*, Zanichelli, Bologna, 1969.

[12] M. Caputo and F. Mainardi, *Linear models of dissipation in anelastic solids*, Riv. Nuovo Cimento (Ser. II)., 1971, 1, 161–198.

[13] J. Choi and R. Maccam y, *Fractional order Volterra equations with applications to elasticity*, J. Math. Anal. Appl., 1989, 139, 448–464.

[14] S. Das, *Functional Fractional Calculus for System Identification and Control*, Springer Science, Business Media, 2011.

[15] I. Elishakoff, *An equation both more consistent and simpler than the Bresse-Timoshenko equation, In: Gilat, R.,Banks-Sills, L. (eds.) Advanced in Mathematical Modeling and Experimental Methods for Materials and Structures*, Berlin. Springer., 2010, pp, 249–254.

[16] I. Elishakoff, *Stepan Prokofievich Timoshenko and America*, ZAMM - Z. Angew. Math. Mech., 2019, 99, 1–18.

[17] I. Elishakoff, *Who developed the so-called Timoshenko beam theory?*, Math. Mech. Solids., 2020, 25, 97–116.

[18] S. A. Faghidian and I. Elishakoff, *The tale of shear coefficients in Timoshenko-Ehrenfest beam theory: 130 years of progress*, Meccanica., 2022, 58, 97–108.

[19] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert space*, Trans. Amer. Math. Soc., 1978, 236, 385–394.

[20] F. L. Huang, *Characteristic conditions for exponential stability of linear dynamical system in Hilbert spaces*, Ann. Differential Equations., 1985, 1, 43–56.

[21] M. Kerdache, M. Kesri and A. Benaissa, *Fractional boundary stabilization for a coupled system of wave equations*, Ann. Univ. Ferrara Sez. VII Sci. Mat., 2021, 67(1), 121–148.

[22] I. Y. Lyubich and Q. P. Vu, *Asymptotic stability of linear differential equations in Banach spaces*, Stud. Math., 1988, 88(1), 37–42.

[23] R. L. Magin, *Fractional calculus in bioengineering*, Begell House, Redding, 2006.

[24] A. I. Manevich and Z. Kolakowski, *Free and forced oscillations of Timoshenko beam made of viscoelastic material*, J. Theor. Appl. Mech., 2011, 49, 3–16.

[25] V. V. Nesterenko, *A theory for transverse vibrations of the Timoshenko beam*, J. Appl. Maths. Mechs., 1993, 51, 669–677.

[26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[27] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering, Academic Press, Cambridge, 1998.

[28] J. Prüss, *On the Spectrum of C_0-Semigroups, Trans.*, Amer. Math. Soc., 1984, 284(2), 847–857.

[29] C. A. Raposo, J. Ferreira, M. L. Santos and N. N. O. Castro, *Exponential stability for the Timoshenko system with two weak dampings*, Appl. Math. Lett., 2005, 18, 535–541.

[30] A. S. Shaikh, I. N. Shaikh and K. S. Nisar, *A mathematical model of COVID-19 using fractional derivative: outbreak in India with dynamics of transmission and control*, Adv. Differ. Equ., 2020, 373, 1–19.

[31] S. Samko, A. Kilbas and O. Marichev, *Integral and Derivatives of Fractional Order*, Gordon Breach, New York, 1993.

[32] A. Soufyane, *Stabilisation de la poutre de Timoshenko*, C. R. Acad. Sci., 1999, 328, 731–734.

[33] H..G. Sun, Y. Z., D. Baleanu, W. Chen and Y. Q. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul., 2018, 64, 213–231.

[34] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Berlin, 2011.

[35] S. P. Timoshenko, *On the correction for shear of the differential equation for transverse vibration of prismatic bars*, Phil. Mag., 1921, 41, 744–746.

[36] O. Zarraga, I. Sarría, J. García-Barruetabeña and F. Cortés, *An analysis of the dynamical behaviour of systems with fractional damping for mechanical engineering applications*, Symmetry., 2019, 1–15.