The modified double shift-splitting preconditioner for nonsymmetric generalized saddle point problems from the time-harmonic Maxwell equations *

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Abstract

Recently, Fan, Zhu and Zheng [*Computational and Applied Mathematics*, 37 (3): 3256-3266] proposed a generalized double shift-splitting (GDSS) preconditioner induced by a new matrix splitting method for nonsymmetric generalized saddle point problems, and gave the corresponding theoretical analysis and numerical experiments. In this paper, based on the generalized double shift-splitting (GDSS) preconditioner, we generalize the GDSS algorithms and further present the modified double shift-splitting (MDSS) preconditioner for nonsymmetric generalized saddle point problems having a nonsymmetric positive definite (1,1)-block and a positive definite (2,2)-block. Moreover, by similar theoretical analysis, we analyze the convergence conditions of the corresponding matrix splitting iteration methods and preconditioning properties of the MDSS preconditioned saddle point matrices. In final, one example

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is provided to confirm the effectiveness.

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1 Introduction

Consider the following 2×2 block saddle point problems

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{1}$$

where $A \in R^{n,n}$ is a positive definite matrix,$B \in R^{m,n}, m \leq n$, is of full rank, $B^T \in R^{n,m}$ is the transpose of B, $C \in R^{m,m}$ is positive definite and $f \in C^n, g \in C^m$ are two given vectors. It appears in many different applications of scientific computing, such as constrained optimization [45], the finite element method for solving the Navier-Stokes equation [27, 28, 30], and constrained least squares problems and generalized least squares problems [1, 34, 40, 41] and so on; see [9-17, 19,20,35,39,40] and references therein.

In recent years, there has been a surge of interest in the saddle point problem of the form (1), and a large number of stationary iterative methods have been proposed. For example, Santos et al. [34] studied preconditioned iterative methods for solving the singular augmented system with $A = I$. Golub et al. [31] presented SOR-like algorithms for solving linear systems (1). Darvishi et al. [26] studied SSOR method for solving the augmented systems. Bai et al. [2, 3, 25, 45] presented GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving linear systems (1). Zhang and Lu [42] showed the generalized symmetric SOR method for augmented systems. Peng and Li [33] studied the unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [4, 5, 7, 8, 9, 32, 36] presented splitting iteration methods such as Hermitian and skew-Hermitian splitting (HSS) iteration scheme and its preconditioned variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on.

Recently, based on a new matrix splitting method, Fan, Zhu and Zheng [29] proposed a generalized double shift-splitting (GDSS) preconditioner induced by a new matrix splitting method for nonsymmetric generalized saddle point problems, and gave the corresponding theoretical analysis and numerical experiments.

For large, sparse or structure matrices, iterative methods are an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The

CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [35].

In this paper, based on the generalized double shift-splitting (GDSS) preconditioner by Fan, Zhu and Zheng [29], we generalize the GDSS algorithms and further present the modified double shift-splitting (MDSS) preconditioner for nonsymmetric generalized saddle point problems having a nonsymmetric positive definite (1,1)-block and a positive definite (2,2)-block. Moreover, by similar theoretical analysis, we analyze the convergence conditions of the corresponding matrix splitting iteration methods and preconditioning properties of the MDSS preconditioned saddle point matrices. In final, one example is provided to confirm the effectiveness.

2 Modified double shift-splitting (MDSS) preconditioner

Recently, for the coefficient matrix of the augmented system (1), Fan, Zhu and Zheng [29] made the following splitting

$$\begin{aligned}\mathcal{A} &= \tfrac{1}{2}(\Sigma+\mathcal{A})-\tfrac{1}{2}(\Sigma-\mathcal{A})\\ &= \tfrac{1}{2}\begin{pmatrix}\alpha\Lambda_1+A & B^T\\ -B & \beta\Lambda_2+C\end{pmatrix}-\tfrac{1}{2}\begin{pmatrix}\alpha\Lambda_1-A & -B^T\\ B & \beta\Lambda_2-C\end{pmatrix},\end{aligned}\tag{2}$$

where $\alpha>0,\beta>0$ are two constant numbers, $\Sigma=\begin{pmatrix}\alpha\Lambda_1 & 0\\ 0 & \beta\Lambda_2\end{pmatrix}$, and the parameter matrices Λ_1 and Λ_2 are both symmetric positive definite. Based on the iteration methods studied in [29], we establish the modified double shift-splitting (MDSS) of the saddle point matrix $\mathcal{A}$, which is as follows:

$$\begin{aligned}\mathcal{A} &= \tfrac{1}{2}(\Sigma+\gamma\mathcal{A})-\tfrac{1}{2}[\Sigma-(2-\gamma)\mathcal{A}]\\ &= \mathcal{P}_{MDSS}-\mathcal{R}_{MDSS}\\ &= \tfrac{1}{2}\begin{pmatrix}\alpha\Lambda_1+\gamma A & \gamma B^T\\ -\gamma B & \beta\Lambda_2+\gamma C\end{pmatrix}-\tfrac{1}{2}\begin{pmatrix}\alpha\Lambda_1-(2-\gamma)A & -(2-\gamma)B^T\\ (2-\gamma)B & \beta\Lambda_2-(2-\gamma)C\end{pmatrix},\end{aligned}\tag{3}$$

where $\alpha>0,\beta>0,\gamma>0$ are three constant numbers, $\Sigma=\begin{pmatrix}\alpha\Lambda_1 & 0\\ 0 & \beta\Lambda_2-C\end{pmatrix}$, and the parameter matrices Λ_1 and Λ_2 are both symmetric positive definite. By this special splitting, the following modified double shift-splitting (MDSS) method can be defined for solving the saddle point problem (1):

Modified double shift-splitting (MDSS) method: Given initial vectors $u^0\in R^{m+n}$, and three relaxed parameters $\alpha>0,\beta>0$ and $\gamma>0$. For $k=0,1,2,...$ until the iteration sequence $\{u^k\}$ converges, compute

$$\frac{1}{2}\begin{pmatrix}\alpha\Lambda_1+\gamma A & \gamma B^T\\ -\gamma B & \beta\Lambda_2+\gamma C\end{pmatrix}u^{k+1}=\frac{1}{2}\begin{pmatrix}\alpha\Lambda_1-(2-\gamma)A & -(2-\gamma)B^T\\ (2-\gamma)B & \beta\Lambda_2-(2-\gamma)C\end{pmatrix}u^k+\begin{pmatrix}f\\ g\end{pmatrix},\tag{4}$$

It is easy to see that the iteration matrix of the MDSS iteration is

$$\Gamma_{MDSS}=\begin{pmatrix}\alpha\Lambda_1+\gamma A & \gamma B^T\\ -\gamma B & \beta\Lambda_2+\gamma C\end{pmatrix}^{-1}\begin{pmatrix}\alpha\Lambda_1-(2-\gamma)A & -(2-\gamma)B^T\\ (2-\gamma)B & \beta\Lambda_2-(2-\gamma)C\end{pmatrix}.\tag{5}$$

If we use a Krylov subspace method such as GMRES (Generalized Minimal Residual) method or its restarted variant to approximate the solution of this system of linear equations, then

$$\mathcal{P}_{MDSS} = \frac{1}{2}\begin{pmatrix} \alpha\Lambda_1 + \gamma A & \gamma B^T \\ -\gamma B & \beta\Lambda_2 + \gamma C \end{pmatrix}, \tag{6}$$

can be served as a preconditioner. We call $\mathcal{P}_{MDSS}$ the MDSS preconditioner for the generalized saddle point matrix $\mathcal{A}$.

In every iteration of the MDSS iteration (4) or the preconditioned Krylov subspace method, we need solve a residual equation

$$\begin{aligned} \mathcal{P}_{MDSS}z &= \tfrac{1}{2}\begin{pmatrix} \alpha\Lambda_1 + \gamma A & \gamma B^T \\ -\gamma B & \beta\Lambda_2 + \gamma C \end{pmatrix} z \\ &= \tfrac{1}{2}\begin{pmatrix} I & B^T(\beta\Lambda_2 + \gamma C)^{-1} \\ 0 & I \end{pmatrix} \times \\ &\quad \begin{pmatrix} G & 0 \\ 0 & \beta\Lambda_2 + \gamma C \end{pmatrix} \begin{pmatrix} I & 0 \\ -(\beta\Lambda_2 + \gamma C)^{-1}B & I \end{pmatrix} z \\ &= r \end{aligned} \tag{7}$$

needs to be solved for a given vector r at each step, where $G = \alpha\Lambda_1 + \gamma A + B^T(\beta\Lambda_2 + \gamma C)^{-1}B$ is called the modified double shift-splitting (MDSS) preconditioner for the saddle point matrix $\mathcal{A}$ and is induced by the modified double shift-splitting iteration (4). Hence, analogous to Algorithm 1 in [29], we can derive the following algorithmic version of the MDSS iteration method.

Algorithm 2.1. For a given vector $r = [r_1^T, r_2^T]^T$, the vector $z = [z_1^T, z_2^T]^T$ can be computed by (7) from the following steps:

Step 1: Solve $(\beta\Lambda_2 + \gamma C)w = 2r_2$ for w;
Step 2: Compute $w_1 = 2r_1 - B^T w$;
Step 3: Solve $(\alpha\Lambda_1 + \gamma A + B^T(\beta\Lambda_2 + \gamma C)^{-1}B)z_1 = w_1$ for z_1;
Step 4: Solve $(\beta\Lambda_2 + \gamma C)v = Bz_1$ for v;
Step 5: Compute $z_2 = v + w$.

Remark 2.1. On the modified double shift-splitting (MDSS) method, when A is symmetric (or nonsymmetric) positive definite, C is positive semidefinite, and $\Lambda_1 = \Lambda_2 = I, \gamma = 1$ with $\alpha = \beta = 0$, the MDSS method reduces to the method in [6]; When $\gamma = 1$ the MDSS method reduces to the GDSS method in [29]. So, the MDSS method is the generalization of existing iteration algorithm.

3 Covergence of MDSS method

Now, we turn to study the convergence of the MDSS iteration for solving saddle point problems (1). It is well known that the iteration method (4) is convergent for every initial guess if and only if $\rho(\Gamma) < 1$, where $\rho(\Gamma)$ denotes the spectral radius of Γ. In [29], based on the GDSS method, Fan, Zhu and Zheng established the spectral properties of the iteration matrix $\mathcal{P}_{GDSS}^{-1}\mathcal{R}$. In this section, we will obtain that the MDSS iteration method is unconditionally convergent.

Lemma 3.1. *Assume that A is positive definite, B has full row rank, and C is positive definite. Let λ be an eigenvalue of the iteration matrix Γ_{MDSS} of the MDSS iteration (5). Then $\lambda \neq \pm 1$.*
Proof. Similar to the proving process of Lemma 2.1 in [29], we obviously can get the above Lemma.

Theorem 3.2. *Let $A \in R^{n,n}$ be positive definite, $B \in R^{n,m}$ be of row full rank matrix, and $C \in R^{m,m}$ be positive definite. Let α, β and γ be positive real numbers. Let Γ_{MDSS} be the iterative matrix defined above. Then*

$$\rho(\Gamma_{MDSS}) < 1,$$

i.e., the modified double shift-splitting iteration method (4) converges unconditionally to the exact solution of the nonsymmetric generalized saddle point problems (1).
Proof. If we let the $u = (x, y)^T$ be an eigenvector corresponding to the eigenvalue λ of Γ_{MDSS}, then we get

$$\mathcal{R}_{MDSS} u = \lambda \mathcal{P}_{MDSS} u,$$

which can be equivalently expanded as follows:

$$\begin{pmatrix} \alpha\Lambda_1 - (2-\gamma)A & -(2-\gamma)B^T \\ (2-\gamma)B & \beta\Lambda_2 - (2-\gamma)C \end{pmatrix} u = \lambda \begin{pmatrix} \alpha\Lambda_1 + \gamma A & \gamma B^T \\ -\gamma B & \beta\Lambda_2 + \gamma C \end{pmatrix} u. \tag{8}$$

Then we have

$$\begin{cases} [\alpha\Lambda_1 - (2-\gamma)A]x - (2-\gamma)B^T y = \lambda(\alpha\Lambda_1 + \gamma A)x + \lambda\gamma B^T y, \\ (2-\gamma)Bx + [\beta\Lambda_2 - (2-\gamma)C]y = -\lambda\gamma Bx + \lambda(\beta\Lambda_2 + \gamma C)y. \end{cases} \tag{9}$$

Left-multiplying both sides of (9) by x^* yields

$$\alpha x^*\Lambda_1 x - \gamma x^* A x - (2-\gamma)(Bx)^* y = \lambda(\alpha x^*\Lambda_1 x + \gamma x^* A x) + \lambda\gamma(Bx)^* y. \tag{10}$$

The cases, $Bx = 0$ and $Bx \neq 0$, are considered.
Suppose $Bx \neq 0$. In this case, from the second formula in Eq. (9), we obtain

$$Bx = \frac{\beta(\lambda - 1)\Lambda_2 y}{2 - \gamma + \lambda\gamma} + Cy. \tag{11}$$

Substituting Eq. (11) into (10) yields

$$(1-\lambda)\alpha x^*\Lambda_1 x - (1+\lambda)\gamma x^* A x = (2-\gamma+\lambda\gamma)\left(\frac{\beta(\bar{\lambda}-1)}{\bar{\lambda}\gamma + 2 - \gamma} y^*\Lambda_2 y + y^* C y\right). \tag{12}$$

We need to go a step further to consider the two cases according to whether the matrix C is symmetric or not.
Case I If C is nonsymmetric. The matrix A is nonsymmetric positive definite. By letting

$$x^* A x = \xi + i\eta, y^* C y = \mu + iv, x^*\Lambda_1 x = s, y^*\Lambda_2 y = \phi,$$

then we can obtain from Eq. (12)

$$\alpha\varphi s + \beta\phi\bar{\varphi} = \varphi'(\xi + i\eta) + \mu + iv, \ \text{with} \ \varphi = \frac{1-\lambda}{2-\gamma+\lambda\gamma}, \varphi' = \frac{1+\lambda}{2-\gamma+\lambda\gamma} \tag{13}$$

Since $\alpha, \beta, \gamma > 0$ and $s, \phi, \mu > 0$, from (13) we can obtain

$$Re(\varphi) = \frac{\varphi^{'}\xi + \mu}{\alpha s + \beta\phi}.$$

So, we have

$$|\lambda| = \left|\frac{1-\varphi}{1+\varphi}\right| = \sqrt{\frac{(1-Re(\varphi))^2 + Im(\varphi)^2}{(1+Re(\varphi))^2 + Im(\varphi)^2}} < 1,$$

where the real part and the imaginary part of a complex number z are denoted as $Re(z)$ and $Im(z)$, respectively.

Case II If C is symmetric. Using the same notation as in Case I, then it is not hard to find that

$$x^*Ax = \xi + i\eta, y^*Cy = \mu, y^*\Lambda_2 y = \phi,$$

then we can obtain from Eq. (12)

$$\alpha\varphi s + \beta\phi\bar{\varphi} = \varphi^{'}(\xi + i\eta) + \mu,\ with\ \varphi = \frac{1-\lambda}{2-\gamma+\lambda\gamma}, \varphi^{'} = \frac{1+\lambda}{2-\gamma+\lambda\gamma} \tag{14}$$

Since $\alpha, \beta, \gamma > 0$ and $s, \phi, \mu > 0$, from (14) we can obtain

$$Re(\varphi) = \frac{\varphi^{'}\xi + \mu}{\alpha s + \beta\phi}.$$

Then we have

$$|\lambda| = \left|\frac{1-\varphi}{1+\varphi}\right| = \sqrt{\frac{(1-Re(\varphi))^2 + Im(\varphi)^2}{(1+Re(\varphi))^2 + Im(\varphi)^2}} < 1,$$

If $Bx = 0$, then Eq. (10) implies

$$|\lambda| = \left|\frac{\alpha x^*\Lambda_1 x - \gamma x^*Ax}{\alpha x^*\Lambda_1 x + \gamma x^*Ax}\right| < 1.$$

Remark 3.1. On the one hand, the MDSS method is the generalization of the GDSS method. On the other hand, when the appropriate parameters are selected, the MDSS method will have better convergence than the GDSS method.

4 Numerical examples

In this section, we give numerical experiments to demonstrate the conclusions drawn above. The numerical experiments were done by using MATLAB 7.1 and the matrix of the numerical experiments were generated based on a two-dimensional time-harmonic Maxwell equations in mixed form, respectively. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least six orders of magnitude (i.e, when $\|b - \mathcal{A}x^k\|_2 \leq 10^{-6}\|b\|_2$).

Example 1. In this section, to further assess the effectiveness of the new preconditioned matrix $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ combined with Krylov subspace methods, we present a sample of numerical examples which are based on a two-dimensional time-harmonic Maxwell equations in mixed form in a square domain ($-1 \leq x \leq 1, -1 \leq y \leq 1$). For the simplicity, we take the generic source: $f = 1$ and a finite element subdivision such as Figure 2 based on uniform grids of triangle elements. Three mesh sizes are considered: $h = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{18}$, and Figure 1 shows a uniform mesh with $h = \frac{\sqrt{2}}{4}$, The solutions of the preconditioned systems in each iteration are computed exactly. Information on the sparsity of relevant matrices on the different meshes is given in Table 1, where $\mathrm{nz}(A)$ denote the nonzero elements of matrix A.

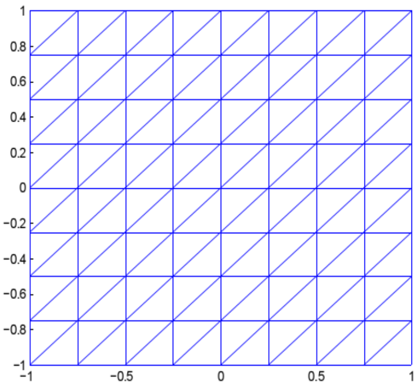

Figure 1: A uniform mesh with $h = \frac{\sqrt{2}}{4}$

Since the new preconditioners have two parameters, in numerical experiments we will test different values. Numerical experiments show the spectrum of the MDSS preconditioned matrix $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ and the GDSS preconditioned matrix $\mathcal{P}_{GDSS}^{-1}\mathcal{A}$ when choosing different parameters, which coincides with theoretical analysis.

In Figures 2, 4 and 6 we display the eigenvalues of the iteration matrix $\mathcal{P}_{MDSS}^{-1}\mathcal{R}$ in the case of $h = \frac{\sqrt{2}}{8}$, $h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters. In Figures 3, 5 and 7 we display the eigenvalues of the iteration matrix $\mathcal{P}_{GDSS}^{-1}\mathcal{R}$ in the case of $h = \frac{\sqrt{2}}{8}$,$h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters. In Tables 2 $\sim$ 4 we show iteration counts about preconditioned matrices $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ and $\mathcal{P}_{GDSS}^{-1}\mathcal{A}$, when choosing different parameters and applying to BICGSTAB and GMRES Krylov subspace iterative methods on three meshes, where $It_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$ and $Res_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ when applying to BICGSTAB Krylov subspace iterative methods, respectively. $It_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$ and $Res_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ when applying to GMRES Krylov subspace iterative methods, respectively. $It_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$, $Res_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$, $It_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$, $Res_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$ are similar definitions.

Remark 4.1. Figures 2 $\sim$ 7 show that the distribution of eigenvalues of the iteration matrix confirm our above theoretical analysis.

Remark 4.2. From Tables 2, 3 and 4, it is very easy to see that the preconditioner $\mathcal{P}_{MDSS}$ and $\mathcal{P}_{GDSS}$ will improve the convergence of BICGSTAB and GMRES iteration efficiently when they are applied to the preconditioned BICGSTAB and GMRES to solove two-dimensional time-harmonic Maxwell equations by choosing different parameters.

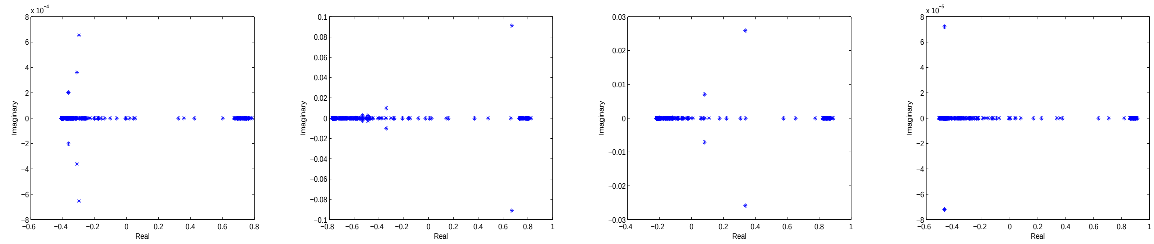

Figure 2: The eigenvalue distribution for the MDSS iteration matrix $\Gamma = \mathcal{P}_{MDSS}^{-1}\mathcal{R}_{MDSS}$ when $\alpha = 0.3, \beta = 0.4, \gamma = 1.4$(the first), $\alpha = 0.4, \beta = 0.8, \gamma = 1.1$(the second),$\alpha = 0.6, \beta = 0.4, \gamma = 1.6$(the third) and $\alpha = 0.8, \beta = 0.2, \gamma = 1.3$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

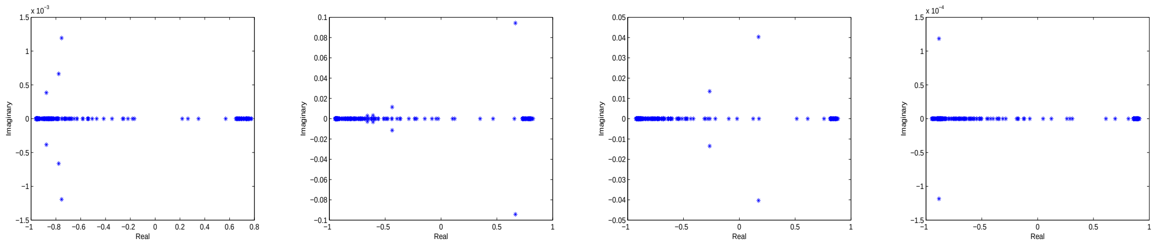

Figure 3: The eigenvalue distribution for the GDSS iteration matrix $\Gamma = \mathcal{P}_{GDSS}^{-1}\mathcal{R}_{GDSS}$ when $\alpha = 0.3, \beta = 0.4$(the first), $\alpha = 0.4, \beta = 0.8$(the second),$\alpha = 0.6, \beta = 0.4$(the third) and $\alpha = 0.8, \beta = 0.2$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

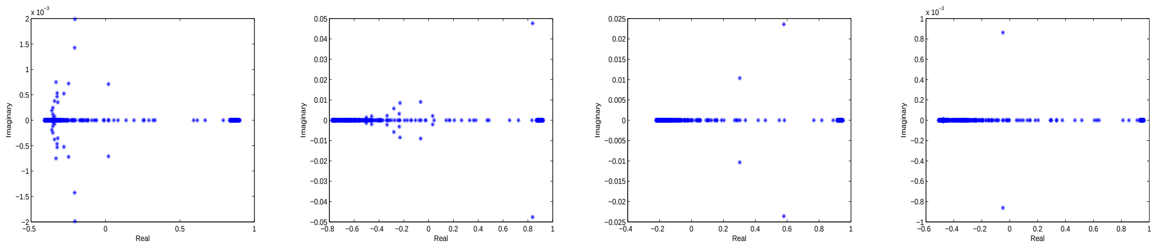

Figure 4: The eigenvalue distribution for the MDSS iteration matrix $\Gamma = \mathcal{P}_{MDSS}^{-1}\mathcal{R}_{MDSS}$ when $\alpha = 0.3, \beta = 0.4, \gamma = 1.4$(the first), $\alpha = 0.4, \beta = 0.8, \gamma = 1.1$(the second),$\alpha = 0.6, \beta = 0.4, \gamma = 1.6$(the third) and $\alpha = 0.8, \beta = 0.2, \gamma = 1.3$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

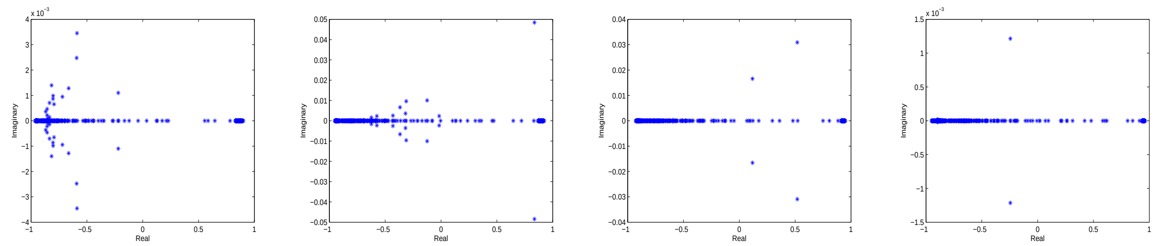

Figure 5: The eigenvalue distribution for the GDSS iteration matrix $\Gamma = \mathcal{P}_{GDSS}^{-1}\mathcal{R}_{GDSS}$ when $\alpha = 0.3, \beta = 0.4$(the first), $\alpha = 0.4, \beta = 0.8$(the second),$\alpha = 0.6, \beta = 0.4$(the third) and $\alpha = 0.8, \beta = 0.2$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

Table 1: datasheet for different grids

Grid	m	n	nz(A)	nz(B)	nz(W)	order of $\mathcal{A}$
8×8	176	49	820	462	217	225
16×16	736	225	3556	2190	1065	961
32×32	3008	961	14788	9486	4681	3969
64×64	12160	3969	60292	39438	19593	16129

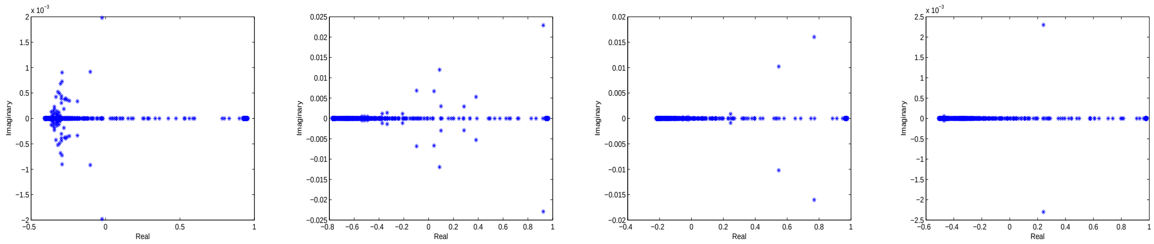

Figure 6: The eigenvalue distribution for the MDSS iteration matrix $\Gamma = \mathcal{P}_{MDSS}^{-1}\mathcal{R}_{MDSS}$ when $\alpha = 0.3, \beta = 0.4, \gamma = 1.4$(the first), $\alpha = 0.4, \beta = 0.8, \gamma = 1.1$(the second),$\alpha = 0.6, \beta = 0.4, \gamma = 1.6$(the third) and $\alpha = 0.8, \beta = 0.2, \gamma = 1.3$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

5 Conclusion

In this paper, based on generalized double shift-splitting (GDSS) preconditioner by Fan, Zhu and Zheng [29], we establish the modified double shift-splitting (MDSS) preconditioner for nonsymmetric generalized saddle point problems. Furthermore, we theoretically verify the convergence conditions of the corresponding matrix splitting iteration methods and preconditioning properties of the MDSS preconditioned saddle point matrices. Finally, numerical examples show the spectrum of the new preconditioned matrix for the different parameters.

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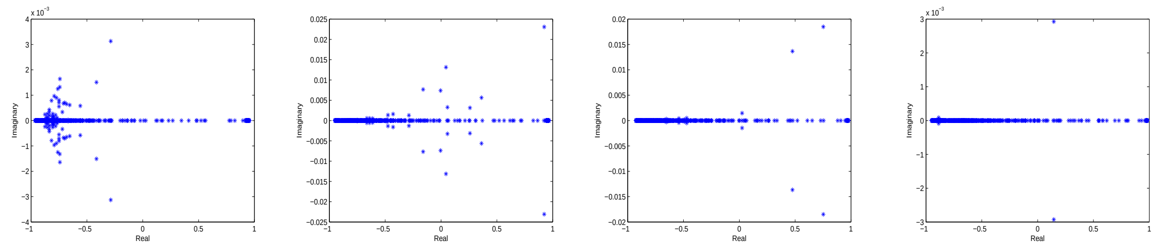

Figure 7: The eigenvalue distribution for the GDSS iteration matrix $\Gamma = \mathcal{P}_{GDSS}^{-1}\mathcal{R}_{GDSS}$ when $\alpha = 0.3, \beta = 0.4$(the first), $\alpha = 0.4, \beta = 0.8$(the second),$\alpha = 0.6, \beta = 0.4$(the third) and $\alpha = 0.8, \beta = 0.2$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

Table 2: Iteration counts and relative residual about preconditioned matrices $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ and $\mathcal{P}_{GDSS}^{-1}\mathcal{A}$ when choosing different parameters, where the unpreconditioned BICGSTAB and GMRES are divergent, respectively. Here, $h = \frac{\sqrt{2}}{8}$ denotes the size of the corresponding grid.

α	β	γ	$It_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	α	β	$It_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$
0.3	0.4	1.4	7.5	2.3518×10^{-7}	0.3	0.4	8	6.1707×10^{-7}
0.4	0.8	1.1	10.5	5.8265×10^{-7}	0.4	0.8	10.5	3.8835×10^{-7}
0.6	0.4	1.6	8	8.8470×10^{-7}	0.6	0.4	11	6.8382×10^{-7}
0.8	0.2	1.3	9	9.7976×10^{-7}	0.8	0.2	12	8.9706×10^{-7}
α	β	γ	$It_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	α	β	$It_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$
0.3	0.4	1.4	13(1)	2.1762×10^{-7}	0.3	0.4	14(1)	7.2630×10^{-7}
0.4	0.8	1.1	16(1)	3.9411×10^{-7}	0.4	0.8	16(1)	9.5468×10^{-7}
0.6	0.4	1.6	13(1)	9.4223×10^{-7}	0.6	0.4	17(1)	5.9267×10^{-7}
0.8	0.2	1.3	15(1)	7.2074×10^{-7}	0.8	0.2	17(1)	7.5318×10^{-7}

Table 3: Iteration counts and relative residual about preconditioned matrices $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ and $\mathcal{P}_{GDSS}^{-1}\mathcal{A}$ when choosing different parameters, where the unpreconditioned BICGSTAB and GMRES are divergent, respectively. Here, $h = \frac{\sqrt{2}}{12}$ denotes the size of the corresponding grid.

α	β	γ	$It_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	α	β	$It_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$
0.3	0.4	1.4	10	7.9753×10^{-7}	0.3	0.4	12.5	2.6569×10^{-7}
0.4	0.8	1.1	14	3.3046×10^{-7}	0.4	0.8	13	8.2725×10^{-7}
0.6	0.4	1.6	11	9.3725×10^{-7}	0.6	0.4	15	3.2842×10^{-7}
0.8	0.2	1.3	19.5	8.6923×10^{-7}	0.8	0.2	15	9.1484×10^{-7}
α	β	γ	$It_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	α	β	$It_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$
0.3	0.4	1.4	17(1)	4.3051×10^{-7}	0.3	0.4	19(1)	6.9077×10^{-7}
0.4	0.8	1.1	22(1)	5.0893×10^{-7}	0.4	0.8	23(1)	5.0511×10^{-7}
0.6	0.4	1.6	18(1)	8.0440×10^{-7}	0.6	0.4	22(1)	9.5648×10^{-7}
0.8	0.2	1.3	21(1)	5.4387×10^{-7}	0.8	0.2	23(1)	8.0100×10^{-7}

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Table 4: Iteration counts and relative residual about preconditioned matrices $\mathcal{P}_{MDSS}^{-1}\mathcal{A}$ and $\mathcal{P}_{GDSS}^{-1}\mathcal{A}$ when choosing different parameters, where the unpreconditioned BICGSTAB and GMRES are divergent, respectively. Here, $h = \frac{\sqrt{2}}{18}$ denotes the size of the corresponding grid.

α	β	γ	$It_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	α	β	$It_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$
0.3	0.4	1.4	14.5	4.8759×10^{-7}	0.3	0.4	19	1.9873×10^{-7}
0.4	0.8	1.1	18	8.8497×10^{-7}	0.4	0.8	17.5	8.2854×10^{-7}
0.6	0.4	1.6	16	8.8967×10^{-7}	0.6	0.4	18.5	9.9327×10^{-7}
0.8	0.2	1.3	19.5	7.4226×10^{-7}	0.8	0.2	20.5	6.6475×10^{-7}
α	β	γ	$It_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MDSS}^{-1}\mathcal{A})}$	α	β	$It_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{GDSS}^{-1}\mathcal{A})}$
0.3	0.4	1.4	23(1)	5.6343×10^{-7}	0.3	0.4	26(1)	9.5003×10^{-7}
0.4	0.8	1.1	31(1)	6.6961×10^{-7}	0.4	0.8	32(1)	8.2485×10^{-7}
0.6	0.4	1.6	24(1)	7.3405×10^{-7}	0.6	0.4	30(1)	8.1096×10^{-7}
0.8	0.2	1.3	29(1)	5.2812×10^{-7}	0.8	0.2	32(1)	9.1635×10^{-7}

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