Variational Approach to Mixed Boundary Value Problems of Fractional Sturm-Liouville Differential Equations with Instantaneous and Non-instantaneous Impulses☆

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Abstract

This paper investigates a class of fractional Sturm-Liouville differential equations with mixed boundary conditions, which are subjected to parameter and impulsive perturbations (including instantaneous and non-instantaneous impulses). By employing the variational methods and critical point theorems, we derive several criteria that guarantee the existence of at least one and two classical solutions, respectively, when the parameters fall within different intervals. Furthermore, we provide an example to demonstrate the effectiveness of our main results.

Keywords: Fractional Sturm-Liouville equation; Mixed boundary condition; Instantaneous impulse; Non-instantaneous impulse; Existence and multiplicity of solution; Critical point theorem

2000 MSC: 34A08, 34B08, 34B15, 34B37.

1. Introduction

The purpose of this paper is to establish the existence and multiplicity of solutions for the following impulsive fractional Sturm-Liouville equation (FSLE for short) supplemented with mixed boundary conditions given by

$$\begin{cases} {}_t\mathfrak{D}_T^{\alpha}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)) + \mathfrak{q}(t)\mathfrak{u}(t) = \lambda\mathfrak{f}_k(t,\mathfrak{u}(t)),\ t\in(s_k,t_{k+1}],\ k=0,1,\cdots,\mathfrak{n}, \\ -\Delta({}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k))) = I_k(\mathfrak{u}(t_k)),\ k=1,2,\cdots,\mathfrak{n}, \\ {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)) = {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^+)),\ t\in(t_k,s_k],\ k=1,2,\cdots,\mathfrak{n}, \\ {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^-)) = {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^+)),\ k=1,2,\cdots,\mathfrak{n}, \\ \mathfrak{u}(0)=0,\ \beta\mathfrak{u}(T) + \frac{1}{\mathfrak{p}(T)}{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(T){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(T)) = \mathfrak{c}, \end{cases} \tag{1}$$

where ${}_0^C\mathfrak{D}_t^{\alpha}, {}_t\mathfrak{D}_T^{\alpha}$ are the left Caputo fractional derivative and the right Riemann-Liouville fractional derivative, respectively, of order $\alpha\in(1/2,1]$, $\mathfrak{p}(t)\in C^1([0,T])$ with $0<\mathfrak{p}_0=\min_{[0,T]}\mathfrak{p}(t)$, $\mathfrak{q}(t)\in C([0,T])$ with $0<\mathfrak{q}_0=\min_{[0,T]}\mathfrak{q}(t)\le\mathfrak{q}(t)\le\mathfrak{q}^0=\max_{[0,T]}\mathfrak{q}(t)$, $\beta,\mathfrak{c}$ are two constants with $\beta>0$, λ is a positive parameter, $0=s_0<t_1<s_1<\cdots<s_{\mathfrak{n}}<t_{\mathfrak{n}+1}=T$, $I_k\in C(\mathbb{R},\mathbb{R})(k{=}1,2,\cdots,\mathfrak{n})$ and $\mathfrak{f}_k\in C((s_k,t_{k+1}]\times\mathbb{R},\mathbb{R})(k{=}0,1,2,\cdots,\mathfrak{n})$,

$$\begin{aligned} &\Delta({}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k))) = {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^+)) - {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^-)), \\ &{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^{\pm})) = \lim_{t\to t_k^{\pm}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)), \\ &{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^{\pm})) = \lim_{t\to s_k^{\pm}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)). \end{aligned}$$

Fractional differential equations are an important extension of integer-order differential equations, introducing the concept of fractional order derivatives. Due to the non-locality of fractional differential operators, fractional differential equations can describe many natural phenomena more accurately. This new mathematical tool provides a fresh perspective, enabling researchers to understand and explore the dynamic behavior of complex systems more deeply. In the past few decades, fractional differential equations have been widely used in many fields, including physics, engineering,

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biology, and economics. For example, in physics, fractional differential equations are used to describe super-diffusion phenomena [1]. In engineering, they are used to model electronic circuits and control systems [2]. In biology, they are used to simulate the dynamic changes of biological populations [3]. In economics, they are used to predict the behavior of financial markets [4]. Therefore, the study of fractional differential equations can not only promote the development of mathematical theory, but also provide powerful tools for solving practical problems.

On the other hand, impulsive differential equations, a unique subset of differential equations, are particularly effective in modeling sudden events or discontinuous behaviors observed in various systems such as signal processing, automatic control, flight object motions, multi-agent systems, time delays, and telecommunications. Comprehensive descriptions and background readings on the origin and development of the theory and applications of impulsive differential equations can be found in the monographs [5, 6]. With the significant advancements in the theory of fractional impulsive differential equations and their extensive applications across multiple fields, recent research has focused on exploring the existence and multiplicity of solutions for fractional boundary value problems influenced by impulsive effects [7–9]. Specifically, in recent years, some scholars have been dedicated to applying variational methods and critical point theory to study the existence and multiplicity of solutions for fractional impulsive differential equations with Dirichet boundary conditions and Sturm-Liouville boundary conditions. For instance, Heidarkhani and Salari [10] used variational methods and critical point theory to prove that a class of fractional differential systems with impulses and Dirichet boundary conditions has three weak solutions. Ledesma and Nyamoradi [11] constructed a variational structure for a class of fractional impulsive differential equations with (k,ψ)-Hilfer fractional derivative operators and Dirichet boundary conditions, and used the linking theorem to study the existence of weak solutions for this problem. Li et al. [12] used the Mountain Pass theorem and iterative techniques to discuss the existence of weak solutions for a class of fractional impulsive differential equations with (p,q)-Laplacian operators and Dirichet boundary conditions. Min and Chen [13] used the critical point theorem and variational methods to prove that a class of fractional impulsive differential equations with p-Laplacian operators and Sturm-Liouville boundary conditions has infinitely many weak solutions. Zhang and Ni [14] used the critical point theorem to prove that a class of fractional p-Laplacian differential equations with instantaneous impulses and non-instantaneous impulses and Sturm-Liouville boundary conditions has three weak solutions.

Mixed boundary conditions are a significant class of boundary conditions for differential equations, characterized by the boundary conditions involving a linear combination of the variable's value and its derivative. In many practical problems, the boundary conditions not only involve the variable's value but also relate to its derivative. This situation is prevalent in various fields of natural science and engineering, such as heat conduction, fluid mechanics, and circuits [15, 16]. The study of mixed boundary value problems can more accurately model these phenomena, providing mathematical models that are more applicable to practical problems. In recent years, the existence of solutions for mixed boundary value problems of fractional differential equations has received widespread attention from scholars. For instance, Łupińska [17] utilized the Banach fixed-point theorem to meticulously explore the existence and uniqueness of solutions for a class of Katugampola fractional differential equations with mixed boundary conditions. Bourguiba et al. [18] employed fixed-point theory and upper and lower solution methods to conduct a detailed study on the existence and multiplicity of solutions for a class of fractional difference equations with mixed boundary conditions. Almeida [19] applied the Banach and Leray-Schauder fixed-point theorems, carried out exhaustive study on the existence and uniqueness of solutions for a class of fractional differential equations with mixed boundary conditions. Carmona et al. [20] used variational methods, conducted profound research on the existence of solutions for a class of fractional elliptic equations with mixed boundary conditions that involve a concave-convex term.

Through an in-depth study of relevant literature, we found that variational methods and critical point theorems are mainly used to handle Dirichlet boundary value problems and Sturm-Liouville boundary value problems for fractional differential equations. However, we have not yet found any literature that uses variational methods and critical point theorem to discuss mixed boundary value problems for impulsive fractional differential equations. Therefore, in this paper, we will use variational methods and critical point theorems to explore the existence and multiplicity of solutions for impulse problem (1). The novelty and significance of our current study compared to pre-existing literature, is manifested in:

- As far as we know, there has been no literature that applies variational methods and critical point theorem to study the existence of solutions to mixed boundary value problems of fractional differential equations. In this paper, we successfully establish the variational structure of the fractional mixed boundary value problem (1) in a new fractional derivative space, and effectively apply variational methods and critical point theorems to discuss the existence of its solutions.

- Most of the existing literature that applies variational methods and critical point theorem to study boundary value problems of fractional differential equations discusses the existence of weak solutions [10–14]. In this paper, we provide a proof for the existence of classical solutions to boundary value problem (1). When the nonlinearity of the equation exhibits different growth at infinity, we apply variational methods, the least action principle and the Mountain Pass Theorem to obtain different parameter ranges to ensure the existence and multiplicity of classical solutions to problem (1).

- The concept of non-instantaneous impulses was first introduced by Hernández and O'Regan in 2013 [21]. These impulses, like instantaneous impulses, have a wide range of applications. In this paper, the problem (1) we study is composed of fractional impulse differential equations and non-homogeneous mixed boundary conditions. This equation takes into account the perturbations of both instantaneous impulses and non-instantaneous impulses. In the special case $t_k{=}s_k(k{=}1, 2, \cdots, \mathfrak{n})$, each interval of non-instantaneous impulses degenerates into a point, and BVP (1) degenerates into a single instantaneous impulse problem. Therefore, compared with the existing literature [22-25], the BVP model we discuss is more general. This generality allows our research to cover a wider range of situations, thus having stronger theoretical significance.

The structure of this paper is organized as follows: In Section 2, we will introduce some basic definitions and preliminary knowledge, which will be used in the subsequent sections. In Section 3, we construct a variational framework for problem (1), provide a proof of regularity, and propose some reasonable assumptions for the nonlinear term of problem (1). In Section 4, based on the assumption conditions proposed in Section 3, using variational methods and the Mountain Pass Theorem, we prove the existence and multiplicity of solutions to problem (1). In Section 5, we provide some examples to illustrate the main results of this paper. The final section is a summary, where we review the research results of this paper and look forward to future research directions.

2. Preliminaries

In this section, we first recall the definitions and related properties of the left and right Riemann-Liouville, as well as Caputo fractional derivatives. Subsequently, we re-examine the definition of the fractional derivative space E^{α}, and within the framework of this space, we propose a new fractional derivative space $\mathbb{E}_0^{\alpha}$ that is suitable for studying problem (1), and provide its related compact embedding results. Then, we elaborate on the definition of the classical solution to problem (1). Finally, we introduce the relevant critical point theorems that will be used in this paper.

Definition 2.1. ([2]) Let $\alpha > 0$, $\mathfrak{u} \in C[0, T]$. Then the left and right Riemann-Liouville fractional integrals ${}_0\mathfrak{D}_t^{-\alpha}\mathfrak{u}(t)$ and ${}_t\mathfrak{D}_T^{-\alpha}\mathfrak{u}(t)$ are respectively defined by

$$
\begin{aligned}
{}_0\mathfrak{D}_t^{-\alpha}\mathfrak{u}(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\mathfrak{u}(\xi)d\xi, \quad t \in [0, T], \\
{}_t\mathfrak{D}_T^{-\alpha}\mathfrak{u}(t) &= \frac{1}{\Gamma(\alpha)}\int_t^T (\xi-t)^{\alpha-1}\mathfrak{u}(\xi)d\xi, \quad t \in [0, T].
\end{aligned}
$$

Definition 2.2. ([2]) Let $\alpha \in (0, 1)$, $\mathfrak{u} \in C[0, T]$. Then the left and right Riemann-Liouville fractional derivatives ${}_0\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)$ and ${}_t\mathfrak{D}_T^{\alpha}\mathfrak{u}(t)$ are respectively defined by

$$
\begin{aligned}
{}_0\mathfrak{D}_t^{\alpha}\mathfrak{u}(t) &= \frac{d}{dt}{}_0\mathfrak{D}_t^{\alpha-1}\mathfrak{u}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}\mathfrak{u}(\xi)d\xi, \quad t \in [0, T], \\
{}_t\mathfrak{D}_T^{\alpha}\mathfrak{u}(t) &= -\frac{d}{dt}{}_t\mathfrak{D}_T^{\alpha-1}\mathfrak{u}(t) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T (\xi-t)^{-\alpha}\mathfrak{u}(\xi)d\xi, \quad t \in [0, T].
\end{aligned}
$$

Definition 2.3. ([2]) Let $\alpha \in (0, 1)$, $\mathfrak{u} \in AC[0, T]$. Then the left and right Caputo fractional derivatives ${}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)$ and

${}_t^C\mathfrak{D}_T^\alpha\mathfrak{u}(t)$ are respectively defined by

$$
{}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t) = {}_0\mathfrak{D}_t^{\alpha-1}\mathfrak{u}'(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}\mathfrak{u}'(\xi)d\xi, \quad t\in[0,T],
$$
$$
{}_t^C\mathfrak{D}_T^\alpha\mathfrak{u}(t) = -{}_t\mathfrak{D}_T^{\alpha-1}\mathfrak{u}'(t) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T (\xi-t)^{-\alpha}\mathfrak{u}'(\xi)d\xi, \quad t\in[0,T].
$$

Lemma 2.1. ([26]) Let $\alpha\in(0,1]$ and $p\in[1,+\infty)$. For any $\mathfrak{u}\in L^p([0,T],\mathbb{R}^N)$,

$$
||{}_0\mathfrak{D}_\xi^{-\alpha}\mathfrak{u}||_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)}||\mathfrak{u}||_{L^p([0,t])}, \;\; \xi\in[0,t], \;\; t\in[0,T].
$$

Lemma 2.2. ([26]) Let $\eta>0$, $p,q\ge1$, $\frac{1}{p}+\frac{1}{q}\le 1+\eta$ or $p\ne1, q\ne1$, $\frac{1}{p}+\frac{1}{q}=1+\eta$, then the following property of fractional integration

$$
\int_0^T [{}_0\mathfrak{D}_t^{-\eta}\mathfrak{x}(t)]\mathfrak{y}(t)dt = \int_0^T [{}_t\mathfrak{D}_T^{-\eta}\mathfrak{y}(t)]\mathfrak{x}(t)dt,
$$

holds, provided that $\mathfrak{x}(t)\in L^p([0,T],\mathbb{R}^N)$, $\mathfrak{y}(t)\in L^q([0,T],\mathbb{R}^N)$.
Definition 2.4. ([29]) Let $\alpha\in(1/2,1]$. The fractional space

$$
E^\alpha = \{\mathfrak{u}\in AC([0,T],\mathbb{R}^N) : {}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t)\in L^2([0,T],\mathbb{R}^N)\},
$$

is defined by the closure of $C^\infty([0,T],\mathbb{R}^N)$ with the norm

$$
||\mathfrak{u}|| = \Big(\int_0^T |\mathfrak{u}(t)|^2dt + \int_0^T |{}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t)|^2dt\Big)^{1/2}. \tag{2}
$$

Remark 2.1. ([29]) For any $\mathfrak{u}\in E^\alpha$, then $\mathfrak{u}\in L^2([0,T],\mathbb{R}^N)$ and ${}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t)\in L^2([0,T],\mathbb{R}^N)$.
Lemma 2.3. ([27]) Let $\mathfrak{h}(t)\in L^\infty([0,T])$, $\mathrm{essinf}_{[0,T]}\mathfrak{h}(t)>0$ and $\mathfrak{a}(t)\in C([0,T])$ is such that $0<\mathfrak{a}_0\le\mathfrak{a}(t)\le\mathfrak{a}^0$, an equivalent norm of (2) in E^α is the following

$$
||\mathfrak{u}||_{\alpha,2} = \Big(\int_0^T \mathfrak{a}(t)|\mathfrak{u}(t)|^2dt + \int_0^T \mathfrak{h}(t)|{}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t)|^2dt\Big)^{1/2}. \tag{3}
$$

Lemma 2.4. ([28]) A closed subspace of a reflexive Banach space is also reflexive.
Definition 2.5. Let $\alpha\in(1/2,1]$. The fractional derivative space

$$
\mathbb{E}_0^\alpha = \{\mathfrak{u}\in E^\alpha : \mathfrak{u}(0)=0\},
$$

is defined by the closure of $C^\infty([0,T],\mathbb{R}^N)$ with the norm

$$
||\mathfrak{u}||_{\alpha,2} = \Big(\int_0^T \mathfrak{q}(t)|\mathfrak{u}(t)|^2dt + \int_0^T \mathfrak{p}(t)|{}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t)|^2dt\Big)^{1/2},
$$

where $\mathfrak{p}(t)\in C^1([0,T])$ with $0<\mathfrak{p}_0=\min_{[0,T]}\mathfrak{p}(t)$, $\mathfrak{q}(t)\in C([0,T])$ with $0<\mathfrak{q}_0=\min_{[0,T]}\mathfrak{q}(t)\le\mathfrak{q}(t)\le\mathfrak{q}^0=\max_{[0,T]}\mathfrak{q}(t)$.
Lemma 2.5. Let $\alpha\in(1/2,1]$. The $\mathbb{E}_0^\alpha$ is a reflexive and separable Banach space.

Proof. Obviously, $\mathbb{E}_0^\alpha$ is a closed subspace of E^α. By the similar method used in (Lemma 4.2, [29]) and combined with Lemma 2.4, we can show that $\mathbb{E}_0^\alpha$ is a reflexive and separable Banach space. □

Lemma 2.6. Assume that $\alpha>1/2$ and the sequence $\{\mathfrak{u}_\mathfrak{n}\}$ converges weakly to $\mathfrak{u}$ in $\mathbb{E}_0^\alpha$, i.e., $\mathfrak{u}_\mathfrak{n}\rightharpoonup\mathfrak{u}$. Then $\mathfrak{u}_\mathfrak{n}\to\mathfrak{u}$ in $C([0,T],\mathbb{R}^N)$, i.e., $||\mathfrak{u}_\mathfrak{n}-\mathfrak{u}||_\infty\to0$ as $\mathfrak{n}\to\infty$.

Proof. The proof of this Lemma, while not overly complex, is omitted here. For a thorough and rigorous demonstration of this Lemma, readers are directed to refer to Proposition 5.5 ([29]). □

Definition 2.6. A function

$$
\mathfrak{u}\in\Big\{\mathfrak{u}\in AC([0,T]) : \int_{s_k}^{t_{k+1}} \big(\mathfrak{q}(t)|\mathfrak{u}(t)|^2 + \mathfrak{p}(t)|{}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t)|^2\big)\,dt<+\infty, \; k=0,1,2,\cdots,\mathfrak{n}\Big\}
$$

is called a classical solution of (1), if u satisfy equations in (1), the limits ${}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u})(t_k^\pm)$ and ${}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u})(s_k^\pm)$ exist and satisfy the impulsive conditions of problem (1) and the boundary conditions

$$
\mathfrak{u}(0)=0, \quad \beta\mathfrak{u}(T)+\frac{1}{\mathfrak{p}(T)}{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(T){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(T))=\mathfrak{c},
$$

hold.

Theorem 2.1. ([30]) Let $\mathfrak{X}$ be a reflexive Banach space and let $\varphi : \mathfrak{X} \to (-\infty, +\infty]$ is weakly lower semi-continuous on $\mathfrak{X}$. If φ has a bounded minimizing sequence, then φ has a minimum on $\mathfrak{X}$.

Remark 2.2. ([30]) If $\varphi : \mathfrak{X} \to (-\infty, +\infty]$ is coercive, Then φ has a bounded minimizing sequence.

Lemma 2.7. ([31, Theorem 38.A]) For the functional $\mathfrak{F} : \mathfrak{M} \subset \mathfrak{X} \to [-\infty, +\infty]$ with $\mathfrak{M} \neq \emptyset$, $\min_{\mathfrak{u}\in\mathfrak{M}}\mathfrak{F}(\mathfrak{u}) = \kappa$ has a solution in case the following hold:

(i) $\mathfrak{X}$ is a real reflexive Banach space.

(ii) $\mathfrak{M}$ is bounded and weak sequentially closed, i.e., by definition, for each sequence $\mathfrak{u}_m$ in $\mathfrak{M}$ such that $\mathfrak{u}_m \rightharpoonup \mathfrak{u}$ as $m \to +\infty$, we always have $\mathfrak{u} \in \mathfrak{M}$.

(iii) F is sequentially weakly lower semi-continuous on $\mathfrak{M}$.

Definition 2.7. ([30]) Let $J : \mathfrak{X} \to \mathbb{R}$ differentiable and $c \in \mathbb{R}$. We say that J satisfies the $(\mathrm{PS})_c$-condition if the existence of a sequence $\mathfrak{u}_m$ in $\mathfrak{X}$ such that

$$J(\mathfrak{u}_m) \to c,\ J'(\mathfrak{u}_m) \to 0,\ \text{as } m \to +\infty,$$

implies that c is a critical value of J.

Definition 2.8. ([30]) Let $\mathfrak{X}$ be a real reflexive Banach space. If any sequence $\mathfrak{u}_m \subset \mathfrak{X}$ for which $J(\mathfrak{u}_m)$ is bounded and $J'(\mathfrak{u}_m) \to 0$ as $m \to 0$ possesses a convergent subsequence, then we say J satisfies Palais-Smale condition ((PS)-condition for short).

Remark 2.3. ([30]) The (PS)-condition implies the $(\mathrm{PS})_c$-condition for each $c \in \mathbb{R}$.

Theorem 2.2. ([30], Theorem 4.10) Let $\mathfrak{X}$ be a Banach space and $J \in C^1(\mathfrak{X}, \mathbb{R})$. Assume that there exist $\mathfrak{u}_0 \in \mathfrak{X}, \mathfrak{u}_1 \in \mathfrak{X}$, and a bounded open neighborhood Ω of $\mathfrak{u}_0$ such that $\mathfrak{u}_1 \in \mathfrak{X}\backslash\Omega$ and

$$\inf_{\partial\Omega} J > \max\{J(\mathfrak{u}_0), J(\mathfrak{u}_1)\}.$$

Let

$$\Gamma = \{\mathfrak{g} \in C([0,1], \mathfrak{X}) : \mathfrak{g}(0) = \mathfrak{u}_0,\ \mathfrak{g}(1) = \mathfrak{u}_1\}$$

and

$$c = \inf_{\mathfrak{g}\in\Gamma} \max_{s\in[0,1]} J(\mathfrak{g}(s)).$$

If J satisfies the $(\mathrm{PS})_c$-condition, then c is a critical value of J and $c > \max\{J(\mathfrak{u}_0), J(\mathfrak{u}_1)\}$.

3. Auxiliary lemmas

In this section, we construct the variational structure of problem (1) in the fractional derivative space $\mathbb{E}_0^\alpha$, and prove the regularity of the solution to problem (1). We also provide proofs for the relevant auxiliary lemmas, and propose a series of reasonable assumptions for the nonlinear terms. These assumptions, along with the proofs of the auxiliary lemmas, serve the main conclusions in the following section. To begin with, we define the fractional space $\mathbb{E}_0^\alpha$ equipped with the norm

$$||\mathfrak{u}||_\alpha = \Big(\int_0^T \mathfrak{p}(t)|{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)|^2 dt + \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \mathfrak{q}(t)|\mathfrak{u}(t)|^2 dt\Big)^{1/2}. \tag{4}$$

Lemma 3.1. For $\mathfrak{u} \in \mathbb{E}_0^\alpha$, the norm $||\mathfrak{u}||_{\alpha,2}$ defined in Lemma 2.3 is equivalent to $||\mathfrak{u}||_\alpha$, that is, there exist two positive constants m_1 and m_2 such that

$$m_1||\mathfrak{u}||_{\alpha,2} \leq ||\mathfrak{u}||_\alpha \leq m_2||\mathfrak{u}||_{\alpha,2},\quad \text{for all } \mathfrak{u} \in \mathbb{E}_0^\alpha.$$

Proof. Obviously, we can obtain $||\mathfrak{u}||_\alpha \leq m_2||\mathfrak{u}||_{\alpha,2}$, by choosing $m_2=1$. On the other hand, for $\mathfrak{u} \in \mathbb{E}_0^\alpha$, by using Lemma 2.1, we have

$$\int_0^T |\mathfrak{u}(t)|^2 dt = \int_0^T |{}_0\mathfrak{D}_t^{-\alpha}({}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))|^2 dt \leq \Big(\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)^2 \int_0^T |{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)|^2 dt$$
$$\leq \Big(\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)^2 \frac{1}{\mathfrak{p}_0} \int_0^T \mathfrak{p}(t)|{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)|^2 dt \leq \Big(\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)^2 \frac{1}{\mathfrak{p}_0}||\mathfrak{u}||_\alpha^2 := \Delta||\mathfrak{u}||_\alpha^2.$$

It follows that

$$||\mathfrak{u}||_{\alpha,2}^2 = ||\mathfrak{u}||_\alpha^2 + \sum_{k=1}^{\mathfrak{n}} \int_{t_k}^{s_k} \mathfrak{q}(t)|\mathfrak{u}(t)|^2 dt$$
$$\leq ||\mathfrak{u}||_\alpha^2 + \mathfrak{q}^0 \int_0^T |\mathfrak{u}(t)|^2 dt \leq (1+\mathfrak{q}^0\Delta)||\mathfrak{u}||_\alpha^2.$$

Take $m_1=(1+\mathfrak{q}^0\Delta)^{-1/2}$, we get $m_1||\mathfrak{u}||_{\alpha,2} \leq ||\mathfrak{u}||_\alpha$. The proof is complete. □

Remark 3.1. If $\alpha \in (1/2, 1]$, then $||\mathfrak{u}||_\infty \leq M||\mathfrak{u}||_\alpha$, where

$$M := \frac{T^{\alpha-(1/2)}\mathfrak{p}_0^{-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}.$$

Proof. For any $\mathfrak{u} \in \mathbb{E}_0^\alpha$, by using the Hölder's inequality, we have

$$|\mathfrak{u}(t)| \leq |{}_0\mathfrak{D}_t^{-\alpha}({}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))| = \frac{1}{\Gamma(\alpha)}\Big|\int_0^t (t-s)^{\alpha-1}{}_0^C\mathfrak{D}_s^\alpha \mathfrak{u}(s)ds\Big|$$
$$\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{2(\alpha-1)}ds\Big)^{1/2}\Big(\int_0^T |{}_0^C\mathfrak{D}_s^\alpha \mathfrak{u}(s)|^2 ds\Big)^{1/2}$$
$$\leq \frac{T^{\alpha-(1/2)}\mathfrak{p}_0^{-1/2}}{\Gamma(\alpha)[(\alpha-1)2+1]^{1/2}}\Big(\int_0^T \mathfrak{p}(t)|{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)|^2 dt\Big)^{1/2} \leq M||\mathfrak{u}||_\alpha,$$

which implies that $||\mathfrak{u}||_\infty \leq M||\mathfrak{u}||_\alpha$. The remark is proved. □

Lemma 3.1. A function $\mathfrak{u} \in \mathbb{E}_0^\alpha$ is a solution of problem (1), then the following identity

$$\begin{aligned}&\int_0^T \mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{v}(t)dt + \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \mathfrak{q}(t)\mathfrak{u}(t)\mathfrak{v}(t)dt - (\mathfrak{c}-\beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T)\\ &= \lambda\sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \mathfrak{f}_k(t,\mathfrak{u}(t))\mathfrak{v}(t)dt + \sum_{k=1}^{\mathfrak{n}} I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k)\end{aligned} \tag{5}$$

holds for any $\mathfrak{v} \in \mathbb{E}_0^\alpha$.

Proof. For $\mathfrak{v} \in \mathbb{E}_0^\alpha$, one has $\mathfrak{v}(0) = 0$. By Lemma 2.2, we have

$$\begin{aligned}&\int_0^T {}_t\mathfrak{D}_T^\alpha(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))\mathfrak{v}(t)dt = -\int_0^T \frac{d}{dt}[{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))]\mathfrak{v}(t)dt\\
&= -\sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \frac{d}{dt}[{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))]\mathfrak{v}(t)dt - \sum_{k=1}^{\mathfrak{n}} \int_{t_k}^{s_k} \frac{d}{dt}[{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))]\mathfrak{v}(t)dt\\
&= -\sum_{k=0}^{\mathfrak{n}} [{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))]\mathfrak{v}(t)\Big|_{s_k^+}^{t_{k+1}^-} + \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))\mathfrak{v}'(t)dt\\
&\quad - \sum_{k=1}^{\mathfrak{n}} [{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))]\mathfrak{v}(t)\Big|_{t_k^+}^{s_k^-} + \sum_{k=1}^{\mathfrak{n}} \int_{t_k}^{s_k} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))\mathfrak{v}'(t)dt\\
&= \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} (\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)){}_0\mathfrak{D}_t^{\alpha-1}\mathfrak{v}'(t)dt + \sum_{k=1}^{\mathfrak{n}} \int_{t_k}^{s_k} (\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)){}_0\mathfrak{D}_t^{\alpha-1}\mathfrak{v}'(t)dt\\
&\quad - \sum_{k=1}^{\mathfrak{n}} I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k) - (\mathfrak{c}-\beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T)\\
&= -\sum_{k=1}^{\mathfrak{n}} I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k) - (\mathfrak{c}-\beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T) + \int_0^T \mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t){}_0^C\mathfrak{D}_t^\alpha \mathfrak{v}(t)dt.\end{aligned} \tag{6}$$

On the other hand, in view of ${}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t))={}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t_k^+))$, $t\in(t_k,s_k]$, $k=1,2,\cdots,\mathfrak{n}$, one has

$$\begin{aligned}&\int_0^T {}_t\mathfrak{D}_T^\alpha(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t))\mathfrak{v}(t)dt\\&=\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}} {}_t\mathfrak{D}_T^\alpha(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t))\mathfrak{v}(t)dt-\sum_{k=1}^{\mathfrak{n}}\int_{t_k}^{s_k}\frac{d}{dt}[{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t))]\mathfrak{v}(t)dt\\&=-\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}\mathfrak{q}(t)\mathfrak{u}(t)\mathfrak{v}(t)dt+\lambda\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}\mathfrak{f}_k(t,\mathfrak{u}(t))\mathfrak{v}(t)dt.\end{aligned}\tag{7}$$

Combining Eqs. (6) and (7), we can get Eq. (5) immediately. The proof is complete. □

Definition 3.1. A function $\mathfrak{u}\in\mathbb{E}_0^\alpha$ is called a weak solution of problem (1), if (5) holds for any $\mathfrak{v}\in E_0^\alpha$.

Define the functional $J:\mathbb{E}_0^\alpha\to\mathbb{R}$ by

$$J(\mathfrak{u})=\Phi(\mathfrak{u})-\lambda\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}F_k(t,\mathfrak{u}(t))dt,$$

where

$$\Phi(\mathfrak{u})=\frac{1}{2}||\mathfrak{u}||_\alpha^2-\sum_{k=1}^{\mathfrak{n}}\int_0^{\mathfrak{u}(t_k)}I_k(x)dx+\frac{\mathfrak{p}(T)}{2\beta}(\mathfrak{c}-\beta\mathfrak{u}(T))^2,$$

and

$$F_k(t,\mathfrak{u})=\int_0^{\mathfrak{u}}\mathfrak{f}_k(t,s)ds\ \text{ for all } (t,\mathfrak{u})\in[s_k,t_{k+1}]\times\mathbb{R}.$$

By using the continuity of $\mathfrak{f}_k$ and I_k, one has that $J\in C^1(\mathbb{E}_0^\alpha,\mathbb{R})$ and, for any $\mathfrak{u},\mathfrak{v}\in\mathbb{E}_0^\alpha$,

$$\begin{aligned}<J'(\mathfrak{u}),\mathfrak{v}>=&\int_0^T\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{v}(t)dt+\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}\mathfrak{q}(t)\mathfrak{u}(t)\mathfrak{v}(t)dt-\lambda\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}\mathfrak{f}_k(t,\mathfrak{u}(t))\mathfrak{v}(t)dt\\&-\sum_{k=1}^{\mathfrak{n}}I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k)-(\mathfrak{c}-\beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T).\end{aligned}\tag{8}$$

Thus, the weak solutions of problem (1) are the critical points of J.

Lemma 3.2. The functional $J:\mathbb{E}_0^\alpha\to\mathbb{R}$ is sequentially weakly lower semi-continuous.

Proof. Let $\{\mathfrak{u}_\mathfrak{m}\}_{\mathfrak{m}=1}^\infty$ be a weakly convergent sequence to $\mathfrak{u}$ in $\mathbb{E}_0^\alpha$. It follows from Lemma 2.6 that $\{\mathfrak{u}_\mathfrak{m}\}_{\mathfrak{m}=1}^\infty$ is convergent uniformly to $\mathfrak{u}$ in $C[0,T]$. On account of the continuity of $\mathfrak{f}_k$ and I_k, yields that

$$\begin{aligned}&\lim_{\mathfrak{m}\to+\infty}-\sum_{k=1}^{\mathfrak{n}}\int_0^{\mathfrak{u}_\mathfrak{m}(t_k)}I_k(\mathfrak{x})d\mathfrak{x}-\lambda\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}F_k(t,\mathfrak{u}_\mathfrak{m}(t))dt+\frac{\mathfrak{p}(T)}{2\beta}(\mathfrak{c}-\beta\mathfrak{u}_\mathfrak{m}(T))^2\\&=-\sum_{k=1}^{\mathfrak{n}}\int_0^{\mathfrak{u}(t_k)}I_k(\mathfrak{x})d\mathfrak{x}-\lambda\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}F_k(t,\mathfrak{u}(t))dt+\frac{\mathfrak{p}(T)}{2\beta}(\mathfrak{c}-\beta\mathfrak{u}(T))^2.\end{aligned}\tag{9}$$

Since $\frac{1}{2}||\mathfrak{u}||_\alpha^2$ is continuous and convex, it follows that $\frac{1}{2}||\mathfrak{u}||_\alpha^2$ is also sequentially weakly lower semi-continuous, that is,

$$\liminf_{\mathfrak{m}\to+\infty}\frac{1}{2}||\mathfrak{u}_\mathfrak{m}||_\alpha^2\geq\frac{1}{2}||\mathfrak{u}||_\alpha^2.$$

Using this property and (9), we conclude that $J(\mathfrak{u})$ is sequentially weakly lower semi-continuous. The proof is complete. □

Lemma 3.3. If $\mathfrak{u}\in\mathbb{E}_0^\alpha$ is a weak solution of problem (1), then $\mathfrak{u}\in\mathbb{E}_0^\alpha$ is a classical solution of problem (1).

Proof. By standard arguments, we have that if $\mathfrak{u}$ is a classical solution of problem (1), then it is also a weak solution of (1). On the other hand, if $\mathfrak{u}\in\mathbb{E}_0^\alpha$ is a weak solution of (1), then $\mathfrak{u}(0)=0$ and $<J'(\mathfrak{u}),\mathfrak{v}>=0$ for all $\mathfrak{v}\in\mathbb{E}_0^\alpha$, i.e.,

$$\begin{aligned}&\int_0^T\mathfrak{p}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{u}(t){}_0^C\mathfrak{D}_t^\alpha\mathfrak{v}(t)dt+\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}\mathfrak{q}(t)\mathfrak{u}(t)\mathfrak{v}(t)dt-(\mathfrak{c}-\beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T)\\&=\lambda\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}\mathfrak{f}_k(t,\mathfrak{u}(t))\mathfrak{v}(t)dt+\sum_{k=1}^{\mathfrak{n}}I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k).\end{aligned}\tag{10}$$

Without loss of generality, let $\mathfrak{v} \in C_0^\infty(s_k, t_{k+1}]$ be such that $\mathfrak{v}(t) \equiv 0$ for all $t \in [0, s_k] \cup (t_{k+1}, T],\ k = 0, 1, 2, \cdots, \mathfrak{n}$. By plugging $\mathfrak{v}(t)$ into (10) and applying Lemma 2.2, we obtain

$$\int_{s_k}^{t_{k+1}} \mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{v}(t)dt + \int_{s_k}^{t_{k+1}} \mathfrak{q}(t)\mathfrak{u}(t)\mathfrak{v}(t)dt = \lambda \int_{s_k}^{t_{k+1}} \mathfrak{f}_k(t, \mathfrak{u}(t))\mathfrak{v}(t)dt, \quad k = 0, 1, 2, \cdots, \mathfrak{n},$$

and

$$\int_{s_k}^{t_{k+1}} \mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{v}(t)dt = \int_{s_k}^{t_{k+1}} {}_t\mathfrak{D}_T^\alpha(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t))\mathfrak{v}(t)dt < +\infty.$$

The above equality implies that

$$ {}_t\mathfrak{D}_T^\alpha(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)) + \mathfrak{q}(t)\mathfrak{u}(t) = \lambda \mathfrak{f}_k(t, \mathfrak{u}(t)), \ \text{ for } a.e.\ t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \cdots, \mathfrak{n}. \tag{11}$$

Since $\mathfrak{u} \in \mathbb{E}_0^\alpha$, we have

$$\int_{s_k}^{t_{k+1}} (\mathfrak{p}(t)|{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)|^2 + q(t)|\mathfrak{u}(t)|^2)dt < +\infty,\ k = 0, 1, 2, \cdots, \mathfrak{n}.$$

Using (11) and the fact that $\mathfrak{f}_k \in C((s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$, we deduce that

$$ {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)) \in AC([s_k, t_{k+1}]).$$

Hence, the following limits exist

$$\begin{aligned} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(s_k^+)) &= \lim_{t\to s_k^+} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)), \\ {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_{k+1})\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_{k+1}^-)) &= \lim_{t\to t_{k+1}^-} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)). \end{aligned}$$

Combining (10) and (11), we obtain

$$\begin{aligned} &\int_0^T \mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{v}(t)dt + \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \frac{d}{dt}({}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)))\mathfrak{v}(t)dt \\ &- (\mathfrak{c} - \beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T) - \sum_{k=1}^{\mathfrak{n}} I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k) = 0, \end{aligned}$$

that is,

$$\begin{aligned} &\sum_{k=0}^{\mathfrak{n}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_{k+1})\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_{k+1}^-))\mathfrak{v}(t_{k+1}^-) - \sum_{k=0}^{\mathfrak{n}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(s_k^+))\mathfrak{v}(s_k^+) \\ &+ \sum_{k=1}^{\mathfrak{n}} \int_{t_k}^{s_k} \mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{v}(t)dt - (\mathfrak{c} - \beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T) - \sum_{k=1}^{\mathfrak{n}} I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k) = 0. \end{aligned} \tag{12}$$

By choosing the test function $\mathfrak{v} \in C_0^\infty(t_k, s_k]$ such that $v(t) \equiv 0$ for $t \in [0, t_k] \cup (s_k, T],\ k = 1, 2, \cdots, \mathfrak{n}$, without loss of generality, we can insert $\mathfrak{v}(t)$ into (12) and obtain ${}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)) = \text{Constant},\ t \in (t_k, s_k],\ k = 1, 2, \cdots, n$, i.e.,

$$\begin{aligned} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_k^+)) &= {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(s_k^-)) \\ &= {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t)),\ t \in (t_k, s_k],\ k = 1, 2, \cdots, \mathfrak{n}. \end{aligned} \tag{13}$$

By plugging (13) into (12), we get

$$\begin{aligned} &\sum_{k=0}^{\mathfrak{n}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_{k+1})\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_{k+1}^-))\mathfrak{v}(t_{k+1}^-) - \sum_{k=0}^{\mathfrak{n}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(s_k^+))\mathfrak{v}(s_k^+) \\ &+ \sum_{k=1}^{\mathfrak{n}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_k^+))\mathfrak{v}(s_k) - \sum_{k=1}^{\mathfrak{n}} {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_k^+))\mathfrak{v}(t_k) \\ &- (\mathfrak{c} - \beta\mathfrak{u}(T))\mathfrak{p}(T)\mathfrak{v}(T) - \sum_{k=1}^{\mathfrak{n}} I_k(\mathfrak{u}(t_k))\mathfrak{v}(t_k) = 0, \end{aligned}$$

that is,

$$\begin{aligned} &\sum_{k=1}^{\mathfrak{n}} [{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_k^-)) - {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_k^+)) - I_k(\mathfrak{u}(t_k))]\mathfrak{v}(t_k) \\ &+ \sum_{k=1}^{\mathfrak{n}} [{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(t_k^+)) - {}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(s_k^+))]\mathfrak{v}(s_k) \\ &+ [{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(T)\,{}_0^C\mathfrak{D}_t^\alpha \mathfrak{u}(T)) - (\mathfrak{c} - \beta\mathfrak{u}(T))\mathfrak{p}(T)]\mathfrak{v}(T) = 0, \end{aligned}$$

From this, it follows that

$$\beta\mathfrak{u}(T)+\frac{1}{\mathfrak{p}(T)}{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(T){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(T))=\mathfrak{c},$$
$${}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^+))-{}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^-))=-I_k(\mathfrak{u}(t_k)),\quad k=1,2,\cdots,\mathfrak{n},$$

and

$${}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(t_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^+))={}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^+)),\quad k=1,2,\cdots,\mathfrak{n}.$$

Combining this with (12), we also obtain

$${}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^-))={}_t\mathfrak{D}_T^{\alpha-1}(\mathfrak{p}(s_k){}_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^+)),\quad k=1,2,\cdots,\mathfrak{n}.$$

Therefore, $\mathfrak{u}$ satisfy the equation, the impulsive conditions, and the boundary conditions of problem (1), i.e., $\mathfrak{u}$ is a classical solution of (1). The proof is complete. □

In this paper, we make the following assumptions:

(H_1) There exist constants $\mathfrak{a}_k>0$, $\mathfrak{b}_k>0$ and $\gamma_k\in[0,1)$, $k=1,2,\cdots,\mathfrak{n}$ such that

$$|I_k(\mathfrak{x})|\le\mathfrak{a}_k+\mathfrak{b}_k|\mathfrak{x}|^{\gamma_k},\quad\text{for every }\mathfrak{x}\in\mathbb{R},\ k=1,2,\cdots,\mathfrak{n}.$$

(H_2) There exist constant $\mu\in[0,2)$ and functions $\tau_1^k(t)\in C([s_k,t_{k+1}])(k=0,1,2,\cdots,\mathfrak{n})$ with $\text{essinf}\tau_1^k(t)>0$ such that

$$\limsup_{|\mathfrak{x}|\to+\infty}\frac{F_k(t,\mathfrak{x})}{|\mathfrak{x}|^{\mu}}<\tau_1^k(t),\ \text{uniformly for almost every } t\in[s_k,t_{k+1}].\tag{14}$$

(H_3) There exist functions $\tau_2^k(t)\in C([s_k,t_{k+1}])(k=0,1,2,\cdots,\mathfrak{n})$ with $\text{essinf}\tau_2^k(t)>0$ such that

$$\limsup_{|\mathfrak{x}|\to+\infty}\frac{F_k(t,\mathfrak{x})}{|\mathfrak{x}|^{2}}<\tau_2^k(t),\ \text{uniformly for almost every } t\in[s_k,t_{k+1}].\tag{15}$$

(H_4) There exists a constant $\sigma>2$ such that $0<\sigma F_k(t,\mathfrak{x})\le\mathfrak{x}\mathfrak{f}_k(t,\mathfrak{x})$ for every $t\in[s_k,t_{k+1}]$ $(k=0,1,2,\cdots,\mathfrak{n})$ and $\mathfrak{x}\in\mathbb{R}\setminus\{0\}$.

Remark 3.1. According to assumptions (H_1)-(H_4), it follows

- For $\mathfrak{u}\in\mathbb{E}_0^{\alpha}$, (H_1) implies that

$$\Phi(\mathfrak{u})\ge\frac{1}{2}||\mathfrak{u}||_{\alpha}^2-\sum_{k=1}^{\mathfrak{n}}\Big(\mathfrak{a}_kM||\mathfrak{u}||_{\alpha}+\frac{\mathfrak{b}_k}{\gamma_k+1}M^{\gamma_k+1}||\mathfrak{u}||_{\alpha}^{\gamma_k+1}\Big),$$

  and

$$\Phi(\mathfrak{u})\le(\frac{1}{2}+\Lambda_1)||\mathfrak{u}||_{\alpha}^2+\sum_{k=1}^{\mathfrak{n}}\Big(\mathfrak{a}_kM||\mathfrak{u}||_{\alpha}+\frac{\mathfrak{b}_k}{\gamma_k+1}M^{\gamma_k+1}||\mathfrak{u}||_{\alpha}^{\gamma_k+1}\Big)+\Lambda_2,$$

  where

$$\Lambda_1=\beta\mathfrak{p}(T)M^2,\quad\Lambda_2=\frac{\mathfrak{p}(T)}{\beta}\mathfrak{c}^2.$$

- Under assumption (H_2), there is a constant $\Lambda_3>0$ such that

$$F_k(t,\mathfrak{x})\le\tau_1^k(t)|\mathfrak{x}|^{\mu}+\Lambda_3,\ \text{for almost every } t\in[s_k,t_{k+1}] \text{ and all } \mathfrak{x}\in\mathbb{R}.\tag{16}$$

- Under assumption (H_3), there is a constant $\Lambda_4>0$ such that

$$F_k(t,\mathfrak{x})\le\tau_2^k(t)|\mathfrak{x}|^{2}+\Lambda_4,\ \text{for almost every } t\in[s_k,t_{k+1}] \text{ and all } \mathfrak{x}\in\mathbb{R}.\tag{17}$$

- Under assumption (H_4), there is a constant $\Lambda_5>0$ such that

$$F_k(t,\mathfrak{x})\le\bar{F}_k|\mathfrak{x}|^{\sigma}\le\bar{F}|\mathfrak{x}|^{\sigma},\quad(t,\mathfrak{x})\in[s_k,t_{k+1}]\times[-1,1],\tag{18}$$

and

$$F_k(t,\mathfrak{x}) \geq \underline{F}_k|\mathfrak{x}|^\sigma - \Lambda_5 \geq \underline{F}|x|^\sigma - \Lambda_5, \quad (t,\mathfrak{x}) \in [s_k, t_{k+1}] \times \mathbb{R}, \tag{19}$$

where

$$\bar{F}_k = \max_{t\in[s_k,t_{k+1}],|\mathfrak{x}|=1} F_k(t,\mathfrak{x}) > 0, \quad \underline{F}_k = \min_{t\in[s_k,t_{k+1}],|\mathfrak{x}|=1} F_k(t,\mathfrak{x}) > 0,$$
$$\bar{F} = \max\{\bar{F}_0, \bar{F}_1, \cdots, \bar{F}_n\}, \quad \underline{F} = \min\{\underline{F}_0, \underline{F}_1, \cdots, \underline{F}_n\}.$$

In fact, in view of (H_1), one has

$$\Big|\int_0^{\mathfrak{u}(t_k)} I_k(\mathfrak{x})dx\Big| \leq \mathfrak{a}_k|\mathfrak{u}(t_k)| + \frac{\mathfrak{b}_k}{\gamma_k+1}|\mathfrak{u}(t_k)|^{\gamma_k+1}.$$

By using Remark 3.1, we obtain

$$\begin{aligned}\Big|\sum_{k=1}^{\mathfrak{n}} \int_0^{\mathfrak{u}(t_k)} I_k(\mathfrak{x})dx\Big| &\leq \sum_{k=1}^{n} \Big|\int_0^{\mathfrak{u}(t_k)} I_k(\mathfrak{x})dx\Big| \\ &\leq \sum_{k=1}^{\mathfrak{n}} \Big[\mathfrak{a}_k|\mathfrak{u}(t_k)| + \frac{\mathfrak{b}_k}{\gamma_k+1}|\mathfrak{u}(t_k)|^{\gamma_k+1}\Big] \\ &\leq \sum_{k=1}^{\mathfrak{n}} \Big(\mathfrak{a}_k M||\mathfrak{u}||_\alpha + \frac{\mathfrak{b}_k}{\gamma_k+1}M^{\gamma_k+1}||\mathfrak{u}||_\alpha^{\gamma_k+1}\Big),\end{aligned} \tag{20}$$

and

$$(\mathfrak{c} - \beta\mathfrak{u}(T))^2 \leq 2(\mathfrak{c}^2 + \beta^2 M^2||\mathfrak{u}||_\alpha^2). \tag{21}$$

It follows from (20) and (21) that

$$\begin{aligned}\Phi(\mathfrak{u}) &= \frac{1}{2}||\mathfrak{u}||_\alpha^2 - \sum_{k=1}^{\mathfrak{n}} \int_0^{\mathfrak{u}(t_k)} I_k(\mathfrak{x})dx + \frac{\mathfrak{p}(T)}{2\beta}(\mathfrak{c} - \beta\mathfrak{u}(T))^2 \\ &\geq \frac{1}{2}||\mathfrak{u}||_\alpha^2 - \Big|\sum_{k=1}^{\mathfrak{n}} \int_0^{\mathfrak{u}(t_k)} I_k(\mathfrak{x})dx\Big| \\ &\geq \frac{1}{2}||\mathfrak{u}||_\alpha^2 - \sum_{k=1}^{\mathfrak{n}} \Big(\mathfrak{a}_k M||\mathfrak{u}||_\alpha + \frac{\mathfrak{b}_k}{\gamma_k+1}M^{\gamma_k+1}||\mathfrak{u}||_\alpha^{\gamma_k+1}\Big),\end{aligned} \tag{22}$$

and

$$\begin{aligned}\Phi(\mathfrak{u}) &= \frac{1}{2}||\mathfrak{u}||_\alpha^2 - \sum_{k=1}^{\mathfrak{n}} \int_0^{\mathfrak{u}(t_k)} I_k(\mathfrak{x})dx + \frac{\mathfrak{p}(T)}{2\beta}(\mathfrak{c} - \beta\mathfrak{u}(T))^2 \\ &\leq \frac{1}{2}||\mathfrak{u}||_\alpha^2 + \Big|\sum_{k=1}^{\mathfrak{n}} \int_0^{\mathfrak{u}(t_k)} I_k(\mathfrak{x})dx\Big| + \frac{\mathfrak{p}(T)}{\beta}(\mathfrak{c}^2 + \beta^2 M^2||\mathfrak{u}||_\alpha^2) \\ &= (\frac{1}{2} + \Lambda_1)||\mathfrak{u}||_\alpha^2 + \sum_{k=1}^{\mathfrak{n}} \Big(\mathfrak{a}_k M||\mathfrak{u}||_\alpha + \frac{\mathfrak{b}_k}{\gamma_k+1}M^{\gamma_k+1}||\mathfrak{u}||_\alpha^{\gamma_k+1}\Big) + \Lambda_2.\end{aligned} \tag{23}$$

By virtue of (H_2), we can find $\Lambda_6 > 0$ such that

$$F_k(t,\mathfrak{x}) \leq \tau_1^k(t)|\mathfrak{x}|^\mu, \text{ for almost every } t \in [s_k, t_{k+1}] \text{ and all } |\mathfrak{x}| \geq \Lambda_6,$$

which, together with the continuity of $F_k(t,\mathfrak{x}) - \tau_1^k(t)|\mathfrak{x}|^\mu$ on $[s_k, t_{k+1}] \times [-\Lambda_6, \Lambda_6]$, leads to (16). Likewise, (H_3) ensures (17). Let us define $\mathfrak{T}_k : (0, +\infty) \to \mathbb{R}$ $(k = 0, 1, 2, \cdots, \mathfrak{n})$ by

$$\mathfrak{T}_k(\mathfrak{z}) = F_k\Big(t, \frac{\mathfrak{x}}{\mathfrak{z}}\Big)\mathfrak{z}^\sigma, \text{ for every } t \in [s_k, t_{k+1}] \text{ and } \mathfrak{x} \neq 0.$$

It follows from (H_4) that

$$\begin{aligned}\mathfrak{T}_k'(\mathfrak{z}) &= \mathfrak{f}_k\Big(t, \frac{\mathfrak{x}}{\mathfrak{z}}\Big)\Big(-\frac{\mathfrak{x}}{\mathfrak{z}^2}\Big)\mathfrak{z}^\sigma + \sigma F_k\Big(t, \frac{\mathfrak{x}}{\mathfrak{z}}\Big)\mathfrak{z}^{\sigma-1} \\ &= -\frac{\mathfrak{x}}{\mathfrak{z}}\mathfrak{f}_k\Big(t, \frac{\mathfrak{x}}{\mathfrak{z}}\Big)\mathfrak{z}^{\sigma-1} + \sigma F_k\Big(t, \frac{\mathfrak{x}}{\mathfrak{z}}\Big)\mathfrak{z}^{\sigma-1} \\ &\leq -\sigma F_k\Big(t, \frac{\mathfrak{x}}{\mathfrak{z}}\Big)\mathfrak{z}^{\sigma-1} + \sigma F_k\Big(t, \frac{\mathfrak{x}}{\mathfrak{z}}\Big)\mathfrak{z}^{\sigma-1} = 0.\end{aligned}$$

This implies that $\mathfrak{T}_k$ is non-increasing on $(0, +\infty)$. Thus, for every $t \in [s_k, t_{k+1}]$,

$$\mathfrak{T}_k(1) = F_k(t, \mathfrak{x}) \leq \mathfrak{T}_k(|\mathfrak{x}|) = F_k\Big(t, \frac{\mathfrak{x}}{|\mathfrak{x}|}\Big)|\mathfrak{x}|^{\sigma}, \text{ if } 0 < |\mathfrak{x}| \leq 1,$$

and

$$\mathfrak{T}_k(1) = F_k(t, \mathfrak{x}) \geq \mathfrak{T}_k(|\mathfrak{x}|) = F_k\Big(t, \frac{\mathfrak{x}}{|\mathfrak{x}|}\Big)|\mathfrak{x}|^{\sigma}, \text{ if } |\mathfrak{x}| \geq 1.$$

Therefore, (18) holds, and

$$F_k(t, \mathfrak{x}) \geq \underline{F_k}|\mathfrak{x}|^{\sigma}, \text{ if } |\mathfrak{x}| \geq 1,$$

which, together with the continuity of $F_k(t, \mathfrak{x}) - \underline{F_k}|\mathfrak{x}|^{\sigma}$ on $[s_k, t_{k+1}] \times [-1, 1]$, implies (19). By (H_4), we also obtain $\bar{F}_k > 0$ and $\underline{F_k} > 0$.

4. Existence result

This section presents the main conclusions, where we have applied the variational method, the least action principle, and the Mountain Pass theorem. As a result, we have determined that there exist at least one and two classical solutions to BVP (1).

Theorem 4.1. Suppose that (H_1) and one of the following conditions hold. Then problem (1) has at least one classical solution.

(C_1) (H_2) holds and $\lambda \in (0, +\infty)$.

(C_2) (H_3) holds and $\lambda \in \Big(0, (2M^2 \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \tau_2^k(t)dt)^{-1}\Big)$.

Proof. If (C_2) is satisfied, then by Remark 3.1 and (17), we obtain

$$\begin{aligned}\sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} F_k(t, \mathfrak{u}(t))dt &\leq \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \tau_2^k(t)|\mathfrak{u}(t)|^2 dt + \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \Lambda_4 dt \\ &\leq M^2||\mathfrak{u}||_{\alpha}^2 \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \tau_2^k(t)dt + \Lambda_4 T,\end{aligned}$$

which, together with (22), implies that

$$\begin{aligned}J(\mathfrak{u}) &= \Phi(\mathfrak{u}) - \lambda \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} F_k(t, \mathfrak{u}(t))dt \\ &\geq \frac{1}{2}||\mathfrak{u}||_{\alpha}^2 - \sum_{k=1}^{\mathfrak{n}} \Big(\mathfrak{a}_k M||\mathfrak{u}||_{\alpha} + \frac{\mathfrak{b}_k}{\gamma_k + 1} M^{\gamma_k+1}||\mathfrak{u}||_{\alpha}^{\gamma_k+1}\Big), \\ &\quad - \lambda M^2||\mathfrak{u}||_{\alpha}^2 \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \tau_2^k(t)dt - \lambda\Lambda_4 T.\end{aligned} \tag{24}$$

Since $\lambda \in \Big(0, (2M^2 \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \tau_2^k(t)dt)^{-1}\Big)$, we have from (24) that

$$\lim_{||\mathfrak{u}||_{\alpha} \to +\infty} J(\mathfrak{u}) = +\infty. \tag{25}$$

Hence, $J(\mathfrak{u})$ admits a bounded minimizing sequence. By Theorem 2.1, Lemma 3.2 and Lemma 3.3, (1) possesses at least one classical solution. The case of (C_1) can be treated analogously. Indeed, (H_2) yields that

$$\begin{aligned}J(\mathfrak{u}) &\geq \frac{1}{2}||u||_{\alpha}^2 - \sum_{k=1}^{\mathfrak{n}} (\mathfrak{a}_k M||\mathfrak{u}||_{\alpha} + \frac{\mathfrak{b}_k}{\gamma_k + 1} M^{\gamma_k+1}||\mathfrak{u}||_{\alpha}^{\gamma_k+1}), \\ &\quad - \lambda M^{\mu}||\mathfrak{u}||_{\alpha}^{\mu} \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \tau_1^k(t)dt - \lambda\Lambda_3 T,\end{aligned} \tag{26}$$

and since $\mu \in [0, 2)$, $\gamma_k \in [0, 1)$, $(k = 0, 1, 2, \cdots, \mathfrak{n})$, (26) leads to (25). The proof is therefore complete. ☐

Corollary 4.1. Suppose that the functions $\mathfrak{f}_k$ and the impulsive functions I_k are bounded, there exists a classical solution for problem (1) for any given $\lambda \in (0, +\infty)$.

Lemma 4.1. If (H_1) and (H_4) holds, then J satisfies the (PS)-condition.

Proof. Let $\{\mathfrak{u}_m\}$ be a sequence in $\mathbb{E}_0^\alpha$ such that $J(\mathfrak{u}_m)$ is bounded and $J'(\mathfrak{u}_m) \to 0$ as $m \to +\infty$. Now we divide the proof into two claims.

Claim 1. $\{\mathfrak{u}_m\}$ is bounded in $\mathbb{E}_0^\alpha$. In fact, it follows from (H_1) and Remark 3.1 that

$$\begin{aligned} I_k(\mathfrak{u}_m(t_k))\mathfrak{u}_m(t_k) &\geq -\mathfrak{a}_k|\mathfrak{u}_m(t_k)| - \mathfrak{b}_k|\mathfrak{u}_m(t_k)|^{\gamma_k+1} \\ &\geq -\mathfrak{a}_k M||\mathfrak{u}_m||_\alpha - \mathfrak{b}_k M^{\gamma_k+1}||\mathfrak{u}_m||_\alpha^{\gamma_k+1}. \end{aligned} \tag{27}$$

Since $\sigma > 2$, one has

$$\begin{aligned} &\sigma\frac{\mathfrak{p}(T)}{2\beta}(\mathfrak{c} - \beta\mathfrak{u}_m(T))^2 + \mathfrak{p}(T)(\mathfrak{c} - \beta\mathfrak{u}_m(T))\mathfrak{u}_m(T) \\ &= \left(\frac{\sigma}{2} - 1\right)\frac{\mathfrak{p}(T)}{\beta}(\mathfrak{c} - \beta\mathfrak{u}_m(T))^2 + \frac{c}{\beta}\mathfrak{p}(T)(\mathfrak{c} - \beta\mathfrak{u}_m(T)) \\ &\geq -|\mathfrak{c}|\frac{\mathfrak{p}(T)}{\beta}|\mathfrak{c} - \beta\mathfrak{u}_m(T)| \\ &\geq -|\mathfrak{c}|\frac{\mathfrak{p}(T)}{\beta}(|\mathfrak{c}| + \beta M||\mathfrak{u}_m||_\alpha). \end{aligned} \tag{28}$$

In view of (H_4), (20), (27) and (28), one obtain

$$\begin{aligned} &\sigma J(\mathfrak{u}_m) - < J'(\mathfrak{u}_m), \mathfrak{u}_m > \\ &= \left(\frac{\sigma}{2} - 1\right)||\mathfrak{u}_m||_\alpha^2 + \lambda\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}[\mathfrak{f}_k(t, \mathfrak{u}_m(t))\mathfrak{u}_m(t) - \sigma F_k(t, \mathfrak{u}_m(t))]dt \\ &\quad - \sigma\sum_{k=1}^{\mathfrak{n}}\int_0^{\mathfrak{u}_m(t_k)} I_k(\mathfrak{x})d\mathfrak{x} + \sum_{k=1}^{\mathfrak{n}} I_k(\mathfrak{u}_m(t_k))\mathfrak{u}_m(t_k) \\ &\quad + \frac{\sigma\mathfrak{p}(T)}{2\beta}(\mathfrak{c} - \beta\mathfrak{u}_m(T))^2 + \mathfrak{p}(T)(\mathfrak{c} - \beta\mathfrak{u}_m(T))\mathfrak{u}_m(T) \\ &\geq \left(\frac{\sigma}{2} - 1\right)||\mathfrak{u}_m||_\alpha^2 - |\mathfrak{c}|\frac{\mathfrak{p}(T)}{\beta}(|\mathfrak{c}| + \beta M||\mathfrak{u}_m||_\alpha) \\ &\quad - \sigma\sum_{k=1}^{\mathfrak{n}}\left(\mathfrak{a}_k M||\mathfrak{u}_m||_\alpha + \frac{\mathfrak{b}_k M^{\gamma_k+1}}{\gamma_k + 1}||\mathfrak{u}_m||_\alpha^{\gamma_k+1}\right) \\ &\quad - \sum_{k=1}^{\mathfrak{n}}(\mathfrak{a}_k M||\mathfrak{u}_m||_\alpha + \mathfrak{b}_k M^{\gamma_k+1}||\mathfrak{u}_m||_\alpha^{\gamma_k+1}), \end{aligned}$$

which implies that $\mathfrak{u}_m$ is bounded in $\mathbb{E}_0^\alpha$.

Claim 2. $\{\mathfrak{u}_m\}$ converges strongly to some $\mathfrak{u}$ in $\mathbb{E}_0^\alpha$. Indeed, given that the sequence $\{\mathfrak{u}_m\}$ is bounded in $\mathbb{E}_0^\alpha$, a subsequence of $\{\mathfrak{u}_m\}$ exists (which we will continue to denote as $\{\mathfrak{u}_m\}$ for simplicity) that weakly converges to some $\mathfrak{u}$ in $\mathbb{E}_0^\alpha$. Consequently, the sequence $\{\mathfrak{u}_m\}$ uniformly converges to $\mathfrak{u}$ in $C[0, T]$. Therefore,

$$\begin{aligned} &\mathfrak{u}_m(T) \to \mathfrak{u}(T), \\ &\sum_{k=1}^{\mathfrak{n}}[I_k(\mathfrak{u}_m(t_k)) - I_k(\mathfrak{u}(t_k))](\mathfrak{u}_m(t_k) - \mathfrak{u}(t_k)) \to 0, \\ &\sum_{k=0}^{\mathfrak{n}}\int_{s_k}^{t_{k+1}}[\mathfrak{f}_k(t, \mathfrak{u}_m(t)) - \mathfrak{f}_k(t, \mathfrak{u}(t))](\mathfrak{u}_m(t) - \mathfrak{u}(t))dt \to 0, \end{aligned} \tag{29}$$

as $m \to +\infty$. Since $\lim\limits_{m \to +\infty} J'(\mathfrak{u}_m) = 0$ and $\{\mathfrak{u}_m\}$ converges weakly to $\mathfrak{u}$, one gets

$$< J'(\mathfrak{u}_m) - J'(\mathfrak{u}), \mathfrak{u}_m - \mathfrak{u} > \to 0, \text{ as } m \to +\infty.$$

It then follows from (8) that

$$\begin{aligned}&< J'(u_m) - J'(u), u_m - u > \\ &= ||u_m - u||_\alpha^2 - \sum_{k=1}^{\mathfrak{n}} [I_k(\mathfrak{u}_m(t_k)) - I_k(\mathfrak{u}(t_k))](\mathfrak{u}_m(t_k) - \mathfrak{u}(t_k)) + \mathfrak{p}(T)(\mathfrak{u}_m(T) - \mathfrak{u}(T))^2\beta \\ &\quad - \lambda \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} [\mathfrak{f}_k(t, \mathfrak{u}_k(t)) - \mathfrak{f}_k(t, \mathfrak{u}(t))](\mathfrak{u}_m(t) - \mathfrak{u}(t))dt.\end{aligned} \tag{30}$$

Combining with (29) and (30), we obtain

$$||\mathfrak{u}_m(t) - \mathfrak{u}(t)||_\alpha \to 0, \text{ as } m \to +\infty,$$

that is, $\{\mathfrak{u}_m\}$ converges strongly to $\mathfrak{u}$ in $\mathbb{E}_0^\alpha$. Therefore, J satisfies the (PS)-condition. The lemma has been proved. □

Theorem 4.2. If $\Lambda > 0$, and conditions (H_1) and (H_4) are satisfied, then problem (1) admits at least two distinct classical solutions for any $\lambda \in (0, \Lambda/\bar{F}T)$, where

$$\Lambda = \frac{1}{2M^2} - \sum_{k=1}^{\mathfrak{n}} \left(\mathfrak{a}_k + \frac{\mathfrak{b}_k}{\gamma_k + 1}\right) - \frac{\mathfrak{c}^2}{2\beta}\mathfrak{p}(T).$$

Proof. Consider the open ball $\mathbb{B}_r$ in $\mathbb{E}_0^\alpha$ with center 0 and radius r, and let $\partial\mathbb{B}_r$ and $\bar{\mathbb{B}}_r$ be its boundary and closure, respectively. Suppose $\{\mathfrak{u}_m\} \subset \bar{\mathbb{B}}_{(1/M)}$ and $\mathfrak{u}_m \rightharpoonup \mathfrak{u}$ as $m \to +\infty$. According to the Mazur Theorem [30, p.4], we can find a sequence of convex combinations

$$\mathfrak{v}_m = \sum_{j=1}^m \ell_{m_j} \mathfrak{u}_j, \quad \sum_{j=1}^m \ell_{m_j} = 1, \quad \ell_{m_j} \geq 0 (m \in \mathbb{N}^+)$$

such that $\mathfrak{v}_m \to \mathfrak{u}$ in $\mathbb{E}_0^\alpha$. Since $\bar{\mathbb{B}}_{(1/M)}$ is a closed convex set, $\mathfrak{v}_m \subset \bar{\mathbb{B}}_{(1/M)}$ and $\mathfrak{u} \in \bar{\mathbb{B}}_{(1/M)}$. Then $\bar{\mathbb{B}}_{(1/M)}$ is bounded and weakly sequentially closed. By Lemma 2.7 and Lemma 3.2, $J(\mathfrak{u})$ admits a local minimum $\mathfrak{u}_0 \in \bar{\mathbb{B}}_{(1/M)}$. Hence

$$J(\mathfrak{u}_0) \leq J(0) = \frac{\mathfrak{c}^2}{2\beta}\mathfrak{p}(T). \tag{31}$$

In view of Remark 3.1, $||\mathfrak{u}||_\alpha \leq 1/M$ implies that $||\mathfrak{u}||_\infty \leq 1$. Owing to (18), one has

$$\sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} F_k(t, \mathfrak{u}(t))dt \leq \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \bar{F}_k |\mathfrak{u}(t)|^\sigma dt \leq M^\sigma \bar{F}T ||\mathfrak{u}||_\alpha^\sigma, \quad ||\mathfrak{u}||_\alpha \leq \frac{1}{M}. \tag{32}$$

Combining this with (22), we obtain that, for any $\mathfrak{u} \in \partial\mathbb{B}_r$ and $r \leq 1/M$,

$$J(\mathfrak{u}) \geq \frac{1}{2}r^2 - \sum_{k=1}^{\mathfrak{n}} \left(\mathfrak{a}_k M r + \frac{\mathfrak{b}_k}{\gamma_k + 1} M^{\gamma_k+1} r^{\gamma_k+1}\right) - \lambda M^\sigma r^\sigma \bar{F}T.$$

Hence

$$J(\mathfrak{u}) \geq \frac{1}{2M^2} - \sum_{k=1}^{\mathfrak{n}} \left(\mathfrak{a}_k + \frac{\mathfrak{b}_k}{\gamma_k + 1}\right) - \lambda \bar{F}T, \quad \text{for any } \mathfrak{u} \in \partial\mathbb{B}_{(1/M)}.$$

Since $\lambda \in (0, \Lambda/\bar{F}T)$ and by (31), we get

$$J(\mathfrak{u}) > J(0) \geq J(\mathfrak{u}_0), \quad \text{for any } \mathfrak{u} \in \partial\mathbb{B}_{(1/M)}.$$

Therefore, $\inf_{\mathfrak{u} \in \partial\mathbb{B}_{(1/M)}} J(\mathfrak{u}) > J(\mathfrak{u}_0)$, and $J(\mathfrak{u})$ admits a local minimum $\mathfrak{u}_0 \in \mathbb{B}_{(1/M)}$. Fix $\xi > 0$, and let $\mathfrak{u} \in \mathbb{E}_0^\alpha$ be such that $||\mathfrak{u}||_\alpha = 1$. Then, from (19)-(21), we have

$$\begin{aligned}J(\xi\mathfrak{u}) \leq \frac{1}{2}\xi^2 + \sum_{k=1}^{\mathfrak{n}} \left(\mathfrak{a}_k M\xi + \frac{\mathfrak{b}_k}{\gamma_k + 1} M^{\gamma_k+1} \xi^{\gamma_k+1}\right) + \frac{\mathfrak{p}(T)}{\beta}(\mathfrak{c}^2 + \beta^2 M^2 \xi^2) \\ - \lambda\Big[\xi^\sigma \sum_{k=0}^{\mathfrak{n}} \int_{s_k}^{t_{k+1}} \underline{F} |u|^\sigma dt\Big] + \lambda\Lambda_5 T.\end{aligned}$$

Given that $\sigma > 2$ and $\gamma_k \in [0, 1)$, it is established that $J(\xi\mathfrak{u}) \to -\infty$ as $\xi \to +\infty$. Consequently, there exists a $\mathfrak{u}_1 > 0$ with $||\mathfrak{u}_1||_\alpha > 1/M$ such that $\inf_{\mathfrak{u} \in \partial\mathbb{B}_{(1/M)}} J(\mathfrak{u}) > J(\mathfrak{u}_1)$. By applying Theorem 2.2 and Lemma 4.1, it can be inferred that there exists a $\mathfrak{u}_2 \in \mathbb{E}_0^\alpha$ such that $J'(\mathfrak{u}_2) = 0$ and $J(\mathfrak{u}_2) > \max\{J(\mathfrak{u}_0), J(\mathfrak{u}_1)\}$. Therefore, $\mathfrak{u}_0$ and $\mathfrak{u}_2$ are two distinct classical solutions of problem (1). The proof is complete. □

5. Example

Example 5.1. Let $\alpha = 0.75$, $T = \beta = 1$, $\mathfrak{n} = 3$. Consider the following mixed boundary value problem of fractional Sturm-Liouville equation with impulsive effects:

$$\begin{cases} {}_t\mathfrak{D}_T^{\alpha}((16+2t)_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)) + (1+2t^2)\mathfrak{u}(t) = \lambda\mathfrak{f}_k(t,\mathfrak{u}(t)),\ t \in (s_k, t_{k+1}],\ k = 0,1,2,3, \\ -\Delta({}_t\mathfrak{D}_T^{\alpha-1}((16+2t_k)_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k))) = I_k(\mathfrak{u}(t_k)),\ k = 1,2,3, \\ {}_t\mathfrak{D}_T^{\alpha-1}((16+2t)_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t)) = {}_t\mathfrak{D}_T^{\alpha-1}((16+2t_k)_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t_k^+)),\ t \in (t_k, s_k],\ k = 1,2,3, \\ {}_t\mathfrak{D}_T^{\alpha-1}((16+2s_k)_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^-)) = {}_t\mathfrak{D}_T^{\alpha-1}((16+2s_k)_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(s_k^+)),\ k = 1,2,3, \\ \mathfrak{u}(0) = 0,\ \mathfrak{u}(1) + \frac{1}{18}{}_t\mathfrak{D}_T^{\alpha-1}((16+2t)_0^C\mathfrak{D}_t^{\alpha}\mathfrak{u}(t))|_{t=1} = \mathfrak{c}. \end{cases} \tag{33}$$

From Remark 3.1, we deduce that $M = \frac{\sqrt{2}}{4\Gamma(3/4)}$. Due to Theorem 4.1 and Theorem 4.2, the following results are obtained:

- Let $\mathfrak{f}_k(t,\mathfrak{u}) = \frac{1}{2}\mathfrak{u}^{1/2}$ $(k = 0,1,2,3)$ and $I_k(\mathfrak{x}) = \frac{1}{4}\mathfrak{x}^{1/2}$, $\mathfrak{a}_k = \mathfrak{b}_k = \frac{1}{4}$, $\gamma_k = \frac{1}{2}$ $(k = 1,2,3)$. It follows that (H_1) and (H_2) are satisfied with $\mu = \frac{3}{2}$. Therefore, for any constant $\mathfrak{c} > 0$, problem (33) has at least one solution for each $\lambda \in (0, +\infty)$.

- Let $\mathfrak{f}_k(t,\mathfrak{u}) = \frac{1}{2}\mathfrak{u}$ $(k = 0,1,2,3)$ and $I_k(\mathfrak{x}) = \frac{1}{4}\mathfrak{x}^{1/2}$, $\mathfrak{a}_k = \mathfrak{b}_k = \frac{1}{4}$, $\gamma_k = \frac{1}{2}$ $(k = 1,2,3)$. It follows that (H_1) and (H_3) are satisfied with $\tau_2^k(t) = \frac{1}{4}$. Therefore, for any constant $\mathfrak{c} > 0$, problem (33) has at least one solution for each $\lambda \in (0, 16\Gamma^2(3/4))$.

- Let $\mathfrak{f}_k(t,\mathfrak{u}) = \frac{1}{2}\mathfrak{u}^3$ $(k = 0,1,2,3)$, $I_k(\mathfrak{x}) = \frac{1}{4}\mathfrak{x}^{1/2}$, $\mathfrak{a}_k = \mathfrak{b}_k = \frac{1}{4}$, $\gamma_k = \frac{1}{2}$ $(k = 1,2,3)$ and $\mathfrak{c} = \frac{1}{3}$. It follows that $\Lambda = \frac{16\Gamma^2(3/4)-9}{4} > 0$, (H_1) and (H_4) are satisfied with $\sigma = 4$. Therefore, problem (33) has at least two distinct solutions for each $\lambda \in (0, 32\Gamma^2(3/4) - 18)$.

6. Conclusion

In recent years, the variational methods and critical point theorems have been widely applied in the qualitative analysis of fractional (impulsive) differential equations with Dirichlet and Sturm-Liouville boundary conditions. This study explores a class of non-homogeneous mixed BVPs of fractional differential equations with parameters and impulsive disturbances. For the first time, we established the variational structure of the fractional mixed BVP in the fractional derivative space $\mathbb{E}_0^{\alpha}$, and used the variational method and critical point theorems to prove the existence and multiplicity of solutions to the proposed problem (1). The results of this study effectively extended the application range of the variational methods in the fractional BVPs, which has important theoretical significance. Based on this study, we can discuss a series of problems, the most direct extension is to extend the results of this study to the quasi-linear operator situation, coupling situation, and BVPs of fractional differential equations with non-local terms, such as Kirchhoff-type fractional BVPs, etc.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Authors' contributions

The authors have made equal contributions to each part of this paper. All the authors read and approved the final manuscript.

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