Applications of variational iteration method to a class of ordinary differential equations

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Abstract

This paper proves the convergence of the variational iteration method for a class of n-th order ordinary differential equations with Lipschitz nonlinearity which can be regarded as a generalization of oscillation equations.

Keywords. Variational iteration method, convergence analysis, oscillation equations

1 Introduction

The variational iteration method was first proposed by He to approximately solve some nonlinear ordinary differential equations and nonlinear partial differential equations without linearization or small perturbation [5–7, 9]. This method was shown by many authors to be superior to many other analytic approximation methods, such as Adomain's decomposition method and perturbation methods. A key point is that a correction functional is constructed by a general Lagrange multiplier [11], which can be identified via variational methods.

In this paper we attempt to use the variational iteration method to get an exact solution of the following n-th order ordinary differential equation

$$\begin{cases} u^{(n)}(t) + a_{n-1}(t)u^{(n-1)}(t) + \cdots + a_1(t)u'(t) + F(t, u(t)) = 0,\ t \in \mathbf{R}, \\ u^{(n-1)}(0) = u_0^{(n-1)}, \cdots, u'(0) = u_0', u(0) = u_0, \end{cases} \tag{1.1}$$

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where $a_i \in C^{n-1}(\mathbf{R}),\ i = 1, 2, 3, \cdots, n-1$, $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous function that is Lipschitz continuous with respect to the second variable, i.e., there exists constant $L > 0$ such that

$$\Big|F(t, u_1) - F(t, u_2)\Big| \le L|u_1 - u_2|,\ \forall\, t, u_1, u_2 \in \mathbf{R}.$$

The motivation for the study of the above equation originated from nonlinear oscillators. Several authors showed the validity of variational iteration method and got approximate solutions of oscillation equations, see for example, [8, 15]. See [1, 3, 4, 12, 16, 17] and the references therein for other new developments of variational iteration methods. The oscillation equations considered in this paper has more general form with Lipschitz nonlinear term. Our main theorem (Theorem 3.1) shows that variational iteration sequence of (1.1) converges uniformly on any finite time interval to an exact solution.

2 Preliminary

In this section, we briefly recall the basic idea of the variational iteration method. Consider the following general nonlinear system

$$L(u(t)) + N(u(t)) = g(t), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and g is a given continuous function.

The essential technique of the variational iteration method is to construct a correction functional for system (2.1) as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s;t)\Big(Lu_n(s) + N\tilde{u}_n(s) - g(s)\Big)ds, \tag{2.2}$$

where λ is a general Lagrange multiplier which can be determined by using the variational approach, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. See [2, 10, 13, 14, 18] for more details on iteration methods.

3 Main Theorem

Theorem 3.1. *For any given $T > 0$, any initial data $v \in C^n([-T, T])$ satisfying the initial condition in (1.1), the variational iteration sequence of (1.1) converges uniformly on $[-T, T]$ to an exact solution.*

Proof. First we derive the variational iteration sequence of (1.1). Let $\lambda = \lambda(s;t)$ be a general Lagrange multiplier to be determined. Then the correction functional for (1.1) is

$$u_{k+1}(t) = u_k(t) + \int_0^t \lambda(s;t)\left(u_k^{(n)}(s) + a_{n-1}(s)u_k^{(n-1)}(s) + \cdots + a_1(s)u_k'(s) + F(s, u_k(s))\right)ds.$$

Making the above correction functional stationary, notice that $\delta u_k(0) = 0$,

$$\begin{aligned}\delta u_{k+1}(t) &= \delta u_k(t) + \delta \int_0^t \lambda(s;t)\Big(u_k^{(n)}(s) + a_{n-1}(s)(\tilde{u}_k)^{(n-1)}(s) + \cdots + a_1(s)(\tilde{u}_k)'(s) + F(s,\tilde{u}_k(s))\Big)\, ds \\ &= \delta u_k(t) + \int_0^t \lambda(s;t)(\delta u_k)^{(n)}(s)ds \\ &= \Big(1 + (-1)^{n-1}\frac{\partial^{n-1}\lambda}{\partial s^{n-1}}(t;t)\Big)\delta u_k(t) + \sum_{j=1}^{n-1}(-1)^{j-1}\frac{\partial^{j-1}\lambda}{\partial s^{j-1}}(t;t)(\delta u_k)^{(n-j)}(t) \\ &\quad + \int_0^t (-1)^n \frac{\partial^n \lambda}{\partial s^n}(s;t)(\delta u_k)(s)ds = 0,\end{aligned}$$

where $\tilde{u}_k$ denotes the restricted variation, i.e., $\delta\tilde{u}_k = 0$.

Thus the arbitrariness of δu_k yields

$$\begin{cases} \dfrac{\partial^n \lambda}{\partial s^n}(s;t) = 0, \ \forall\ s \in [0,t], \\ \dfrac{\partial^{j-1}\lambda}{\partial s^{j-1}}(t;t) = 0, \ \forall\ j = 1,2,3,\cdots n-1, \\ \dfrac{\partial^{n-1}\lambda}{\partial s^{n-1}}(t;t) = (-1)^n, \end{cases}$$

which can be readily solved to obtain $\lambda(s;t) = \frac{(-1)^n}{(n-1)!}(s-t)^{n-1}$. So the variational iteration sequence of (1.1) reads

$$u_{k+1}(t) = u_k(t) + \int_0^t \frac{(-1)^n}{(n-1)!}(s-t)^{n-1}\Big(u_k^{(n)}(s) + a_{n-1}(s)u_k^{(n-1)}(s) + \cdots + a_1(s)u_k'(s) + F(s,u_k(s))\Big)ds.$$

Now we define an operator $\mathbb{A}_T : C^0([-T,T]) \to C^0([-T,T])$ by

$$\mathbb{A}_T[u](t) := \sum_{j=1}^{n-1}(-1)^j \int_0^t \left(\frac{(-1)^n(s-t)^{n-1}}{(n-1)!}a_j(s)\right)^{(j)} u(s)ds + \int_0^t \frac{(-1)^n(s-t)^{n-1}}{(n-1)!}F(s,u(s))ds.$$

It can be easily seen that $\mathbb{A}_T$ is well-defined. Also defined $\tilde{\mathbb{A}}_T : C^n([-T,T]) \to C^n([-T,T])$ by

$$\tilde{\mathbb{A}}_T[u](t) := u(t) + \int_0^t \frac{(-1)^n(s-t)^{n-1}}{(n-1)!}\Big(u^{(n)}(s) + a_{n-1}(s)u^{(n-1)}(s) + \cdots + a_1(s)u'(s) + F(s,u(s))\Big)ds.$$

The classical Cauchy-Lipschitz theorem and extension theorem guarantee that the unique exact solution $\varphi(t)$ of (1.1) exists on $[-T,T]$. It is clear that φ is a fixed point of $\tilde{\mathbb{A}}_T$, i.e., $\tilde{\mathbb{A}}_T[\varphi] = \varphi$.

Therefore for any $v \in C^n([-T,T])$ satisfying the initial condition in (1.1), we have

$$\begin{aligned}
&\left|\tilde{\mathbb{A}}_T[v](t)-\varphi(t)\right| = \left|\tilde{\mathbb{A}}_T[v](t)-\tilde{\mathbb{A}}_T[\varphi](t)\right| \\
= \;& \Big|(v-\varphi)(t)+\int_0^t \frac{(-1)^n(s-t)^{n-1}}{(n-1)!}\Big((v^{(n)}-\varphi^{(n)})(s)+\sum_{j=1}^{n-1}a_j(s)(v^{(j)}-\varphi^{(j)})(s) \\
& +F(s,v(s))-F(s,\varphi(s))\Big)ds\Big| \\
= \;& \Big|\sum_{j=1}^{n-1}(-1)^j\int_0^t \left(\frac{(-1)^j(s-t)^{n-1}}{(n-1)!}a_j(s)\right)^{(j)}(v-\varphi)(s) \\
& +\int_0^t \frac{(-1)^n(s-t)^{n-1}}{(n-1)!}\Big(F(s,v(s))-F(s,\varphi(s))\Big)ds\Big| \\
= \;& \left|\mathbb{A}_T[v](t)-\mathbb{A}_T[\varphi](t)\right| \\
\leq \;& C(n,T,M)\Big|\int_0^t \left|v-\varphi_T\right|(s)ds\Big| \leq C(n,T,M)\|v-\varphi\|_{C^0([-T,T])}|t|, \ \forall\, t\in[-T,T]
\end{aligned} \tag{3.1}$$

where $C(n,T,M)$ is a positive constant depending on $n,T,M := \max\{\|a_j\|_{C^{n-1}([-T,T])}; 1 \leq j \leq n-1\}$ and L.

One more step iteration gives

$$\begin{aligned}
\left|\tilde{\mathbb{A}}_T^2[v](t)-\varphi(t)\right| &= \left|\tilde{\mathbb{A}}_T\Big(\tilde{\mathbb{A}}_T[v]\Big)(t)-\tilde{\mathbb{A}}_T\Big(\tilde{\mathbb{A}}_T[\varphi]\Big)(t)\right| \\
&\leq C(n,T)\Big|\int_0^t \left|\tilde{\mathbb{A}}_T[v](s)-\tilde{\mathbb{A}}_T[\varphi](s)\right|ds\Big| \\
&= C(n,T)\Big|\int_0^t \left|\mathbb{A}_T[v](s)-\mathbb{A}_T[\varphi](s)\right|ds\Big| \\
&\leq C(n,T,M)^2\Big|\int_0^t \|v-\varphi\|_{C^0([-T,T])}s ds\Big| \\
&= \frac{C(n,T,M)^2\|v-\varphi\|_{C^0([-T,T])}}{2!}t^2, \ \forall\, t\in[-T,T].
\end{aligned}$$

Inductively we have

$$\left|\tilde{\mathbb{A}}_T^N[v](t)-\varphi(t)\right| \leq \frac{C(n,T,M)^N\|v-\varphi\|_{C^0([-T,T])}}{N!}t^N, \ \forall\, t\in[-T,T].$$

Thus

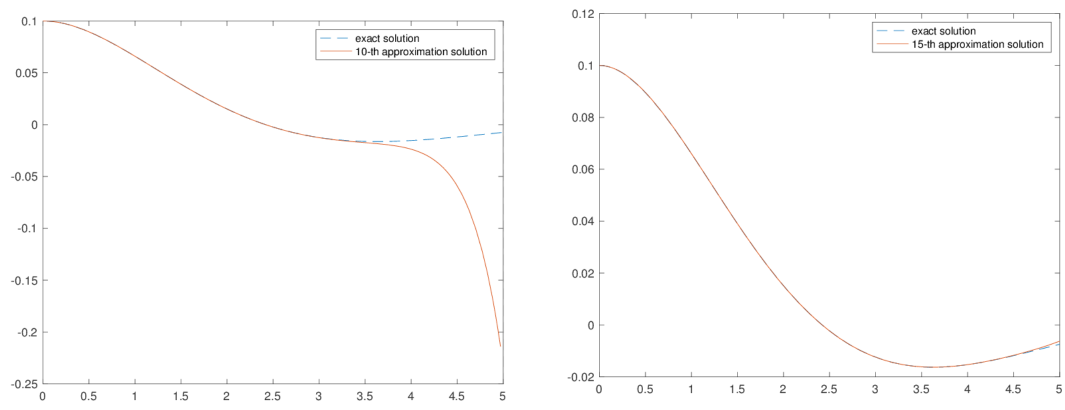
$$\left\|\tilde{\mathbb{A}}_T^N[v]-\varphi\right\|_{0,[-T,T]} \leq \frac{C(n,T,M)^N\|v-\varphi\|_{C^0([-T,T])}}{N!}(2T)^N \to 0, \ N\to\infty.$$

The proof is complete. □

Example 3.1. (Dissipative pendulum)

Consider the equation of dissipative pendulum with external force

$$\begin{cases} u''(t)+u'(t)+\sin(u(t))=0, \ t\in\mathbf{R}, \\ u(0)=A>0, u'(0)=0. \end{cases} \tag{3.2}$$



(a) (b)

Figure 1

Here $F(u) = \sin u$ is a Lipschitz function with Lipschitz constant 1. The variational iteration formula for this equation reads

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\Big(u_n''(s) + u_n'(s) + \sin(u_n(s))\Big)ds.$$

Rewrite (3.2) into planar system

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -v - \sin(u). \end{cases} \tag{3.3}$$

It can be easily seen that $(0, 0)$ is a saddle point, so for $A > 0$ small, the orbit of (3.3) will spiral towards the origin. For $A = 0.1$, initial approximation guess $v(t) = A\cos(t)$ and time interval $[0, 5]$. (a) and (b) in Figure 1 show a comparison between the exact solutions versus 10-th and 15-th approximation solutions respectively. Graphically, the more we iterate, the closer, in an uniform way, these two solutions with each other.

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