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GLOBAL BOUNDEDNESS IN A QUASILINEAR CHEMOTAXIS-CONSUMPTION SYSTEM WITH SIGNAL-DEPENDENT MOTILITY AND SUPER-QUADRATIC DAMPING

Chi Xu[1,*]

Abstract In this paper, we consider a quasilinear chemotaxis-consumption model

$$\begin{cases} u_t = \Delta(v^\alpha u^m) + ru - \mu u^l, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0 \end{cases}$$

within a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ under homogeneous Neumann boundary conditions, where the parameters $\alpha, r, \mu > 0$ and $l, m > 1$. For any sufficiently regular initial data and parameters $l, m > 1$ with $l > m + 1$, it is shown that the aforementioned system possesses at least one global weak solution with a boundedness property

$$\|u(\cdot,t)\|_{L^p(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C$$

for all $p \ge 2$ and $t > 0$. This finding indicates the regularizing effect of super-quadratic damping of a logistic-type source under strong degeneracy of signal-dependent motility, even though the cross-diffusion is simultaneously enhanced.

Keywords Chemotaxis; Weak solution; Degenerate diffusion; Boundedness.

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1. Introduction

Chemotaxis, which refers to the directed movement toward the area with higher chemical concentrations, is essential for the growth and survival of cells or bacteria. The chemotactic movement can be described by the following reaction-diffusion system

$$\begin{cases} u_t = \nabla \cdot (\gamma(u,v)\nabla u) + \nabla \cdot (\phi(u,v)u\nabla v) + g(u,v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + f(u,v), & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

which is also known as the Keller-Segel model. Here, u and v represent the density of cells and chemicals, respectively, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth

[1]School of Mathematics and Big data, Anhui University of Science and Technology, 232001, Huainan, P. R. China.
[*]The author was supported by the Key Program of the University Natural Science Research Fund of Anhui Provence (2022AH050826)
Email: XuChi1993@126.com(Chi Xu)

boundary. In the signal equation, $f(u,v)$ reflects the dynamics of cells and significantly influences the behavior of the solution. For example, with the prototypical choice $f(u,v) = -v + u$, cells aggregate due to the movement of the chemotaxis induced by the chemical they secrete. In this scenario, the dynamics of (1.1) is highly dependent on the spatial dimension, with the phenomenon of critical mass observed when $n = 2$ under conditions $\gamma(u,v) = \phi(u,v) = 1$ and $g(u,v) = 0$ (see survey [1]). In contrast, when $f(u,v) = -uv$, cells are primitive bacteria and the movement of the chemotaxis is mainly a result of the cross-diffusion movement toward the oxygen they consume. However, compared to the case where the chemical signal is produced, the chemotaxis consumption system is dominated by random diffusion of bacteria, and the solution will eventually approach its homogeneous steady state as time tends to infinity ([16]).

As a simplification of (1.1), the Keller-Segel model with signal-dependent motility

$$\begin{cases} u_t = \Delta(\phi(v)u^m) + g(u,v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + f(u,v), & x \in \Omega,\ t > 0 \end{cases} \tag{1.2}$$

was proposed in [4] to model the stripe formation structure observed in experiments. When the diffusion is Brownian, i.e., $m = 1$, in this case (1.2) can be written as

$$\begin{cases} u_t = \Delta(\phi(v)u) + g(u,v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.3}$$

When $g(u,v) = 0$, the blow-up phenomena in (1.1) can be suppressed if the motility function $\phi(v)$ exhibits algebraic decay at infinity ([3, 5, 6]). However, a similar critical mass phenomenon, which firstly detected in Keller-Segel model, was also identified in [5, 7, 8] if the motility function is an exponential decay function $e^{-\chi v}$. These findings indicate that, on the one hand, the signal-dependent motility can bring a regularizing effect to suppress the blow-up of solutions when $\phi(v)$ is an algebraic decay function, but, on the other hand, the blow-up suppression of signal-dependent motility might be invalid if $\phi(v)$ is a fast decay function such as an exponential function. When considering cell proliferation, the prototypical choice of $g(u,v)$ is a logistic source $g(u,v) = ru - \mu u^l$. When $l = 2$ and the spatial dimension $n = 2$, for any $\phi(s) \in C^3([0,\infty))$ such that $\phi'(s) < 0$, $\lim_{s\to\infty} \phi(s) = 0$, and the limit $\lim_{s\to\infty} \frac{\phi'(s)}{\phi(s)}$ exists, (1.2) will admit a unique global classic solution, which eventually tends towards the constant equilibrium of (1.2) if $\mu > \frac{1}{16} \max_{0\le s<+\infty} \frac{|\phi'(s)|^2}{\phi(s)}$ ([9]). Subsequently, a boundedness property in higher dimensions is obtained in [29] with sufficiently large μ. Nevertheless, pattern formation can occur if μ is sufficiently small ([22]). For further results on super-quadratic damping, we refer to [19, 20].

However, if $m > 1$, the diffusion of cells is not Brownian and the literature on this topic is far from complete. For example, the following quasi-linear chemotaxis-production system

$$\begin{cases} u_t = \Delta(\phi(v)u^m), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

where the motility function generalizes the prototype $\phi(v) = v^{-\alpha}$, admits a global bounded weak solution under several constraints on α and the additional condition $m > \frac{n}{2}$ ([36]).

In the signal consumption scenario with $m = 1$, system (1.2) can readily take the form

$$\begin{cases} u_t = \Delta(\phi(v)u) + g(u,v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0. \end{cases} \tag{1.5}$$

When $g(u,v) = 0$, rigorous analytical research in the early stages mainly concentrates on the case where $\phi(s)$ has a uniform positive lower bound for $s \in [0,\infty)$, and the findings suggest that a non-degenerate motility function does not significantly impact the dynamics of (1.2) when cells consume the signal ([10, 13, 14, 26]). In fact, under the condition $\phi(s) > 0,\ s \in [0,\infty)$, the existence of a classical solution was established with a smallness condition on the initial data ([10]). Later, this smallness condition was removed for spatial dimension $n = 2$ ([26]). Recently, global solvability was established in a generalized framework, and this solution will eventually stabilize to the homogeneous steady state of (1.6) ([13,14]). If the motility function is given by $\phi(s) = s^{-\alpha}$, there exists a weak-strong solution for (1.6), which can transform into a standard weak solution under conditions $2 \le n \le 5$ and $\alpha > \frac{n-2}{6-n}$. Moreover, the solution can be converted into a classical solution in spatial dimension $n = 1$ ([25]). But, the qualitative asymptotic analysis in the case of $\phi(s) = s^{-\alpha}$ is still lacking. When considering the degeneracy of $\phi(s)$ in $s = 0$, several profound studies reveal that this degeneracy can essentially complicate both the theory of the solution and its asymptotic behavior. For example, the constant equilibrium, which is asymptotically stable for any non-degenerate $\phi(s)$, will eventually lose its stability, causing the solution to approach a non-constant equilibrium as time tends to infinity ([11, 34, 35]). When $g(u,v) \neq 0$, the system (1.2) possesses a globally bounded classical solution with non-degenerate $\phi(s)$ if the source term is either a super-quadratic degradation term or a standard logistic source with large $\mu > 0$ ([27]). Other similar variants with logistic-type source can be found in [21, 28].

When $m > 1$ and $g(u,v) = 0$, an analogous result in the chemotaxis-consumption system

$$\begin{cases} u_t = \Delta(\phi(v)u^m), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0 \end{cases} \tag{1.6}$$

was achieved in [15] with a non-degenerate motility function $\phi(s)$. These findings illustrate that porous medium-type diffusion can regularize the signal-dependent Keller-Segel model in both signal production and consumption scenarios, provided there is sufficiently strong nonlinear enhancement, even though cross-diffusion is simultaneously enhanced. Recently, the existence of a global weak solution was established in [2] when $\phi(s)$ is a singular motility function $s^{-\alpha}$ under constraints between m and α, but the results regarding the well-posedness of (1.6) under the degeneracy of $\phi(s)$ at $s = 0$ are still quite limited, not to mention the influence of considering of proliferation effect of cells. To the best of our knowledge, if $g(u,v) = 0$ and the motility function nearly take the form $\phi(s) = s^{\alpha}$, system (1.6) possesses a global weak solution under the condition $a \in [1, 2m)$, while in one-dimensional setting, the existence of global weak solution can also be obtained for arbitrary $\alpha > 0$, and that solution remains uniformly bounded when $\alpha \ge 1$. Moreover, the uniformly bounded solution will eventually converge to a nonhomogeneous steady state as time passes to infinity ([12]). However, as far as we know, no rigorous analysis exists for the scenario where $g(u,v) \neq 0$.

Main result. Based on the preceding discussion, this study primarily investigates the impact of cell proliferation on the system (1.6). To this end, by incorporating a logistic source term $ru - \mu u^l$ in (1.6), the objective of this paper is to determine to what extent the damping effect of the logistic source can regularize (1.6) when $\phi(s)$ potentially degenerates at $s = 0$. Specifically, we consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta(v^\alpha u^m) + ru - \mu u^l, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega,\ t > 0, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary and ν is the outward unit normal vector of Ω. The parameters $r, \mu, \alpha > 0$ and $m > 1$. We assume the initial data satisfies

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega),\ u_0 \geq 0 \text{ with } u_0 \not\equiv 0 \text{ and}, \\ v_0 \in W^{1,\infty}(\Omega),\ v_0 > 0 \text{ in } \overline{\Omega}. \end{cases} \tag{1.8}$$

Our main result can be stated as follows

Theorem 1.1. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that the parameters $r, \mu, \alpha > 0$ and $m > 1$. For any initial data satisfies* (1.8) *and $m, l > 1$ fulfills*

$$l > m + 1, \tag{1.9}$$

the problem (1.7) *exists at least one global weak solution in the sense of Definition 2.1 below with additional boundedness property*

$$\|u(\cdot,t)\|_{L^p(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \tag{1.10}$$

with constant $C > 0$, for all $p \geq 2$ and $t > 0$.

Remark 1.1. 1. If $l = m + 1$, the sufficiently large $\mu > 0$ can also guarantee the existence of the global weak solution.
2. Given that $l > m + 1 > 2$, our findings suggest that the superquadratic damping effect of the logistic source can effectively regularize the system (1.6), even when $\phi(s)$ degenerates at $s = 0$.

Main idea. Our intention of constructing a global weak solution in the sense of Definition 2.1 below is based on the analysis of functional of energy type

$$\mathcal{F}(t) = \int_\Omega u^p(\cdot,t) + \int_\Omega v^{-2p-2m+3}|\nabla v(\cdot,t)|^{2p+2m-2} \tag{1.11}$$

for a certain regularized system of (1.7). The functional $\mathcal{F}(t)$ actually admits a quasi-energy structure for any suitably large p (Lemma 3.1). Thereafter, by introducing the transformation

$$w = -\ln \frac{v}{\|v_0\|_{L^\infty(\Omega)}},$$

the positive lower bound of v can be obtained through an argument of heat semigroup along with aforementioned L^p-boundedness property (Lemma 3.3). Then the local L^∞-boundedness of solution can readily be obtained through standard Moser iteration (Lemma 3.4). Now, we collect all aforementioned estimates to derive a time regularity of $u^m v^\alpha$ in regularized system to gain the compactness features of the solution of regularized system through Aubin-Lions Lemma (Lemma 3.6) and the weak solution can be constructed through a standard extraction procedure (Lemma 4.1).

2. Preliminaries

We firstly form the concept of weak solution as follows

Definition 2.1. Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. Suppose that $r, \mu > 0$, that $l \geq 1$ and that (1.8) holds for any initial value of (1.7). Then, a pair of functions

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \\ v \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \bigcap L^1_{loc}((0,\infty); W^{1,1}(\Omega)), \end{cases} \tag{2.1}$$

is called a global weak solution for system (1.7), if (u, v) satisfies

$$u^l \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \ u^m v^\alpha \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \tag{2.2}$$

and

$$\begin{aligned} &-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) \\ =& \int_0^\infty \int_\Omega u^m v^\alpha \Delta\varphi + r\int_0^\infty \int_\Omega u\varphi - \mu \int_0^\infty \int_\Omega u^l \varphi \end{aligned} \tag{2.3}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ with $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0$, as well as

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega uv\varphi \tag{2.4}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

To appropriately start the approximation procedures, the regularized system for (1.7) can be constructed by

$$\begin{cases} u_{\varepsilon t} = \Delta(v_\varepsilon^\alpha u_\varepsilon^m) + ru_\varepsilon - \mu u_\varepsilon^l, & x \in \Omega, \ t > 0, \\ v_\varepsilon = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x \in \Omega, \ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u_\varepsilon(\cdot,0) = u_0 + \varepsilon, \quad v_\varepsilon(\cdot,0) = v_0, & x \in \Omega, \end{cases} \tag{2.5}$$

and thereby the local existence and the extensibility for (2.5) can be established by applying the argument in [32]. Therefore, the proof is omitted.

Lemma 2.1. *Let $n \geq 1$, r, $\mu > 0$ and assume (1.8) holds. Then for every $\varepsilon \in (0,1)$, there exists*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \bigcap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ v_\varepsilon \in \bigcap_{q \geq 1} C^0([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)) \bigcap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \end{cases} \tag{2.6}$$

with $T_{\max,\varepsilon} \in (0, \infty]$ such that $(u_\varepsilon, v_\varepsilon)$ solves (2.5) in the classical sense in $\Omega \times (0, T_{\max,\varepsilon})$ with $u_\varepsilon > 0$, $v_\varepsilon > 0$. Moreover,

$$\limsup_{t \to T_{\max,\varepsilon}} \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} = \infty, \quad \text{if } T_{\max,\varepsilon} < \infty \tag{2.7}$$

Then some basic estimates can be readily at hand.

Lemma 2.2. *Let $n \geq 1$, r, $\mu > 0$ and assume (1.8) holds. Then for $t \in (0, T_{\max,\varepsilon})$, the solution $(u_\varepsilon, v_\varepsilon)$ satisfies*

$$\int_\Omega u_\varepsilon(\cdot, t) \leq m := \max\left\{ \int_\Omega u_0, \left(\frac{r}{\mu}\right)^{\frac{1}{l-1}} |\Omega| \right\}, \tag{2.8}$$

and

$$\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} < \|v_0\|_{L^\infty(\Omega)}. \tag{2.9}$$

Proof. An integration along with the Hölder inequality in the first equation of (2.5) can produce the following result.

$$\frac{d}{dt}\int_\Omega u_\varepsilon = r\int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^2 \leq r\int_\Omega u_\varepsilon - \mu|\Omega|^{\frac{1}{l-1}} \left(\int_\Omega u_\varepsilon\right)^l. \tag{2.10}$$

Hence (2.8) is a direct consequence of the above inequality through standard comparison argument. Then we invoke the maximum principal on the second equation of (2.5) and the positivity of the solution to obtain (2.9). □

Finally, we collect some basic inequalities presented in [33], which can be treated as a preparation for the analysis of the functional energy in the latter section.

Lemma 2.3. *Let $n \geq 1$, r, μ, $\alpha > 0$ and assume (1.8) holds. For each $\varepsilon \in (0,1)$ and any choice of $\eta > 0$ as well as $q > 2$, there exist $C = C(q, \eta)$ such that for all $t \in (0, T_{\max,\varepsilon})$*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^q + q\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2 \\ &\leq \frac{q}{2}\int_{\partial\Omega} v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} + q(q-2+\sqrt{n})\int_\Omega u_\varepsilon v_\varepsilon^{-q+2}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|, \end{aligned} \tag{2.11}$$

and

$$\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2} \leq (q+\sqrt{n})^2\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2, \tag{2.12}$$

and

$$\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2 v_\varepsilon|^2 \leq (q+\sqrt{n}+1)^2\int_\Omega v_\varepsilon^{-q+3}|\nabla v_\varepsilon|^{q-2}|D^2 \ln v_\varepsilon|^2, \tag{2.13}$$

as well as

$$\begin{aligned}
&\int_{\partial\Omega} v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}\\
&\le \eta\int_\Omega v_\varepsilon^{-q+1}|\nabla v_\varepsilon|^{q-2}|D^2v_\varepsilon|^2+\eta\int_\Omega v_\varepsilon^{-q-1}|\nabla v_\varepsilon|^{q+2}+C\int_\Omega v_\varepsilon.
\end{aligned}\tag{2.14}$$

Proof. The proof of all the above inequalities can be found in [33] and thereby we omit it directly. □

3. A priori estimates

3.1. Global L^p-boundedness of solution

Our procedures will start with the analysis of a suitably designed functional

$$\mathcal{F}_\varepsilon(t)=\int_\Omega u_\varepsilon^p(\cdot,t)+\int_\Omega v_\varepsilon^{-2p-2m+3}(\cdot,t)|\nabla v_\varepsilon(\cdot,t)|^{2p+2m-2},\tag{3.1}$$

with suitably large $p>2$, which genuinely admits a quasi-energy structure by fully utilizing the outcomes of Lemma 2.2. Our core lemma can be stated as follows

Lemma 3.1. *Let $n\ge 1$, r, μ, $\alpha>0$ and assume that* (1.8)-(1.9) *hold. Then for all $\varepsilon\in(0,1)$ and any $p>2$ satisfying*

$$p>\left(\frac{1}{\alpha}-m\right)_+,$$

one can find a constant $C=C(p)>0$ such that

$$\frac{d}{dt}\mathcal{F}_\varepsilon(t)+\mathcal{F}_\varepsilon(t)+\frac{mp(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla u_\varepsilon|^2+\frac{\mu}{2}\int_\Omega u_\varepsilon^{p+l-1}\le C\tag{3.2}$$

for all $t\in(0,T_{\max})$.

Proof. At first, multiplying the first equation of (2.5) by u_ε^{p-1} with $p>2$ and utilizing Young inequality, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p+m(p-1)\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla u_\varepsilon|^2+\mu\int_\Omega u_\varepsilon^{p+l-1}\\
&=\alpha(p-1)\int_\Omega u_\varepsilon^{m+p-2}v_\varepsilon^{\alpha-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon+r\int_\Omega u_\varepsilon^p\\
&\le\frac{m(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla u_\varepsilon|^2+\frac{\alpha^2(p-1)}{2m}\int_\Omega u_\varepsilon^{m+p-1}v_\varepsilon^{\alpha-2}|\nabla v_\varepsilon|^2+r\int_\Omega u_\varepsilon^p
\end{aligned}\tag{3.3}$$

Then, we apply Young inequality again to find $c_1>0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p+\frac{m(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla u_\varepsilon|^2+\frac{\mu}{2}\int_\Omega u_\varepsilon^{p+1}\\
&\le\frac{\alpha^2(p-1)}{2m}\int_\Omega u_\varepsilon^{p+m-1}v_\varepsilon^{\alpha-2}|\nabla v_\varepsilon|^2+c_1.
\end{aligned}\tag{3.4}$$

Now recalling the outcomes of Lemma 2.2, we set $q = 2m + 2p - 2 > 0$ to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega v_\varepsilon^{-2p-2m+3}|\nabla v_\varepsilon|^{2p+2m-2} \\
&+(2p+2m-2)\int_\Omega v_\varepsilon^{-2p-2m+5}|\nabla v_\varepsilon|^{2p+2m-4}|D^2\ln v_\varepsilon|^2 \\
\le\ & (p-m+1)\int_{\partial\Omega} v_\varepsilon^{-2m-2p+3}|\nabla v_\varepsilon|^{2p+2m-4}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \\
&+(2p+2m-2)(2p+2m-4+\sqrt{n})\int_\Omega u_\varepsilon v_\varepsilon^{-2m-2p+4}|\nabla v_\varepsilon|^{2p+2m-4}|D^2 v_\varepsilon|.
\end{aligned}
\tag{3.5}
$$

At this position, we set two constant

$$
c_2 := \frac{p+m-1}{(2p+2m-1+\sqrt{n})^2}
$$

and

$$
c_3 := \frac{p+m-1}{(2p+2m-2+\sqrt{n})^2},
$$

then (2.14) along with (2.12) as well as (2.13) readily yields

$$
\begin{aligned}
&(p+m-1)\int_{\partial\Omega} v_\varepsilon^{-2p-2m+3}|\nabla v_\varepsilon|^{2p+2m-4}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \\
\le\ & \frac{c_2}{2}\int_\Omega v_\varepsilon^{-2p-2m+3}|\nabla v_\varepsilon|^{2p+2m-4}|D^2 v_\varepsilon|^2 + \frac{c_3}{2}\int_\Omega v_\varepsilon^{-2p-2m+1}|\nabla v_\varepsilon|^{2p+2m} \\
&+C\int_\Omega v_\varepsilon, \quad \text{for all } t\in(0,T_{\max,\varepsilon}) \text{ and } \varepsilon\in(0,1)
\end{aligned}
\tag{3.6}
$$

with some positive content $C = C(c_2, c_3)$. Noting the fact that

$$
\begin{aligned}
&(2p+2m-2)\int_\Omega v_\varepsilon^{-2p-2m+5}|\nabla v_\varepsilon|^{2p+2m-4}|D^2\ln v_\varepsilon|^2 \\
\ge\ & c_2\int_\Omega v_\varepsilon^{-2m-2p+3}|\nabla v_\varepsilon|^{2p+2m-4}|D^2 v_\varepsilon|^2 + c_3\int_\Omega v_\varepsilon^{-2m-2p+1}|\nabla v_\varepsilon|^{2p+2m}.
\end{aligned}
\tag{3.7}
$$

Hence, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega v_\varepsilon^{-2p-2m+3}|\nabla v_\varepsilon|^{2p+2m-2} \\
&+\frac{c_2}{2}\int_\Omega v_\varepsilon^{-2m-2p+3}|\nabla v_\varepsilon|^{2p+2m-4}|D^2 v_\varepsilon|^2 + \frac{c_3}{2}\int_\Omega v_\varepsilon^{-2m-2p+1}|\nabla v_\varepsilon|^{2p+2m} \\
\le\ & (2p+2m-2)(2p+2m-4+\sqrt{n})\int_\Omega u_\varepsilon v_\varepsilon^{-2m-2p+4}|\nabla v_\varepsilon|^{2p+2m-4}|D^2 v_\varepsilon|.
\end{aligned}
\tag{3.8}
$$

Now, combining (3.3) with (3.8), we have

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}_\varepsilon(t)+\mathcal{F}_\varepsilon(t)+\frac{mp(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^{\alpha}|\nabla v_\varepsilon|^2\\
&+\frac{c_2}{2}\int_\Omega v_\varepsilon^{-2m-2p+3}|\nabla v_\varepsilon|^{2p+2m-4}|D^2v_\varepsilon|^2\\
&+\frac{c_3}{2}\int_\Omega v_\varepsilon^{-2m-2p+1}|\nabla v_\varepsilon|^{2p+2m}+\mu\int_\Omega u_\varepsilon^{p+l-1}\\
\le\ & c_4\int_\Omega u_\varepsilon v_\varepsilon^{-2m-2p+4}|\nabla v_\varepsilon|^{2p+2m-4}|D^2v_\varepsilon|+c_5\int_\Omega u_\varepsilon^{p+m-1}v_\varepsilon^{\alpha-2}|\nabla v_\varepsilon|^2\\
&+(r+1)\int_\Omega u_\varepsilon^p+\int_\Omega v_\varepsilon^{-2p-2m+3}|\nabla v_\varepsilon|^{2p+2m-2}+C\int_\Omega v_\varepsilon
\end{aligned}
\tag{3.9}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$, where

$$c_4=(2p+2m+2)(2p+2m-4+\sqrt{n})$$

and

$$c_5=\frac{\alpha^2p(p-1)}{2m}.$$

Then, by applying Young inequality, we can obtain

$$
\begin{aligned}
&c_5\int_\Omega u_\varepsilon^{p+m-1}v_\varepsilon^{\alpha-2}|\nabla v_\varepsilon|^2\\
\le\ &\frac{c_3}{6}\int_\Omega v_\varepsilon^{-2p-2m+1}|\nabla v_\varepsilon|^{2p+2m}+\left(\frac{6}{c_3}\right)^{\frac{1}{p+m-1}}c_5^{\frac{p+m}{p+m-1}}\int_\Omega u_\varepsilon^{p+m}v_\varepsilon^{\frac{p+m}{p+m-1}(\alpha-\frac{1}{p+m})}
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
&c_4\int_\Omega u_\varepsilon v_\varepsilon^{-2m-2p+4}|\nabla v_\varepsilon|^{2m+2p-4}|D^2v_\varepsilon|\\
\le\ &\frac{c_2}{2}\int_\Omega v_\varepsilon^{-2m-2p+3}|\nabla v_\varepsilon|^{2m+2p-4}|D^2v_\varepsilon|^2+\frac{c_4^2}{2c_2}\int_\Omega u_\varepsilon^2v_\varepsilon^{-2m-2p+5}|\nabla v_\varepsilon|^{2p+2m-4}\\
\le\ &\frac{c_2}{2}\int_\Omega v_\varepsilon^{-2p-2m+3}|\nabla v_\varepsilon|^{2p+2m-4}|D^2v_\varepsilon|^2+\frac{c_3}{6}\int_\Omega v_\varepsilon^{2m-2p-3}|\nabla v_\varepsilon|^{2p-2m+4}\\
&+\left(\frac{6}{c_3}\right)^{\frac{m+p-2}{2}}\left(\frac{c_4^2}{2c_2}\right)^{\frac{p+m}{2}}\int_\Omega u_\varepsilon^{p+m}v_\varepsilon
\end{aligned}
\tag{3.11}
$$

as well as

$$
\begin{aligned}
&\int_\Omega v_\varepsilon^{-2p-2m+3}|\nabla v_\varepsilon|^{2p+2m-2}\\
\le\ &\frac{c_3}{6}\int_\Omega v_\varepsilon^{-2p-2m+1}|\nabla v_\varepsilon|^{2p+2m}+\left(\frac{6}{c_2}\right)^{p+m-1}\int_\Omega v_\varepsilon
\end{aligned}
\tag{3.12}
$$

for all $\varepsilon\in(0,1)$. Hence, one can immediately find constant $c_6,\ c_7>0$ as well as

$c_8 > 0$ such that

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}_\varepsilon(t) + \mathcal{F}_\varepsilon(t) + \frac{mp(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla u_\varepsilon|^2 + \mu\int_\Omega u_\varepsilon^{p+l-1}\\&\le c_6\int_\Omega u_\varepsilon^{p+m}v_\varepsilon^{\frac{p+m}{p+m-1}(\alpha-\frac{1}{p+m})} + c_7\int_\Omega u_\varepsilon^{p+m} + c_8\end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Now for $\alpha > 0$, by the choice of $p > 0$, we conclude

$$p > \left(\frac{1}{\alpha} - m\right)_+ > \frac{1}{\alpha} - m$$

which yields

$$\alpha > \frac{1}{p+m} \tag{3.14}$$

for such choice of p. Thus (2.9) together with (3.14) can immediately yield

$$c_6\int_\Omega u_\varepsilon^{p+m}v_\varepsilon^{\frac{p+m}{p+m-1}(\alpha-\frac{1}{p+m})} \le c_6\|v_0\|_{L^\infty(\Omega)}^{\frac{p+m}{p+m-1}(\alpha-\frac{1}{p+m})}\int_\Omega u_\varepsilon^{p+m}. \tag{3.15}$$

Then, this fact shows that

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}_\varepsilon(t) + \mathcal{F}_\varepsilon(t) + \frac{mp(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla v_\varepsilon|^2 + \mu\int_\Omega u_\varepsilon^{p+l-1}\\&\le \left(c_6\|v_0\|^{\frac{p+m}{p+m-1}(\alpha-\frac{1}{p+m})} + c_7\right)\int_\Omega u_\varepsilon^{p+m} + c_8.\end{aligned} \tag{3.16}$$

Because

$$p + l - 1 - (p+m) = l - (m+1) > 0.$$

Then Young inequality entails

$$\frac{d}{dt}\mathcal{F}_\varepsilon(t) + \mathcal{F}_\varepsilon(t) + \frac{mp(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla u_\varepsilon|^2 + \frac{\mu}{2}\int_\Omega u_\varepsilon^{p+l-1} \le c_9 \tag{3.17}$$

with constant $c_9 > 0$ for every $\varepsilon \in (0,1)$, which directly yields (3.2). □

From the outcome of Lemma 3.1, a standard comparison argument can be applied to obtain the boundedness property as follows:

Lemma 3.2. *Let $n \ge 1$, r, μ, $\alpha > 0$ and assume* (1.8)-(1.9) *hold. Then for arbitrary $p > 2$, one can find $C(p) > 0$ such that for all $t \in (0, T_{\max,\varepsilon})$*

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p). \tag{3.18}$$

Proof. For every sufficiently large p satisfying

$$p > \left(\frac{1}{\alpha} - m\right)_+,$$

by dropping the positive term of left-hand side in (3.2), we can obtain

$$\frac{d}{dt}\mathcal{F}_\varepsilon(t) + \mathcal{F}_\varepsilon(t) \le C, \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \tag{3.19}$$

which immediately yields

$$\int_\Omega u_\varepsilon^p \le \max\{\mathcal{F}_\varepsilon(0), C\} = \max\left\{\int_\Omega u_0^p + \int_\Omega v_0^{-2m-2p+3}|\nabla v_0|^{2m+2p-2}, C\right\} \quad (3.20)$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. But for p satisfying

$$1 < p \le \left(\frac{1}{\alpha} - m\right)_+,$$

Hölder inequality, along with the fact that Ω is bounded, can readily yield

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le c_1\|u_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)}, \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

with constant $c_1 > 0$ and $p_0 > \left(\frac{1}{\alpha} - m\right)_+$, which finish the proof. $\square$

3.2. Local L^∞-boundedness of solution

Due to the presence of degeneracy in (2.5), the Moser-type iteration cannot be applied to convert the L^p-boundedness property presented in Lemma 3.2 into a L^∞-bound of solution. Hence, by following the variable of change used in [30], we set

$$w_\varepsilon = -\ln\frac{v_\varepsilon}{\|v_0\|_{L^\infty(\Omega)}}$$

to avoid the difficulty arising from the degeneracy. The result can be stated as follows.

Lemma 3.3. *Let $n \ge 1$, r, μ, $\alpha > 0$ and assume that condition* (1.8)-(1.9) *hold. Then, one can find constant $C = C(T) > 0$ with $T \in (0, T_{\max,\varepsilon})$ such that*

$$v_\varepsilon(x,t) \ge C(T), \quad \textit{for all } x \in \Omega,\ t \in (0,T) \textit{ and } \varepsilon \in (0,1). \quad (3.21)$$

Proof. By setting

$$w_\varepsilon = -\ln\frac{v_\varepsilon}{\|v_0\|_{L^\infty(\Omega)}},$$

we rewrite the second equation of (2.5) in form

$$\begin{cases} w_{\varepsilon t} = \Delta w_\varepsilon - |\nabla w_\varepsilon|^2 + u_\varepsilon, & x \in \Omega,\ t \in (0,T), \\ \frac{\partial w_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0,T), \\ w_\varepsilon(x,0) = -\ln\frac{v_0}{\|v_0\|_{L^\infty(\Omega)}}, & x \in \Omega. \end{cases} \quad (3.22)$$

with some $T \in (0, T_{\max,\varepsilon})$. Based on a semigroup argument (eg. [31]), we fix $p > n$ to obtain

$$\begin{aligned} \|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le & \|e^{t\Delta}w_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \\ \le & \|w_0\|_{L^\infty(\Omega)} + c(p)\int_0^t (1 + (t-s)^{-\frac{n}{2p}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s) - \overline{u}_\varepsilon(s)\|_{L^p(\Omega)}ds \\ & + \int_0^t \overline{u}_\varepsilon(s)ds, \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \end{aligned}$$

with $c(p) > 0$, where λ_1 is the first nonzero eigenvalue of $-\Delta$ under homogeneous Neumann boundary condition. Now, recalling the boundedness property of solution in $L^p(\Omega)$ with arbitrary $p > 2$, one can immediately find $c_1,\ c_2 > 0$ such that

$$\|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 + c_2 \int_0^t (1+\sigma^{-\frac{n}{2p}})e^{-\lambda_1\sigma}d\sigma + c_1 T \tag{3.23}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$. Then the fixed $p > n$ can entail

$$\int_0^\infty (1+\sigma^{-\frac{n}{2p}})e^{-\lambda_1\sigma}d\sigma < \infty$$

and thereby we can estimate

$$\|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} < c_3(T), \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1) \tag{3.24}$$

with constant $c_3(T) > 0$. Hence (3.21) can be readily obtained through the above estimation as well as the definition of w_ε. □

Then the following L^∞-bound of solution will be immediately at hand through a Moser-type iteration.

Lemma 3.4. *Let $n \geq 1,\ r,\ \mu,\ \alpha > 0$ and assume (1.8)-(1.9) hold. Then one can find a constant $C = C(T)$ with some $T \in (0, T_{\max,\varepsilon})$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C(T), \quad \textit{for all } \varepsilon \in (0,1) \textit{ and } t \in (0,T). \tag{3.25}$$

Proof. We first invoke the heat semigroup regularities to estimate

$$\begin{aligned}
\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| \nabla e^{t\Delta}v_0 - \int_0^t \nabla e^{(t-s)\Delta}(u_\varepsilon(\cdot,s)v_\varepsilon(\cdot,s))ds \right\|_{L^\infty(\Omega)} \\
&\leq c_1\|\nabla v_0\|_{L^\infty(\Omega)} + c_2 \int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}})e^{-\lambda_1(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \\
&\leq c_3, \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),
\end{aligned} \tag{3.26}$$

where λ_1 is the first non-zero eigenvalue of $-\Delta$ under homogeneous Neumann boundary condition. Then, by multiplying pu_ε^{p-1} on the first equation of (2.5) and integrating over Ω, one can employ integration by parts and Young inequality to derive

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{mp(p-1)}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^\alpha|\nabla u_\varepsilon|^2 + \mu\int_\Omega u_\varepsilon^{p+1} \\
\leq\ & \frac{\alpha^2 p(p-1)}{2m}\int_\Omega u_\varepsilon^{p+m-1}v_\varepsilon^{\alpha-2}|\nabla v_\varepsilon|^2 + r\int_\Omega u_\varepsilon^p.
\end{aligned} \tag{3.27}$$

Now, due to (3.21), (3.27) can convert into

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + c_4(T)\int_\Omega u_\varepsilon^{m+p-3}|\nabla u_\varepsilon|^2 + \mu\int_\Omega u_\varepsilon^{p+1} \\
\leq\ & c_5(T)\int_\Omega u_\varepsilon^{p+m-1}|\nabla v_\varepsilon|^2 + r\int_\Omega u_\varepsilon^p, \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

with constant $c_4(T),\ c_5(T) > 0$. Therefore, together with (3.26), we invoke the well-established Moser-type iteration (see e.g. [24]) to derive (3.25), thereby completing the proof. □

We can now assert that the maximal interval $T_{\max,\varepsilon}$ extends to infinity.

Corollary 3.1. *Let $n \geq 1$, $r,\ \mu,\ \alpha > 0$ and assume* (1.8) *hold. Then we have*

$$T_{\max,\varepsilon} = \infty. \tag{3.28}$$

Proof. If $T_{\max,\varepsilon} < \infty$, from Lemma 3.4, the boundedness property of u_ε contradicts (2.7), which finishes the proof. □

3.3. Further ε-dependent regularities

Now we should deduce the time-derivative regularity of $u_\varepsilon^p v_\varepsilon^\alpha$ with $p > 0$, and thereafter prepare an argument based on the application of Aubin-Lions Lemma, which is the objective of the next two lemmas. We first concentrate on the regularity of spatial gradient involving u_ε.

Lemma 3.5. *Let $n \geq 1$ and assume* (1.8)-(1.9) *hold with $r,\ \mu,\ \alpha > 0$. Then for all $p > 0$ and any $T > 0$, there exist $C(T) > 0$ such that*

$$\int_0^T \int_\Omega u_\varepsilon^{m+p-3} v_\varepsilon^\alpha |\nabla u_\varepsilon|^2 \leq C(p,T), \quad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{3.29}$$

Proof. Based on the assumption on the initial data, we combine the outcomes of Lemma 3.3 and Lemma 3.4 to deduce

$$v_\varepsilon(x,t) \geq c_1(T), \quad u(x,t) \leq c_2(T), \quad \text{in } \Omega \times (0,T) \text{ for every } \varepsilon \in (0,1), \tag{3.30}$$

and

$$\int_\Omega |\nabla v_\varepsilon(x,t)|^2 \leq c_3, \quad \text{in } \Omega \times (0,T) \text{ for every } \varepsilon \in (0,1) \tag{3.31}$$

with any given $T > 0$ and constants $c_1,\ c_2,\ c_3 > 0$, which relying on T. Then (3.30) can warrant that it will be sufficient to prove (3.29) is valid when $p \in (0,1)$. Hence by multiplying $-u_\varepsilon^p$ on the first equation of (2.5), we have

$$\begin{aligned} &-\frac{1}{p}\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{m(1-p)}{2}\int_\Omega u_\varepsilon^{m+p-3} v_\varepsilon^\alpha |\nabla u_\varepsilon|^2 - \mu \int_\Omega u_\varepsilon^{p+l-1} \\ &\leq \frac{\alpha^2(1-p)}{2m}\int_\Omega u_\varepsilon^{p+m-1} v_\varepsilon^{\alpha-2}|\nabla v_\varepsilon|^2 - r\int_\Omega u_\varepsilon^p. \end{aligned} \tag{3.32}$$

Then an integration on the above inequality will yield

$$\begin{aligned} &\frac{m(1-p)}{2}\int_0^T \int_\Omega u_\varepsilon^{m+p-3} v_\varepsilon^\alpha |\nabla u_\varepsilon|^2 \\ &\leq \mu \int_0^T \int_\Omega (u_\varepsilon + \varepsilon)^{p+l-1} + \frac{1}{p}\int_\Omega (u_\varepsilon+\varepsilon)^p \\ &\quad + \frac{\alpha^2(1-p)}{2m}\int_0^T\int_\Omega (u_\varepsilon+\varepsilon)^{p+m-1} v_\varepsilon^{\alpha-2}|\nabla v_\varepsilon|^2. \end{aligned}$$

Now, along with (3.30) and (3.31), above estimates can readily imply (3.29). □

By applying the results in Lemma 3.5, we immediately obtain the following regularity feature involving the time derivative.

Lemma 3.6. *Let $p>0$ and suppose that (1.8)-(1.9) hold with $\alpha,\ \mu,\ r>0$. Then for all $T>0$ and $k>\frac{n}{2}$, there exist $C(T)>0$ such that*

$$\int_0^T \|\partial_t\left(u_\varepsilon^p(\cdot,t)v_\varepsilon^\alpha(\cdot,t)\right)\|_{(W^{k,2}(\Omega))^\star}dt \le C(T) \tag{3.33}$$

and

$$\int_0^T \|\partial_t v_\varepsilon(\cdot,t)\|^2_{(W^{k,2}(\Omega))^\star}dt \le C(T) \tag{3.34}$$

for all $t\in(0,T)$ and $\varepsilon\in(0,1)$.

Proof. By fixing $k>\frac{n}{2}$, the continuity of embedding $W^{k,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ can allow us to find $c_1>0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le c_1\|\varphi\|_{W^{k,2}(\Omega)}$$

for all $\varphi\in C^\infty(\overline{\Omega})$. Thus, for $\varphi\in C^\infty(\overline{\Omega})$ with $\|\varphi\|_{W^{k,2}(\Omega)}\le 1$, a testing procedure implies that

$$\begin{aligned}
&\left|\int_\Omega \partial_t(u_\varepsilon^p v_\varepsilon^\alpha)\cdot\varphi\right| \\
\le\ & c_2\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^{2\alpha}|\nabla u_\varepsilon|^2 + c_2\int_\Omega u_\varepsilon^{m+p-2}v_\varepsilon^{2\alpha-1}|\nabla u_\varepsilon||\nabla v_\varepsilon| \\
&+c_2\int_\Omega u_\varepsilon^{m+p-1}v_\varepsilon^{2\alpha-2}|\nabla v_\varepsilon|^2 + c_2\int_\Omega u_\varepsilon^{m+p-2}v_\varepsilon^{2\alpha}|\nabla u_\varepsilon| \\
&+c_2\int_\Omega u_\varepsilon^{m+p-1}v_\varepsilon^{2\alpha-1}|\nabla v_\varepsilon|, \quad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1),
\end{aligned} \tag{3.35}$$

with a constant $c_2>0$. Now for given $T>0$, let us once more recall Lemma 3.3, Lemma 3.4 and (3.26) to find $c_3(T),\ c_4(T)$ and c_5 such that

$$v_\varepsilon(x,t)\ge c_3(T),\quad u_\varepsilon(x,t)\le c_4(T),\quad \text{for every } \varepsilon\in(0,1) \text{ in } \Omega\times(0,T) \tag{3.36}$$

and

$$\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 \le c_5,\quad \text{for every } \varepsilon\in(0,1) \text{ in } \Omega\times(0,T). \tag{3.37}$$

Hence, by utilizing Young inequality, one can see that

$$\begin{aligned}
\int_\Omega u_\varepsilon^{m+p-2}v_\varepsilon^{2\alpha-1}|\nabla u_\varepsilon||\nabla v_\varepsilon| &\le \frac{1}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^{\alpha}|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega u_\varepsilon^{m+p-1}v_\varepsilon^{3\alpha-2}|\nabla v_\varepsilon|^2 \\
&\le \frac{1}{2}\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^{\alpha}|\nabla u_\varepsilon|^2 + \|v_0\|^{3\alpha}_{L^\infty(\Omega)}c_3^{-2}(T)c_4^{m+p-1}(T)|\Omega|
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
\int_\Omega u_\varepsilon^{m+p-2}v_\varepsilon^{2\alpha}|\nabla u_\varepsilon| &\le \int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^{\alpha}|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^{m+p-1}v_\varepsilon^{3\alpha} \\
&\le \int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^{\alpha}|\nabla u_\varepsilon|^2 + c_4^{m+p-1}(T)\|v_0\|^{3\alpha}_{L^\infty(\Omega)}|\Omega|.
\end{aligned} \tag{3.39}$$

Then, we insert (3.38) and (3.39) into (3.35) to obtain

$$\|\partial_t(u_\varepsilon^p v_\varepsilon^\alpha)\|_{(W^{k,2}(\Omega))^\star} \le c_6(T)\int_\Omega u_\varepsilon^{m+p-3}v_\varepsilon^{\alpha}|\nabla u_\varepsilon|^2 + c_6(T) \tag{3.40}$$

for all $t \in (0,T)$ and $\varepsilon \in (0,1)$ with constant $c_6(T) > 0$, which readily yield (3.33) through an integration upon interval $[0,T]$.

As for (3.34), we once more choose $\varphi \in C^\infty(\overline{\Omega})$ with $\|\varphi\|_{W^{k,2}(\Omega)} \le 1$ and invoke the continuity of embedding $W^{k,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ to see

$$\left|\int_\Omega v_{\varepsilon t} \cdot \varphi\right| \le \int_\Omega |\nabla v_\varepsilon| + c_1 \int_\Omega u_\varepsilon v_\varepsilon \le c_7(T) \tag{3.41}$$

with constant $c_7(T) > 0$, which readily yield (3.34) through integration. □

Now, for turning the compactness feature of $u_\varepsilon^p v_\varepsilon^\alpha$ into component u_ε, the obstacle therein seems to be the information on the positivity of weight function v_ε. Hence, the following observation, which guarantee the positivity of v_ε, is necessary when accomplishing subsequent extraction procedure.

Lemma 3.7. *Let $n \ge 1$ and assume* (1.8)-(1.9) *hold with $r,\ \mu,\ \alpha > 0$. Then, for given $T > 0$, there exist $C(T) > 0$ such that*

$$\int_\Omega \ln \frac{\|v_0\|_{L^\infty(\Omega)}}{v_\varepsilon(\cdot,t)} \le C(T), \quad \textit{for all } t \in (0,T) \textit{ and each } \varepsilon \in (0,1). \tag{3.42}$$

Proof. Based on an argument to the second equation of (2.5), we invoke Young inequality and (2.8) to obtain

$$\frac{d}{dt}\int_\Omega \ln \frac{\|v_0\|_{L^\infty(\Omega)}}{v_\varepsilon} = -\frac{d}{dt}\int_\Omega \ln v_\varepsilon = -\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \int_\Omega u_\varepsilon \le m. \tag{3.43}$$

Then an integration in the time interval $[0,T]$ can yield (3.42) along with the positivity of v_0 in $\overline{\Omega}$. □

4. Passing to the limit

In this section, we combine all of the above estimates, especially the compactness property thus implied, to accomplish the main step towards the existence of the global weak solution by passing to the limit $\varepsilon \searrow 0$.

Lemma 4.1. *Let $n \ge 1$ and assume* (1.8) *hold with $r,\ \mu,\ \alpha > 0$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ as well as negative functions*

$$\begin{cases} u \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)), \\ v \in L^\infty((0,\infty); W^{1,\infty}(\Omega)), \end{cases} \tag{4.1}$$

such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that $\varepsilon = \varepsilon_j \searrow 0$, we have

$$u_\varepsilon \to u, \quad \textit{in } \textstyle\bigcap_{p\ge1} L^p_{loc}(\overline{\Omega} \times [0,\infty)) \textit{ and a.e. in } \Omega \times (0,\infty), \tag{4.2}$$

$$v_\varepsilon \to v, \quad \textit{in } \textstyle\bigcap_{p\ge1} L^p_{loc}(\overline{\Omega} \times [0,\infty)) \textit{ and a.e. in } \Omega \times (0,\infty), \tag{4.3}$$

$$\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v, \quad \textit{in } L^\infty(\overline{\Omega} \times [0,\infty)). \tag{4.4}$$

Moreover, $v > 0$ in $\Omega \times (0,\infty)$ and (u,v) forms a global weak solution of (1.7) *in the sense of Definition 2.1.*

Proof. By fixing $p > \frac{m-1}{2}$ and $k \in \mathbb{N}$ such that $k > \frac{n}{2}$, we infer from (3.33) that for any given $T > 0$,

$$(\partial_t(u_\varepsilon^p v_\varepsilon^\alpha))_{\varepsilon\in(0,1)} \text{ is bounded in } L^1((0,T);(W^{k,2}(\Omega))^\star).$$

Now according to (2.9), (3.26), (3.29) and Lemma 3.3 as well as Lemma 3.4, one can find a constant $C(T) > 0$ such that

$$\begin{aligned}\int_0^t \|\nabla(u^p(\cdot,s)v_\varepsilon^\alpha(\cdot,s))\|_{L^2(\Omega)}^2 ds &\leq 2p\int_\Omega u_\varepsilon^{2p-2}v_\varepsilon^{2\alpha}|\nabla u_\varepsilon|^2 + 2\alpha^2\int_\Omega u_\varepsilon^{2p}v_\varepsilon^{2\alpha-2}|\nabla v_\varepsilon|^2 \\ &\leq C(T).\end{aligned} \tag{4.5}$$

with such $p > \frac{m-1}{2}$ and $T > 0$. Hence, this fact actually implies that

$$(u_\varepsilon^p v_\varepsilon^\alpha)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T);(W^{1,2}(\Omega))).$$

Thereafter, from (2.9), (3.34) and (3.26), it can readily be verified that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^\infty((0,\infty);W^{1,\infty}(\Omega))$$

as well as

$$(v_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,\infty);(W^{k,2}(\Omega))^\star).$$

Then, two applications of Aubin-Lions Lemma can allow us to pick a subsequence $\varepsilon = \varepsilon_j \searrow 0$ with two nonnegative functions $z \in L^1_{loc}(\overline{\Omega}\times[0,\infty))$ and $v \in L^\infty((0,\infty);W^{1,\infty}(\Omega))$ such that (4.3), (4.4) hold as well as

$$z_\varepsilon = u_\varepsilon^p v_\varepsilon \to z, \quad \text{a.e. in } \Omega\times(0,\infty) \text{ and in } L^1_{loc}(\overline{\Omega}\times[0,\infty)) \tag{4.6}$$

when $\varepsilon = \varepsilon_j \searrow 0$. Since $v \leq \|v_0\|_{L^\infty(\Omega)}$ due to (2.9) and (4.3), Lemma 3.7 along with Fatou's Lemma guarantees that v is positive a.e. in $\Omega\times(0,\infty)$ and therefore, we define $u = (\frac{z}{v^\alpha})^{\frac{1}{p}}$, which is nonnegative and $u_\varepsilon = (\frac{z_\varepsilon}{v_\varepsilon^\alpha})^{\frac{1}{p}} \to u$ a.e. in $\Omega\times(0,\infty)$. Since u_ε is bounded in $L^\infty_{loc}(\overline{\Omega}\times[0,\infty))$, u must belong to this space and satisfy (4.2) as a consequence of Vitali convergence theorem.

Now, the verification of the claimed weak solution of (u,v) is quite straightforward. In fact, the integral identities (2.3)-(2.4) can be derived by standard argument from the corresponding weak formations in (2.5) after letting $\varepsilon = \varepsilon_j \searrow 0$ and using (4.2)-(4.5) as well. As for the rest of the integrability features required in this lemma, it can be immediately obtained from (4.3). □

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