Solvability of Sequential Fractional Differential Equations with Functional Boundary Value Conditions at Resonance *†

Bingzhi Sun[1],Shuqin Zhang[2],Shanshan Li[3]

[1] School of Mathematical Sciences, Luoyang Normal University, China

[2]Department of Mathematics, China University of Mining and Technology, Beijing

[3]School of Mathematics and Information Sciences, Yantai University, 264005 Yantai, China

Abstract: The purpose of this paper is to develop the existence theory for a functional boundary problem of sequential fractional differential equations involving Caputo fractional derivatives of order $\alpha + 1$ with $n - 1 < \alpha \leq n$. The main goal of the current contribution is to use Mawhin's coincidence degree theory and a few novel operators to derive sufficient criteria for the existence of solutions to the resonance problems at hand. An example that is relevant is given to support the findings.

MSC code Primary 34A08; Secondary 34B15

Keywords: sequential derivative; functional boundary condition; fractional differential equations; resonance.

1 Introduction

The subject of perfect fractional derivatives is a long-standing topic that is still being widely researched. Fractional calculus has grown in popularity and relevance as a result of its widespread use in engineering sciences, economics, physics, quantum mechanics, and biology. The common applications in several domains have inadvertently contributed to the theoretical study of fractional derivatives. As a result, we will look at the sequential fractional derivatives listed sequentially below.

Miller and Ross introduced the concept of this derivative in [1], and it is widely recognized as a generalized expression. Since sequential fractional derivatives and non-sequential fractional

*

†* Correspondence: Bingzhi Sun[1]. E-mail addresses: bingzhi93@qq.com

derivatives are closely related ([2, 3]), researchers have focused on finding solutions to sequential fractional differential equations with various initial and boundary value conditions [4–12]. We will discuss some of their works here.

In [6], the authors considered the existence of minimal and maximal solutions and uniqueness of solution of the initial value problem for fractional differential equation involving Riemann-Liouville sequential fractional derivative, using the method of upper and lower solutions and its associated monotone iterative method.

$$\begin{cases} \big(\mathcal{D}^{2\alpha}_{0+}y\big)(x) = f\big(x, y(x), \mathcal{D}^{\alpha}_{0+}y(x)\big), x \in (0, T], \\ x^{1-\alpha}y(x)|_{x=0} = y_0, x^{1-\alpha}(\mathcal{D}^{\alpha}_{0+}y)(x)|_{x=0} = y_1, \end{cases}$$

where $0 < T < +\infty$ and $f \in C([0, T] \times \mathbb{R} \times \mathbb{R})$.

Zhang and Su[10] obtained the existence and uniqueness results for a periodic boundary value problem of nonlinear sequential fractional differential equations by the method of upper and lower solutions, together with the monotone iterative technique.

$$\begin{cases} \mathcal{D}^{2\alpha}x(t) = f\big(t, x(t), \mathcal{D}^{\alpha}x(t)\big), t \in (0, 1], 1 < \alpha \leq 1, \\ x(0) = x(1), \mathcal{D}^{\alpha}x(0) = \mathcal{D}^{\alpha}x(1), \end{cases}$$

where $f(t, x, y)$ is a continuous E-value function on $[0, 1] \times E \times E$, $\mathcal{D}^{\alpha}$ is the conformable fractional derivative of order α,and $\mathcal{D}^{2\alpha} = \mathcal{D}^{\alpha}\mathcal{D}^{\alpha}$ is the conformable sequential fractional derivative.

As for sequential fractional differential equations associated with boundary value conditions, we refer the reader to a series of papers[13–29]. For example, in [14], the authors are concerned with the existence and uniqueness of solutions for a coupled system of Caputo-type sequential fractional differential equations supplemented with nonlocal Riemann-Liouville integral boundary conditions via Leray-Schauder's alternative and Banach's contraction principle.

$$\begin{cases} ({}^{C}D^{q} + k{}^{C}D^{q-1})x(t) = f\big(t, x(t), y(t)\big), t \in [0, 1], 2 < q \leq 3, k > 0, \\ ({}^{C}D^{p} + k{}^{C}D^{p-1})y(t) = g\big(t, x(t), y(t)\big), t \in [0, 1], 2 < p \leq 3, k > 0, \end{cases}$$

supplemented with coupled nonlocal integral boundary conditions

$$\begin{cases} x(0) = 0, x'(0) = 0, x(\zeta) = a\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}x(s)ds, \beta > 0, 0 < \eta < \zeta < 1, \\ y(0) = 0, y'(0) = 0, y(z) = b\int_0^{\theta} \frac{(\theta-s)^{\gamma-1}}{\Gamma(\gamma)}y(s)ds, \gamma > 0, 0 < \theta < z < 1, \end{cases}$$

where ${}^{C}D^{q}$, ${}^{C}D^{p}$ denote the Caputo fractional derivatives of order q and p respectively, $f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and a, b are real constants.

Ahmad and Ntouyas[18] studied a nonlinear three-point boundary value problem of sequential fractional differential equations of order α with $1 < \alpha \leq 2$.

$$\begin{cases} {}^CD^{\alpha}(D+\lambda)x(t) = f\big(t, x(t)\big), 0 < t < 1, \\ x(0) = 0, x'(0) = 0, x(1) = \beta x(\eta), 0 < \eta < 1, \end{cases}$$

where ${}^CD^{\alpha}$ is the Caputo fractional derivative, D is the ordinary derivative, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, λ is a positive real number and β is a real number such that $\beta \neq \frac{\lambda + e^{-\lambda} - 1}{\lambda\eta + e^{-\lambda\eta} - 1}$.

Salem and Almaghamsi[29] showed the existence of a solution for the boundary value problem by using the coincidence degree theory due to Mawhin[30].

$$\begin{cases} {}^CD^{\alpha}(D+\lambda)x(t) = g\big(t, x(t), x'(t), {}^CD^{\alpha-1}x(t)\big) + e(t), t \in [0,1], \\ x(0) = 0, x'(0) = 0, x(1) = \beta x(\eta), 0 < \eta < 1, \end{cases}$$

where ${}^CD^{\alpha}$ represents the Caputo derivative of fractional order $1 < \alpha \leq 2$, while D denotes the first derivative, $g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a function verifying with the Carath$\acute{e}$odory conditions, $e(t) \in L^1[0,1], \lambda \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ such that $\beta = \frac{\lambda + e^{-\lambda} - 1}{\lambda\eta + e^{-\lambda\eta} - 1}$.

Recently, much interest[31–35] has developed related to the existence of solutions for fractional differential equations when subjected to functional boundary conditions. The valuable point is that the boundary value conditions can generalize recent work on multi-point and integral boundary value conditions. [33] discussed the existence of solutions for resonant functional problems involving both left Riemann-Liouville and right Caputo-type fractional derivatives, relying on the coincidence degree theory due to Mawhin.

$$\begin{cases} -{}^CD^{\alpha}_{1^-}D^{\beta}_{0^+}u(t) + f(t, u(t), D^{\beta}_{0^+}u(t), D^{\beta+1}_{0^+}u(t)) = 0, t \in (0,1), \\ D^{\beta-1}_{0^+}u(0) = 0, I^{2-\beta}_{0^+}u(0) = 0, T_1(u) = 0, T_2(u) = 0, \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$, $1 < \alpha, \beta \leq 2$, such that $\alpha + \beta > 3$, T_1, T_2 are continuous linear functionals with the resonance condition: $T_1(t^{\beta+1})T_2(t^{\beta}) = T_1(t^{\beta})T_2(t^{\beta+1})$.

To the best of our knowledge, the problem of sequential operators of high order with functional boundary value conditions has rarely been explored, and based on this perspective and the motivation of the above papers, we establish the existence of solutions for the following nonlinear sequential fractional differential equation subject to functional boundary value conditions.

$$\begin{cases} \big({}^CD^{\alpha+1}_{0^+} + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1}\ {}^CD^{\alpha-i}_{0^+}\big)u(t) = f\big(t, u(t), u'(t), \ldots, u^{(n)}(t)\big), t \in [0,1], \\ u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, B(u) = 0. \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $n-1<\alpha\leq n$, C_n^{i+1} is the usual notation for the binomial coefficients, μ is a positive real number, and $B: C^n[0,1]\to\mathbb{R}$ is a continuous linear functional.

A boundary value problem is said to be at resonance if the corresponding homogeneous boundary value problem has a non-trivial solution, which means that the linear operator $Lu = \big({}^{C}D_{0^+}^{\alpha+1}+\sum\limits_{i=0}^{n-1}C_n^{i+1}\mu^{i+1}\ {}^{C}D_{0^+}^{\alpha-i}\big)u$ corresponding to (1.1) has nontrivial solutions(more details can be found in Lemma 3.4 below). So, the resonance condition: $B(\Phi(t))=0(\Phi(t)=1-e^{-\mu t}\sum\limits_{i=0}^{n-1}\dfrac{(\mu t)^i}{i!}$ will be obtained. We will always suppose $f:[0,1]\times\mathbb{R}^{n+1}\to\mathbb{R}$ satisfies the following conditions:

(1) $f(\cdot,u)$ is measurable for each fixed $u\in\mathbb{R}^{n+1}$, $f(t,\cdot)$ is continuous for a.e.,$t\in[0,1]$.

(2) $\sup\{|f(t,u)|: u\in D_0\}<+\infty$, for any compact set $D_0\in\mathbb{R}^{n+1}$.

The present study is novel in the given configuration and enriches the literature on boundary value problems of sequential fractional differential equations, which has a high degree of generality. It includes the following features: we analyze the functional boundary conditions and treat the details in the paper with various innovations, like the design of the projection operator Q, in addition to improving the order of the sequential derivatives. We can attempt to locate the resonance solution using different approaches in the future, such as the fixed point theorem, monotonic iteration techniques, etc., in addition to the methods described in this paper. Of course, one can also investigate the characteristics of the solution.

This paper is structured as follows: the next part covers some introductions and fundamental ideas related to linear operators, the coincidence degree continuation theorem, and fractional calculus. In Section 3, we discusses two types of problems. Subsection 3.1, by means of the Banach fixed point theorem, discusses the non-resonant case and yields solvability results of the problem. Subsection 3.2 discusses the existence of solution in the resonance sense by Mawhin's coincidence theory's extension theory. A numerical example is provided in section four to demonstrate our key theorems.

2 Preliminaries

Definition 2.1 *(see [1, 36, 37]) The Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $y \in L^1(0,1)$ is given by*

$$I^{\alpha}_{a+}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds,$$

where the right side is pointwise defined on $(a, +\infty)$.

Definition 2.2 *(see [1, 36, 37])The Caputo fractional derivatives of order $\alpha > 0$ of a function $y \in AC^n[a,b]$ is given by*

$${}^{C}D^{\alpha}_{a_+}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}y^{(n)}(s)ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, and the right side is pointwise defined on $(a, +\infty)$.

Lemma 2.3 *(see [1, 36, 37]) Let $\alpha > 0$. If $u(t) \in AC^n[a,b]$ or $u(t) \in C^n[a,b]$, then the fractional differential equation ${}^{c}D^{\alpha}_{a+}u(t) = 0$ (or ${}^{c}D^{\alpha}_{b_-}u(t) = 0$) has solution*

$$u(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_{n-1}(t-a)^{n-1},$$

where $c_i = \dfrac{u^{(i)}(a)}{i!} \in \mathbb{R}, i = 0, 1, \ldots, n-1$, and $n = [\alpha] + 1$.

Lemma 2.4 *(see [1, 36, 37])*

(1) Let $\alpha > 0$; If $u(t) \in AC^n[a,b]$ or $u(t) \in C^n[a,b]$, then one has

$$I^{\alpha}_{a+}\,{}^{c}D^{\alpha}_{0+}u(t) = u(t) - c_0 - c_1(t-a) - c_2(t-a)^2 - \ldots - c_{n-1}(t-a)^{n-1},$$

where $c_i = \dfrac{u^{(i)}(a)}{i!} \in \mathbb{R}, i = 0, 1, \ldots, n-1$, and $n = [\alpha] + 1$;

(2) The equality ${}^{c}D^{\beta}_{0+}I^{\beta}_{0+}y = y$ holds for every $\beta > 0$ and $y \in L^1[0,1]$.

Definition 2.5 *([30, 38]) Let X, Z be real Banach spaces, $L : domL \subset X \to Z$ be a linear operator. X is said to be the Fredholm operator of index zero provided that:*

(i) $\operatorname{Im} L$ *is a closed subset of* Y*;*

(ii) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

Let $P : X \to X$, $Q : Z \to Z$ are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is reversible. We denote the inverse of the mapping by K_P(generalized inverse operator of L). If Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the mapping $N : X \to Z$ will be called $L-$*compact* on $\overline{\Omega}$, if $QN(\overline{\Omega})$ and $K_P(I-Q)N : \overline{\Omega} \to X$ are continuous and compact.

Theorem 2.6 *(see[30, 38] Mawhin continuation theorem) Let $L : \operatorname{dom} L \subset X \to Z$ be a Fredholm operator of index zero and $N : X \to Z$ is L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(\mathrm{dom}L \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0, 1)$;

(ii) $Nu \notin \mathrm{Im}L$ for every $u \in \mathrm{Ker}L \cap \partial\Omega$;

(iii) $\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$, where $Q : Z \to Z$ is a continuous projection such that $\mathrm{Im}L = \mathrm{Ker}Q$.

Then the equation $Lu = Nu$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$.

Take $X = C^n[0,1]$ with the norm $\|u\| = \max\Big\{\|u\|_\infty, \|u'\|_\infty, \ldots, \|u^{(n)}\|_\infty\Big\}$, where $\|u\|_\infty = \max\limits_{t\in[0,1]} |u(t)|$.

Let $Y = L^1[0,1]$ be endowed with the norm $\|y\|_1 = \displaystyle\int_0^1 |y(t)|dt$. Obviously, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ are Banach spaces.

3 Main results

We will discuss two types of solutions to (1.1), *i.e.*, one for the non-resonance case and the other for the resonance case.

Define the operators $L : domL \subset X \to Y, N : X \to Y$ as follows:

$$Lu = \big({}^C D_{0^+}^{\alpha+1} + \sum_{i=0}^{n-1} C_n^{i+1} \mu^{i+1C} D_{0^+}^{\alpha-i}\big)u(t), \quad Nu = f\big(t, u(t), u'(t), \ldots, u^{(n)}(t)\big),$$

where $domL = \{u \in X :^C D_{0^+}^{\alpha+1-i} u \in Y, i = 0, 1, \ldots, n, u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, B(u) = 0\}$. So the problem (1.1) becomes $Lu = Nu$.

3.1 Non-resonance case

If $B(\Phi(t)) \neq 0$ holds, then $\mathrm{Ker}L = \{0\}$. It is so-called non-resonance case. As to this case, the problem (1.1) can be transformed into an operator equation.

Lemma 3.1 *If $B(\Phi(t)) \neq 0$ holds, then the boundary value problem (1.1) has a unique solution if and only if the following operator $T : X \to X$ has a unique fixed point, where*

$$\begin{aligned} T(u)(t) =& \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}f(s,u(s),u'(s),\ldots,u^{(n)}(s))ds \\ &- \frac{B\Big(\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}f(s,u(s),u'(s),\ldots,u^{(n)}(s))ds\Big)}{B(\Phi(t))}\Phi(t). \end{aligned} \tag{3.1}$$

Proof. If u is a solution to $Tu = u$, we get

$$\big({}^CD_{0^+}^{\alpha+1} + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1}\, {}^CD_{0^+}^{\alpha-i}\big)u(t) = f\big(t,u(t),u'(t),\ldots,u^{(n)}(t)\big).$$

Considering $u \in C^n[0,1]$ and $\Phi(t) = 1 - e^{-\mu t}\sum_{i=0}^{n-1}\frac{(\mu t)^i}{i!}$, we have $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$. Based on the linearly of B and (3.1), we have

$$\begin{aligned} B(u) =& B\Big(\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}f(s,u(s),u'(s),\ldots,u^{(n)}(s))ds\Big) \\ &- \frac{B\Big(\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}f(s,u(s),u'(s),\ldots,u^{(n)}(s))ds\Big)}{B(\Phi(t))}B(\Phi(t)) \\ =& 0. \end{aligned}$$

So, we have u, which is a solution to BVP(1.1). If u is a solution to BVP(1.1), then

$$\begin{aligned} T(u)(t) =& \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}f(s,u(s),u'(s),\ldots,u^{(n)}(s))ds \\ &- \frac{B\Big(\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}f(s,u(s),u'(s),\ldots,u^{(n)}(s))ds\Big)}{B(\Phi(t))}\Phi(t) \\ =& \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}({}^CD_{0^+}^{\alpha+1} + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1}\, {}^CD_{0^+}^{\alpha-i})u(s))ds \\ &- \frac{B\Big(\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}e^{-\mu(t-s)}(I_{0^+}^{\alpha-n+1}({}^CD_{0^+}^{\alpha+1} + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1}\, {}^CD_{0^+}^{\alpha-i})u(s))ds\Big)}{B(\Phi(t))}\Phi(t) \\ =& u(t) - \frac{u^{(n)}(0)}{\mu^n}\Phi(t) + \frac{u^{(n)}(0)}{\mu^n}\Phi(t) = u(t). \end{aligned}$$

From the above two arguments, we get that BVP(1.1) has a unique solution in X if and only if the operator equation $Tu = u$ has a unique solution in X. $\square$

By making use of lemma 3.1, we can obtain the following existence theorem for BVP(1.1) at non-resonance.

Theorem 3.2 *Let $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a Carathéodory function. Assume $B(\Phi(t)) \neq 0$ and the following conditions hold:*

(C_1): $\sum_{i=0}^{k} \frac{C_k^i |\mu|^{k-i} e^{-\mu t}}{\Gamma(\alpha - i + 2)} + \frac{B(t^{\alpha+1})}{\Gamma(\alpha+2)|B(\Phi(t))|} \max\{1 + n2^k \mu^{n+k-1}, 1 + n2^k \mu^k\} < 1.$

(C_2): *For almost every $t \in [0,1]$, then, $\forall (u_1, u_2, \ldots, u_{n+1}), (v_1, v_2, \ldots, v_{n+1}) \in \mathbb{R}^{n+1}$,*

$$|f(t, u_1, u_2, \ldots, u_{n+1}) - f(t, v_1, v_2, \ldots, v_{n+1})| \leq \max\{|u_1 - v_1|, |u_2 - v_2|, \ldots, |u_{n+1} - v_{n+1}|\}.$$

(C_3): If each $u_1, u_2 \in X$ satisfy $|u_1(t)| \leq |u_2(t)|, \forall t \in [0,1]$, then $|B(u_1)| \leq |B(u_2)|$.

Then BVP(1.1) has a unique solution in X.

Proof. We shall prove that $Tu = u$ has a unique solution in X. By Leibniz product rule and derivative forms of each order of $\Phi(t)$,

$$\begin{aligned} T^{(k)}(u)(t) =& \sum_{i=0}^{k} C_k^i (e^{-\mu t})^{(k-i)} I_{0+}^{n-i} (e^{\mu t} I_{0+}^{\alpha-n+1} f(t, u(t), u'(t), \ldots, u^{(n)}(t))) \\ &- \frac{B\Big(\frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} e^{-\mu(t-s)} (I_{0+}^{\alpha-n+1} f(s, u(s), u'(s), \ldots, u^{(n)}(s)) ds\Big)}{B(\Phi(t))} \Phi^{(k)}(t), \end{aligned}$$

where $k = 0, 1, \ldots, n$.

For each $u, v \in X$, by making use of $(C_2) - (C_3)$ and the linearly of B, we have

$$\begin{aligned} &|T^{(k)}(u)(t) - T^{(k)}(v)(t)| \\ =& \Big| \sum_{i=0}^{k} C_k^i (e^{-\mu t})^{(k-i)} I_{0+}^{n-i} (e^{\mu t} I_{0+}^{\alpha-n+1} \big(f(t, u(t), u'(t), \ldots, u^{(n)}(t)) - f(t, v(t), v'(t), \ldots, v^{(n)}(t))\big)) \\ &- \frac{B\Big(e^{-\mu t} I_{0+}^{n} e^{\mu t} \big(I_{0+}^{\alpha-n+1} (f(t, u(t), u'(t), \ldots, u^{(n)}(t)) - f(t, v(t), v'(t), \ldots, v^{(n)}(t)))\big)\Big)}{B(\Phi(t))} \Phi^{(k)}(t) \Big| \\ \leq& \sum_{i=0}^{k} \frac{C_k^i |\mu|^{k-i} e^{-\mu t}}{\Gamma(\alpha - i + 2)} \|u - v\| + \frac{B(t^{\alpha+1}) |\Phi^{(k)}(t)|}{\Gamma(\alpha+2)|B(\Phi(t))|} \|u - v\| \\ \leq& \|u - v\| \Big(\sum_{i=0}^{k} \frac{C_k^i |\mu|^{k-i} e^{-\mu t}}{\Gamma(\alpha - i + 2)} + \frac{B(t^{\alpha+1})}{\Gamma(\alpha+2)|B(\Phi(t))|} \max\{1 + n2^k \mu^{n+k-1}, 1 + n2^k \mu^k\} \Big) \end{aligned}$$

Since

$$|\Phi^{(k)}(t)| \leq \begin{cases} 1 + n2^k\mu^{n+k-1}, \mu > 1, \\ 1 + n2^k\mu^k, \mu \leq 1, \ k = 0, 1, 2, \ldots, n. \end{cases}$$

Considering (C_1), the above inequality implies that T is a contraction. By using *Banach's contaction principle*, $Tu = u$ has a unique solution in X. From lemma 3.1, BVP(1.1) has a unique solution in X.

3.2 Resonance case

In this part, noting that if $B(\Phi(t)) = 0$ holds, then $\mathrm{Ker}L = \{c\Phi(t) : c \in \mathbb{R}\}$. It is so-called resonance case. To obtain our main results, we will introduce the following conditions:

(H_0) The functional $B : X \to \mathbb{R}$ is linear continuous with the norm β, that is, $|B(u)| \leq \beta\|u\|$. In addition, $B(e^{-\mu t}I^n_{0^+}(e^{\mu t}I^{\alpha-n+1}_{0^+}1)) \neq 0$.

(H_1) There exist nonnegative functions $p_i(t), q(t) \in Y$ such that

$$|f(t, u_1, u_2, \ldots, u_{n+1})| \leq \sum_{i=1}^{n+1} p_i(t)|u_i(t)| + q(t), \ \forall (t, u_1, u_2, \ldots, u_{n+1}) \in [0, 1] \times \mathbb{R}^{n+1},$$

where if $\mu > 1$, $A(e^{\mu} \cdot n\mu^{n-1} + 1)\sum\limits_{i=0}^{n} \|p_{i+1}\|_1 < 1$; If $\mu \leq 1$, $A(e^{\mu} \cdot n + 1)\sum\limits_{i=0}^{n} \|p_{i+1}\|_1 < 1$; $A := \dfrac{1}{\Gamma(\alpha - n + 2)} + \sum\limits_{k=0}^{n-1} C^k_n \mu^{n-k} \sum\limits_{i=0}^{k} \dfrac{1}{\Gamma(\alpha - k + i + 2)} C^i_k \mu^i$.

(H_2) There exists a constant $M_1 > 0$ such that if $|u^{(n)}(t)| > M_1$, for $t \in [0, 1]$, then

$$B\Big(e^{-\mu t}I^n_{0^+}\big(e^{\mu t}I^{\alpha-n+1}_{0^+}f(t, u(t), u'(t), \ldots, u^{(n)})\big)\Big) \neq 0.$$

(H_3) There exists a constant $a > 0$ such that if $|c| > a$, then either

$$cB\Big(e^{-\mu t}I^n_{0^+}\big(e^{\mu t}I^{\alpha-n+1}_{0^+}N(c\Phi(t))\big)\Big) > 0, \tag{3.2}$$

or

$$cB\Big(e^{-\mu t}I^n_{0^+}\big(e^{\mu t}I^{\alpha-n+1}_{0^+}N(c\Phi(t))\big)\Big) < 0. \tag{3.3}$$

Theorem 3.3 *Suppose that $(H_0) - (H_3)$ are satisfied, then the functional boundary value problem (1.1) has at least one solution.*

In order to prove Theorem 3.3, we next state our main lemmas.

Lemma 3.4 *Assume that* (H_0) *holds, then* $L : \operatorname{dom} L \subset X \to Z$ *is a Fredholm operator of index zero. Moreover,*

$$\operatorname{Ker} L = \{c\Phi(t) : c \in \mathbb{R}\}, \qquad \dim \operatorname{Ker} L = 1. \tag{3.4}$$

and

$$ImL = \Big\{y \in Y | B\Big(e^{-\mu t} I_{0^+}^{n}\big(\big(e^{\mu t} I_{0^+}^{\alpha-n+1} y(t)\big)\Big) = 0\Big\}. \tag{3.5}$$

Proof Let $y \in \operatorname{Im} L$, then there exists $u \in \operatorname{dom} L$ such that $Lu = y$, that is, ${}^{C}D_{0^+}^{\alpha+1}\big(u(t) + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1} I_{0^+}^{i+1}\big)u(t) = y(t)$, we can write its solution as

$$u(t) + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1} I_{0^+}^{i+1} u(t) = I_{0^+}^{\alpha+1} y(t) + \sum_{i=0}^{n} c_i t^i, \tag{3.6}$$

where $c_i = \dfrac{u^{(i)}(0)}{i!}, i = 0, 1, \ldots, n$. Now, differentiating (3.6), we obtain

$$u'(t) + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1} I_{0^+}^{i} u(t) = I_{0^+}^{\alpha} y(t) + \sum_{i=1}^{n} c_i i t^{i-1}. \tag{3.7}$$

Next, deriving (3.7) $n-1$ times, we hold

$$u^{(n)}(t) + \sum_{i=0}^{n-1} C_n^{i+1}\mu^{i+1} u^{(n-1-i)}(t) = I_{0^+}^{\alpha-n+1} y(t) + c_n n!, \tag{3.8}$$

which can alternatively be written as

$$(u(t)e^{\mu t})^{(n)} = e^{\mu t}(I_{0^+}^{\alpha-n+1} y(t) + u^{(n)}(0)). \tag{3.9}$$

Integrating n times from 0 to t, we have

$$u(t) = e^{-\mu t}\sum_{i=0}^{n-1}\frac{d_i t^i}{i!} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} e^{-\mu(t-s)} I_{0^+}^{\alpha-n+1} y(s)ds + u^{(n)}(0))e^{-\mu t} I_{0^+}^{n} e^{\mu t}, \tag{3.10}$$

where d_i are arbitrary constants. Substituting the values of $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$ in (3.10) yields the solution

$$\begin{aligned} u(t) &= \frac{u^{(n)}(0)}{\mu^n}(1 - e^{-\mu t}\sum_{0}^{n-1}\frac{(\mu t)^i}{i!}) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} e^{-\mu(t-s)} I_{0^+}^{\alpha-n+1} y(s)ds \\ &= \frac{u^{(n)}(0)}{\mu^n}\Phi(t) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} e^{-\mu(t-s)} I_{0^+}^{\alpha-n+1} y(s)ds. \end{aligned} \tag{3.11}$$

Considering resonance condition $B(\Phi(t))=0$, we have $B\Big(e^{-\mu t}I_{0^+}^{n}\big((e^{\mu t}I_{0^+}^{\alpha-n+1}y(t)\big)\Big)=0$. That is,

$$\operatorname{Im}L\subseteq\Big\{y\in Y|B\Big(e^{-\mu t}I_{0^+}^{n}\big((e^{\mu t}I_{0^+}^{\alpha-n+1}y(t)\big)\Big)=0\}.$$

If $y\in\Big\{y\in Y|B\Big(e^{-\mu t}I_{0^+}^{n}\big((e^{\mu t}I_{0^+}^{\alpha-n+1}y(t)\big)\Big)=0\}$, take

$$u(t)=\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}e^{-\mu(t-s)}I_{0^+}^{\alpha-n+1}y(s)ds.$$

By a simple calculation, we get $\big({}^{C}D_{0^+}^{\alpha+1}+\sum_{i=0}^{n-1}C_n^{i+1}\mu^{i+1}\,{}^{C}D_{0^+}^{\alpha-i}\big)u(t)=y(t)$, $u(0)=u'(0)=\cdots=u^{(n-1)}(0)=0$, and $B(u)=0$. That is, $y\in\operatorname{Im}L, i.e.,$

$$\Big\{y\in Y|B\Big(e^{-\mu t}I_{0^+}^{n}\big((e^{\mu t}I_{0^+}^{\alpha-n+1}y(t)\big)\Big)=0\}\subseteq\operatorname{Im}L.$$

Therefore, we obtain (3.5).

If $u\in \operatorname{Ker}L, i.e., Lu(t)=\big({}^{C}D_{0^+}^{\alpha+1}+\sum_{i=0}^{n-1}C_n^{i+1}\mu^{i+1}\,{}^{C}D_{0^+}^{\alpha-i}\big)u(t)=0$, and $u(0)=u'(0)=\cdots=u^{(n-1)}(0)=0$, we have $u(t)=c(1-e^{-\mu t}\sum_{i=0}^{n-1}\frac{(\mu t)^i}{i!}):=c\Phi(t)$.

Based on the boundary condition $B(u)=0$, one has $B(u(t))=cB(\Phi(t))=0$. So,

$$\operatorname{Ker}L=\{c\Phi(t):c\in\mathbb{R}\},\qquad \dim\operatorname{Ker}L=1,$$

i.e., (3.4) holds. □

Take a projector $P:X\to X$ and an operator $Q:Y\to Y$ as follows:

$$Pu(t)=\frac{u^{(n)}(0)}{\mu^n}\Phi(t),\quad Qy=\frac{\Gamma(\alpha-n+2)B\Big(e^{-\mu t}I_{0^+}^{n}\big(e^{\mu t}I_{0^+}^{\alpha-n+1}y\big)\Big)}{B\Big(e^{-\mu t}I_{0^+}^{n}\big(e^{\mu t}I_{0^+}^{\alpha-n+1}1\big)\Big)}.$$

We can easily check that $Q^2y=Qy$, and $Q:Y\to Y$ is a linear projector. For $y\in Y$, we have $y=y-Qy+Qy, Qy\in\operatorname{Im}Q, y-\operatorname{Im}Q\in\operatorname{Ker}Q=\operatorname{Im}L$. So, we obtain $Y=\operatorname{Im}Q+\operatorname{Im}L$. Let $y_0\in\operatorname{Im}Q$ means that $y_0=c, c\in\mathbb{R}$. At the same time, by $y_0\in\operatorname{Im}L$, $y_0\equiv0$. Thus, $Y=\operatorname{Im}Q\oplus\operatorname{Im}L$, and $\dim\operatorname{Ker}L=\operatorname{codim}\operatorname{Im}L<+\infty$. Observing that $\operatorname{Im}L$ is a closed subspace of Y ; L is a Fredholm operator of index zero. □

Noting that P is a continuous projector and $\operatorname{Ker}P=\{u\in X:u^{(n)}(0)=0\}$. For $u\in X$, set $u=u-Pu+Pu, i.e., X=\operatorname{Ker}L+\operatorname{Ker}P$. It is easy to check that

$$P^2u(t)=P(Pu(t))=P(\frac{u^{(n)}(0)}{\mu^n}\Phi(t))=\frac{\frac{u^{(n)}(0)}{\mu^n}\Phi^{(n)}(0)}{\mu^n}\Phi(t)=\frac{u^{(n)}(0)}{\mu^n}\Phi(t)=Pu(t), u\in X,$$

since $\Phi^{(n)}(t)=-e^{\mu t}\mu^n\sum\limits_{i=0}^{n-1}C_n^i(-1)^{n-i}\sum\limits_{j=i}^{n-1}\frac{(\mu t)^{j-i}}{(j-i)!}$, and

$$\Phi^{(n)}(0)=-\mu^n\sum_{i=0}^{n-1}C_n^i(-1)^{n-i}=-\mu^n(\sum_{i=0}^{n}C_n^i(-1)^{n-i}-1)=\mu^n.$$

Take $u_0\in\operatorname{Ker}L, i.e., u_0=c\Phi(t), c\in\mathbb{R}$. If $u_0\in\operatorname{Ker}P$, then $c\Phi^{(n)}(0)=c\mu^n=0$, which implies that $c\equiv 0$. Thus, $X=\operatorname{Ker}L\oplus\operatorname{Ker}P$. $\square$

Lemma 3.5 *The mapping* $K_p:\operatorname{Im}L\to\operatorname{dom}L\cap\operatorname{Ker}P$ *can be defined by*

$$K_py(t)=\frac{1}{\Gamma(\alpha-n+1)(n-1)!}\int_0^t(t-s)^{n-1}e^{-\mu(t-s)}\int_0^s(s-\tau)^{\alpha-n}y(\tau)d\tau ds,$$

is the generalized inverse operator of L.

Proof For $y\in\operatorname{Im}L$, we have $B\Big(e^{-\mu t}I_{0^+}^n\big((e^{\mu t}I_{0^+}^{\alpha-n+1}y(t)\big)\Big)=0, i.e., B(K_py)=0$.

From the definition of K_p, by a simple calculation,

$$(K_py)^{(k)}(t)=\sum_{i=0}^{k}C_k^i(-\mu)^ie^{-\mu t}I_{0^+}^{n-k+i}(e^{\mu t}I_{0^+}^{\alpha-n+1}y)(t),\ 0\le k\le n-1, \tag{3.12}$$

$$(K_py)^{(n)}(t)=I_{0^+}^{\alpha-n+1}y(t)-\sum_{k=0}^{n-1}C_n^k\mu^{n-k}(K_py)^{(k)}(t). \tag{3.13}$$

Obviously, $(K_py)^{(k)}(0)=0$ and $(K_py)^{(n)}(0)=0$. Therefore, $K_py\in\operatorname{dom}L\cap\operatorname{Ker}P, y\in\operatorname{Im}L$.

Now, we will prove K_p is the inverse of $L|_{\operatorname{dom}L\cap\operatorname{Ker}P}$.

In fact, if $y\in\operatorname{Im}L$, by Lemma 3.4, then

$$(LK_py)(t)=\big({}^CD_{0^+}^{\alpha+1}+\sum_{i=0}^{n-1}C_n^{i+1}\mu^{i+1C}D_{0^+}^{\alpha-i}\big)(K_py)(t)=y(t).$$

If $u\in\operatorname{dom}L\cap\operatorname{Ker}P$, $u^{(i)}(0)=0, i=0,1,\dots,n$, from (3.11), we have

$$\begin{aligned}(K_pLu)(t)&=e^{-\mu t}I_{0+}^ne^{\mu t}I_{0+}^{\alpha-n+1}Lu\\&=e^{-\mu t}I_{0+}^ne^{\mu t}I_{0+}^{\alpha-n+1}\sum_{i=0}^{n}C_n^i\mu^iI_{0^+}^{n-\alpha}\frac{d^{n-i+1}u}{dt^{n-i+1}}\\&=e^{-\mu t}I_{0+}^ne^{\mu t}\sum_{i=0}^{n}C_n^i\mu^iI_{0+}^1u^{(n-i+1)}(t)\\&=e^{-\mu t}I_{0+}^n\sum_{i=0}^{n}C_n^i\mu^ie^{\mu t}u^{(n-i)}(t)\\&=e^{-\mu t}I_{0+}^n(e^{\mu t}u)^{(n)}(t)=u(t),\end{aligned}$$

so, $K_p=(L|_{\operatorname{dom}L\cap\operatorname{Ker}P})^{-1}$.

Lemma 3.6 *Assume* (H_0) *hold,* $\Omega \subset X$ *is an open bounded set and* $\operatorname{dom} L \cap \Omega \neq \emptyset$*.Then* N *is* $L-$*compact on* $\overline{\Omega}$*.*

Proof For convenience, denote $v(t) := e^{-\mu t} I_{0^+}^{n} \left(e^{\mu t} I_{0^+}^{\alpha-n+1} Nu \right)$.

We will prove the $QN(\overline{\Omega})$ is continuous and bounded.

It follows (H_0) that $QNu = CB(v(t))$, where $C := \frac{\Gamma(\alpha-n+2)}{B\left(e^{-\mu t} I_{0^+}^{n}\left(e^{\mu t} I_{0^+}^{\alpha-n+1} 1\right)\right)}$.

Since $\Omega \subset X$ is bounded, for $u \in \overline{\Omega}$. By the condition (2) on the function f, there exists a constant $M > 0$ such that $\sup |f(t, u(t), u'(t), \ldots, u^{(n)}(t))| \leq M, t \in [0,1], u \in \overline{\Omega}$. From formulas (3.12) and (3.13), we obtain

$$v^{(k)}(t) = \sum_{i=0}^{k} C_k^i (-\mu)^i e^{-\mu t} I_{0^+}^{n-k+i} (e^{\mu t} I_{0^+}^{\alpha-n+1} Nu)(t),\ 0 \leq k \leq n-1, and$$

$$v^{(n)}(t) = I_{0^+}^{\alpha-n+1} Nu(t) - \sum_{i=0}^{n-1} C_n^k \mu^{n-k} v^{(k)}(t).$$

Then

$$\begin{aligned}
|v^{(k)}(t)| &\leq \sum_{i=0}^{k} C_k^i \mu^i |I_{0^+}^{\alpha-k+i+1} Nu| \\
&\leq \sum_{i=0}^{k} C_k^i \mu^i \frac{1}{\Gamma(\alpha-k+i+1)} \int_0^t (t-s)^{\alpha-k+i} |Nu(s)| ds \\
&\leq \sum_{i=0}^{k} \frac{M C_k^i \mu^i}{\Gamma(\alpha-k+i+2)} \\
&\leq \frac{M}{\Gamma(\alpha-k+2)} \sum_{i=0}^{k} C_k^i \mu^i = \frac{M(1+\mu)^k}{\Gamma(\alpha-k+2)} < +\infty, \\
|v^{(n)}(t)| &= |I_{0^+}^{\alpha-n+1} Nu(t) - \sum_{k=0}^{n-1} C_n^k v^{(k)}(t) \mu^{n-k}| \\
&\leq |I_{0^+}^{\alpha-n+1} Nu(t)| + \sum_{i=0}^{n-1} C_n^k |v^{(k)}(t)| \mu^{n-k} \\
&\leq \frac{M}{\Gamma(\alpha-n+2)} + \frac{(1+\mu)^n - 1}{\Gamma(\alpha-k+2)} (1+\mu)^k M < +\infty.
\end{aligned}$$

So, $\|v\| < +\infty$. Then $|QNu| \leq |C|\beta\|v\| < +\infty, |(I-Q)Nu| < +\infty$, and $\|QNu\|_1 < +\infty$, $i.e., QN(\overline{\Omega})$ is bounded. Of course, by the above discussions, it is not difficult to verify that $K_p(I-Q)Nu : (\overline{\Omega})$ is also bounded.

In view of (1) on the function f and the Lebesgue dominated convergence theorem, we can easily show that that QN and $K_p(I-Q)Nu : \overline{\Omega} \to Y$ are continuous.

Now, we will prove that $K_p(I-Q)Nu : (\overline{\Omega})$ is compact.

For the simplicity of the following mathematical formulas, it may be assumed that there is a constant $M' > 0$ such that $\|(I-Q)Nu\|_1 \leq M'$. Again, using formulas (3.12) and (3.13), for $0 \leq t_1 < t_2 \leq 1, u \in \overline{\Omega}$, we obtain

$$
\begin{aligned}
&|I_{0^+}^{\alpha-n+1}Nu(t_2) - I_{0^+}^{\alpha-n+1}Nu(t_1)| \\
&= \Big|\frac{1}{\Gamma(\alpha-n+1)}\int_0^{t_2}(t_2-s)^{\alpha-n}(I-Q)Nu(s)ds - \frac{1}{\Gamma(\alpha-n+1)}\int_0^{t_1}(t_1-s)^{\alpha-n}(I-Q)Nu(s)ds\Big| \\
&\leq \frac{M'}{\Gamma(\alpha-n+1)}\Big|\int_0^{t_1}\big((t_2-s)^{\alpha-n}-(t_1-s)^{\alpha-n}\big)ds\Big| + \frac{M'}{\Gamma(\alpha-n+1)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-n}ds\Big| \\
&= \frac{t_2^{\alpha-n+1}-t_1^{\alpha-n+1}+2(t_2-t_1)^{\alpha-n+1}}{\Gamma(\alpha-n+2)}M', \\
&|(K_p(I-Q)Nu)^{(k)}(t_2) - (K_p(I-Q)Nu)^{(k)}(t_1)|,\ 0 \leq k \leq n-1, \\
&= \Big|\sum_{i=0}^{k} C_k^i(-\mu)^i e^{-\mu t} I_{0^+}^{n-k+i}(e^{\mu t}I_{0^+}^{\alpha-n+1}(I-Q)Nu)\big|_{t=t_2} \\
&\quad - \sum_{i=0}^{k} C_k^i(-\mu)^i e^{-\mu t} I_{0^+}^{n-k+i}(e^{\mu t}I_{0^+}^{\alpha-n+1}(I-Q)Nu)\big|_{t=t_1}\Big| \\
&= \sum_{i=0}^{k} C_k^i\mu^i(e^{-\mu t_1}-e^{-\mu t_2})\Big|I_{0^+}^{n-k+i}(e^{\mu t}I_{0^+}^{\alpha-n+1}(I-Q)Nu)\big|_{t=t_2}\Big| \\
&+ \sum_{i=0}^{k} C_k^i\mu^i e^{-\mu t_1}\Big|I_{0^+}^{n-k+i}e^{\mu t}I_{0^+}^{\alpha-n+1}(I-Q)Nu\big|_{t=t_2} - I_{0^+}^{n-k+i}e^{\mu t}I_{0^+}^{\alpha-n+1}(I-Q)Nu\big|_{t=t_1}\Big| \\
&\leq \sum_{i=0}^{k} C_k^i\mu^i|1-e^{\mu(t_2-t_1)}|\frac{M'}{\Gamma(\alpha-k+i+2)} + \sum_{i=0}^{k} C_k^i\mu^i e^{-\mu t_1}\Big|\frac{1}{\Gamma(n-k+i)}\int_0^{t_2}(t_2-s)^{n-k+i-1} \\
&\quad e^{\mu s}I_{0^+}^{\alpha-n+1}(I-Q)Nu(s)ds - \frac{1}{\Gamma(n-k+i)}\int_0^{t_1}(t_1-s)^{n-k+i-1}e^{\mu s}I_{0^+}^{\alpha-n+1}(I-Q)Nu(s)ds\Big| \\
&\leq \sum_{i=0}^{k} C_k^i\mu^i|1-e^{\mu(t_2-t_1)}|\frac{M'}{\Gamma(\alpha-k+i+2)} \\
&\quad + \frac{(t_2-t_1)^{n-k+i}+t_2^{n-k+i}-t_1^{n-k+i}}{\Gamma(n-k+i+1)}\frac{(1+\mu)^kM'}{\Gamma(\alpha-n+2)} + \frac{(t_2-t_1)^{n-k+i}e^{\mu(t_2-t_1)}}{\Gamma(n-k+i+1)}\frac{(1+\mu)^kM'}{\Gamma(\alpha-n+2)},
\end{aligned}
$$

and

$$
\begin{aligned}
&|(K_p(I-Q)Nu)^{(n)}(t_2)-(K_p(I-Q)Nu)^{(n)}(t_1)|\\
=&|I_{0^+}^{\alpha-n+1}Nu(t_2)-I_{0^+}^{\alpha-n+1}Nu(t_1)+\sum_{k=0}^{n-1}C_n^k\mu^{n-k}[(K_p(I-Q)Nu)^{(k)}(t_2)-(K_p(I-Q)Nu)^{(k)}(t_1)]|\\
\leq&|I_{0^+}^{\alpha-n+1}Nu(t_2)-I_{0^+}^{\alpha-n+1}Nu(t_1)|+\sum_{k=0}^{n-1}C_n^k\mu^{n-k}|(K_p(I-Q)Nu)^{(k)}(t_2)-(K_p(I-Q)Nu)^{(k)}(t_1)|.
\end{aligned}
$$

Since $t, t^{\alpha-n+1}$ and t^{n-k+i} are uniformly continuous on $[0,1]$, we obtain that $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous. By the Arzela-Ascoli theorem, $K_P(I-Q)N:(\overline{\Omega})$ is compact. Thus, N is L-compact.□

Lemma 3.7 *The set* $\Omega_1=\{u\in \operatorname{dom}L\setminus \operatorname{Ker}L: Lu=\lambda Nu, \lambda\in[0,1]\}$ *is bounded, if conditions* $(H_0)-(H_2)$ *are satisfied.*

Proof For $u\in\Omega_1$, we have $QNu=0, i.e., B\Big(e^{-\mu t}I_{0^+}^{n}\big(e^{\mu t}I_{0^+}^{\alpha-n+1}f(u(t),u'(t),\dots,u^{(n)}(t))\big)\Big)=0$. By (H_2), there exists a constant $t_0\in[0,1]$ such that $|u^{(n)}(t_0)|\leq M_1$.

From boundary conditions $u(0)=u'(0)=\ldots=u^{(n-1)}(0)=0$, we get $u^{(i)}(t)=\int_0^t u^{(i+1)}(s)ds$, $i=0,1,,\dots,n-1$, and

$$
\|u\|_\infty\leq\|u'\|_\infty\leq\cdots\leq\|u^{(n)}\|_\infty. \tag{3.14}
$$

By $Lu=\lambda Nu$, we hold

$$
u(t)=c\Phi(t)+\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}e^{-\mu(t-s)}I_{0^+}^{\alpha-n+1}\lambda f(u(s),u'(s),\dots,u^{(n)}(s))ds,
$$

furthermore,

$$
u^{(n)}(t)=c\Phi^{(n)}(t)+I_{0^+}^{\alpha-n+1}\lambda Nu(t)-\sum_{k=0}^{n-1}C_n^k\mu^{n-k}\sum_{i=0}^{k}C_k^i(-\mu)^ie^{-\mu t}I_{0^+}^{n-k+i}(e^{\mu t}I_{0^+}^{\alpha-n+1}\lambda Nu)(t).
$$

Let $v_\lambda(t):=e^{-\mu t}I_{0^+}^{n}\big(e^{\mu t}I_{0^+}^{\alpha-n+1}\lambda Nu\big)$, then $u^{(n)}(t)=c\Phi^{(n)}(t)+v_\lambda^{(n)}(t)$.

In view of (3.12), (3.13) and $|u^{(n)}(t_0)|\leq M_1$,

$$
\begin{aligned}
|v^{(n)}(t)|&=|I_{0^+}^{\alpha-n+1}\lambda Nu(t)-\sum_{k=0}^{n-1}C_n^k\mu^{n-k}\sum_{i=0}^{k}C_k^i(-\mu)^ie^{-\mu t}I_{0^+}^{n-k+i}(e^{\mu t}I_{0^+}^{\alpha-n+1}\lambda Nu)(t)|\\
&\leq\frac{\|Nu\|_1}{\Gamma(\alpha-n+2)}+\sum_{k=0}^{n-1}C_n^k\mu^{n-k}\sum_{i=0}^{k}C_k^i\mu^i\frac{\|Nu\|_1}{\Gamma(\alpha-k+i+2)}=A\|Nu\|_1.
\end{aligned}
$$

And $|c| \le \dfrac{|v^{(n)}(t_0)| + |u^{(n)}(t_0)|}{\Phi^{(n)}(t_0)} \le \dfrac{A\|Nu\|_1 + M}{(2^n-1)\mu^n e^{-\mu}}$, where
$A := \dfrac{1}{\Gamma(\alpha-n+2)} + \sum\limits_{k=0}^{n-1} C_n^k \mu^{n-k} \sum\limits_{i=0}^{k} C_k^i \mu^i \dfrac{1}{\Gamma(\alpha-k+i+2)}$,
since $|\Phi^{(n)}(t_0)| = e^{-\mu t_0}\mu^n \sum\limits_{i=0}^{n-1} C_n^i \sum\limits_{j=i}^{n-1} \frac{(\mu t_0)^{j-i}}{(j-i)!} \ge e^{-\mu}\mu^n \sum\limits_{i=0}^{n-1} C_n^i = (2^n-1)\mu^n e^{-\mu}$.
Therefore, $|u^{(n)}(t)| \le |c||\Phi^{(n)}(t)| + |v^{(n)}(t)| \le (A\|Nu\|_1 + M_1)e^{\mu} \sum\limits_{k=0}^{n-1} \mu^k + A\|Nu\|_1$,
since

$$
\begin{aligned}
|\Phi^{(n)}(t)| &= \Big| - e^{\mu t}\mu^n \sum_{i=0}^{n-1} C_n^i (-1)^{n-i} \sum_{j=i}^{n-1} \frac{(\mu t)^{j-i}}{(j-i)!}\Big| \\
&\le \mu^n \sum_{i=0}^{n-1} C_n^i \sum_{j=i}^{n-1} \frac{(\mu t_0)^{j-i}}{(j-i)!} \\
&\le \mu^n \sum_{i=0}^{n-1} C_n^i \sum_{k=0}^{n-1} \mu^k = (2^n-1)\mu^n \sum_{k=0}^{n-1} \mu^k.
\end{aligned}
$$

From (H_1) and (3.14), we know that $\|Nu\|_1 = \int_0^1 |Nu(s)|ds \le \sum\limits_{i=0}^{n} \|p_{i+1}\|_1 \|u^{(n)}\|_\infty + \|q\|_1$.
At the same time, if $\mu > 1$, then $\sum\limits_{k=0}^{n-1} \mu^k \le n\mu^{n-1}$, and $|u^{(n)}(t)| \le (e^{\mu} \cdot n\mu^{n-1} + 1)A\|Nu\|_1 + n\mu^{n-1}e^{\mu}M_1$, so, $\|u^{(n)}\| \le \dfrac{A(e^{\mu} \cdot n\mu^{n-1} + 1)\|q\|_1 + e^{\mu}M_1 \cdot n\mu^{n-1}}{1 - A(e^{\mu} \cdot n\mu^{n-1} + 1) \sum\limits_{i=0}^{n} \|p_{i+1}\|_1}$.
Similarly, if $\mu \le 1$, then $\sum\limits_{k=0}^{n-1} \mu^k \le n$, and $\|u^{(n)}\| \le \dfrac{A(e^{\mu} \cdot n + 1)\|q\|_1 + e^{\mu}M_1 \cdot n}{1 - A(e^{\mu} \cdot n + 1) \sum\limits_{i=0}^{n} \|p_{i+1}\|_1}$.

These, together with condition (H_1), mean that Ω_1 is bounded in X.

Lemma 3.8 *The set $\Omega_2 = \{u \in \operatorname{Ker} L : Nu \in \operatorname{Im} L\}$ is bounded if (H_3) hold.*

Proof: Let $u_c \in \Omega_2$, then $u_c(t) \equiv c\Phi(t),\ c \in \mathbb{R}$ and $QNu_c(t) = 0$. By (H_3), we have $|c| \le a$. Since

$$
|\Phi(t)| = \big|1 - e^{-\mu t} \sum_{i=0}^{n-1} \frac{(\mu t)^i}{i!}\big| \le \begin{cases} 1 + n\mu^{n-1}, \mu > 1, \\ 1 + n, \mu \le 1, \end{cases}
$$

$$
|\Phi^{(k)}(t)| = \big| - e^{-\mu t}\mu^k \sum_{i=0}^{k} C_k^i (-1)^{k-i} \sum_{j=i}^{n-1} \frac{(\mu t)^{j-i}}{(j-i)!}\big| \le \begin{cases} n2^k\mu^{n+k-1}, \mu > 1, \\ n2^k\mu^k, \mu \le 1,\ k = 1, 2, \ldots, n-1, \end{cases}
$$

$$
|\Phi^{(n)}(t)| = \big| - e^{-\mu t}\mu^n \sum_{i=0}^{n-1} C_n^i (-1)^{n-i} \sum_{j=i}^{n-1} \frac{(\mu t)^{j-i}}{(j-i)!}\big| \le \begin{cases} n(2^n-1)\mu^{2n-1}, \mu > 1, \\ n(2^n-1)\mu^n, \mu \le 1, \end{cases}
$$

Taking into account the finiteness of fixed variables n and μ, we hold that $\|u_c\| < +\infty$, *i.e.*, Ω_2 is bounded.

Lemma 3.9 *The set* $\Omega_3 = \{u \in KerL : \rho\lambda Ju + (1-\lambda)QNu = 0, \lambda \in [0,1]\}$ *is bounded if conditions* (H_3) *is satisfied, where* $J : \operatorname{Ker} L \to \operatorname{Im} Q$ *is a homeomorphism with* $J(c\Phi(t)) = \dfrac{c}{B\Big(e^{-\mu t} I_{0^+}^{n}\big(e^{\mu t} I_{0^+}^{\alpha-n+1} 1\big)\Big)}, c \in \mathbb{R}$, *where*

$$\rho = \begin{cases} 1, & if\ (3.2)\ holds; \\ -1, & if\ (3.3)\ holds. \end{cases} \tag{3.15}$$

Proof: Suppose that $u' \in \Omega_3$, we have $u'(t) = c\Phi(t), c \in R$ and $\lambda Ju' + \rho(1-\lambda)QNu' = 0$. If $\lambda = 0$, we have $QNu' = 0$. By (H_5), one has $|c| \leq a$, which follows from the proof of boundedness of Ω_2 that $\|u'\| < +\infty$. If $\lambda = 1$, then $c = 0, i.e., u' = 0$. If $\lambda \in (0,1)$, taking $|c| > a$, we have

$$\rho\lambda c + (1-\lambda)B\Big(e^{-\mu t} I_{0^+}^{n}\big(e^{\mu t} I_{0^+}^{\alpha-n+1} N(c\Phi(t))\big)\Big) = 0.$$

Hence,

$$\rho\lambda c^2 = -(1-\lambda)cB\Big(e^{-\mu t} I_{0^+}^{n}\big(e^{\mu t} I_{0^+}^{\alpha-n+1} N(c\Phi(t))\big)\Big).$$

According to the condition (H_3), we can easily get the contradiction. So Ω_3 is bounded.

Proof of Theorem 3.3 Let Ω be a bounded open subset of X such that $\{0\} \cup \bigcup_{j=1}^{3} \overline{\Omega}_j \subset \Omega$. By Lemma 3.6, we know that N is L-compact on $\overline{\Omega}$. According to the Lemma 3.7 and Lemma 3.8, we have :

(i) $Lu \neq \lambda Nu$, for every $(u, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(ii) $Nu \notin \operatorname{Im} L$, for every $u \in \operatorname{Ker} L \cap \partial\Omega$;

At last, we will prove that (iii) of Theorem 2.6 is satisfied.

Let $H(u,\lambda) = \rho\lambda Ju + (1-\lambda)QNu$. Noting that Lemma 3.9 and $\Omega_3 \subset \Omega$, we have $H(u,\lambda) \neq 0$ for every $u \in \partial\Omega \cap \operatorname{Ker} L$. Thus, by the homotopic property of degree, we know that

$$\begin{aligned} deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= deg\big(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\big) \\ &= deg\big(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\big) \\ &= deg\big(\pm J, \Omega \cap \operatorname{Ker} L, 0\big) \neq 0. \end{aligned}$$

The assumption (iii) of Theorem 2.6 is verified and the proof is completed.

Then by the Theorem 2.6, the functional boundary value problem (1.1) has at least one solution in X. The proof of the Theorem 3.3 is also completed.

4 Example

Now, we illustrate Theorem 3.3 by the following example. Consider the functional boundary value problem

$$\begin{cases} ({}^C D_{0+}^{\frac{3}{2}} + 2{}^C D_{0+}^{\frac{1}{2}})u(t) = f(t, u(t), u'(t)), t \in [0, 1], \\ u(0) = 0, B(u) = 2e^2 u(1) - (e^2 - 1)u'(0) = 0, \end{cases}$$

where $\alpha = \frac{1}{2}$,$\mu = 2, \Phi(t) = 1 - e^{-2t}$, and $f(t, u(t), u'(t)) = t - 1 + \frac{1}{50}\sin u(t) + \frac{1}{50}u'(t)$.

Then the functional problem is at resonance with $B(\Phi(t)) = 2e^2(1 - e^{-2}) - 2(e^2 - 1) = 0$. In this case, $KerL = \{c(1 - e^{-2t})|c \in \mathbb{R}\}$, $|B(u)| \le (3e^2 + 1)\|u\|$,

$$B(e^{-2t}I_{0+}^1(e^{2t}I_{0+}^{\frac{1}{2}}1)) = \frac{2}{\Gamma(\frac{3}{2})}\int_0^1 t^{\frac{1}{2}}e^{2t}dt = \frac{4}{\sqrt{\pi}}\int_0^1 t^{\frac{1}{2}}e^{2t}dt \approx \frac{4}{\sqrt{\pi}} \cdot 2.5123 \ne 0.$$

$p_1(t) = \frac{1}{50}, p_2(t) = \frac{1}{50}, q(t) = 1$. It is easy to check that

$$A := \frac{1}{\Gamma(\alpha - n + 2)} + \sum_{k=0}^{n-1} C_n^k \mu^{n-k} \sum_{i=0}^{k} \frac{1}{\Gamma(\alpha - k + i + 2)} C_k^i \mu^i = \frac{14}{3\sqrt{\pi}},$$

moreover, $A(e^{\mu} \cdot n\mu^{n-1} + 1)\sum_{i=0}^{n}\|p_{i+1}\|_1 = A(e^2 + 1)\frac{2}{50} \approx 0.8835 < 1$.Conditions (H_0) and (H_1) are satisfied.

Take $M_1 = 52$. If $u'(t) > 52$, then $f(t, u(t), u'(t)) > -1 - \frac{1}{50} + \frac{M_1}{50} = \frac{1}{50} > 0$, and if $u'(t) < -52$, then $f(t, u(t), u'(t)) < \frac{1}{50} - \frac{M_1}{50} = \frac{-51}{50} < 0$. Hence, if $|u'(t)| > M = 52$, then

$$B(e^{-2t}I_{0+}^1(e^{2t}I_{0+}^{\frac{1}{2}}f(t, u(t), u'(t)))) = \frac{2}{\Gamma(\frac{1}{2})}\int_0^1 e^{2t}\int_0^t (t - s)^{-\frac{1}{2}} f(s, u(s), u'(s))dsdt \ne 0.$$

Thus (H_2) is satisfied.

Finally, take $u \in KerL$ and $u(t) = c\Phi(t) = c(1 - e^{-2t})$, one choose $|c| > a = 189$,

$$\begin{aligned} & cB\Big(e^{-2t}I_{0+}^1(e^{2t}I_{0+}^{\frac{1}{2}}N\big(c(\Phi(t))\big))\Big) \\ &= \frac{2c}{\Gamma(\frac{1}{2})}\int_0^1 e^{2t}\int_0^t (t - s)^{-\frac{1}{2}} f(s, c\Phi(s), c\Phi'(s))dsdt \\ &= \frac{2}{\Gamma(\frac{1}{2})}\int_0^1 e^{2t}\int_0^t (t - s)^{-\frac{1}{2}} cf(s, c\Phi(s), c\Phi'(s))dsdt > 0, \end{aligned}$$

since $cf(s, c\Phi(s), c\Phi'(s)) = c(s - 1 + \frac{1}{50}\sin(c(1 - e^{-2s})) + \frac{2c}{50}e^{-2t}) > -|c| - \frac{|c|}{50} + \frac{2c^2}{50e^2} > 0$, $|c| > 189$, then condition (H_3) is satisfied. It follows from Theorem 3.3 that there must be at least one solution in X.

5 Conclusion

This work examines a category of higher-order sequential operator problems with functional boundary conditions. First, it is reasonable to regard sequential operators as a generic statement. Second, we examine non-resonance and resonance issues for sequential operators of order $n-1<\alpha\leq n$. These considerations are specific improvements and complements of non-resonant boundary value problems (BVPs) of lower order or resonance problems, as found, for example, in the literature [25, 29]. In [29], the authors studied a nonlinear three-point boundary value problem of sequential fractional differential equations of order α with $1<\alpha\leq 2$ at the resonance case, but we explore the resonant BVPs(1.1) that firstly can be lifted from the order $1<\alpha\leq 2$ of the fractional operator to $n-1<\alpha\leq n$, and secondly the boundary condition $B(u)=0$ can contain the original conditions $x(1)=\beta x(\eta)$. These mean that some similar results can be expanded.

In [25], the authors studied the existence of solutions to the nonlinear sequential fractional differential equation at resonance with the order $0<\alpha\leq 1$. Again, we generalize both in terms of the order of the operators and in terms of the boundary conditions. So, we study the resonance problem for the order $n-1<\alpha\leq n$ sequential operators with functional boundary conditions, which gives a better generalization based on the above problems in terms of the choice of sequential operators, the order of the differential operators, and the boundary value conditions.

Data Availability

No data were used to support this study.

Competing interests

The authors declare that they have no competing interests.

Statements and Declarations

The authors would like to thank the handling editors for the help in the processing of the paper. This work was supported by the Natural Science Foundation of China(12071302),National Natural Science Foundation of China (Grant No. 12372013), Natural Science Foundation of Henan (Grant No. 242300421166), Program for Science and Technology Innovation Talents in Universities of Henan Province, China (Grant No. 24HASTIT034), Key Scientific Research Projects of Universities in Henan Province(Grant No. 25B110004), Natural Science Foundation of Henan (Grant No. 232300420122). All results belong to Sun.B, Zhang.S and Li.S.

References

[1] K.S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[2] Baghani H, Alzabut J, Farokhi-Ostad J, Nieto, J.J. Existence and uniqueness of solutions for a coupled system of sequential fractional differential equations with initial conditions. J. Pseudo-Differ. Oper. Appl. 2020, 11, 1731-1741.

[3] Z. Wei, W. Dong. Periodic boundary value problems for Riemann-Liouville sequential fractional differential equations, Electron. J. Qual.Theory Differ.Equ. 87 (2011) 1-13.

[4] Alsaedi A, Ahmad B, Aljoudi S, Ntouyas S.K. A study of a fully coupled two-parameter system of sequential fractional integro-differential equations with nonlocal integro-multipoint boundary conditions. Acta Math. Sci. Ser. B 2019, 39, 927-944.

[5] Saengthong W, Thailert E, Ntouyas S.K. Existence and uniqueness of solutions for system of Hilfer-Hadamard sequential fractional differential equations with two point boundary conditions. Adv. Differ. Equ. 2019, 2019, 525.

[6] Wei Z, Li Q, Che J. Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative. , J. Math. Anal. Appl., 2010, 367(1): 260-272.

[7] Ahmad B, Alghanmi M, Alsaedi A, Nieto J.J. Existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions. Appl. Math. Lett. 2021, 116, 107018.

[8] Guida K, Ibnelazyz L, Hilal K, Melliani S. Existence and uniqueness results for sequential Ψ-Hilfer fractional pantograph differential equations with mixed nonlocal boundary conditions. AIMS Math. 6(8), 8239-8255 (2021).

[9] Wongcharoen A, Ntouyas S.K, Wongsantisuk P, Tariboon J. Existence results for a nonlocal coupled system of sequential fractional differential equations involving Ψ-Hilfer fractional derivatives. Adv. Math. Phys. 2021, 2021, 5554619.

[10] Su X, Zhang S, Zhang L. Periodic boundary value problem involving sequential fractional derivatives in Banach space. AIMS Math. 2020, 5, 7510-7530.

[11] Fazli H, Nieto J.J, Bahrami F. On the existence and uniqueness results for nonlinear sequential fractional differential equations, Appl. Comput. Math. 2018, 17, 36-47.

[12] Ahmad B, R Luca. Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions, Appl. Math. Comput., 339 (2018), 516-534.

[13] C Nuchpong, S. K. Ntouyas, A. Samadi, J. Tariboon. Boundary value problems for hilfer type sequential fractional differential equations and inclusions involving riemann-stieltjes integral multi-strip boundary conditions, Adv. Differ. Equ., 2021 (2021), 268.

[14] Ahmad B, Ntouyas S K. Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, Appl. Math. Comput., 2015, 266: 615-622.

[15] N. Phuangthong, S. K. Ntouyas, J. Tariboon, K. Nonlaopon. Nonlocal sequential boundary value problems for hilfer type fractional integro-differential equations and inclusions, Mathematics, 9(2021), 615.

[16] Z. Baitiche et al., Monotone Iterative Technique for a New Class of Nonlinear Sequential Fractional Differential Equations with Nonlinear Boundary Conditions under the Ψ-Caputo Operator. Mathematics 10 (2022), 1173.

[17] Alruwaily Y, Ahmad B, Ntouyas S K, et al. Existence results for coupled nonlinear sequential fractional differential equations with coupled Riemann-Stieltjes integro-multipoint boundary conditions. Fractal & Fractional, 2022, 6(2): 123.

[18] Ahmad B, J. J. Nieto. Sequential fractional differential equations with three point boundary conditions. Comput. Math. Appl. 64 (2012), 3046-3052.

[19] Khan A, Alshehri H M, Gómez-Aguilar J F, et al. A predator-prey model involving variable-order fractional differential equations with Mittag-Leffler kernel. Adv. Differ. Equ., 2021, 2021(1): 1-18.

[20] Morsy A, Nisar K S, Ravichandran C, et al. Sequential fractional order Neutral functional Integro differential equations on time scales with Caputo fractional operator over Banach spaces. AIMS Math., 2023, 8(3): 5934-5949.

[21] Lazreg J E, Benkhettou N, Benchohra M, et al. Neutral functional sequential differential equations with Caputo fractional derivative on time scales. Fixed Point Theory and Algorithms for Sciences and Engineering, 2022, 2022(1): 1-16.

[22] Sitho S, Ntouyas S K, Sudprasert C, et al. Integro-differential boundary conditions to the sequential ψ_1-Hilfer and ψ_2-Caputo fractional differential equations. Mathematics, 2023, 11(4): 867.

[23] Djaout A, Benbachir M, Lakrib M, et al. Solvability and stability analysis of a coupled system involving generalized fractional derivatives. AIMS Math., 2023, 8(4): 7817-7839.

[24] Alkhazzan A, Jiang P, Baleanu D, et al. Stability and existence results for a class of nonlinear fractional differential equations with singularity. Math. Methods Appl. Sci. 2018, 41(18): 9321-9334.

[25] Baitiche Z, Guerbait K, Hammouche H, et al. Sequential fractional differential equations at resonance. Funct. Differ. Equ, 2020, 26: 167-184.

[26] Alsaedi A, Ntouyas S K, Agarwal R P, et al. On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Adv. Difference Equ., 2015, 2015: 1-12.

[27] Houas M, Samei M E. Existence and Mittag-Leffler-Ulam-Stability Results for Duffing Type Problem Involving Sequential Fractional Derivatives. INT J AP MAT COM-POL, 2022, 8(4): 185.

[28] Lazreg J E, Benkhettou N, Benchohra M, et al. Neutral functional sequential differential equations with Caputo fractional derivative on time scales. Fixed Point Theory Algorithm. Sci. Eng, 2022, 2022(1): 1-16.

[29] Salem A, Almaghamsi L. Solvability of Sequential Fractional Differential Equation at Resonance. Mathematics, 2023, 11(4): 1044.

[30] R.Gaines and J.Mawhin,*Coincidence Degree and Nonlinear Differential Equation*, Springer, Berlin, 1977.

[31] N. Kosmatov, Jiang W, Second-order functional problems with a resonance of dimension one, Differ. Equ. Appl. **8**, No. 3 (2016),349-365.

[32] Sun B, Jiang W, Zhang S. Solvability of Fractional Differential Equations with p-Laplacian and Functional Boundary Value Conditions at Resonance. Mediterr. J. Math, 2022, 19: 1-18.

[33] Zhang S, Sun B. Nonlinear differential equations involving mixed fractional derivatives with functional boundary data. Math. Method. Appl. Sci., 2022, 45(10): 5930-5944.

[34] Zou Y ,Cui Y.Existence results for a functional boundary value problem of fractional differential equations, Adv. Difference Equ., **2013** 2013: 233, 25 pp.

[35] Kosmatov N, Jiang W. Resonant functional problems of fractional order. Chaos, Solitons & Fractals, 2016, 91: 573-579.

[36] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V, Amsterdam, 2006.

[37] I. Podlubny,*Fractional Differential Equations*, Academic Press, New York, London, 1999.

[38] J. Mawhin, Topological degree methods in nonlinear boundary value problems, in NSF-CBMS Regional Conference Series in Math., No. **40**, Amer. Math. Soc., Providence, RI, 1979.