Spatial dynamics of the advective reaction-diffusion equation on funnel-shaped domains

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Abstract

This paper is concerned with the entire solution of the advective reaction-diffusion equation with the bistable nonlinear reaction term on funnel-shaped domains. We focus on the well-posedness and long-time behavior of the entire solution. Because of the impact of advection, the previous super and sub-solutions are no longer applicable, so we study the existence of the entire solution behaving as a planar front by constructing appropriate super-solutions and sub-solutions. In addition, we show the uniqueness and Lyapunov stability of the entire solution. This is probably the first study of the advective reaction-diffusion on funnel-shaped domains.

Keywords: Reaction-diffusion equation; Advective; Bistable; Entire solution.

AMS Subject Classification (2010): 35B08, 35C07, 35K40, 35K57

1 Introduction

In this paper, we study the spatial dynamic properties of the following equation

$$\begin{cases} u_t = \Delta u - a \cdot \nabla u + f(u), & (t,x) \in \mathbb{R} \times \overline{\Omega}, \\ \nu \cdot \nabla u = 0, & (t,x) \in \mathbb{R} \times \partial\Omega. \end{cases} \tag{1.1}$$

Here $u \in [0,1]$ is bounded, Δu is the diffusion term, ∇u is the advective term, a is a constant which is called the diffusion coefficient. ν is the outward

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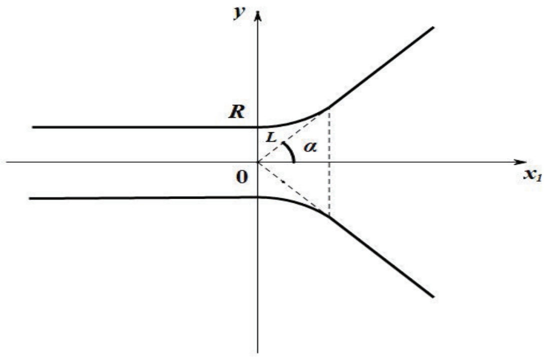

Figure 1: The funnel-shaped domain Ω

unit normal on the boundary $\partial\Omega$. $x = (x_1, y) \in \mathbb{R}^N$, with $y \in \mathbb{R}^{N-1}$. $f \in C^{1,1}([0,1], \mathbb{R})$ is a bistable nonlinear reaction term satisfying

$$\begin{aligned} &f(0) = f(\theta) = f(1) = 0,\ f'(0) < 0,\ f'(\theta) > 0,\ f'(1) < 0, \\ &f(u) < 0 \ \text{ in } (0,\theta),\ f(u) > 0 \ \text{ in } (\theta, 1),\ \int_0^1 f(s)ds > 0. \end{aligned} \tag{1.2}$$

It can be extended to $\mathbb{R}$ by

$$f(s) = f'(0)s \ \text{ for } s < 0, \quad f(s) = f'(1)(s-1) \ \text{ for } s > 1. \tag{1.3}$$

$\Omega \subset \mathbb{R}^N (N \geq 2)$ is the funnel-shaped domain satisfying

$$\Omega = \{x = (x_1, y) : x_1 \in \mathbb{R},\ |y| < h(x_1)\}, \tag{1.4}$$

where $|\cdot|$ signifies the Euclidean norm, $h : \mathbb{R} \to \mathbb{R}^+$ is a function in $C^{2,\beta}(\mathbb{R})$ ($0<\beta<1$) which satisfies

$$\begin{cases} h = R \text{ in } (-\infty, 0], & \text{for some radius } R > 0, \\ 0 \leq h' \leq \tan\alpha \text{ in } \mathbb{R}, & \text{for some angle } \alpha \in [0, \frac{\pi}{2}), \\ h(x_1) = x_1 \tan\alpha \text{ in } [L\cos\alpha, +\infty), & \text{for some } L > R \text{ when } \alpha \neq 0, \end{cases}$$

The domain Ω is rotationally invariant concerning the x_1-axis and its image is shown in Figure 1. In addition, we suppose that there exists a pair of (c,ϕ) ($c > 0$) satisfying

$$\begin{cases} \phi''(z) + c\phi'(z) + f(\phi(z)) = 0 \text{ in } \mathbb{R}, \\ \phi(-\infty) = 1,\ \phi(+\infty) = 0, \\ 0 < \phi(z) < 1 \text{ in } \mathbb{R},\ \phi(0) = \theta. \end{cases} \tag{1.5}$$

The research on the advective reaction-diffusion equation originated from practical applications. In the practical problems, the advection term can reflect the influence of the river [28, 29], chemotactic movement of organisms [32], stirred reactions [11], and bird movement [10] on the research object. In [2], Berestycki showed the influence of the advection term on the propagation of fronts of the model

$$u_t - \Delta u + A\alpha(y)\frac{\partial u}{\partial x_1} = f(u)$$

Berestycki and Nirenberg made the pioneering work of the entire solution of the advective reaction-diffusion equation in [1]. They are concerned with the entire solution of the following type of equation

$$\Delta u - (c + \alpha(y))\partial_1 u + f(u) = 0$$

with three kinds of nonlinearity. Therefore, much work has been devoted to studying the entire solution of the advective reaction-diffusion equation, such as Bu and Wang [8], Indekeu and Smets [17] and Zhao et al. [33]. In addition to the general entire solution, Li et al. [19] attention to the interaction between traveling wave solutions. They construct the new types of entire solutions for the model

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - \alpha(y)\frac{\partial u(x,t)}{\partial x_1} + f(u(x,t))$$

with monostable or ignition temperature nonlinearities, whose propagation phenomenon is similar to that of two traveling wave solutions interacting. Then in [22], they added relevant results on bistable reaction terms. In [23], the existence, uniqueness, and stability of the entire solution for bistable reaction-advection-diffusion equations in heterogeneous media were proved. For the study of this type of entire solution, there are Liu et al. [24], Li et al. [20], Wang and Li [27] and Ma and Wang [25].

The funnel-shaped domains were first defined in [16]. A classical reaction-diffusion equation

$$u_t = \Delta u + f(u)$$

was considered to the large time dynamics of entire solutions. Except for the funnel-shaped domains, other high-dimensional regions are being studied. The infinite cylinders are the regions that have been studied the most frequently, like [14, 15, 19, 20, 22, 23, 25, 30, 31, 34], they are all studying different equations in infinite cylinders. Similar to the infinite cylinders, there are also the "cylinder-like" domains. In [6, 26], the "cylinder-like" domains were considered to analyze the existence of the entire solution and conditions of propagation or blocking phenomenon. Moreover, there are domains with smooth compact obstacles [5, 7], multiple cylindrical branches [13, 21], multiple asymptotically cylindrical branches [12] and the domains are periodic

[3, 4] or the succession of two semi-infinite infinite cylinders with square cross sections [9].

Except that infinite cylinder and periodic domains, advective reaction-diffusion equations are less studied in other regions. Inspired by [5, 16, 18], we consider a more general problem (1.1) in this paper. Due to the introduction of the advection term and the particularity of the region, it leads to the previous super and sub-solutions not being applicable. To solve this problem, we construct new super-solutions and sub-solutions, and obtain the following results.

Theorem 1.1. *For any $R > 0$ and $\alpha \in [0, \frac{\pi}{2})$, (1.1) exists an entire solution $u(t, x)$ which satisfies $u_t > 0$ and $0 < u < 1$ for each $(t, x) \in \mathbb{R} \times \overline{\Omega}$, and that*

$$\begin{cases} u(t,x) - \phi(x_1 - ct - at) \to 0, & a \geq 0, \\ u(t,x) - \phi(-x_1 - ct + at) \to 0, & a < 0, \end{cases} \tag{1.6}$$

as $t \to -\infty$, uniformly in $x \in \overline{\Omega}$. Furthermore, $u(t, x)$ is symmetric with respect to the x_1 axis for any $t \in \mathbb{R}$, that is, u is only related to x_1 and $|y|$.

Theorem 1.2. *For any $(t, x) \in \mathbb{R} \times \overline{\Omega}$, the entire solution $u(t, x)$ converges to a classical solution $u_\infty(x)$ of*

$$\begin{cases} \Delta u_\infty - a \cdot \nabla u_\infty + f(u_\infty) = 0, & \text{in } \overline{\Omega}, \\ \nu \cdot \nabla u_\infty = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

as $t \to +\infty$ in $C^2_{loc}(\overline{\Omega})$. Moreover, the $u_\infty(x)$ satisfies $0 < u_\infty(x) \leq 1$ and

$$\begin{cases} \lim\limits_{x_1 \to -\infty} u_\infty(x) = 1, & a \geq 0, \\ \lim\limits_{x_1 \to +\infty} u_\infty(x) = 1, & a < 0. \end{cases}$$

Theorem 1.3. *For any $R > 0$ and $\alpha \in [0, \frac{\pi}{2})$, $u(t, x)$ is the entire solution satisfying (1.6) in Theorem 1.1. If there is a function $U(x) \in C^2(\overline{\Omega})$ of the (1.7) such that $0 < U(x) \leq 1$ and*

$$\begin{cases} \lim\limits_{x_1 \to -\infty} U(x) = 1 & a \geq 0, \\ \lim\limits_{x_1 \to +\infty} U(x) = 1, & a < 0. \end{cases}$$

Then $u(t, x) \leq U(x)$ for any $(t, x) \in \mathbb{R} \times \overline{\Omega}$.

Theorem 1.1, Theorem 1.2 and Theorem 1.3 are related to the existence of the entire solution of (1.1). The following theorem is about the uniqueness and stability of the entire solution.

Theorem 1.4. *For any $R > 0$ and $\alpha \in [0, \frac{\pi}{2})$, the entire solution $u(t, x)$ of* (1.1) *satisfying* (1.6) *is unique and Lyapunov stable.*

This paper is organized as follows. In Section 2, we prove the existence of the entire solution of (1.1) and then we show some theorems related to existence, that is Theorem 1.2 and Theorem 1.3. We will see that the entire solution of (1.1) is unique and Lyapunov stable in Section 3.

2 Existence of the Entire Solution

In this section, we prove the existence of the entire solution by using the upper and lower solution methods. Motivated by[5] and [16], the key to this method is to construct a suitable super-solution and sub-solution. Therefore, in Subsection 2.1, we give the basic estimates that will be used later. Then we construct the super and sub-solution in Subsection 2.2 and complete the proof of Theorem 1.2 in Subsection 2.3. In Subsection 2.4 and Subsection 2.5, we show the proof of Theorem 1.2 and 1.3.

2.1 Basic Estimates

Suppose that the function $\xi(t)$ satisfies

$$\begin{cases} \xi'(t) = Me^{\mu^*(ct+\xi)} \quad \text{for } t \leq T, \\ \xi(-\infty) = 0, \end{cases} \tag{2.1}$$

where the constant $M > 0$, $T < 0$ are given later, μ^* is the positive root of the equation

$$\mu^2 + c\mu + f'(0) = 0,$$

that is,

$$\mu^* = \frac{-c + \sqrt{c^2 - 4f'(0)}}{2} > 0.$$

From a simple calculation of (2.1), we obtain

$$\xi(t) = \frac{1}{\mu^*} \ln \frac{1}{1 - c^{-1}Me^{\mu^* ct}}$$

and $1 - c^{-1}Me^{\mu^* ct} > 0$. In addition, we suppose that

$$ct + \xi(t) \leq 0 \quad \text{for } t \in (-\infty, T],$$

therefore, we have

$$T := \frac{1}{\mu^* c} \ln \frac{c}{c + M} < 0.$$

The traveling wave solution ϕ decays exponentially and satisfies

$$\begin{cases} b_1 e^{-\mu^* z} \le \phi(z) \le B_1 e^{-\mu^* z}, & z \ge 0, \\ b_2 e^{\mu_* z} \le 1 - \phi(z) \le B_2 e^{\mu_* z}, & z < 0, \end{cases}$$

with $\mu_* = \dfrac{c + \sqrt{c^2 - 4f'(1)}}{2} > 0$, b_1, b_2, B_1, B_2 are positive constants, and the $\phi'(z)$ satisfies

$$\begin{cases} b_3 e^{-\mu^* z} \le -\phi'(z) \le B_3 e^{-\mu^* z}, & z \ge 0, \\ b_4 e^{\mu_* z} \le -\phi'(z) \le B_4 e^{\mu_* z}, & z < 0, \end{cases}$$

with b_3, b_4, B_3, B_4 are positive constants.

The reaction term f satisfies

$$|f(u+v) - f(u) - f(v)| \le Luv, \ 0 \le u, v \le 1. \tag{2.2}$$

where L is a non-negative constant.

2.2 Construction of the super-solution and sub-solution

Before we construct the super-solution and sub-solution, we first show the definition of them. We denote that

$$\mathscr{L}w := w_t - \Delta w + a \cdot \nabla w - f(w). \tag{2.3}$$

Then we can define

Definition 2.1. (The Definition of super-solution and sub-solution) *The function w is called the super-solution of* (1.1) *in* $\mathbb{R} \times \overline{\Omega}$*, if*

$$\begin{cases} \mathscr{L}w \ge 0, & (t,x) \in \mathbb{R} \times \overline{\Omega}, \\ \nu \cdot \nabla w \ge 0, & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases}$$

The function w is called the sub-solution of (1.1) *in* $\mathbb{R} \times \overline{\Omega}$*, if*

$$\begin{cases} \mathscr{L}w \le 0, & (t,x) \in \mathbb{R} \times \overline{\Omega}, \\ \nu \cdot \nabla w \le 0, & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases}$$

Since $a \in \mathbb{R}$, we discuss a in two cases: $a \geqslant 0$ and $a < 0$. First in the range $a \geqslant 0$. We assume that there are two functions

$$w^+(t,x) = \phi(x_1 - ct - at) \tag{2.4}$$

and

$$w^-(t,x) = \begin{cases} \phi(x_1 - ct - at + \xi(t)) - \phi(-x_1 - ct - at + \xi(t)), & x_1 < 0, \\ 0, & x_1 \ge 0. \end{cases} \tag{2.5}$$

Then, we have the following

Lemma 2.2. *w^+ is a super-solution of the* (1.1) *in* $\mathbb{R}\times\overline{\Omega}$ *and for any* $M > 0$, *there exists some* $T_1 \in (-\infty, T]$ *such that* w^- *is a generalized sub-solution of* (1.1) *in* $(-\infty, T_1] \times \overline{\Omega}$.

Proof. It is obvious these conditions are satisfied when $x \in \partial\Omega$. Then we prove whether $\mathscr{L}w^+ \geq 0$ or $\mathscr{L}w^- \leq 0$ when $x \in \Omega$.

First, we show the proof of the super-solution w^+. According to (2.4), we have

$$\begin{aligned} {w^+}_t &= (-c-a)\phi', \\ \nabla w^+ &= \phi', \\ \Delta w^+ &= \phi''. \end{aligned} \tag{2.6}$$

Then substitute (2.6) into (2.3), and there is

$$\begin{aligned} \mathscr{L}w^+ &= (-c-a)\phi' - \phi'' + a\phi' - f(\phi) \\ &= 0 \geq 0. \end{aligned}$$

The proof of the super-solution is completed.

Next, we prove that w^- is a generalized sub-solution of (1.1). It is obvious when $x_1 \geq 0$ because of (2.5), therefore we prove that $\mathscr{L}w^- \leq 0$ in the case $x_1 < 0$.

We suppose that $z_1 := x_1 - ct - at + \xi(t)$, $z_2 := -x_1 - ct - at + \xi(t)$. A straightforward computation shows that

$$\begin{aligned} {w^-}_t &= (-c-a+\xi'(t))(\phi'(z_1) - \phi'(z_2)), \\ \nabla w^- &= \phi'(z_1) + \phi'(z_2), \\ \Delta w^- &= \phi''(z_1) - \phi''(z_2). \end{aligned} \tag{2.7}$$

Substitute (2.7) into (2.3), we have

$$\begin{aligned} \mathscr{L}w^- &= (-c-a+\xi'(t))(\phi'(z_1) - \phi'(z_2)) - \phi''(z_1) + \phi''(z_2) \\ &\quad + a(\phi'(z_1) + \phi'(z_2)) - f(\phi(z_1) - \phi(z_2)) \\ &= \xi'(t)\phi'(z_1) + (2a - \xi'(t))\phi'(z_2) + F(x_1, t) \\ &\leq \xi'(t)(\phi'(z_1) - \phi'(z_2)) + F(x_1, t), \end{aligned} \tag{2.8}$$

where $F(x_1, t) = f(\phi(z_1)) - f(\phi(z_2)) - f(\phi(z_1) - \phi(z_2))$. Owing to the (2.2), then

$$F(x_1, t) \leq L\phi(z_2)(\phi(z_1) - \phi(z_2)).$$

Therefore, the third inequality in (2.8) is transformed into

$$\mathscr{L}w^- \leq \xi'(t)(\phi'(z_1) - \phi'(z_2)) + L\phi(z_2)(\phi(z_1) - \phi(z_2)). \tag{2.9}$$

In order to further verify $\mathscr{L}w^- \leq 0$, we discuss two cases $ct + at - \xi(t) \leq x_1 < 0$ and $x_1 < ct + at - \xi(t)$ respectively.

Case 1. In the case $ct + at - \xi(t) \leq x_1 < 0$, that is, $0 \leq z_1 < z_2$.

Since the definition of ϕ in (1.5), one then infers that

$$\phi(z) \leq \phi(0) = \theta \quad \text{ for } z_1 \leq z \leq z_2,$$

which yeilds $f(\phi(z)) \leq 0$. Thus we have

$$\begin{aligned} \phi'(z_1) - \phi'(z_2) &= -\int_{z_1}^{z_2} \phi''(z)dz = \int_{z_1}^{z_2} (c\phi'(z) + f(\phi(z)))dz \\ &\leq \int_{z_1}^{z_2} c\phi'(z)dz \\ &= c(\phi(z_2) - \phi(z_1)), \end{aligned} \tag{2.10}$$

It follows from the previous (2.9) and the estimates of the $\phi(z)$ that

$$\begin{aligned} \mathscr{L}w^- &\leq c\xi'(t)(\phi(z_2) - \phi(z_1)) + L\phi(z_2)(\phi(z_1) - \phi(z_2)) \\ &= (L\phi(z_2) - c\xi'(t))(\phi(z_1) - \phi(z_2)) \\ &\leq (LB_1 e^{-\mu^*(-x_1 - ct - at + \xi(t))} - cMe^{\mu^*(ct+\xi(t))})(\phi(z_1) - \phi(z_2)) \\ &= e^{\mu^*(ct+\xi(t))}(\phi(z_1) - \phi(z_2))(LB_1 e^{-\mu^*(-x_1 - at + 2\xi(t))} - cM). \end{aligned}$$

Therefore we have $\mathscr{L}w^- \leq 0$ provided that $T_1 \in (-\infty, T]$ is chosen sufficiently negative so that

$$LB_1 e^{-\mu^*(-x_1 - at + 2\xi(t))} - cM \leq 0 \quad \text{ for } -\infty < t \leq T_1. \tag{2.11}$$

Case 2. In the case $x_1 < ct + at - \xi(t)$, that is, $z_1 < 0 < z_2$.

By using estimates of ϕ and ϕ', we have

$$\begin{aligned} \mathscr{L}w^- &\leq \xi'(t)(\phi'(z_1) - \phi'(z_2)) + L\phi(z_2)(\phi(z_1) - \phi(z_2)) \\ &\leq Me^{\mu^*(ct+\xi(t))}(-b_4 e^{\mu_*(x_1 - ct - at + \xi(t))} + B_3 e^{-\mu^*(-x_1 - ct - at + \xi(t))}) \\ &\quad + LB_1 e^{-\mu^*(-x_1 - ct - at + \xi(t))} \\ &= e^{\mu^*(ct+\xi(t))}[M(-b_4 e^{\mu_*(x_1 - ct - at + \xi(t))} + B_3 e^{-\mu^*(-x_1 - ct - at + \xi(t))}) \\ &\quad + LB_1 e^{-\mu^*(-x_1 - at + 2\xi(t))}]. \end{aligned}$$

Therefore we have $\mathscr{L}w^- \leq 0$ provided that $T_1 \in (-\infty, T]$ is chosen sufficiently negative so that

$$M(-b_4 e^{\mu_*(x_1 - ct - at + \xi(t))} + B_3 e^{-\mu^*(-x_1 - ct - at + \xi(t))}) \leq -LB_1 e^{-\mu^*(-x_1 - at + 2\xi(t))} \tag{2.12}$$

for $t \in (-\infty, T_1]$.

In conclusion, we show that w^- is a sub-solution of (1.1) for any $M > 0$ when $T_1 \in (-\infty, T]$ satisfies (2.11) and (2.12). $\square$

In the range $a < 0$, we assume that there are other two functions

$$\widetilde{w}^{+}(t,x) = \phi(-x_1 - ct + at)$$

and

$$\widetilde{w}^{-}(t,x) = \begin{cases} 0, & x_1 \le 0, \\ \phi(-x_1 - ct + at + \xi(t)) - \phi(x_1 - ct + at + \xi(t)), & x_1 > 0. \end{cases}$$

Using the same method, we obtain the following lemma

Lemma 2.3. *$\widetilde{w}^{+}$ is a super-solution of the (1.1) in $\mathbb{R}\times\overline{\Omega}$ and for any $M > 0$, there exists some $\widetilde{T_1} \in (-\infty, T]$ such that $\widetilde{w}^{-}$ is a generalized sub-solution of the (1.1)in $(-\infty, \widetilde{T_1}] \times \overline{\Omega}$.*

2.3 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1 by using the method of upper and lower solutions. Since the proof process is similar in both cases $a \ge 0$ and $a < 0$, we will take the case $a \ge 0$ as an example here.

For any $n \in \mathbb{N}$ and $n > -T_1$, let $u_n(t,x)$ be the solution of the Cauchy problem

$$\begin{cases} (u_n)_t = \Delta u_n - a \cdot \nabla u_n + f(u_n), & (t,x) \in [-n, +\infty) \times \overline{\Omega}, \\ \nu \cdot \nabla u_n = 0, & (t,x) \in [-n, +\infty) \times \partial\Omega, \\ u_n(-n,x) = w^{-}(-n,x) \in [0,1), & x \in \Omega, \end{cases} \tag{2.13}$$

of (1.1). From the strong maximum principle and the well-posedness of Cauchy problem (2.13), we note that

$$0 < u_n(t,x) < 1 \quad \text{for } (t,x) \in [-n, +\infty) \times \overline{\Omega}. \tag{2.14}$$

Because of the axial symmetry of Ω concerning x_1, $u_n(t,x)$ is symmetric about the x_1 axis. Since $w^{-}(t,x)$ is the generalized sub-solution of (1.1) in $(-\infty, T_1] \times \overline{\Omega}$, $w^{+}(t,x)$ is the super-solution of the (1.1) in $\mathbb{R} \times \overline{\Omega}$, we note that

$$w^{-}(t,x) \le u_n(t,x) \le w^{+}(t,x) \quad \text{for } (t,x) \in [-n, T_1] \times \overline{\Omega}. \tag{2.15}$$

Then we plug in $t = -n+1$ to this inequality, we can obtain

$$u_n(-n+1,x) \ge w^{-}(-n+1,x) = u_{n-1}(-n+1,x) \quad \text{for } x \in \overline{\Omega}.$$

Owing to the maximum principle, we have

$$u_n(t,x) \ge u_{n-1}(t,x) \quad \text{for } (t,x) \in [-n+1, +\infty) \times \overline{\Omega},$$

which can see that the sequence $u_n(t,x)$ is non-decreasing in n for $n > -T_1$. Putting $n \to +\infty$, one gets that

$$u_n(t,x) \to u(t,x) \quad \text{in } C^{1,2}_{(t,x);loc}(\mathbb{R} \times \overline{\Omega})$$

from the standard parabolic estimates and monotone bounded theorem. And that u is the entire solution of (1.1).

On account of the axial symmetry of u_n, we know that u is symmetric about the x_1 axis. The comparison principle and (2.14) show that $0 \leq u(t,x) \leq 1$ for $(t,x) \in \mathbb{R} \times \overline{\Omega}$. Combining $u \not\equiv 0$ and $u \not\equiv 1$, we have

$$0 < u(t,x) < 1 \quad \text{for } (t,x) \in \mathbb{R} \times \overline{\Omega}$$

from the strong parabolic maximum principle. Letting $n \to \infty$, (2.15) implies

$$w^-(t,x) \leq u(t,x) \leq w^+(t,x) \quad \text{for } (t,x) \in (-\infty, T_1] \times \overline{\Omega},$$

where we can know

$$u(t,x) - \phi(x_1 - ct - at) \to 0, \quad \text{as } t \to -\infty.$$

Finally, we show that

$$u_t(t,x) > 0 \quad \text{for } (t,x) \in \mathbb{R} \times \overline{\Omega}. \tag{2.16}$$

We first claim that $w^-(t,x)$ is non-decreasing with respect to t when t is sufficiently small. According to the definition of $w^-(t,x)$, we have

$$w_t{}^-(t,x) = \begin{cases} (-c - a + \xi'(t))(\phi'(z_1) - \phi'(z_2)), & x_1 < 0, \\ 0, & x_1 \geq 0. \end{cases}$$

Letting $t \to -\infty$, then $z_1 \to +\infty > 0$, $z_2 \to +\infty > 0$, $-c - a + \xi'(t) < 0$. Combining the fourth inequality of (2.10) and the monotonicity of ϕ, we obtain that

$$w_t{}^-(t,x) \geq 0,$$

when t is sufficiently small. For any $t \in [-n, T_1]$, $x \in \overline{\Omega}$, setting

$$g(t,x) = u_n(t,x) - w^-(t,x),$$

which yields that

$$g_t(t,x) = (u_n)_t(t,x) - w_t{}^-(t,x).$$

If $(u_n)_t(t,x) < 0$, then $g_t(t,x) < 0$, which means that $g(t,x)$ decreases monotonically with respect to t. Since $g(-n,x) = 0$, the function $g(t,x) = u_n(t,x) - w^-(t,x) < 0$ for any $t > -n$, this conflicts with (2.15).

Therefore, $(u_n)_t(t,x) \geq 0$ for $(t,x) \in [-n,+\infty)\times\overline{\Omega}$ and it yields $(u_n)_t(-n,x) \geq 0$ for $x \in \overline{\Omega}$ and all sufficiently large n. Then, we get

$$(u_n)_t(t,x) \geq 0 \quad \text{for } (t,x) \in [-n,+\infty) \times \overline{\Omega}$$

by the maximum principle. Letting $n \to +\infty$, we have

$$u_t(t,x) \geq 0 \quad \text{for } (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

Since $u_t \not\equiv 0$, we use the strong maximum principle can obtain (2.16).

2.4 Proof of Theorem 1.2

In this subsection, we study the long-time behavior of the entire solution $u(t,x)$ of the (1.1), that is Theorem 1.2, and show the proof of it. First, in the case $a \geq 0$, the parabolic estimates and (2.16) imply that

$$u(t,x) \to u_\infty(x) \quad \text{as } t \to +\infty \text{ uniformly in } x \in \overline{\Omega}$$

in $C^2_{loc}(\overline{\Omega})$. Since u is the entire solution of (1.1), the limit u_∞ satisfies

$$\begin{cases} \Delta u_\infty - a \cdot \nabla u_\infty + f(u_\infty) = 0, & \text{in } \overline{\Omega}, \\ \nu \cdot \nabla u_\infty = 0, & \text{on } \partial\Omega. \end{cases}$$

Taking limits simultaneously on both sides of the inequality for $0 < u(t,x) < 1$, we obtain that $0 \leq u_\infty(x) \leq 1$. If $u_\infty(x) = 0$, that is, $\lim_{t\to+\infty} u(t,x) = 0$, owing to (2.16), when $t < +\infty$, we have $u < 0$, but this is impossible because

$$0 < u(t,x) < 1 \quad \text{for } (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

by Theorem 1.1. Therefore, we have

$$0 < u_\infty(x) \leq 1 \ \text{ for } x \in \overline{\Omega}.$$

Since u increases monotonically with respect to t, one then infers

$$w^-(t,x) \leq u(t,x) < u_\infty(x) \leq 1 \quad \text{for } (t,x) \in (-\infty, T_1] \times \overline{\Omega}.$$

It follows from the inequality above that

$$\begin{aligned} w^-(T_1,x) &= \phi(x_1 - cT_1 - aT_1 + \xi(T_1)) - \phi(-x_1 - cT_1 - aT_1 + \xi(T_1)) \\ &\leq u_\infty(x) \leq 1 \end{aligned} \tag{2.17}$$

when $x \in \overline{\Omega}$ and $x_1 < 0$. Hence, we can see that

$$\lim_{x_1\to-\infty} u_\infty(x) = 1.$$

In the case $a < 0$, we can use the same method to obtain $\lim_{x_1\to+\infty} u_\infty(x) = 1$.

2.5 Proof of Theorem 1.3

When $a \geq 0$, according to the definition of the nonlinear term f, we can see that $f(1) = 0$ and $f'(1) < 0$ by (1.2), f can be extended to $(1, +\infty)$ by $f(s) = f'(1)(s-1)$ for $s > 1$ from (1.3). Setting $\varsigma > 0$ such that $f' < 0$ in $[1-\varsigma, +\infty)$ and $C > 0$ satisfying

$$1 - \varsigma \leq U(x) \leq 1 \quad \text{for } x \in \overline{\Omega} \text{ with } x_1 \leq -C. \tag{2.18}$$

If there exists $T_1 \in (-\infty, T]$ such that for any $t \leq T_1$ and $x \in \overline{\Omega}$, we have $w^-(t,x) \leq U(x)$. Using the parabolic maximum principle and the relationship between w^- and u_n, then we can prove that $u(t,x) \leq U(x)$ for any $(t,x) \in \mathbb{R} \times \overline{\Omega}$, which is what we ultimately want to prove. Therefore, we then claim the following lemma.

Lemma 2.4. *There exists $T_1 \in (-\infty, T]$ such that*

$$w^-(t,x) \leq U(x) \quad \textit{for } t \leq T_1 \textit{ and } x \in \overline{\Omega} \textit{ with } x_1 \geq -C. \tag{2.19}$$

Proof. According to (2.18), if $x_1 \geq -C$, then we find that $U(x) \leq 1 - \varsigma$ or $U(x) \geq 1$. Owing to the assumption $0 < U(x) \leq 1$, we have $0 < U(x) \leq 1-\varsigma$ or $U(x) = 1$.

When $U(x) = 1$, according to the definition of w^-, we can see that $0 \leq w^- < 1$. It is obvious that $w^-(t,x) \leq U(x)$ for $(t,x) \in (-\infty, T_1) \times \overline{\Omega}$ with $x_1 \geq -C$.

When $0 < U(x) \leq 1 - \varsigma$, we use the method of proof by contradiction to prove. If for any $T_1' \in (-\infty, T]$, there is $w^-(t,x) > U(x)$ for $t \leq T_1'$ and $x \in \overline{\Omega}$ with $x_1 \geq -C$. The inequality (2.17) infers that there exists $T_1 \in (-\infty, T]$ such that

$$w^-(T_1, x) \leq u_\infty(x) \leq 1$$

for $x \in \overline{\Omega}$ and $0 < u_\infty(x) \leq 1$ is the solution of (1.7) satisfying the limit $\lim_{x_1 \to -\infty} u_\infty(x) = 1$. However, $0 < U(x) \leq 1 - \varsigma$ satisfies all conditions of $u_\infty(x)$, which is a contradiction with the assumption. □

Proof of Theorem 1.3. We first claim that $w^-(t,x) \leq U(x)$ for any $t \leq T_1$ and $x \in \overline{\Omega}$. We define that

$$\varrho^* = \min\left\{\varrho \geq 0 : w^-(t,x) \leq U(x) + \varrho, t \leq T_1, x \in \overline{\Omega}\right\}. \tag{2.20}$$

Then we will prove that $\varrho^* = 0$.

If $\varrho^* > 0$, then there exists $(t_*, x_*) \in (-\infty, T_1] \times \overline{\Omega}$ with $(x_1)_* < -C$, such that

$$w^-(t_*, x_*) = U(x_*) + \varrho^*. \tag{2.21}$$

Otherwise, for any $(t,x) \in (-\infty, T_1] \times \overline{\Omega}$ with $x_1 < -C$, there is $w^-(t,x) \neq U(x) + \varrho^*$, then we have

$$w^-(t,x) < U(x) + \varrho^* \tag{2.22}$$

for any $(t,x) \in (-\infty, T_1] \times \overline{\Omega}$ with $x_1 < -C$ by the (2.20). Owing to Lemma 2.4, we see that

$$w^-(t,x) \leq U(x) < U(x) + \varrho^* \tag{2.23}$$

for any $(t,x) \in (-\infty, T_1] \times \overline{\Omega}$ with $x_1 \geq -C$. Combining (2.22) and (2.23), we obtain

$$w^-(t,x) < U(x) + \varrho^* \quad \text{for } (t,x) \in (-\infty, T_1] \times \overline{\Omega}.$$

It follows from the density that there exists $\varrho_1^* \geq 0$, such that $w^-(t,x) < U(x) + \varrho_1^* < U(x) + \varrho^*$. This is a contradiction with the definition of ϱ^* that ϱ^* is the minimum number making $w^-(t,x) \leq U(x) + \varrho$.

Since $1-\varsigma \leq U(x) \leq 1$ for $x \in \overline{\Omega}$ with $x_1 \leq -C$, we know that $1-\varsigma+\varrho^* \leq U(x) + \varrho^* \leq 1+\varrho^*$. The assumption that $f' < 0$ in $[1-\varsigma, +\infty)$ denotes that f monotonically decreases in $[1-\varsigma, +\infty)$. That is

$$f(U(x) + \varrho^*) \leq f(U(x))$$

for any $x \in \overline{\Omega}$ with $x_1 \leq -C$, which implies $U(x) + \varrho^*$ is the super-solution of the (1.7) in $x \in \overline{\Omega}$ with $x_1 \leq -C$.

Notice that w^- is a generalized sub-solution of the (1.1) in $(-\infty, T_1] \times \overline{\Omega}$ by Lemma 2.2, then from the strong parabolic maximum principle and (2.21), we have

$$w^-(t,x) = U(x) + \varrho^*$$

for any $t \leq t_*$ and $x \in \overline{\Omega}$ with $x_1 \leq -C$.

When $x_1 \to -\infty$, $w^-(t,x) = U(x) + \varrho^* \to 1 + \varrho^* > 1$ is a contradiction with $w^-(t,x) < 1$. Therefore, $\varrho^* = 0$ and it yields that

$$w^-(t,x) \leq U(x) \tag{2.24}$$

for any $t \leq T_1$ and $x \in \overline{\Omega}$.

Then we claim that $u(t,x) \leq U(x)$ for all $(t,x) \in \mathbb{R}\times\overline{\Omega}$. Since $u_n(-n,x) = w^-(-n,x) \in [0,1)$ for $n \in \mathbb{N}$ with $n \geq -T_1$ in $x \in \Omega$ from (2.13), according to (2.24), we have

$$u_n(-n,x) = w^-(-n,x) \leq U(x) \quad \text{for } x \in \overline{\Omega}.$$

According to the parabolic maximum principle, we know $u_n(t,x) \leq U(x)$ for $t \geq -n$ and $n \in \mathbb{N}$ with $n \geq -T_1$ in $(t,x) \in [-n, +\infty) \times \overline{\Omega}$. Let $n \to +\infty$, we obtain

$$u(t,x) \leq U(x) \quad \text{for } (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

In the case of $a < 0$, we can use the similar arguments to verify $u(t,x) \leq U(x)$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$. The proof of this theorem is complete.

3 Uniqueness and Stability of the Entire Solution

In this section, we begin to prove Theorem 1.4. We divide the proof into two parts. The first part is to prove the uniqueness of the entire solution, and the second part is to prove the stability of the entire solution.

3.1 Uniqueness of the Entire Solution

First is the uniqueness of the entire solution $u(t, x)$. Before starting the proof, we provide a lemma. Assuming

$$G_\gamma(t) := \{x \in \Omega : \gamma \leq u(t, x) \leq 1 - \gamma\}, \quad \gamma \in (0, \frac{1}{2}].$$

By (1.6), we can know that for any $\gamma \in (0, \frac{1}{2}]$, there exists $T_\gamma \in \mathbb{R}$ and $M_\gamma \in (0, +\infty)$, such that for any $t \in (-\infty, T_\gamma]$, when $a \geq 0$,

$$G_\gamma(t) \subset \{x \in \Omega : |x_1 - ct - at| \leq M_\gamma\} \subset \left\{x \in \mathbb{R}^N : x_1 \leq -1\right\}, \tag{3.1}$$

when $a < 0$,

$$G_\gamma(t) \subset \{x \in \Omega : |-x_1 - ct + at| \leq M_\gamma\} \subset \left\{x \in \mathbb{R}^N : x_1 \geq 1\right\}. \tag{3.2}$$

Lemma 3.1. *$u(t, x)$ is the entire solution of* (1.1) *satisfying* (1.6). *Then for any $\gamma \in (0, \frac{1}{2}]$, there exists $\delta_0 > 0$ such that $u_t(t, x) \geq \delta_0$ for any $t \in (-\infty, T_\gamma]$ and $x \in G_\gamma(t)$.*

Proof. We use the method of proof by contradiction to prove this lemma. Suppose there exists $\gamma_0 \in (0, \frac{1}{2}]$, for any $\delta > 0$, there exists the sequence $t_k \in (-\infty, T_{\gamma_0}]$ and $x_k := (x_{k1}, x_{k2}, \cdots, x_{kN}) \in G_{\gamma_0}(t)$, such that

$$u_t(t_k, x_k) \to 0 \quad \text{as } k \to +\infty.$$

The proof process for $a \geq 0$ and $a < 0$ is similar, let's take $a \geq 0$ as an example.

Case 1. If $t_k \to t^* \in (-\infty, T_{\gamma_0}]$ as $k \to +\infty$.

The condition $x_k \in G_{\gamma_0}(t)$ yields that the sequence x_{k1} is bounded. On the base of the monotonic bounded principle, we know that

$$x_{k1} \to x_1^* \quad \text{as } k \to +\infty.$$

Consider that

$$u_k(t, x) := u(t, x + x_k),$$

then u_k is defined in $(t, x) \in (-\infty, T_{\gamma_0}] \times \overline{\Omega}$, hence (1.4) and (3.1). By parabolic estimates, we select a subsequence and denote it again as $\{u_k\}$, then we can obtain

$$u_k(t, x) \to u^*(t, x) \quad \text{as } k \to +\infty$$

in $C^{1,2}_{loc}((-\infty, T_{\gamma 0}] \times \overline{\Omega})$ and $u^*(t,x)$ satisfies (1.1) for any $(t,x) \in (-\infty, T_{\gamma 0}] \times \overline{\Omega}$. Based on the monotonicity of u with respect to t in $\mathbb{R}$ and the convergence of u_k, it can be inferred that

$$u_t^*(t^*, 0) = 0,\ u_t^*(t,x) \geq 0 \quad \text{for } (t,x) \in (-\infty, T_{\gamma 0}] \times \overline{\Omega}.$$

Therefore, by the strong maximum principle, we know

$$u_t^*(t,x) \equiv 0 \quad \text{for } t \leq t^*.$$

But when $t \to -\infty$, there is a contradiction between this equation and

$$u^*(t,x) - \phi(x_1 + x^* - ct - at) \to 0 \quad \text{as } t \to -\infty$$

uniformly in $x \in \overline{\Omega}$ from (1.6).
Case 2. If $t_k \to -\infty$ as $k \to +\infty$.

Similarly, we consider that

$$u_k(t,x) := u(t + t_k, x + x_k).$$

Then, we can also find a subsequence $u_k(t,x) \to u^*(t,x)$ when $k \to +\infty$ and $u^*(t,x)$ satisfies $u_t^*(0,0) = 0$. Whence

$$u_t^*(t,x) \equiv 0 \quad \text{for } t \leq 0.$$

This is impossible because there exists a $\gamma_1 \in [-M_{\gamma 0}, M_{\gamma 0}]$ such that

$$u^*(t,x) = \phi(x_1 - ct - at + \gamma_1).$$

The proof of this lemma is complete. $\square$

Now, we prove the uniqueness of $u(t,x)$. Assuming there exists another entire solution v of (1.1), take $\kappa \in (0, \frac{1}{4})$ as small as possible, such that

$$f'(s) \leq -\varepsilon \quad \text{for } s \in (-\infty, 2\kappa] \cup [1 - 2\kappa, +\infty) \tag{3.3}$$

for $\varepsilon > 0$. Then for $\epsilon \in (0, \kappa)$, there is $t_\epsilon < 0$, such that

$$\|v(t,x) - u(t,x)\|_{L^\infty(\Omega)} < \epsilon \quad \text{for } (t,x) \in (-\infty, t_\epsilon] \times \overline{\Omega}. \tag{3.4}$$

Suppose $t_0 \in (-\infty, t_\epsilon)$, we define that

$$\begin{aligned} W^+(t,x) &= u(t + \sigma\epsilon(1 - e^{-\varepsilon(t-t_0)}), x) + \epsilon e^{-\varepsilon(t-t_0)}, \\ W^-(t,x) &= u(t - \sigma\epsilon(1 - e^{-\varepsilon(t-t_0)}), x) - \epsilon e^{-\varepsilon(t-t_0)}, \end{aligned} \tag{3.5}$$

where the $\sigma > 0$ will be provided later. By (3.4), we obtain

$$W^-(t_0, x) \leq v(t_0, x) \leq W^+(t_0, x) \quad \text{for } x \in \overline{\Omega}.$$

We then claim that $W^+(t,x)$ and $W^-(t,x)$ are the generalized super-solution and sub-solution of (1.1) in $[t_0,t_\epsilon]\times\overline{\Omega}$.

First is the super-solution. From a calculation of the (3.5), we can see

$$\begin{aligned}
{W^+}_t\ &= (1+\sigma\epsilon\varepsilon e^{-\varepsilon(t-t_0)})u_t - \epsilon\varepsilon e^{-\varepsilon(t-t_0)},\\
\nabla W^+ &= \nabla u,\\
\Delta W^+ &= \Delta u.
\end{aligned}$$

It follows from the equations above that

$$\begin{aligned}
\mathscr{L}W^+ &= (1+\sigma\epsilon\varepsilon e^{-\varepsilon(t-t_0)})u_t - \epsilon\varepsilon e^{-\varepsilon(t-t_0)} - \Delta u + a\nabla u - f(W^+)\\
&= \sigma\epsilon\varepsilon e^{-\varepsilon(t-t_0)}u_t - \epsilon\varepsilon e^{-\varepsilon(t-t_0)} + f(u) - f(u+\epsilon e^{-\varepsilon(t-t_0)})\\
&= \sigma\epsilon\varepsilon e^{-\varepsilon(t-t_0)}u_t - \epsilon\varepsilon e^{-\varepsilon(t-t_0)} - \epsilon e^{-\varepsilon(t-t_0)}f'(u+\vartheta\epsilon e^{-\varepsilon(t-t_0)})\\
&= \epsilon e^{-\varepsilon(t-t_0)}[\sigma\varepsilon u_t - \varepsilon - f'(u+\vartheta\epsilon e^{-\varepsilon(t-t_0)})],
\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}
&\vartheta = \vartheta(t,x)\in(0,1),\\
&u = u(t+\sigma\epsilon(1-e^{-\varepsilon(t-t_0)}),x),\\
&u_t = u_t(t+\sigma\epsilon(1-e^{-\varepsilon(t-t_0)}),x).
\end{aligned}$$

Case 1. If $x\in G_\kappa(t+\sigma\epsilon(1-e^{-\varepsilon(t-t_0)}))$, there are $\kappa\le u\le 1-\kappa$ and

$$\kappa+\vartheta\epsilon e^{-\varepsilon(t-t_0)} \le u+\vartheta\epsilon e^{-\varepsilon(t-t_0)} \le 1-\kappa+\vartheta\epsilon e^{-\varepsilon(t-t_0)}.$$

Since $\kappa>0$, $0<\vartheta<1$ and $0<\epsilon<\kappa$, one then infers $\kappa+\vartheta\epsilon e^{-\varepsilon(t-t_0)}>0$. Because of $0<\vartheta<1$ and $\varepsilon>0$, we have

$$0<\vartheta\epsilon e^{-\varepsilon(t-t_0)} < \epsilon e^{-\varepsilon(t-t_0)} < \epsilon < \kappa, \tag{3.7}$$

which infers that $1-\kappa+\vartheta\epsilon e^{-\varepsilon(t-t_0)}<1$. Hence, we can get

$$0<u+\vartheta\epsilon e^{-\varepsilon(t-t_0)}<1.$$

Applying the lemma 3.1 to the fourth equation of (3.6), we know

$$\begin{aligned}
\mathscr{L}W^+ &\ge \epsilon e^{-\varepsilon(t-t_0)}(\sigma\varepsilon\delta_0 - \varepsilon - \max_{0\le s\le 1} f'(s))\\
&\ge \epsilon\sigma\varepsilon\delta_0 e^{-\varepsilon(t-t_0)}
\end{aligned}$$

by (3.3). Therefore, $\mathscr{L}W^+\ge 0$ when σ is sufficiently large.

Case 2. If $x\notin G_\kappa(t+\sigma\epsilon(1-e^{-\varepsilon(t-t_0)}))$, there are $u<\kappa$ or $u>1-\kappa$. Then with $0<u<1$ and (3.7), one sees that $u+\vartheta\epsilon e^{-\varepsilon(t-t_0)}\in[0,2\kappa]\cup[1-\kappa,1+\kappa]$, which yields that

$$f'(u+\vartheta\epsilon e^{-\varepsilon(t-t_0)})\le -\varepsilon.$$

Since $u_t > 0$, $\sigma > 0$ and $\varepsilon > 0$, we know

$$\begin{aligned}\mathscr{L}W^+ &= \epsilon e^{-\varepsilon(t-t_0)}[\sigma\varepsilon u_t - \varepsilon - f'(u + \vartheta\epsilon e^{-\varepsilon(t-t_0)})] \\ &\geq \epsilon e^{-\varepsilon(t-t_0)}[-\varepsilon - f'(u + \vartheta\epsilon e^{-\varepsilon(t-t_0)})] \\ &\geq 0.\end{aligned}$$

In summary, when σ is sufficiently large, $\mathscr{L}W^+ \geq 0$ for $t \in [t_0, t_\epsilon]$, $x \in \overline{\Omega}$. Similarly, we can prove $\mathscr{L}W^- \leq 0$. Hence, we have

$$W^+(t,x) \leq v(t,x) \leq W^-(t,x) \quad \text{ for } (t,x) \in [t_0, t_\epsilon] \times \overline{\Omega},$$

where $t_0 \in (-\infty, t_\epsilon)$. Substituting (3.5) into the above equation and letting $t_0 \to -\infty$, then

$$u(t - \sigma\epsilon, x) \leq v(t,x) \leq u(t + \sigma\epsilon, x) \quad \text{ for } (t,x) \in (-\infty, t_\epsilon] \times \overline{\Omega}.$$

Applying the comparison principle, the above inequalities hold in $(t,x) \in \mathbb{R} \times \overline{\Omega}$. Letting $\epsilon \to 0$, we have $v(t,x) \equiv u(t,x)$, which proves the uniqueness of the entire solution of (1.1).

3.2 Stability of the Entire Solution

This subsection is devoted to proving the stability of the entire solution. We are concerned with the Lyapunov stability of the entire solution $u(t,x)$ satisfying (1.6) of (1.1). Considering the Cauchy problem of (1.1)

$$\begin{cases} u_t = \Delta u - a \cdot \nabla u + f(u), & (t,x) \in (0, +\infty) \times \overline{\Omega}, \\ \nu \cdot \nabla u = 0, & (t,x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x; u_0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.8}$$

Set $u(t,x;u_0)$ be the classical solution of (3.8), $u(t,x)$ be the entire solution of (1.1). Therefore, we have the following definition

Definition 3.2. (The Definition of Lyapunov Stability)
The entire solution $u(t,x)$ is Lyapunov stability, if for any $\delta_* > 0$, there exists a constant $\varepsilon_* > 0$, such that

$$\|u(t,x;u_0) - u(t,x)\|_{L^\infty((0,+\infty)\times\overline{\Omega})} \leq \delta_*$$

when $\|u_0(x) - u(0,x)\|_{L^\infty(\Omega)} \leq \varepsilon_*$.

Then we assume that there are two functions

$$\begin{aligned} U^+(t,x) &= u(t + \sigma_1\varepsilon_*(1 - e^{-\varepsilon_1 t}), x) + \varepsilon_* e^{-\varepsilon_1 t}, \\ U^-(t,x) &= u(t - \sigma_1\varepsilon_*(1 - e^{-\varepsilon_1 t}), x) - \varepsilon_* e^{-\varepsilon_1 t}, \end{aligned} \tag{3.9}$$

where the constant $\varepsilon_1 > 0$, and the constant $\sigma_1 > 0$ will be provided later.

According to the method of proving the super-solution and sub-solution $W^\pm$, when σ_1 is sufficiently large, we can obtain the following lemma

Lemma 3.3. *For any $\varepsilon_* \in (0, \kappa]$ with $0 < \kappa < \frac{1}{4}$ given in* (3.3)*, $U^+(t,x)$ and $U^-(t,x)$ are respectively a super-solution and a sub-solution of* (1.1) *for $(t,x) \in [0,+\infty) \times \overline{\Omega}$.*

Now, we prove the Lyapunov stability of $u(t,x)$. According to Definition 3.2, we set that for any $\delta_* > 0$, there exists a constant $\varepsilon_* > 0$, such that

$$\|u_0(x) - u(0,x)\|_{L^\infty(\Omega)} \le \varepsilon_*.$$

Then we need to verify $\|u(t,x;u_0) - u(t,x)\|_{L^\infty((0,+\infty)\times\overline{\Omega})} \le \delta_*$.

By the Lemma 3.3, $U^\pm$ are the super-solution and sub-solution of (1.1) which imply

$$U^-(0,x) = u(0,x) - \varepsilon_* \le u_0(x) \le u(0,x) + \varepsilon_* = U^+(0,x)$$

for $x \in \Omega$ and

$$\nu \cdot \nabla U^- = \nu \cdot \nabla u = \nu \cdot \nabla U^+ = 0 \quad \text{ for } x \in \partial\Omega.$$

Owing to the comparison principle, we have

$$U^-(t,x) \le u(t,x;u_0) \le U^+(t,x) \quad \text{ for } (t,x) \in [0,+\infty) \times \overline{\Omega}. \tag{3.10}$$

With the definitions of $U^+(t,x)$ and $U^-(t,x)$, for all $x \in \overline{\Omega}$ and $t \ge 0$, the (3.10) yields

$$\begin{aligned} u(t,x;u_0) \le U^+(t,x) \le u(t + \sigma_1\varepsilon_*(1 - e^{-\varepsilon_1 t}), x) + \varepsilon_*, \\ u(t,x;u_0) \ge U^-(t,x) \ge u(t - \sigma_1\varepsilon_*(1 - e^{-\varepsilon_1 t}), x) - \varepsilon_*, \end{aligned} \tag{3.11}$$

Since $0 < 1 - e^{-\varepsilon_1 t} < 1$ for $t > 0$ and $u_t(t,x) > 0$ for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, using the Lagrange middle-value theorem implies

$$\begin{aligned} u(t - \sigma_1\varepsilon_*(1 - e^{-\varepsilon_1 t}), x) - \varepsilon_* &\ge u(t - \sigma_1\varepsilon_*, x) - \varepsilon_* \\ &= u(t,x) - \sigma_1\varepsilon_* u_t(t - \sigma_1\varepsilon_* e^{-\varepsilon_1 t}, x) - \varepsilon_* \\ &\ge u(t,x) - \sigma_1\varepsilon_* \sup_{(t,x)\in\mathbb{R}\times\overline{\Omega}} u_t(t,x) - \varepsilon_* \end{aligned}$$

for $(t,x) \in (0,+\infty) \times \overline{\Omega}$. Therefore, from (3.11), we know

$$u(t,x;u_0) \ge u(t,x) - \sigma_1\varepsilon_* \sup_{(t,x)\in\mathbb{R}\times\overline{\Omega}} u_t(t,x) - \varepsilon_*. \tag{3.12}$$

Similarly, we can obtain the following inequality

$$\begin{aligned} u(t + \sigma_1\varepsilon_*(1 - e^{-\varepsilon_1 t}), x) + \varepsilon_* &\le u(t + \sigma_1\varepsilon_*, x) + \varepsilon_* \\ &= u(t,x) + \sigma_1\varepsilon_* u_t(t + \sigma_1\varepsilon_*(1 - e^{-\varepsilon_1 t}), x) + \varepsilon_* \\ &\le u(t,x) + \sigma_1\varepsilon_* \sup_{(t,x)\in\mathbb{R}\times\overline{\Omega}} u_t(t,x) + \varepsilon_* \end{aligned}$$

for $(t,x)\in(0,+\infty)\times\overline{\Omega}$. Then from (3.11) again, we can get that

$$u(t,x;u_0)\le u(t,x)+\sigma_1\varepsilon_*\sup_{(t,x)\in\mathbb{R}\times\overline{\Omega}}u_t(t,x)+\varepsilon_*. \tag{3.13}$$

Combining (3.12) and (3.13), for any $\delta_*>0$ and $(t,x)\in(0,+\infty)\times\overline{\Omega}$, it is obvious that $|u(t,x;u_0)-u(t,x)|\le(1+\sigma_1\sup_{(t,x)\in\mathbb{R}\times\overline{\Omega}}u_t(t,x))\varepsilon_*\le\delta_*$ with $\varepsilon_*\le\min\left\{\kappa,\frac{\delta_*}{1+\sigma_1\sup_{(t,x)\in\mathbb{R}\times\overline{\Omega}}u_t(t,x)}\right\}$.

To sum up, for any $\delta_*>0$, there exists a constant $\varepsilon_*>0$, such that

$$\|u(t,x;u_0)-u(t,x)\|_{L^\infty((0,+\infty)\times\overline{\Omega})}\le\delta_*$$

when $\|u_0(x)-u(0,x)\|_{L^\infty(\Omega)}\le\varepsilon_*$.

Combining the Subsection 3.1 and 3.2, the proof of Theorem 1.4 is complete.

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