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ASYMPTOTIC BEHAVIOR OF A DELAYED NONLOCAL DISPERSAL LOTKA-VOLTERRA COMPETITIVE SYSTEM*

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Abstract This paper investigates the asymptotic behavior of a nonlocal dispersal Lotka-Volterra competitive system with time delay across the entire $\mathbb{R}^N$. We establish $L^\infty-$decay estimates of solutions of linear systems converging to equilibria utilizing the Fourier transform method applied to the fundamental solution and the Fourier splitting technique. For the nonlinear time-delayed nonlocal dispersal Lotka-Volterra competitive system, we leverage the results from linear systems and obtain the long-time behavior of solutions of the nonlinear system manifesting as the form of time-exponential. More precisely, we further deduce $L^\infty-$decay estimates of solutions of the original nonlinear system through the properties of convolution and Hölder inequality. Additionally, numerical simulations are presented to bolster the principal theoretical results and illustrate that the time delay impedes species growth.

Keywords Asymptotic stability; Nonlocal dispersal; Competitive system; Fourier transform

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1. Introduction

The exploration of nonlocal dispersal problems has sparked significant interest in understanding the impact of nonlocal interactions on diffusion through mathematical models, tracing back to the pioneering work of Kolmogorov–Petrovskii–Piscounov (KPP) [22]. In the realm of cell biology and the behavior of species, Lee et al. [23] introduced the notion of nonlocal and substantiated it theoretically with concrete examples. Under the circumstances of local and nonlocal spatially, they investigated

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explicit models of mathematical biology and showed that nonlocal models are appropriate simplifications of local models when the rate differences between local fast and slow dynamics are sufficiently large. Further, this model and its deformations have been applied to a wide range of applications such as heat conduction in materials with long-range interactions, diffusion of species in biologically heterogeneous environments, and fluid dynamics, see [1, 20, 24, 31, 32, 38, 45, 49]. For instance, in biological populations, individuals move in extensive spatial domains rather than being restricted to small neighborhoods, which leads to spatially nonlocal interactions. Consequently, the study of nonlocal dispersal problems holds paramount significance.

To begin with, in terms of the Lotka-Volterra competitive reaction diffusion system, Ruan and Zhao [36] studied two types of two species reaction diffusion model with time delay, namely, predator-prey system and competitive system. They proved uniform persistence and global extinction of systems. Wang and Lv [41] investigated the entire solution of a diffusive Lotka–Volterra competitive system with nonlocal delays. Both the comparing argument and the upper-lower solution method were applied to derive the existence of the entire solution. In [44], Xu et al. demonstrated that the dynamics of anti-symmetric Lotka-Volterra system is determined by some linear inequalities. Furthermore, researches on the Lotka–Volterra competitive system have been extensive which can be found in [6, 12, 27, 28, 39, 42, 47, 51, 52].

In recent decades, the field of population dynamics has been enriched by numerous nonlocal dispersal mathematical models. For a scalar nonlocal dispersal equation, Pan et al. [35] dedicated their efforts to the study of a nonlocal dispersal single species model with time delay. They showed that the existence, uniqueness and asymptotic behavior of traveling wavefronts can be obtained by the upper-lower solution method and the squeezing technique. Subsequently, Huang et al. [14] considered the nonlocal dispersal equations with time delay and monostable nonlinearity in N-dimensional space. They demonstrated the global stability of traveling waves, including the exponential rate of convergence for noncritical plane wavefronts and the algebraic rate of convergence for critical wavefronts. These analyses were facilitated by the Fourier transform method and the weighted energy method with appropriate selection of weighting functions. Recently, Li et al. [26] studied the dynamics of a population model with nonlocal dispersal in a shifting environment characterized by a shrinking favorable region. They elucidated the conditions conducive to species survival and the relationship between the speed of the shifting habitat edge and the wave speed. In addition, we can refer to [2, 10, 16, 25, 29, 50] and the references therein for more related works. In what follows, we focus on literatures with respect to the nonlocal dispersal system. Yu and Yuan [48] proposed a nonlocal dispersal system with delays as follows,

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(x,t) = d_1(J_1 * u_1 - u_1)(x,t) + f_1(u_1(x,t+\theta), u_2(x,t+\theta)), \\ \dfrac{\partial u_2}{\partial t}(x,t) = d_2(J_2 * u_2 - u_2)(x,t) + f_2(u_1(x,t+\theta), u_2(x,t+\theta)), \end{cases} \tag{1.1}$$

where $d_i > 0$ $(i = 1, 2)$, $-\tau \leq \theta \leq 0$ and τ is the maximal time delay. Further, f_i represents the continuous function and the kernel J_i is nonnegative, symmetric and unit for $i = 1, 2$. They got the existence of traveling wave solutions as well as asymptotic behavior of the system (1.1) via the cross-iteration technique and the

Schauder's fixed point theorem. Bao et al. [3] were concerned with the spreading speed and linear determinacy of the nonlocal dispersal competitive system. They derived the upper and lower bounds of spreading speed intervals for the nonlocal dispersal competitive system. Meanwhile, linear determinacy for this system was also obtained by drawing on conclusions from the cooperative system, along with the principal Lyapunov exponent and the principal Floquent bundle theory. For more works on the study of the nonlocal dispersal system, we refer to [4, 9, 11, 13, 19, 40, 43, 46, 53] and the references therein.

When it comes to the essential problem in the theory of properties of solutions of nonlocal dispersal equations, it is extremely important to study the asymptotic behavior of solutions of nonlocal dispersal equations. For the nonlocal dispersal equation

$$u_t = J * u - u + f(u), \tag{1.2}$$

Coville and Dupaigne [8] deduced the variational formulas used to express the speed of traveling wavefronts and decay rates of solutions of the equation (1.2). Chasseigne et al. [5] studied the asymptotic behavior of the nonlocal dispersal equation (1.2) for Cauchy problems. It was proved that the behavior of the Fourier transform of J near the origin determines the decay rate of solutions of the equation, which is associated with the behavior of J as time t tends to infinity. Moreover, they showed that if the Fourier transform of $J(x)$ satisfies $\hat{J}(\omega) = 1 - k\,|\eta|^{2\beta} + o(|\eta|^{2\beta})$ ($\beta \in (0, 1]$), then the asymptotic behavior coincides with that of the evolutionary solution given by the β fractional power of the Laplacian. In a bounded smooth domain, Cortazar et al. [7] considered a nonlocal dispersal equation with Dirichlet boundary conditions. It was shown that the solution of the nonlocal problem is uniformly similar to the solution of the corresponding Dirichlet problem of the classical heat equation. Ignat and Rossi [17] studied the following nonlocal equation with convective effects,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - (J * u - u)(x,t) = (G * (f(u)) - f(u))(x,t), & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where J denotes radially symmetric while G is not necessarily symmetric. $J * u - u$ represents the nonlocal diffusion operator and $G * (f(u)) - f(u)$ signifies the convective effect. It was proved that appropriately scaling the kernels J and G leads to the convergence of solutions of nonlocal equations to solutions of local diffusion equations with convection. In addition, the asymptotic behavior of solutions has been analyzed for the case when $f(u) = |u|^{q-1}\,u$ for $q > 1$. Ignat and Rossi [18] further extended their findings, establishing that $\|u\|_{L^q(\mathbb{R}^N)} \leq Ct^{-\alpha}$ under appropriate assumptions.

Recently, Huang et al. [15] introduced the delayed nonlocal equation as follows which can be applied to the Nicholson's blowflies model,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - D(J * u - u)(x,t) + du(x,t) = b(u(t-\tau, x)), & x \in \mathbb{R}^N, t > 0, \\ u(x,s) = u_0(x,s), & x \in \mathbb{R}^N, s \in [-\tau, 0], \end{cases} \tag{1.3}$$

where $u(x,t)$ denotes the density of mature population at position x and time t. $D > 0$ represents the diffusion rate. Moreover, du is the death rate function and

$b(u)$ is the birth rate function which is taken as a nonlinear function $b(u) = pue^{-au}$. Here p is the maximum production rate while $\frac{1}{a}$ corresponds to the population size. They have obtained the threshold convergence results of optimal decay estimates with respect to the asymptotic behavior of solutions in the whole $\mathbb{R}^N$.

Inspired by the study of asymptotic behavior for the scalar nonlocal dispersal equation (1.3), we intend to extend the scalar equation to a nonlocal dispersal system. Therefore, in this paper, we study the asymptotic behavior of a delayed nonlocal dispersal Lotka-Volterra competitive system in the whole $\mathbb{R}^N$ as follows,

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(x,t) = d_1(J * u_1 - u_1)(x,t) + r_1 u_1(x,t)[1 - u_1(x,t) - b_1 u_2(x, t-\tau)], \\ \dfrac{\partial u_2}{\partial t}(x,t) = d_2(J * u_2 - u_2)(x,t) + r_2 u_2(x,t)[1 - b_2 u_1(x, t-\tau) - u_2(x,t)], \\ u_i(x,s) = u_{i0}(x,s), \end{cases} \tag{1.4}$$

for $x \in \mathbb{R}^N$, $t > 0$, $s \in [-\tau, 0]$, where $u_1(x,t)$ and $u_2(x,t)$ represent the densities of two competitors at location x and time t, d_i is the diffusion coefficient for species, r_i is intrinsic growth rate of species, b_i represents the competition coefficient of species $(i = 1, 2)$. In addition,

$$J * u_i(x,t) = \int_{\mathbb{R}^N} J(y) u_i(x-y, t) \mathrm{d}y, \quad i = 1, 2.$$

In this paper, $J(x)$ is the nonlocal kernel function and satisfies the following assumptions,

(J1) $J(x)$ is nonnegative, symmetric and unit, i.e.,

$$J \geq 0, \quad J(x) = J(-x) \quad \text{and} \quad \int_{\mathbb{R}^N} J(x) \mathrm{d}x = 1,$$

(J2) Fourier transform of $J(x)$ satisfies $\hat{J}(\eta) = 1 - k|\eta|^{2\beta} + o(|\eta|^{2\beta})$ as $\eta \to 0$ with $k > 0$ and $0 < \beta \leq 1$, where $|\eta| = (\sum_{i=1}^N \eta_i^2)^{\frac{1}{2}}$.

Assumption **(J1)** is natural from the perspective of the nonlocal diffusion model. As to assumption **(J2)**, we have $\hat{J}(\eta) - 1 \sim -k|\eta|^{2\beta}$ which implies that the nonlocal dispersal operator $u \mapsto J*u - u$ approaches to the Laplacian operator. Furthermore, based on Riemann–Lebesgue lemma, we see $\hat{J}(\eta) - 1 \sim -1$ as $|\eta| \to +\infty$ such that the nonlocal dispersal operator is bounded, which is a completely different property from the local dispersal operator. In particular, we set $J(x) = \frac{1}{(4\pi a^2)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4a^2}}$, $x \in \mathbb{R}^N$, then its Fourier transform $\hat{J}(\eta) = e^{-\frac{a^2|\eta|^2}{4}} \sim 1 - \frac{a^2|\eta|^2}{4}$ as $\eta \to 0$.

In what follows, we examine the situation when two competing species do not experience nonlocal dispersal and the time delay effect. As a result, the system (1.4) is reduced to

$$\begin{cases} u_1'(t) = r_1 u_1(t)(1 - u_1(t) - b_1 u_2(t)), \\ u_2'(t) = r_2 u_2(t)(1 - b_2 u_1(t) - u_2(t)). \end{cases} \tag{1.5}$$

The system (1.5) has a trivial equilibrium $E_0^* = (0,0)$, two semi-trivial equilibria $E_1^* = (0,1)$ and $E_2^* = (1,0)$ and the coexistence equilibrium $E_3^* = (\frac{1-b_1}{1-b_1b_2}, \frac{1-b_2}{1-b_1b_2})$ when $b_1b_2 \neq 1$. By a phase plane diagram, we determine the qualitative behavior of the orbit of the solution (see [34]) and show the following asymptotic behavior of the solution as $t \to +\infty$,

(i): $(u_1(t), u_2(t)) \to (0,1)$ if $0 < b_2 < 1 < b_1$;

(ii): $(u_1(t), u_2(t)) \to (1,0)$ if $0 < b_1 < 1 < b_2$;

(iii): $(u_1(t), u_2(t)) \to (\frac{1-b_1}{1-b_1b_2}, \frac{1-b_2}{1-b_1b_2})$ if $0 < b_1, b_2 < 1$;

(iv): $(u_1(t), u_2(t)) \to$ one of $(0,1)$, $(1,0)$ (depending on the initial data) if $b_1, b_2 > 1$.

In this paper, we discuss cases (i), (ii) and (iii). For one thing, we will derive the asymptotic behavior of solutions of linear nonlocal problems corresponding to the nonlocal Lotka-Volterra competitive system (1.4). The primary purpose is to obtain the convergence rates of solutions at equilibria and the adopted methods are the energy estimate and Fourier transform. More precisely, we establish L^∞-decay estimates for linear problems. Then, for the nonlinear problem (1.4), we consider the L^∞-decay estimates for solutions that converge globally to constant equilibria. Taking advantage of the conclusion that higher order parts decay faster than linear parts, we can ignore the higher order terms in decay estimates of solutions.

Our paper is structured as follows. In Section 2, we introduce some necessary notations and present main results concerning the asymptotic behavior of the nonlocal dispersal Lotka-Volterra competitive system. In Section 3, we are dedicated to deriving decay estimates for solutions of linear systems with time delay by the fundamental solution formulation, the Fourier transform and the Fourier splitting method. In Section 4, the convergence rates of solutions for the nonlinear problem are derived using the results of linear problems. In Section 5, we give some numerical simulations to illustrate the asymptotic behavior of the nonlocal dispersal system (1.4) in different cases and the effect of time delay on species growth. Finally, Section 6 is a summary of this paper.

2. Preliminaries and main results

In the beginning of this section, we elaborate on the following necessary notations throughout this paper. Let $\|\cdot\|$ represents $1-$norm of the vector or the matrix. Further, it can be defined as $\|A\| := \max_{1\leq j\leq n}\sum_{i=1}^{m}\|a_{ij}\|$, where $A = (a_{ij})_{m\times n}$. $C > 0$ is a generic constant while $C_i > 0$ $(i = 0, 1, 2, \cdots)$ denotes a specific constant. Besides, let Ω be a domain, typically $\Omega = \mathbb{R}^N$. $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a multi-index with nonnegative integers $\alpha_i \geq 0$ $(i = 1, 2, \cdots, N)$ and $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_N$. The derivatives for multivariable function are denoted by

$$\partial_x^\alpha f(x) := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N} f(x).$$

$L^p(\mathbb{R}^N)$ $(p \geq 1)$ that defined on $\mathbb{R}^N$ denotes the Lebesgue space of integrable functions and $W^{m,p}(\mathbb{R}^N)$ $(m \geq 0, p \geq 1)$ is a Sobolev space which can be defined by

$$W^{m,p}(\mathbb{R}^N) = \left\{ f(x) \in L^p(\mathbb{R}^N) \mid \partial_x^\alpha f(x) \in L^p(\mathbb{R}^N),\ |\alpha| \leq m \right\}$$

with the norm

$$\|f\|_{W^{m,p}} = \left(\sum_{|\alpha|=0}^{m} \int_{\Omega} |\partial_x^\alpha f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

For a multivariable function $f(x)$, its Fourier transform is defined as

$$\mathcal{F}[f](\eta) = \hat{f}(\eta) := \int_{\mathbb{R}^N} e^{-\mathrm{i}x\cdot\eta} f(x)\mathrm{d}x, \quad \mathrm{i} := \sqrt{-1},$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{\mathrm{i}x\cdot\eta} \hat{f}(\eta)\mathrm{d}\eta.$$

In what follows, we recall the solution formula and the decay rate for the linear differential equation with time delay, which plays a vital role in proofs of asymptotic behavior in Section 3.

Lemma 2.1 ([21, 30]). *Let $u(t)$ be the solution of the delayed differential system as follows,*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t) + Bu(t-\tau), & t > 0, \\ u(s) = u_0(s), & s \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $u(t) = (u_1(t), u_2(t), \cdots, u_N(t))^\top$, the time delay $\tau > 0$ and A, $B \in \mathbb{C}^{N\times N}$. If the initial data $u_0(s) \in C^1([-\tau, 0], \mathbb{C}^N)$, then the solution of the system (2.1) turns out that

$$u(t) = e^{A(t+\tau)} e_\tau^{B_1 t} u_0(-\tau) + \int_{-\tau}^{0} e^{A(t-s)} e_\tau^{B_1(t-s-\tau)} [u_0'(s) - Au_0(s)]\mathrm{d}s,$$

where $B_1 = Be^{-A\tau}$ and $e_\tau^{B_1 t}$ is the so-called delayed exponential function in the following form

$$e_\tau^{B_1 t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \le t < 0, \\ 1 + B_1 t, & 0 \le t < \tau, \\ 1 + B_1 t + \frac{B_1^2}{2!}(t-\tau)^2, & \tau \le t < 2\tau, \\ \vdots & \vdots \\ 1 + B_1 t + \frac{B_1^2}{2!}(t-\tau)^2 + \cdots + \frac{B_1^m}{m!}[t-(m-1)\tau]^m, & (m-1)\tau \le t < m\tau, \\ \vdots & \vdots \end{cases}$$

Furthermore, if $\zeta(A) := \frac{\zeta_1(A)+\zeta_\infty(A)}{2} < 0$ and $\sigma(B) := \frac{\|B\|+\|B\|_\infty}{2} \le -\zeta(A)$, then there exists a decreasing function

$$\varepsilon_\tau = \varepsilon(\tau) = \frac{1}{(1+\tau)(1+\|B\|_\infty) + 2\omega\tau} \in (0, 1),$$

with $\tau > 0$ such that any solution of the system (2.1) satisfies

$$\|u(t)\| \le C_1 e^{-\varepsilon_\tau \omega t}, \quad t > 0,$$

where the matrix measure of A induced by the matrix $1-$norm $\|\cdot\|$ and $\infty-$norm $\|\cdot\|_\infty$ are denoted by $\zeta_1(A)$ and $\zeta_\infty(A)$, respectively. In addition, C_1 is a positive constant depending on the initial data $u_0(s), s \in [-\tau, 0]$ and $\omega = |\zeta(A)| - \sigma(B)$. Specifically, we have

$$\|e^{At} e_\tau^{B_1 t}\| \le C_1 e^{-\varepsilon_\tau \omega t}, \quad t > 0.$$

In what follows, we present the main theorems in this paper.

Theorem 2.1. *When $0 < b_2 < 1 < b_1$, assume that the initial data satisfies*

$$0 \le u_{i0}(x, s) \le 1, \quad (x, s) \in \mathbb{R}^N \times [-\tau, 0], \quad i = 1, 2,$$

$$\tilde{u}_{i0}(\cdot, s) \in C([-\tau, 0], W^{m,1}(\mathbb{R}^N)), \quad i = 1, 2,$$

and

$$\partial_s \tilde{u}_{i0}(\cdot, s) \in L^1([-\tau, 0], W^{m,1}(\mathbb{R}^N)), \quad i = 1, 2,$$

with $m \ge 0$, where $\tilde{u}_{10} := u_{10}$ and $\tilde{u}_{20} := u_{20} - 1$. Then the unique solution of the nonlinear system (1.4) satisfies

$$(0, 0) \le (u_1(x, t), u_2(x, t)) \le (1, 1), \quad (x, t) \in \mathbb{R}^N \times (0, \infty).$$

Moreover, if assumptions **(J1)** *and* **(J2)** *hold, for $|\alpha| = 0, 1, 2, \cdots, [m]$, $N - 1 > 2\beta$ and $\delta_1 := r_2(1 - b_2) > 0$, we obtain that the solution $(u_1(x, t), u_2(x, t))$ converges globally to $E_1^* = (0, 1)$ time-exponentially*

$$\|\partial_x^\alpha u_1(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} + \|\partial_x^\alpha (u_2 - 1)(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \le C \rho_{\tilde{u}}^\alpha e^{-\tilde{\varepsilon} t} (t + 1)^{-\frac{N-1}{2\beta}}, \quad t > 0,$$

for some constant $\xi \in (0, a_0)$, where a_0 is mentioned in (3.10), $\tilde{\varepsilon} = \varepsilon_1 \xi^\beta \sqrt{k \delta_1 d_2}$, $\varepsilon_1 = \varepsilon_1(\tau) \in (0, 1)$ represents a decreasing function,

$$\rho_{\tilde{u}}^\alpha = \sum_{i=1}^{2} \left(\|\tilde{u}_{i0}(\cdot, -\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\mathcal{F}[\tilde{u}_{i0}](\cdot, -\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \int_{-\tau}^{0} \tilde{M}(s) \mathrm{d}s \right), \tag{2.2}$$

and

$$\begin{aligned} \tilde{M}(s) = \sum_{i=1}^{2} \Bigg(& \left\| \frac{\mathrm{d}}{\mathrm{d}s} \tilde{u}_{i0}(\cdot, s) \right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\tilde{u}_{i0}(\cdot, s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \\ & + \left\| \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{F}[\tilde{u}_{i0}](\cdot, s) \right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\mathcal{F}[\tilde{u}_{i0}](\cdot, s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \Bigg). \end{aligned} \tag{2.3}$$

Theorem 2.2. *Under the condition of $0 < b_1 < 1 < b_2$, assume that the initial data satisfies*

$$0 \le u_{i0}(x, s) \le 1, \quad (x, s) \in \mathbb{R}^N \times [-\tau, 0], \quad i = 1, 2,$$

$$\bar{u}_{i0}(\cdot,s) \in C([-\tau,0], W^{m,1}(\mathbb{R}^N)), \quad i=1,2,$$

and

$$\partial_s \bar{u}_{i0}(\cdot,s) \in L^1([-\tau,0], W^{m,1}(\mathbb{R}^N)), \quad i=1,2,$$

with $m \geq 0$*, where* $\bar{u}_{10} := u_{10} - 1$ *and* $\bar{u}_{20} := u_{20}$*. Then the unique solution of the nonlinear system (1.4) satisfies*

$$(0,0) \leq (u_1(x,t), u_2(x,t)) \leq (1,1), \quad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Furthermore, if assumptions **(J1)** *and* **(J2)** *hold, for* $|\alpha| = 0,1,2,\cdots,[m]$*,* $N-1 > 2\beta$ *and* $\delta_2 := r_1(1-b_1) > 0$*, we obtain that the solution* $(u_1(x,t), u_2(x,t))$ *converges globally to* $E_2^* = (1,0)$ *time-exponentially*

$$\|\partial_x^\alpha (u_1 - 1)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} + \|\partial_x^\alpha u_2(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C\rho_{\bar{u}}^\alpha e^{-\bar{\varepsilon}t}(t+1)^{-\frac{N-1}{2\beta}}, \quad t>0,$$

with some constant $\nu \in (0,a_0)$*, where* a_0 *is mentioned in (3.10),* $\bar{\varepsilon} = \varepsilon_2 \nu^\beta \sqrt{k\delta_2 d_1}$ *and* $\varepsilon_2 = \varepsilon_2(\tau) \in (0,1)$ *represents a decreasing function.*

Theorem 2.3. *In the case of* $0 < b_1, b_2 < 1$*, suppose that the initial data satisfies*

$$0 \leq u_{i0}(x,s) \leq 1, \quad (x,s) \in \mathbb{R}^N \times [-\tau,0], \quad i=1,2,$$

$$\check{u}_{i0}(\cdot,s) \in C([-\tau,0], W^{m,1}(\mathbb{R}^N)), \quad i=1,2,$$

and

$$\partial_s \check{u}_{i0}(\cdot,s) \in L^1([-\tau,0], W^{m,1}(\mathbb{R}^N)), \quad i=1,2,$$

with $m \geq 0$*, where* $\check{u}_{10} := u_{10} - \frac{1-b_1}{1-b_1b_2}$ *and* $\check{u}_{20} := u_{20} - \frac{1-b_2}{1-b_1b_2}$*. Then the unique solution of the nonlinear system (1.4) satisfies*

$$(0,0) \leq (u_1(x,t), u_2(x,t)) \leq (1,1), \quad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Besides, if $r_1 > 4r_2b_2(1-b_2)$ *and assumptions* **(J1)** *and* **(J2)** *hold, for* $|\alpha| = 0,1,2,\cdots,[m]$*,* $N-1 > 2\beta$*,* $\delta_3 := \frac{r_1(1-b_1)^2}{1-b_1b_2} > 0$*, we obtain that the solution* $(u_1(x,t), u_2(x,t))$ *converges globally to* $E_3^* := (m_1, m_2) = (\frac{1-b_1}{1-b_1b_2}, \frac{1-b_2}{1-b_1b_2})$ *time-exponentially*

$$\sum_{i=1}^{2} \|\partial_x^\alpha (u_i - m_i)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C\rho_{\check{u}}^\alpha e^{-\check{\varepsilon}t}(t+1)^{-\frac{N-1}{2\beta}},$$

for some constant $\vartheta \in (0,a_0)$*, where* a_0 *is mentioned in (3.10),* $\check{\varepsilon} = \varepsilon_3 \vartheta^\beta \sqrt{k\delta_3 d_1}$ *and* $\varepsilon_3 = \varepsilon_3(\tau) \in (0,1)$ *represents a decreasing function.*

Remark 2.1. Our decay estimates indicate that the convergence rates of the solution of the nonlocal dispersal system are related to the dispersal coefficient d_i $(i=1,2)$, which implies that introducing nonlocal dispersal into Lotka-Volterra competitive system makes a difference. In addition, as d_i $(i=1,2)$ approaches to 0, a transition occurs in the convergence rate of the system: from time-exponential stability to time-algebraic stability.

Remark 2.2. We offer the following biological explanations for $L^\infty-$decay estimates employed in this paper. (i) Such estimates provide an upper bound that describes the maximum number of species that may be reached under conditions of abundant resources or when there are fewer competitors. This upper bound helps researchers understand the growth potential of a given species and the densest state it may reach under given conditions. (ii) The $L^\infty-$norm provides an estimate of the maximum density of a species under competitive conditions and reflects the survivability and adaptability of a population in a high-stress environment. Competition among species often leads to a decrease in the population of a given species or, under conditions of limited resources, species are forced to adjust their growth strategies. Therefore, the $L^\infty-$norm describes the upper limit of the maximum population size that a species can reach under the pressure of competition from other species. If the $L^\infty-$norm indicates that the upper density limit of a species is high, it means that the species is more adaptable and resilient to competition, and vice versa, it indicates that the species may be at a competitive disadvantage. (iii) The $L^\infty-$norm can also help determine whether there is a risk of excessive competition in an ecosystem. If the maximum density of a species is too high, it may indicate that the demand for resources by that species is greater than the rate of renewal of the resources, thus triggering the depletion of the resources, which in turn affects the survival of other species. Biologically, the estimation of $L^\infty-$norms can identify which species may be at risk of destabilizing the system through excessive resource depletion in competition, thus providing a basis for ecological conservation and resource management.

Remark 2.3. Notice that $\varepsilon_i = \varepsilon_i(\tau) \in (0,1)$ denotes the decreasing function $(i = 1,2,3)$, we know the time delay inhibits the decay rates of the solution of the nonlocal dispersal Lotka-Volterra competitive system which is illustrated in numerical experiments in Section 5.

3. Linear time-delayed nonlocal dispersal system

In this section, we focus on the asymptotic behavior of the linear nonlocal dispersal system with time delay, which is of great significance to the exploration of the nonlinear problem (1.4). Based on the fundamental solution formula of linear problems, we mainly deduce the solution formula for the linear system (3.1) and decay rates of solutions in the case of different equilibria by the Fourier splitting method which was proposed by Schonbek in [37].

Consider the following linear time-delayed nonlocal dispersal system, namely,

$$\begin{cases} \dfrac{\partial U}{\partial t}(x,t) = D(J * U - U)(x,t) + KU(x,t) + LU(x,t-\tau), \\ U(x,s) = U_0(x,s), \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^N$, $t > 0$, $s \in [-\tau, 0]$, $U(x,t) = (u_1(x,t), u_2(x,t))^\top$, $U(x,t-\tau) = (u_1(x,t-\tau), u_2(x,t-\tau))^\top$, $D = \mathrm{diag}\{d_1, d_2\}$ and $K, L \in \mathbb{R}^{2\times 2}$.

Taking the Fourier transform to both sides of (3.1), the system (3.1) can be transformed into

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{U}(\eta,t) = A(\eta)\hat{U}(\eta,t) + L\hat{U}(\eta,t-\tau), \\ \hat{U}(\eta,s) = \hat{U}_0(\eta,s), \qquad\qquad s\in[-\tau,0], \end{cases} \tag{3.2}$$

where

$$A(\eta) = D(\hat{J}(\eta)-1) + K,$$

and

$$\hat{U}(\eta,t) = \mathcal{F}[U](\eta,t) = \int_{\mathbb{R}^N} e^{-\mathrm{i}x\cdot\eta}U(x,t)\mathrm{d}x.$$

By Lemma 2.1, we have

$$\begin{aligned} \hat{U}(\eta,t) =& \int_{-\tau}^{0} e^{A(\eta)(t-s)}e_{\tau}^{\bar{L}(\eta)(t-s-\tau)}\left[\frac{\mathrm{d}}{\mathrm{d}s}\hat{U}_0(\eta,s) - A(\eta)\hat{U}_0(\eta,s)\right]\mathrm{d}s \\ &+ e^{A(\eta)(t+\tau)}e_{\tau}^{\bar{L}(\eta)t}\hat{U}_0(\eta,-\tau) \\ :=& \hat{F}(\eta,t)\hat{U}_0(\eta,-\tau) + \int_{-\tau}^{0}\hat{F}(\eta,t-s-\tau)\left[\frac{\mathrm{d}}{\mathrm{d}s}\hat{U}_0(\eta,s) - A(\eta)\hat{U}_0(\eta,s)\right]\mathrm{d}s, \end{aligned} \tag{3.3}$$

where

$$\bar{L}(\eta) = Le^{-A(\eta)\tau},$$

and

$$\hat{F}(\eta,t) = e^{A(\eta)(t+\tau)}e_{\tau}^{\bar{L}(\eta)t}.$$

Then, we carry out the inverse Fourier transform on both sides of (3.3) and obtain that

$$\begin{aligned} &U(x,t) \\ =&F(\cdot,t) * U_0(\cdot,-\tau) \\ &+ \int_{-\tau}^{0} F(\cdot,t-s-\tau) * \left[\frac{\mathrm{d}}{\mathrm{d}s}U_0(s) - D(J*U_0-U_0)(s) - KU_0(s)\right]\mathrm{d}s \\ =&\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{\mathrm{i}x\cdot\eta}e^{A(\eta)(t+\tau)}e_{\tau}^{\bar{L}(\eta)t}\hat{U}_0(\eta,-\tau)\mathrm{d}\eta \\ &+ \frac{1}{(2\pi)^N}\int_{-\tau}^{0}\int_{\mathbb{R}^N} e^{\mathrm{i}x\cdot\eta}e^{A(\eta)(t-s)}e_{\tau}^{\bar{L}(\eta)(t-s-\tau)}\left[\frac{\mathrm{d}}{\mathrm{d}s}\hat{U}_0(\eta,s) - A(\eta)\hat{U}_0(\eta,s)\right]\mathrm{d}\eta\mathrm{d}s, \end{aligned}$$

where $F(\cdot,t)$ is the inverse Fourier transform to $\hat{F}(\eta,t)$. By direct calculations, the derivatives of $U(x,t)$ can be shown as

$$\begin{aligned}
&\partial_x^\alpha U(x,t)\\
&= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} e^{\mathrm{i}x\cdot\eta} e^{A(\eta)(t+\tau)} e_\tau^{\bar{L}(\eta)t}\hat{U}_0(\eta,-\tau)\mathrm{d}\eta\\
&\quad + \frac{1}{(2\pi)^N}\int_{-\tau}^0\int_{\mathbb{R}^N}\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} e^{\mathrm{i}x\cdot\eta} e^{A(\eta)(t-s)} e_\tau^{\bar{L}(\eta)(t-s-\tau)}\\
&\qquad\qquad \times\left[\frac{\mathrm{d}}{\mathrm{d}s}\hat{U}_0(\eta,s) - A(\eta)\hat{U}_0(\eta,s)\right]\mathrm{d}\eta\mathrm{d}s
\end{aligned}\tag{3.4}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ with nonnegative integers α_i, $i = 1, 2, \cdots, N$.

In what follows, we aim to derive decay rates at which the solution of the linear system (3.1) converges to equilibria E_1^*, E_2^* and E_3^*, respectively.

3.1. Case 1: convergence to the equilibrium E_1^*

To start with, we consider the linearization result of the system (1.4) around the constant equilibrium E_1^*. Let

$$\begin{cases}\tilde{u}_1(x,t) = u_1(x,t),\\ \tilde{u}_2(x,t) = u_2(x,t) - 1.\end{cases}\tag{3.5}$$

By substituting (3.5) into the system (1.4), for $i = 1, 2$, we obtain

$$\begin{cases}\dfrac{\partial \tilde{u}_1}{\partial t}(x,t) = d_1(J * \tilde{u}_1 - \tilde{u}_1)(x,t) + r_1\tilde{u}_1(x,t)[1 - b_1 - \tilde{u}_1(x,t) - b_1\tilde{u}_2(x,t-\tau)],\\ \dfrac{\partial \tilde{u}_2}{\partial t}(x,t) = d_2(J * \tilde{u}_2 - \tilde{u}_2)(x,t) + r_2(\tilde{u}_2(x,t) + 1)[-b_2\tilde{u}_1(x,t-\tau) - \tilde{u}_2(x,t)],\\ \tilde{u}_i(x,s) = \tilde{u}_{i0}(x,s),\end{cases}\tag{3.6}$$

for $x \in \mathbb{R}^N$, $t > 0$ and $s \in [-\tau, 0]$. Inspired by the linear system (3.1), it is obvious that the linearization of the system (3.6) can be shown as

$$\begin{cases}\dfrac{\partial \tilde{U}}{\partial t}(x,t) = D(J * \tilde{U} - \tilde{U})(x,t) + K_1\tilde{U}(x,t) + L_1\tilde{U}(x,t-\tau),\\ \tilde{U}(x,s) = \tilde{U}_0(x,s),\end{cases}\tag{3.7}$$

for $x \in \mathbb{R}^N$, $t > 0$, $s \in [-\tau, 0]$, where $\tilde{U}(x,t) = (\tilde{u}_1(x,t), \tilde{u}_2(x,t))^\top$, $\tilde{U}_0(x,s) = (\tilde{u}_{10}(x,t), \tilde{u}_{20}(x,t))^\top$ and

$$K_1 = \begin{pmatrix} r_1(1-b_1) & 0\\ 0 & -r_2\end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0\\ -r_2b_2 & 0\end{pmatrix}.$$

Thus, we have

$$A_1(\eta) = D(\hat{J}(\eta) - 1) + K_1 = \begin{pmatrix} d_1\hat{J}(\eta) - d_1 + r_1(1-b_1) & 0 \\ 0 & d_2\hat{J}(\eta) - d_2 - r_2 \end{pmatrix}.$$

In view of definitions of matrix measure and matrix norm, it follows that

$$\begin{aligned} \omega_1 &:= |\zeta(A_1(\eta))| - \sigma(L_1) \\ &= \left|\max\left\{d_1(\hat{J}(\eta)-1) + r_1(1-b_1), d_2(\hat{J}(\eta)-1) - r_2\right\}\right| - r_2 b_2 \\ &\geq r_2 + d_2(1-\hat{J}(\eta)) - r_2 b_2 = r_2(1-b_2) + d_2(1-\hat{J}(\eta)). \end{aligned} \tag{3.8}$$

By Lemma 2.1 and (3.8), there exists a decreasing function $\varepsilon_1 = \varepsilon_1(\tau) \in (0,1)$ such that

$$\|e^{A_1(\eta)t} e_\tau^{\bar{L}_1(\eta)t}\| \leq C_2 e^{-\varepsilon_1(|\zeta(A_1(\eta))| - \sigma(L_1))t} \leq C_2 e^{-\varepsilon_1 \delta_1 t} e^{-\varepsilon_1 d_2(1-\hat{J}(\eta))t}, \tag{3.9}$$

where $\bar{L}_1(\eta) = L_1 e^{-A_1(\eta)\tau}$. C_2 represents a positive constant relying on $\tilde{U}_0(s)$ with $s \in [-\tau, 0]$ and $\delta_1 := r_2(1-b_2) > 0$ under the condition of $0 < b_2 < 1 < b_1$. According to the assumption **(J2)**, we see $\hat{J}(\eta) = 1 - k|\eta|^{2\beta} + o(|\eta|^{2\beta})$ as $\eta \to 0$ with $k > 0$ and $0 < \beta \leq 1$ where $|\eta| = (\sum_{i=1}^2 \eta_i^2)^{\frac{1}{2}}$. As a result, there exist $a_0 > 0$ and $\gamma = \frac{k}{2} a_0^{2\beta} \in (0,1)$ such that

$$\begin{cases} 1 - \hat{J}(\eta) \geq \frac{k}{2}|\eta|^{2\beta}, & |\eta| \leq a_0, \\ 1 - \hat{J}(\eta) \geq \gamma, & |\eta| \geq a_0. \end{cases} \tag{3.10}$$

Supposed that the conditions in Theorem 2.1 hold, the following lemma indicates the decay rate of $\tilde{u}_i(x,t)$ in $\mathbb{R}^N \times (0, +\infty]$ $(i = 1, 2)$.

Lemma 3.1. *For $N - 2 > 2\beta$, the solution of the linearized system (3.7) corresponding to the equilibrium E_1^* satisfies*

$$\sum_{i=1}^{2} \|\partial_x^\alpha \tilde{u}_i(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C \rho_{\tilde{u}}^\alpha e^{-\varepsilon_1 t \sqrt{k\delta_1 d_2 \xi^{2\beta}}} (t+1)^{-\frac{N-1}{2\beta}},$$

where $\varepsilon_1 = \varepsilon_1(\tau) = \frac{1}{(1+\tau)(1+\|L_1\|_\infty) + 2\omega_1\tau} \in (0,1)$, $\delta_1 = r_2(1-b_2) > 0$, $\xi \in (0, a_0)$ represents some constant and $\rho_{\tilde{u}}^\alpha$ is defined in (2.2) and (2.3).

Proof. Note that the higher order derivatives of the solution of the system (3.7) have the similar form to (3.4), further we divide it into two parts for simplicity. Denote

$$P_1(\eta, s) := \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[\tilde{U}_0](\eta, s) - A_1(\eta)\mathcal{F}[\tilde{U}_0](\eta, s).$$

That is to say,

$$\begin{aligned} &\partial_x^\alpha \tilde{U}(x,t) \\ &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \prod_{i=1}^{N} (\mathrm{i}\eta_i)^{\alpha_i} e^{\mathrm{i}x\cdot\eta} e^{A_1(\eta)(t+\tau)} e_\tau^{\bar{L}_1(\eta)t} \mathcal{F}[\tilde{U}_0](\eta, -\tau) \mathrm{d}\eta \end{aligned}$$

$$+\frac{1}{(2\pi)^N}\int_{-\tau}^{0}\int_{\mathbb{R}^N}\prod_{i=1}^{N}(\mathrm{i}\eta_i)^{\alpha_i}e^{\mathrm{i}x\cdot\eta}e^{A_1(\eta)(t-s)}e_{\tau}^{\bar{L}_1(\eta)(t-s-\tau)}P_1(\eta,s)\mathrm{d}\eta\mathrm{d}s$$

$$:=\mathcal{F}^{-1}[I_1](x,t)+\int_{-\tau}^{0}\mathcal{F}^{-1}[I_2](x,t-s-\tau)\mathrm{d}s, \tag{3.11}$$

where

$$I_1=\prod_{i=1}^{N}(\mathrm{i}\eta_i)^{\alpha_i}e^{A_1(\eta)(t+\tau)}e_{\tau}^{\bar{L}_1(\eta)t}\mathcal{F}[\tilde{U}_0](\eta,-\tau)$$

and

$$I_2=\prod_{i=1}^{N}(\mathrm{i}\eta_i)^{\alpha_i}e^{A_1(\eta)(t-s)}e_{\tau}^{\bar{L}_1(\eta)(t-s-\tau)}\left[\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[\tilde{U}_0](\eta,s)-A_1(\eta)\mathcal{F}[\tilde{U}_0](\eta,s)\right].$$

In order to complete the proof of Lemma 3.1, we estimate the $L^\infty-$decay rate for (3.11) using Fourier splitting method. At the beginning, we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^N}\|\mathcal{F}^{-1}[I_1](x,t)\|\\
&=\sup_{x\in\mathbb{R}^N}\left\|\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\prod_{i=1}^{N}(\mathrm{i}\eta_i)^{\alpha_i}e^{\mathrm{i}x\cdot\eta}e^{A_1(\eta)(t+\tau)}e_{\tau}^{\bar{L}_1(\eta)t}\mathcal{F}[\tilde{U}_0](\eta,-\tau)\mathrm{d}\eta\right\|\\
&\le Ce^{-\varepsilon_1\delta_1 t}\left(\int_{\mathbb{R}^N}e^{-\varepsilon_1 d_2(1-\hat{J}(\eta))t}\left\|\prod_{i=1}^{N}(\mathrm{i}\eta_i)^{\alpha_i}\mathcal{F}[\tilde{U}_0](\eta,-\tau)\right\|\mathrm{d}\eta\right)\\
&\le Ce^{-\varepsilon_1\delta_1 t}\left(\int_{|\eta|\le a_0}+\int_{|\eta|\ge a_0}\right)e^{-\varepsilon_1 d_2(1-\hat{J}(\eta))t}\left\|\partial_x^\alpha\mathcal{F}[\tilde{U}_0](\eta,-\tau)\right\|\mathrm{d}\eta\\
&:=I_3(t)+I_4(t).
\end{aligned}$$

By virtue of properties of the Fourier transform, we have

$$\begin{aligned}
\sup_{\eta\in\mathbb{R}^N}\left\|\partial_x^\alpha\mathcal{F}[\tilde{U}_0](\eta,-\tau)\right\|&\le\int_{\mathbb{R}^N}\left\|\partial_x^\alpha\tilde{U}_0(x,-\tau)\right\|\mathrm{d}x=\sum_{i=1}^{2}\|\partial_x^\alpha\tilde{u}_{i0}(\cdot,-\tau)\|_{L^1(\mathbb{R}^N)}\\
&\le\sum_{i=1}^{2}\|\tilde{u}_{i0}(\cdot,-\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}.
\end{aligned}$$

Therefore, the following estimate holds,

$$\begin{aligned}
I_3(t)&\le Ce^{-\varepsilon_1\delta_1 t}\int_{|\eta|\le a_0}e^{-\frac{k}{2}\varepsilon_1 d_2|\eta|^{2\beta}(t+1)}\left\|\partial_x^\alpha\mathcal{F}[\tilde{U}_0](\eta,-\tau)\right\|\mathrm{d}\eta\\
&\le Ce^{-\varepsilon_1\delta_1 t}\sup_{\eta\in\mathbb{R}^N}\left\|\partial_x^\alpha\mathcal{F}[\tilde{U}_0](\eta,-\tau)\right\|\int_{|\eta|\le a_0}e^{-\frac{k}{2}\varepsilon_1 d_2|\eta|^{2\beta}(t+1)}\mathrm{d}\eta\\
&\le C(t+1)^{-\frac{N-1}{2\beta}}e^{-\varepsilon_1\delta_1 t}e^{-\frac{k}{2}\varepsilon_1 d_2\xi^{2\beta}t}\sum_{i=1}^{2}\|\tilde{u}_{i0}(\cdot,-\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)},
\end{aligned}\tag{3.12}$$

where $\xi \in (0, a_0)$ represents some constant. Here we use the mean value theorems for definite integrals in the following derivation, namely,

$$\begin{aligned}
&\int_{|\eta|\le a_0} e^{-\frac{k}{2}\varepsilon_1 d_2 |\eta|^{2\beta}(t+1)} \mathrm{d}\eta \\
&\le \left(\prod_{i=1}^{N-1} \int_{\mathbb{R}} e^{-\frac{k}{2}\varepsilon_1 d_2 \eta_i^{2\beta}(t+1)} \mathrm{d}\eta_i\right) \left(2\int_0^{a_0} e^{-\frac{k}{2}\varepsilon_1 d_2 \eta_N^{2\beta} t} \mathrm{d}\eta_N\right) \\
&= \left[\prod_{i=1}^{N-1} \int_{\mathbb{R}} e^{-\frac{k}{2}\varepsilon_1 d_2 (\eta_i (t+1)^{\frac{1}{2\beta}})^{2\beta}} (t+1)^{-\frac{1}{2\beta}} \mathrm{d}(\eta_i (t+1)^{\frac{1}{2\beta}})\right] \left(2a_0 e^{-\frac{k}{2}\varepsilon_1 d_2 \xi^{2\beta} t}\right) \\
&\le C(t+1)^{-\frac{N-1}{2\beta}} e^{-\frac{k}{2}\varepsilon_1 d_2 \xi^{2\beta} t}. \qquad (3.13)
\end{aligned}$$

On the other hand, we consider the decay estimate of $I_4(t)$. One have

$$\begin{aligned}
I_4(t) &\le C e^{-\varepsilon_1 \delta_1 t} \int_{|\eta|\ge a_0} e^{-\varepsilon_1 d_2 \gamma t} \left\| \partial_x^\alpha \mathcal{F}[\tilde{U}_0](\eta, -\tau) \right\| \mathrm{d}\eta \\
&\le C e^{-\varepsilon_1 \delta_1 t} e^{-\varepsilon_1 d_2 \gamma t} \int_{\mathbb{R}^N} \left\| \partial_x^\alpha \mathcal{F}[\tilde{U}_0](\eta, -\tau) \right\| \mathrm{d}\eta \\
&\le C e^{-\varepsilon_1 \delta_1 t} e^{-\varepsilon_1 d_2 \gamma t} \sum_{i=1}^{2} \|\mathcal{F}[\tilde{u}_{i0}](\cdot, -\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}. \qquad (3.14)
\end{aligned}$$

Note that

$$\begin{aligned}
C e^{-\varepsilon_1 d_2 \gamma t} &= C e^{-\frac{k}{2}\varepsilon_1 d_2 a_0^{2\beta} t} \le C e^{-\frac{k}{2}\varepsilon_1 d_2 \xi^{2\beta} t} e^{-\frac{k}{2}\varepsilon_1 d_2 (a_0-\xi)^{2\beta}(t+1)} \\
&\le C e^{-\frac{k}{2}\varepsilon_1 d_2 \xi^{2\beta} t} (t+1)^{-\frac{N-1}{2\beta}}. \qquad (3.15)
\end{aligned}$$

Namely, by (3.12), (3.14), (3.15) and the mean value inequality, it can be concluded that

$$\begin{aligned}
\sup_{x\in\mathbb{R}^N} \|\mathcal{F}^{-1}[I_1](x,t)\| &\le C e^{-\varepsilon_1 \delta_1 t} \Bigg[(t+1)^{-\frac{N-1}{2\beta}} e^{-\frac{k}{2}\varepsilon_1 d_2 \xi^{2\beta} t} \sum_{i=1}^{2} \|\tilde{u}_{i0}(\cdot, -\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \\
&\qquad + e^{-\varepsilon_1 d_2 \gamma t} \sum_{i=1}^{2} \|\mathcal{F}[\tilde{u}_{i0}](\cdot, -\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \Bigg] \\
&\le C e^{-\varepsilon_1 t \sqrt{k\delta_1 d_2 \xi^{2\beta}}} (t+1)^{-\frac{N-1}{2\beta}} \sum_{i=1}^{2} \Big(\|\tilde{u}_{i0}(\cdot, -\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \\
&\qquad + \|\mathcal{F}[\tilde{u}_{i0}](\cdot, -\tau)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \Big). \qquad (3.16)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^N} \|\mathcal{F}^{-1}[I_2](x, t-s-\tau)\| \\
&= \sup_{x\in\mathbb{R}^N} \left\| \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \prod_{i=1}^{N} (\mathrm{i}\eta_i)^{\alpha_i} e^{\mathrm{i}x\cdot\eta} e^{A_1(\eta)(t-s)} e_\tau^{\bar{L}_1(\eta)(t-s-\tau)} P_1(\eta, s) \mathrm{d}\eta \right\|
\end{aligned}$$

$$\le C\int_{\mathbb{R}^N} e^{-\varepsilon_1 d_2(1-\hat{J}(\eta))(t-s)}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} P_1(\eta,s)\right\| \mathrm{d}\eta$$
$$\le Ce^{-\varepsilon_1\delta_1(t-s)}\int_{|\eta|\le a_0} e^{-\frac{k}{2}\varepsilon_1 d_2|\eta|^{2\beta}(t-s)}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} P_1(\eta,s)\right\| \mathrm{d}\eta$$
$$+ Ce^{-\varepsilon_1\delta_1(t-s)}\int_{|\eta|\ge a_0} e^{-\frac{k}{2}\varepsilon_1 d_2\gamma(t-s)}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} P_1(\eta,s)\right\| \mathrm{d}\eta$$
$$:= I_5(t) + I_6(t).$$

Notice that

$$\sup_{\eta\in\mathbb{R}^N}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i}\left[\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[\tilde{U}_0](\eta,s)\right]\right\| = \sup_{\eta\in\mathbb{R}^N}\left\|\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[\partial_x^\alpha\tilde{U}_0](\eta,s)\right\|$$
$$\le \sum_{i=1}^2\int_{\mathbb{R}^N}\left|\partial_x^\alpha\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u}_{i0}(x,s)\right|\mathrm{d}x = \sum_{i=1}^2\left\|\partial_x^\alpha\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u}_{i0}(\cdot,s)\right\|_{L^1(\mathbb{R}^N)}$$
$$\le \sum_{i=1}^2\left\|\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u}_{i0}(\cdot,s)\right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \tag{3.17}$$

and

$$\sup_{\eta\in\mathbb{R}^N}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} A_1(\eta)\mathcal{F}[\tilde{U}_0](\eta,s)\right\| = C\sup_{\eta\in\mathbb{R}^N}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i}\hat{J}(\eta)\mathcal{F}[\tilde{U}_0](\eta,s)\right\|$$
$$= C\sup_{\eta\in\mathbb{R}^N}\left\|\mathcal{F}[J*\partial_x^\alpha\tilde{U}_0](\eta,s)\right\| \le C\sum_{i=1}^2\|J(\cdot)\|_{L^1(\mathbb{R}^N)}\|\partial_x^\alpha\tilde{u}_{i0}(\cdot,s)\|_{L^1(\mathbb{R}^N)}$$
$$\le C\sum_{i=1}^2\|\tilde{u}_{i0}(\cdot,s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}. \tag{3.18}$$

Naturally, by (3.13), (3.17) and (3.18), the following estimates hold,

$$I_5(t) \le Ce^{-\varepsilon_1\delta_1(t-s)}\sup_{\eta\in\mathbb{R}^N}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} P_1(\eta,s)\right\|\int_{|\eta|\le a_0} e^{-\frac{k}{2}\varepsilon_1 d_2|\eta|^{2\beta}(t-s)}\mathrm{d}\eta$$
$$\le Ce^{-\varepsilon_1\delta_1(t-s)}e^{-\frac{k}{2}\varepsilon_1 d_2\xi^{2\beta}(t-s)}(t-s)^{-\frac{N-1}{2\beta}}$$
$$\times\sum_{i=1}^2\left(\left\|\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u}_{i0}(\cdot,s)\right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\tilde{u}_{i0}(\cdot,s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}\right) \tag{3.19}$$

and

$$I_6(t) \le Ce^{-\varepsilon_1\delta_1(t-s)}e^{-\frac{k}{2}\varepsilon_1 d_2\gamma(t-s)}\int_{\mathbb{R}^N}\left\|\prod_{i=1}^N (\mathrm{i}\eta_i)^{\alpha_i} P_1(\eta,s)\right\|\mathrm{d}\eta$$
$$\le Ce^{-\varepsilon_1\delta_1(t-s)}e^{-\frac{k}{2}\varepsilon_1 d_2\gamma(t-s)}\sum_{i=1}^2\left(\left\|\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[\tilde{u}_{i0}](\cdot,s)\right\|_{W^{|\alpha|,1}(\mathbb{R}^N)}\right. \tag{3.20}$$

$$+ \|\mathcal{F}[\tilde{u}_{i0}](\cdot,s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}\Bigg).$$

In view of (3.19) and (3.20), we have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^N} \|\mathcal{F}^{-1}[I_2](x,t-s-\tau)\| \\
&\le Ce^{-\varepsilon_1\delta_1(t-s)}e^{-\frac{k}{2}\varepsilon_1 d_2\xi^{2\beta}(t-s)}(t-s)^{-\frac{N-1}{2\beta}} \\
&\qquad\times \sum_{i=1}^{2}\left(\left\|\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u}_{i0}(\cdot,s)\right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\tilde{u}_{i0}(\cdot,s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}\right) \\
&\quad + Ce^{-\varepsilon_1\delta_1(t-s)}e^{-\frac{k}{2}\varepsilon_1 d_2\gamma(t-s)} \\
&\qquad\times \sum_{i=1}^{2}\left(\left\|\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[\tilde{u}_{i0}](\cdot,s)\right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\mathcal{F}[\tilde{u}_{i0}](\cdot,s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}\right) \\
&\le Ce^{-\varepsilon_1(t-s)\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t-s)^{-\frac{N-1}{2\beta}}\sum_{i=1}^{2}\Bigg(\left\|\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u}_{i0}(\cdot,s)\right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\tilde{u}_{i0}(\cdot,s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)} \\
&\quad + \left\|\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}[\tilde{u}_{i0}](\cdot,s)\right\|_{W^{|\alpha|,1}(\mathbb{R}^N)} + \|\mathcal{F}[\tilde{u}_{i0}](\cdot,s)\|_{W^{|\alpha|,1}(\mathbb{R}^N)}\Bigg).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\int_{-\tau}^{0}\sup_{x\in\mathbb{R}^N}\|\mathcal{F}^{-1}[I_2](x,t-s-\tau)\|\mathrm{d}s \\
&\le C\rho_{\tilde{u}}^{\alpha}\int_{-\tau}^{0} e^{-\varepsilon_1(t-s)\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t-s)^{-\frac{N-1}{2\beta}}\mathrm{d}s \\
&\le C\rho_{\tilde{u}}^{\alpha}e^{-\varepsilon_1 t\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}}\int_{-\tau}^{0}\left(\frac{t-s}{t+1}\right)^{-\frac{N-1}{2\beta}}\mathrm{d}s \\
&\le C\rho_{\tilde{u}}^{\alpha}e^{-\varepsilon_1 t\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}}.
\end{aligned}$$

In conclusion, we obtain

$$\sum_{i=1}^{2}\|\partial_x^{\alpha}\tilde{u}_i(x,t)\|_{L^\infty(\mathbb{R}^N)} \le C\rho_{\tilde{u}}^{\alpha}e^{-\varepsilon_1 t\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}},$$

where $\varepsilon_1 = \varepsilon_1(\tau) \in (0,1)$, $\xi \in (0,a_0)$ and $\rho_{\tilde{u}}^{\alpha}$ is defined in (2.2). This finishes the proof. □

3.2. Case 2: convergence to the equilibrium E_2^*

Now, we are in the position to demonstrate the decay rate of the solution of linearized system around the equilibrium E_2^*. To begin with, we make the change of variable

$$\begin{cases} \bar{u}_1(x,t) = u_1(x,t) - 1, \\ \bar{u}_2(x,t) = u_2(x,t). \end{cases} \tag{3.21}$$

Then, from (1.4) and (3.21), for $i = 1, 2$, we obtain

$$\begin{cases} \dfrac{\partial \bar{u}_1}{\partial t}(x,t) = d_1(J * \bar{u}_1 - \bar{u}_1)(x,t) + r_1(\bar{u}_1(x,t)+1)[-\bar{u}_1(x,t) - b_1\bar{u}_2(x,t-\tau)], \\ \dfrac{\partial \bar{u}_2}{\partial t}(x,t) = d_2(J * \bar{u}_2 - \bar{u}_2)(x,t) + r_2\bar{u}_2(x,t)[1 - b_2 - b_2\bar{u}_1(x,t-\tau) - \bar{u}_2(x,t)], \\ \bar{u}_i(x,s) = \bar{u}_{i0}(x,s), \end{cases} \tag{3.22}$$

for $x \in \mathbb{R}^N$, $t > 0$ and $s \in [-\tau, 0]$. Hence, the system (3.22) can be linearized in the following form,

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial t}(x,t) = D(J * \bar{U} - \bar{U})(x,t) + K_2\bar{U}(x,t) + L_2\bar{U}(x,t-\tau), & x \in \mathbb{R}^N, \quad t > 0, \\ \bar{U}(x,s) = \bar{U}_0(x,s), & x \in \mathbb{R}^N, \quad s \in [-\tau, 0], \end{cases} \tag{3.23}$$

where $\bar{U}(x,t) = (\bar{u}_1(x,t), \bar{u}_2(x,t))^\top$, $\bar{U}_0(x,s) = (\bar{u}_{10}(x,s), \bar{u}_{20}(x,s))^\top$ and

$$K_2 = \begin{pmatrix} -r_1 & 0 \\ 0 & r_2(1-b_2) \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & -r_1 b_1 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$A_2(\eta) = D(\hat{J}(\eta) - 1) + K_2 = \begin{pmatrix} d_1\hat{J}(\eta) - d_1 - r_1 & 0 \\ 0 & d_2\hat{J}(\eta) - d_2 + r_2(1-b_2) \end{pmatrix}.$$

Next, we show that

$$\begin{aligned} \omega_2 &:= |\zeta(A_2(\eta))| - \sigma(L_2) \\ &= \left|\max\left\{d_1(\hat{J}(\eta)-1) - r_1, d_2(\hat{J}(\eta)-1) + r_2(1-b_2)\right\}\right| - r_1 b_1 \\ &\geq r_1 + d_1(1-\hat{J}(\eta)) - r_1 b_1 = r_1(1-b_1) + d_1(1-\hat{J}(\eta)). \end{aligned} \tag{3.24}$$

Let

$$\delta_2 := r_1(1-b_1) > 0$$

under the condition of $0 < b_1 < 1 < b_2$. Furthermore, by Lemma 2.1 and (3.24), we derive

$$\|e^{A_2(\eta)t}e_\tau^{\bar{L}_2(\eta)t}\| \leq C_3 e^{-\varepsilon_2(|\zeta(A_2(\eta))| - \sigma(L_2))t} \leq C_3 e^{-\varepsilon_2\delta_2 t} e^{-\varepsilon_2 d_1(1-\hat{J}(\eta))t}, \tag{3.25}$$

where $\varepsilon_2 = \varepsilon_2(\tau) \in (0,1)$ is a decreasing function and C_3 denotes a positive constant depending on $\bar{U}_0(s)$ for $s \in [-\tau, 0]$. By the same proof as the Lemma 3.1, it is proved that the following lemma holds.

Lemma 3.2. *Assume that the conditions in Theorem 2.2 hold, there exists a decreasing function* $\varepsilon_2 = \varepsilon_2(\tau) = \frac{1}{(1+\tau)(1+\|L_2\|_\infty)+2\omega_2\tau} \in (0,1)$ *such that*

$$\sum_{i=1}^{2} \|\partial_x^\alpha \bar{u}_i(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C\rho_{\bar{u}}^\alpha e^{-\varepsilon_2 t\sqrt{k\delta_2 d_1 \nu^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}}$$

for some constant $\nu \in (0, a_0)$ *and* $\delta_2 = r_1(1-b_1) > 0$.

3.3. Case 3: convergence to the equilibrium E_3^*

In this case, we show the decay estimate of the linearized system of solution around the equilibrium E_3^*. Denote $E_3^* = (m_1, m_2) = \left(\frac{1-b_1}{1-b_1b_2}, \frac{1-b_2}{1-b_1b_2}\right)$.

Set

$$\begin{cases} \check{u}_1(x,t) = u_1(x,t) - m_1, \\ \check{u}_2(x,t) = u_2(x,t) - m_2. \end{cases} \tag{3.26}$$

Moreover, the system (1.4) becomes as

$$\begin{cases} \dfrac{\partial \check{u}_1}{\partial t} = d_1(J * \check{u}_1 - \check{u}_1) + r_1(\check{u}_1 + m_1)[1 - m_1 - b_1 m_2 - \check{u}_1 - b_1 \check{u}_2^\tau], \\ \dfrac{\partial \check{u}_2}{\partial t} = d_2(J * \check{u}_2 - \check{u}_2) + r_2(\check{u}_2 + m_2)[1 - b_2 m_1 - m_2 - b_2 \check{u}_1^\tau - \check{u}_2], \end{cases} \tag{3.27}$$

where $\check{u}_i = \check{u}_i(x,t)$ and $\check{u}_i^\tau = \check{u}_i(x, t-\tau)$ with $(x,t) \in \mathbb{R}^N \times (0, +\infty)$ and the initial data

$$\check{u}_i(x,s) = \check{u}_{i0}(x,s), \quad (x,s) \in \mathbb{R}^N \times [-\tau, 0], \quad i = 1, 2.$$

Then, the vector-valued function $\check{U}(x,t)$ satisfies the following linearized system

$$\begin{cases} \dfrac{\partial \check{U}}{\partial t}(x,t) = D(J * \check{U} - \check{U})(x,t) + K_3 \check{U}(x,t) + L_3 \check{U}(x, t-\tau), \\ \check{U}(x,s) = \check{U}_0(x,s), \end{cases} \tag{3.28}$$

for $x \in \mathbb{R}^N$, $t > 0$, $s \in [-\tau, 0]$, where $\check{U}(x,t) = (\check{u}_1(x,t), \check{u}_2(x,t))^\top$, $\check{U}_0(x,s) = (\check{u}_{10}(x,t), \check{u}_{20}(x,t))^\top$ and

$$K_3 = \begin{pmatrix} \frac{r_1(b_1-1)}{1-b_1b_2} & 0 \\ 0 & \frac{r_2(b_2-1)}{1-b_1b_2} \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & \frac{r_1 b_1(b_1-1)}{1-b_1b_2} \\ \frac{r_2 b_2(b_2-1)}{1-b_1b_2} & 0 \end{pmatrix}.$$

It follows that

$$A_3(\eta) = D(\hat{J}(\eta) - 1) + K_3 = \begin{pmatrix} d_1 \hat{J}(\eta) - d_1 + \frac{r_1(b_1-1)}{1-b_1b_2} & 0 \\ 0 & d_2 \hat{J}(\eta) - d_2 + \frac{r_2(b_2-1)}{1-b_1b_2} \end{pmatrix}.$$

As a consequence, we have

$$\begin{aligned} \omega_3 &:= |\zeta(A_3(\eta))| - \sigma(L_3) \\ &= \left| \max\left\{ d_1(\hat{J}(\eta) - 1) + \frac{r_1(b_1-1)}{1-b_1b_2}, d_2(\hat{J}(\eta) - 1) + \frac{r_2(b_2-1)}{1-b_1b_2} \right\} \right| \\ &\quad - \left| \max\left\{ \frac{r_1 b_1(1-b_1)}{1-b_1b_2}, \frac{r_2 b_2(1-b_2)}{1-b_1b_2} \right\} \right| \end{aligned}$$

$$\geq \begin{cases} \frac{r_1(1-b_1)^2}{1-b_1b_2} + d_1(1-\hat{J}(\eta)), & \frac{r_1b_1(1-b_1)}{r_2b_2(1-b_2)} > 1, \\ \frac{r_2(1-b_2)^2}{1-b_1b_2} + d_2(1-\hat{J}(\eta)), & \frac{r_1b_1(1-b_1)}{r_2b_2(1-b_2)} < 1. \end{cases}$$

That is to say,

$$\omega_3 \geq \begin{cases} \delta_3 + d_1(1-\hat{J}(\eta)), & r_1 > 4r_2b_2(1-b_2), \\ \delta_4 + d_2(1-\hat{J}(\eta)), & r_2 > 4r_1b_1(1-b_1), \end{cases} \tag{3.29}$$

where $0 \leq b_1, b_2 \leq 1$, $\delta_3 := \frac{r_1(1-b_1)^2}{1-b_1b_2} > 0$ and $\delta_4 := \frac{r_2(1-b_2)^2}{1-b_1b_2} > 0$. By virtue of the symmetry in form of the two cases contained in (3.29), we only need to consider the first condition. Then, by Lemma 2.1 and the first inequality in (3.29), we get

$$\|e^{A_3(\eta)t} e_{\tau}^{\bar{L}_3(\eta)t}\| \leq C_4 e^{-\varepsilon_3\delta_3 t} e^{-\varepsilon_3 d_1(1-\hat{J}(\eta))t}, \quad r_1 > 4r_2b_2(1-b_2), \tag{3.30}$$

where $\varepsilon_3 = \varepsilon_3(\tau) \in (0,1)$ is a decreasing function and C_4 denotes the positive constant relying on $\check{U}_0(s)$ with $s \in [-\tau, 0]$, respectively. It can be followed from ideas for the proof of Lemma 3.1 that a similar lemma is also obtained in this case as follows.

Lemma 3.3. *Provided that the conditions in Theorem 2.3 hold, there exists a decreasing function* $\varepsilon_3 = \varepsilon_3(\tau) = \frac{1}{(1+\tau)(1+\|L_3\|_\infty)+2\omega_3\tau} \in (0,1)$ *such that*

$$\sum_{i=1}^{2} \|\partial_x^\alpha \check{u}_i(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C\rho_{\check{u}}^\alpha e^{-\varepsilon_3 t\sqrt{k\delta_3 d_1 \vartheta^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}},$$

for some constant $\vartheta \in (0, a_0)$ *and* $\delta_3 := \frac{r_1(1-b_1)^2}{1-b_1b_2} > 0$.

4. Nonlinear stability

In this section, due to decay rates of solutions of linear problems are estimated, we make full use of the above conclusions to obtain decay estimates of solutions of the nonlinear problem (1.4) around equilibria. As a matter of fact, the global existence and uniqueness of the solution of the Cauchy problem (1.4) can be established through the standard energy method and continuity extension method [14,33]. Hence, we show the proposition as follows.

Proposition 4.1 (Global Existence and Uniqueness). *If the initial data* $u_{i0}(x,s)$ *satisfies* $u_{i0}(x,s) \in C([-\tau,0]; C(\mathbb{R}^N))$ *and*

$$0 \leq u_{i0}(x,s) \leq 1, \quad (x,s) \in \mathbb{R}^N \times [-\tau, 0], \quad i = 1,2,$$

then the solution of the system (1.4) uniquely and globally exists with

$$u_i \in C^1((0,\infty), C(\mathbb{R}^N))$$

and satisfies

$$0 \leq u_i(x,t) \leq 1, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \quad i = 1,2.$$

4.1. Case of convergence to the equilibrium E_1^*

Let's go back to the investigation of the nonlinear problem (1.4).
Proof of Theorem 2.1. From (3.7), the system (3.6) can be rewritten as

$$\begin{cases} \dfrac{\partial \tilde{U}}{\partial t}(x,t) = D(J * \tilde{U} - \tilde{U})(x,t) + K_1 \tilde{U}(x,t) + L_1 \tilde{U}(x,t-\tau) + \tilde{R}, \\ \tilde{U}(x,s) = \tilde{U}_0(x,s), \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^N$, $t > 0$, $s \in [-\tau, 0]$ and

$$\tilde{R} = \tilde{R}(x,t) = \begin{pmatrix} -r_1 \tilde{u}_1^2(x,t) - r_1 b_1 \tilde{u}_1(x,t)\tilde{u}_2(x,t-\tau) \\ -r_2 \tilde{u}_2^2(x,t) - r_2 b_2 \tilde{u}_2(x,t)\tilde{u}_1(x,t-\tau) \end{pmatrix}.$$

Applying the fundamental solution formula to (4.1), we obtain

$$\begin{aligned} \tilde{U}(x,t) = {} & G(\cdot,t) * \tilde{U}_0(\cdot,-\tau) + \int_0^t G(\cdot,t-s-\tau) * \tilde{R}(\cdot,s)\mathrm{d}s \\ & + \int_{-\tau}^0 G(\cdot,t-s-\tau) * \left[\frac{\mathrm{d}}{\mathrm{d}s}\tilde{U}_0(\cdot,s) - D(J * \tilde{U}_0 - \tilde{U}_0)(\cdot,s) - K_1 \tilde{U}_0(\cdot,s)\right]\mathrm{d}s \\ \le {} & G(\cdot,t) * \tilde{U}_0(\cdot,-\tau) \\ & + \int_{-\tau}^0 G(\cdot,t-s-\tau) * \left[\frac{\mathrm{d}}{\mathrm{d}s}\tilde{U}_0(\cdot,s) - D(J * \tilde{U}_0 - \tilde{U}_0)(\cdot,s) - K_1 \tilde{U}_0(\cdot,s)\right]\mathrm{d}s, \end{aligned}$$

where $G(\cdot,t)$ represents the fundamental solution of the following time-delayed dispersal equation

$$\frac{\partial \tilde{U}}{\partial t}(x,t) = D(J * \tilde{U} - \tilde{U})(x,t) + K_1 \tilde{U}(x,t) + L_1 \tilde{U}(x,t-\tau).$$

According to the definition of 1−norm of vector, the properties of convolution and Lemma 3.1, we have

$$\begin{aligned} & \sup_{x\in\mathbb{R}^N} \|\tilde{U}(x,t)\| \\ \le {} & \|G(\cdot,t) * \tilde{U}_0(\cdot,-\tau)\| \\ & + \int_{-\tau}^0 \left\| G(\cdot,t-s-\tau) * \left[\frac{\mathrm{d}}{\mathrm{d}s}\tilde{U}_0(s) - D(J * \tilde{U}_0 - \tilde{U}_0)(s) - K_1 \tilde{U}_0(s)\right]\right\| \mathrm{d}s \\ \le {} & C \sup_{x\in\mathbb{R}^N} \|G(x,t)\| \|\tilde{U}_0(\cdot,-\tau)\| \\ & + \int_{-\tau}^0 \sup_{x\in\mathbb{R}^N} \|G(x,t-s-\tau)\| \left\| \left[\frac{\mathrm{d}}{\mathrm{d}s}\tilde{U}_0(s) - D(J * \tilde{U}_0 - \tilde{U}_0)(s) - K_1 \tilde{U}_0(s)\right]\right\| \mathrm{d}s \\ \le {} & C\rho_{\tilde{u}}^0 e^{-\varepsilon_1 t\sqrt{k\delta_1 d_2 \xi^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}}. \end{aligned}$$

Then, we get

$$\partial_x^\alpha \tilde{U}(x,t)$$

$$= \partial_x^\alpha G(\cdot,t) * \tilde{U}_0(\cdot,-\tau) + \int_0^t \partial_x^\alpha G(\cdot,t-s-\tau) * \tilde{R}(\cdot,s)\mathrm{d}s$$
$$+ \int_{-\tau}^0 \partial_x^\alpha G(\cdot,t-s-\tau) * \left[\frac{\mathrm{d}}{\mathrm{d}s}\tilde{U}_0(\cdot,s) - D(J*\tilde{U}_0 - \tilde{U}_0)(\cdot,s) - K_1\tilde{U}_0(\cdot,s)\right]\mathrm{d}s.$$

Notice that $0 \le u_i(x,t) \le 1$, then we have

$$\begin{aligned}\|\tilde{R}(\cdot,s)\| &= r_1\|\tilde{u}_1^2(\cdot,s) + b_1\tilde{u}_1(\cdot,s)\tilde{u}_2^\tau\|_{L^\infty(\mathbb{R}^N)} + r_2\|\tilde{u}_2^2(\cdot,s) + b_2\tilde{u}_2(\cdot,s)\tilde{u}_1^\tau\|_{L^\infty(\mathbb{R}^N)} \\ &\le (r_1+r_2)\sup_{x\in\mathbb{R}^N}\|\tilde{U}(x,t)\| + r_1b_1\left(\|\tilde{u}_2^\tau\|_{L^\infty(\mathbb{R}^N)} + \frac{r_2b_2}{r_1b_1}\|\tilde{u}_1^\tau\|_{L^\infty(\mathbb{R}^N)}\right) \\ &\le (r_1+r_2)\sup_{x\in\mathbb{R}^N}\|\tilde{U}(x,t)\| + \max\left\{1,\frac{r_2b_2}{r_1b_1}\right\}\sup_{x\in\mathbb{R}^N}\|\tilde{U}(x,t)\| \\ &\le C_4\rho_{\tilde{u}}^0 e^{-\varepsilon_1(s-\tau)\sqrt{k\delta_1 d_2\xi^{2\beta}}}(s-\tau+1)^{-\frac{N-1}{2\beta}}. \end{aligned} \tag{4.2}$$

where $\tilde{u}_i^\tau = \tilde{u}_i(\cdot,s-\tau)$. Based on the Hölder inequality, the properties of convolution and (4.2), we obtain

$$\begin{aligned}&\sup_{x\in\mathbb{R}^N}\left\|\partial_x^\alpha\tilde{U}(x,t)\right\| \\ \le& \left\|\partial_x^\alpha G(\cdot,t)*\tilde{U}_0(\cdot,-\tau)\right\| + \int_0^t\left\|\partial_x^\alpha G(\cdot,t-s-\tau)*\tilde{R}(\cdot,s)\right\|\mathrm{d}s \\ &+ \int_{-\tau}^0\left\|\partial_x^\alpha G(\cdot,t-s-\tau)*\left[\frac{\mathrm{d}}{\mathrm{d}s}\tilde{U}_0(s) - D(J*\tilde{U}_0-\tilde{U}_0)(s) - K_1\tilde{U}_0(s)\right]\right\|\mathrm{d}s \\ \le& \sup_{x\in\mathbb{R}^N}\|\partial_x^\alpha G(\cdot,t)\|\,\|\tilde{U}_0(\cdot,-\tau)\| + \int_0^t\sup_{x\in\mathbb{R}^N}\|\partial_x^\alpha G(\cdot,t-s-\tau)\|\|\tilde{R}(\cdot,s)\|\mathrm{d}s \\ &+ \int_{-\tau}^0\sup_{x\in\mathbb{R}^N}\|\partial_x^\alpha G(\cdot,t-s-\tau)\|\left\|\left[\frac{\mathrm{d}}{\mathrm{d}s}\tilde{U}_0(s) - D(J*\tilde{U}_0-\tilde{U}_0)(s) - K_1\tilde{U}_0(s)\right]\right\|\mathrm{d}s \\ \le& C\rho_{\tilde{u}}^\alpha\left(e^{-\varepsilon_1 t\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}} + e^{-\varepsilon_1(t-s)\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t-s+1)^{-\frac{N-1}{2\beta}}\|\tilde{R}(\cdot,s)\|\mathrm{d}s\right) \\ \le& C\rho_{\tilde{u}}^\alpha e^{-\varepsilon_1 t\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}} \\ &+ C\rho_{\tilde{u}}^\alpha\int_0^t e^{-\varepsilon_1(t-s)\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t-s+1)^{-\frac{N-1}{2\beta}}e^{-\varepsilon_1(s-\tau)\sqrt{k\delta_1 d_2\xi^{2\beta}}}(s-\tau+1)^{-\frac{N-1}{2\beta}}\mathrm{d}s \\ \le& C\rho_{\tilde{u}}^\alpha e^{-\varepsilon_1 t\sqrt{k\delta_1 d_2\xi^{2\beta}}}(t+1)^{-\frac{N-1}{2\beta}},\end{aligned}$$

where we use the fact that

$$\int_0^t (t-s+1)^{-\frac{N-1}{2\beta}}(s+1)^{-\frac{N-1}{2\beta}}\mathrm{d}s \le C(t+1)^{-\frac{N-1}{2\beta}}$$

with $N-1 > 2\beta$. That is to say, for some constant $\xi \in (0,a_0)$, we have

$$\|\partial_x^\alpha u_1(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} + \|\partial_x^\alpha(u_2-1)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\rho_{\tilde{u}}^\alpha e^{-\tilde{\varepsilon}t}(t+1)^{-\frac{N-1}{2\beta}},$$

where $\tilde{\varepsilon} = \varepsilon_1\xi^\beta\sqrt{k\delta_1 d_2}$ and $\varepsilon_1 = \varepsilon_1(\tau) \in (0,1)$ represents a decreasing function. This finishes the proof. □

4.2. Case of convergence to the equilibrium E_2^*

This subsection will be completed on the basis of the second case in the Section 3.
Proof of Theorem 2.2. The nonlinear system (1.4) can be transformed into

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial t}(x,t) = D(J * \bar{U} - \bar{U})(x,t) + K_2\bar{U}(x,t) + L_2\bar{U}(x,t-\tau) + \bar{R}, \\ \bar{U}(x,s) = \bar{U}_0(x,s), \end{cases} \tag{4.3}$$

where $x \in \mathbb{R}^N$, $t > 0$, $s \in [-\tau, 0]$ and

$$\bar{R} = \bar{R}(x,t) = \begin{pmatrix} -r_1\bar{u}_1^2(x,t) - r_1b_1\bar{u}_1(x,t)\bar{u}_2(x,t-\tau) \\ -r_2\bar{u}_2^2(x,t) - r_2b_2\bar{u}_2(x,t)\bar{u}_1(x,t-\tau) \end{pmatrix}.$$

It is noted that the nonlinear problem (4.3) exhibits the similar form to the nonlinear problem (4.1) and their difference is only the symbols caused by transformations. In addition, according to the boundedness of the solution of the nonlinear problem (1.4), the fundamental solution formula to the system (4.3) and Lemma 3.2, we also have a decay estimate of $\bar{R}(x,t)$. Thus, similar to the decay estimate of $(u_1(x,t), u_2(x,t))$ convergence to E_1^*, we derive that

$$\|\partial_x^\alpha (u_1 - 1)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} + \|\partial_x^\alpha u_2(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\rho_{\bar{u}}^\alpha e^{-\bar{\varepsilon}t}(t+1)^{-\frac{N-1}{2\beta}},$$

for some constant $\nu \in (0, a_0)$, $\bar{\varepsilon} = \varepsilon_2\nu^\beta\sqrt{k\delta_2 d_1}$ and $\varepsilon_2 = \varepsilon_2(\tau) \in (0,1)$ is a decreasing function. Up to now, we have got Theorem 2.2. □

4.3. Case of convergence to the equilibrium E_3^*

Proof of Theorem 2.3. By the linearized system (3.28), the original problem (1.4) can be expressed by

$$\begin{cases} \dfrac{\partial \check{U}}{\partial t}(x,t) = D(J * \check{U} - \check{U})(x,t) + K_3\check{U}(x,t) + L_3\check{U}(x,t-\tau) + \check{R}, \\ \check{U}(x,s) = \check{U}_0(x,s), \end{cases} \tag{4.4}$$

where $x \in \mathbb{R}^N$, $t > 0$, $s \in [-\tau, 0]$ and

$$\check{R} = \check{R}(x,t) = \begin{pmatrix} -r_1\check{u}_1^2(x,t) - r_1b_1\check{u}_1(x,t)\check{u}_2(x,t-\tau) \\ -r_2\check{u}_2^2(x,t) - r_2b_2\check{u}_2(x,t)\check{u}_1(x,t-\tau) \end{pmatrix}.$$

By the same token, the decay rate of the solution of the nonlinear problem (4.4) is determined by its linear part. That is to say, one have

$$\sum_{i=1}^{2} \|\partial_x^\alpha (u_i - m_i)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\rho_{\check{u}}^\alpha e^{-\check{\varepsilon}t}(t+1)^{-\frac{N-1}{2\beta}}, \quad r_1 > 4r_2b_2(1-b_2),$$

for some constant $\vartheta \in (0, a_0)$, $\check{\varepsilon} = \varepsilon_3\vartheta^\beta\sqrt{k\delta_3 d_1}$ and $\varepsilon_3 = \varepsilon_3(\tau) \in (0,1)$ is a decreasing function $(i = 1, 2)$. Thus, this finishes the proof of Theorem 2.3. □

5. Numerical simulations

In this section, we are concerned with several numerical simulations to verify the theoretical results of the asymptotic behavior of the nonlocal dispersal Lotka-Volterra competitive system using the finite difference method and the numerical iteration technique by Matlab software. Moreover, we reveal the inhibition of time delay on decay rates of the solution. In what follows, we take the kernel $J(x) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}, x \in \mathbb{R}$ and a finite spatial domain $\Omega = [-300, 300]$.

(i): Case of convergence to the equilibrium $E_1^* = (0, 1)$.

Parameter values are as follows: $\tau = 0.5$, $d_1 = d_2 = 1$, $r_1 = 0.8$, $r_2 = 0.5$, $b_1 = 1.5$ and $b_2 = 0.6$ which satisfy the condition $0 < b_2 < 1 < b_1$. Then we choose the initial data

$$\begin{cases} u_{10}(x,s) = 0.4 + 0.2\cos(0.08x + 0.2), \\ u_{20}(x,s) = 0.4 + 0.08\cos(0.5x + 0.3), \end{cases} \quad x \in [-300, 300], \quad s \in [-0.5, 0].$$

As is shown in Figure 1, the density of two species converges to 0 and 1, which implies asymptotic behavior of the solution of the nonlocal dispersal system (1.4). According to Remark 2.2, Figure 2(a) gives a description of the $L^\infty-$norm of the density of two species. Further, in Figure 2(b), we consider the influence of the time delay on density of species and the results show that the time delay can inhibit the species growth. That is to say, we verify the conclusion in Theorem 2.1.

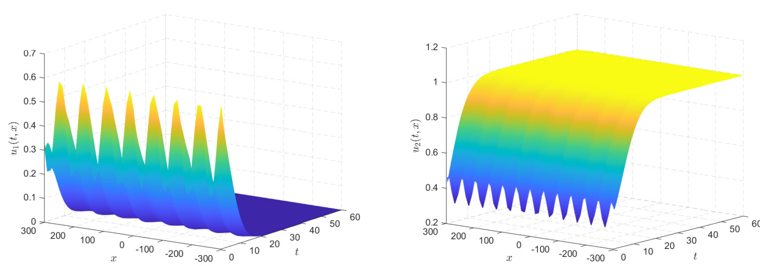


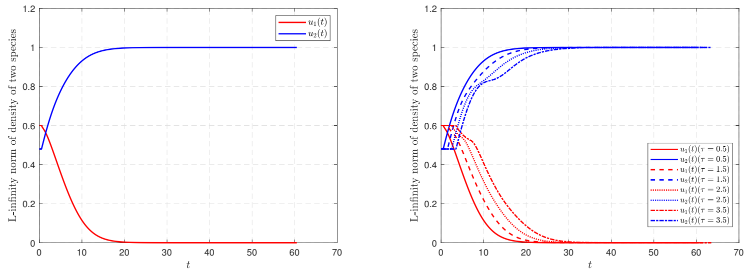
Figure 1. The asymptotic behavior of the solution convergence to E_1^* of the nonlocal dispersal Lotka-Volterra competitive system.

(ii): Case of convergence to the equilibrium $E_2^* = (1, 0)$.

Parameter values are as follows: $\tau = 0.5$, $d_1 = d_2 = 1$, $r_1 = r_2 = 1$, $b_1 = 0.375$ and $b_2 = 1.125$ which satisfy the condition $0 < b_1 < 1 < b_2$. We take the initial data

$$\begin{cases} u_{10}(x,s) = 0.2 + 0.15\cos(0.1x + 0.12), \\ u_{20}(x,s) = 0.5 + 0.08\cos(0.5x + 0.3), \end{cases} \quad x \in [-300, 300], \quad s \in [-0.5, 0].$$

It is obvious that the solution of the nonlocal dispersal Lotka-Volterra competitive system converges to the equilibrium $E_2^* = (1, 0)$ (see Figure 3). We also show the $L^\infty-$norm of the density of two species in Figure 4(a) which is consistent with



(a) (b)

Figure 2. (a) $L^\infty-$norm of density of two species. (b) The influence of time delay on density of species where $\tau = 0.5,\ 1.5,\ 2.5,\ 3.5$.

Theorem 2.2. If we take the time delay as different values such as $\tau = 1, 2, 3, 4$, as shown in Figure 4(b), which means that the time delay hinders the species growth.

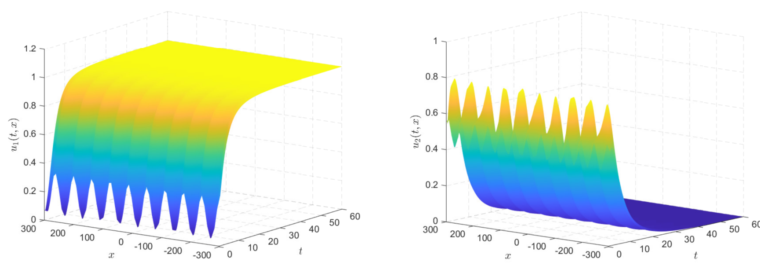
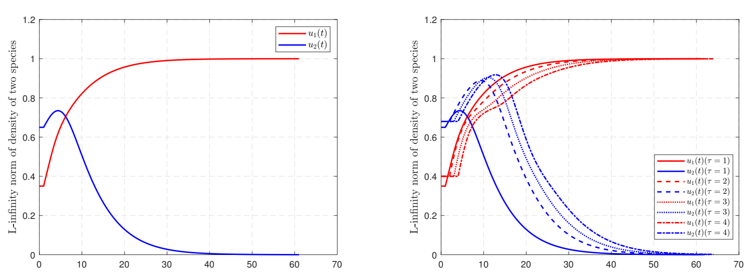


Figure 3. The asymptotic behavior of the solution convergence to E_2^* of the nonlocal dispersal Lotka-Volterra competitive system.



(a) (b)

Figure 4. (a) $L^\infty-$norm of density of two species. (b) The influence of time delay on density of species where $\tau = 1,\ 2,\ 3,\ 4$.

(iii): Case of convergence to the equilibrium $E_3^* = (\frac{1-b_1}{1-b_1 b_2}, \frac{1-b_2}{1-b_1 b_2})$.

Parameter values are as follows: $\tau = 0.5$, $d_1 = 0.5$, $d_2 = 0.6$, $r_1 = 0.5$, $r_2 = 0.3$, $b_1 = 0.2$ and $b_2 = 0.3$ which satisfy the condition $0 < b_1, b_2 < 1$ and $r_1 > 4r_2 b_2(1 -$

b_2). The initial data is

$$\begin{cases} u_{10}(x,s) = 0.2 + 0.2\cos(0.1x + 0.35), \\ u_{20}(x,s) = 0.2 + 0.1\cos(0.5x + 0.15), \end{cases} \quad x \in [-300, 300], \quad s \in [-0.5, 0].$$

Similar to *(i)* and *(ii)*, the asymptotic behavior of the nonlocal dispersal Lotka-Volterra competitive system is illustrated in Figure 5. Namely, the density of two species converges to $E_3^* = (0.85, 0.75)$. Then, we obtain the $L^\infty-$norm of the density of two species and restriction of the density of species by introducing the time delay (see Figure 6). In brief, these numerical results confirm Theorem 2.3.

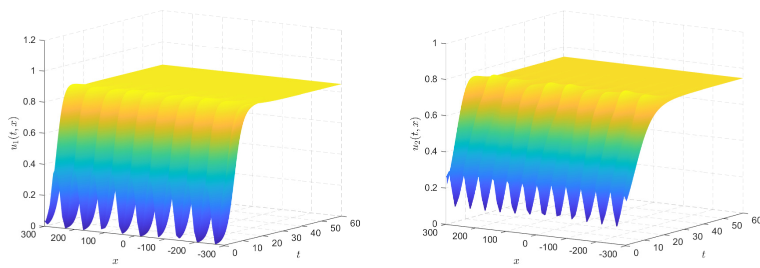
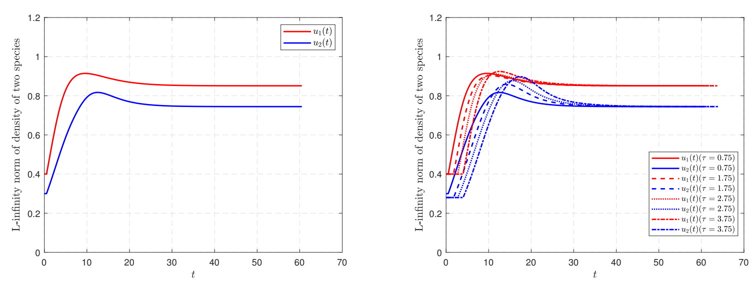


Figure 5. The asymptotic behavior of the solution convergence to E_3^* of the nonlocal dispersal Lotka-Volterra competitive system.



(a) (b)

Figure 6. (a) $L^\infty-$norm of density of two species. (b) The influence of time delay on density of species where $\tau = 0.75,\ 1.75,\ 2.75,\ 3.75$.

(iv): Spatial periodicity.

Parameter values are as follows: $\tau = 0.5$, $d_1 = d_2 = 1$, $r_1 = r_2 = 0.6$, $b_1 = 4$ and $b_2 = 3$. The initial data is

$$\begin{cases} u_{10}(x,s) = 0.4 + 0.1\cos(0.08x + 0.1), \\ u_{20}(x,s) = 0.2 + 0.08\cos(0.5x + 0.15), \end{cases} \quad x \in [-300, 300], \quad s \in [-0.5, 0].$$

Notice that the parameter values $b_1 > 1$ and $b_2 > 1$ which do not meet the conditions of Theorem 2.1, Theorem 2.2 and Theorem 2.3, the solution of the system (1.4) occurs periodic oscillation (see Figure 7), which implies that different regions have different ecological conditions, resulting in periodic species distribution patterns.

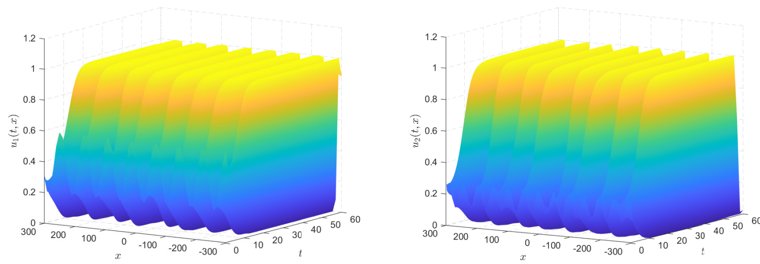


Figure 7. The space periodic solution

(iv): Time periodicity.

Parameter values are as follows: $\tau = 0.5$, $d_1 = 0.8$, $d_2 = 0.7$, $r_1 = r_2 = 0.6$, $b_1 = 2.5$ and $b_2 = 2.1$, which implies that the parameter values $b_1 > 1$ and $b_2 > 1$ do not satisfy the conditions of Theorem 2.1, Theorem 2.2 and Theorem 2.3. The initial data is

$$\begin{cases} u_{10}(x,s) = 0.4 + 0.2\cos(0.08x + 0.1), \\ u_{20}(x,s) = 0.2 + 0.08\cos(0.5x + 0.2), \end{cases} \qquad x \in [-300, 300], \quad s \in [-0.5, 0].$$

In Figure 8, the solution of the system (1.4) is periodic with respect to time. For example, in the case of seasonal environmental changes, the reproduction, migration and survival strategies of biological populations usually show periodic changes, which leads to periodic fluctuations in the density of species with time.

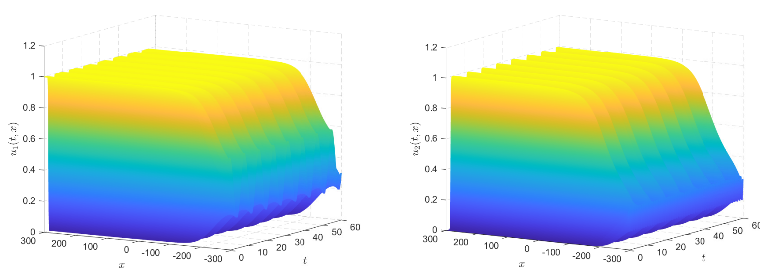


Figure 8. The time periodic solution.

6. Conclusion

In this paper, we investigate the asymptotic behavior of the solution of the nonlocal dispersal Lotka-Volterra system with time delay in the whole $\mathbb{R}^N$. More precisely, we are concerned with $L^\infty-$decay estimates of the solution of the nonlinear system (1.4) converging to the equilibria E_1^*, E_2^* and E_3^*.

To begin with, we consider linear time-delayed nonlocal dispersal systems corresponding to the nonlinear system (1.4) at equilibria. We adopt the method of Fourier transform for the fundamental solution and combine with the Fourier splitting method which divides the Fourier transform into high-frequency and low-frequency parts. It is worth noting that Lemma 2.1 plays a key role in decay estimates of solutions of linear systems. In this part, we obtain $L^\infty-$decay estimates of solutions of linear systems. Due to conclusions of linear systems and boundedness of the solution of the nonlinear system (1.4), we obtain the $L^\infty-$decay estimates of the solution of the nonlinear system (1.4) using the properties of convolution and Hölder inequality.

Finally, in various cases, we product some numerical simulations to verify the theoretical results and get the restrictive effect of the time delay on growth of the density of species.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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