An effective numerical algorithm for solving the Lane-Emden type equation based on the variational iteration method coupled with the homotopy analysis method

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Abstract

This research introduces an effective numerical algorithm to determine the numerical solution of the Lane-Emden equation. This method is based on the variational iteration method coupled with the homotopy analysis method. We also included the convergence study of the proposed algorithm. Eight application problems of the Lane-Emden type equation of various kinds with several types of initial and boundary conditions are included to demonstrate the efficacy and accuracy of the proposed algorithm. The numerical outcomes are contrasted with those obtained by other methods [12, 20, 21] and the exact solution. Unlike other methods, the proposed algorithm does not require discretization or perturbation and can be applied easily and accurately. The proposed method can solve complex problems with less computational work and computation time.

Keywords: Astrophysics, Lane-Emden equation, Homotopy analysis method, Variational iteration method.

1 Introduction

In recent years, singular Lane-Emden equations (LEE) have arisen in many real-world applications in the fields of engineering and science. It arises in the modeling of stellar structure, clusters of galaxies, the catalytic diffusion process, thermal explosions, the behavior of gas clouds, population evolution, etc. In this article, we consider the general Lane-Emden equation that arises in astrophysics

$$\frac{d^2V(t)}{dt^2} + \frac{\alpha}{t}\frac{dV(t)}{dt} + f(t, V(t)) = 0, \quad 0 < t \leq 1, \alpha \geq 0 \tag{1.1}$$

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with the initial and boundary conditions

$$V(0) = a, \quad V'(0) = b, \quad a_1 V(1) + b_1 V'(1) = c \tag{1.2}$$

where α is a shape factor that describes the geometry of a gas vessel and f represents a non-linear function.

It is difficult to capture a solution close to the singular point of any nonlinear differential equation because the coefficients in the differential equation blow up around singularities. Wazwaz [23] presented two different approaches for solving LEEs using the Adomian decomposition technique. Yildirim and Ozis [24] proposed a homotopy perturbation method to solve IVPs of LEEs. In [8], Ozturk and Gulsu proposed an approximation algorithm using Hermite polynomials for solving LEEs arising in astrophysics and engineering. In [22], Gorder et al. presented an analytic solution using the homotopy analysis method and standard power series approach of LEE that describes the thermal behaviour of a spherical gas cloud acting under the mutual attraction of its molecules. Al-Hayani et al. [2] proposed an algorithm using the homotopy analysis method to find the numerical solution for IVPs of LEE. In [14], Singh et al. presented an algorithm for solving LEEs with various boundary conditions using the Haar wavelet collocation method. In [3], Dizicheh et al. presented a spectral method for finding the approximate solution of LEEs using the Legendre wavelet. In [10], Sabir et al. proposed an algorithm based on a Morlet wavelet neural network to solve second-order LEE. In [19], Tiwari et al. proposed an orthogonal polynomial wavelet method for solving strongly nonlinear LEE. In [13], Singh proposed a scheme using Green's function and decomposition technique to solve coupled LEEs. In [11], Saha and Singh developed a new method for dealing with IVPs of second-order Emden Fowler pantograph differential equations using Laguerre polynomials. In [17], Sinha and Maroju developed an algorithm for solving nonlinear LEEs using the variational iteration method and the quasilinearization method. In [16], Sinha and Maroju introduce an algorithm to solve coupled Lane-Emden type equations using the homotopy analysis method by embedding the quasilinearization technique. In [1], Ahmed described a numerical algorithm to obtain the approximate solution of singular LEEs using the first kind of shifted Chebyshev polynomials. In [7], Malele et al. used a high-order compact-finite-difference scheme for solving LEEs with various boundary conditions.

In this research, we are going to introduce an effective numerical algorithm called the homotopy variational iteration method (HVIM), which is based on the variational iteration method [4, 15, 18] coupled with the homotopy analysis method [5, 6, 16] to determine the numerical solution of the Lane-Emden type equation. Also, the convergence study of HVIM is addressed under general conditions. We included eight different kinds of nonlinear LEEs arises in astrophysics, where two are boundary value problems and the remaining six are initial value problems. We consider the LEEs containing nonlinear functions as exponential, trigonometric, and hyperbolic, which is difficult to solve due to their strong nonlinearity, which can be solved by means of the proposed HVIM. The numerical results obtained by the proposed HVIM are compared with advanced Adomian decomposition method [20, 21], homotopy perturbation method [12] and available exact solutions to check the reliability of the method. Unlike other methods, the proposed algorithm does not require discretization or perturbation and can be applied easily and accurately. The proposed method can solve complex problems with less computational work and computation time.

2 The VIM and the Lagrange multiplier for LEEs

According to variational theory [4, 15], correction functional for Lane-Emden equation (1.1) can be constructed as

$$V_{k+1}(t) = V_k(t) + \int_0^t \lambda(x) \left[\frac{d^2 V_k(x)}{dx^2} + \frac{\alpha}{x} \frac{dV_k(x)}{dx} + f(x, \tilde{V}_k) \right] dx, \; k = 0, 1, 2, ...$$

where, λ is a general Lagrange multiplier and, $f(x, \tilde{V}_k)$ denote restricted variation, i.e. $\delta f(x, \tilde{V}_k) = 0$.

$$\delta V_{k+1}(t) = \delta V_k(t) + \delta \int_0^t \lambda(\xi) \left[\frac{d^2 V_k(x)}{dx^2} + \frac{\alpha}{x} \frac{dV_n(x)}{dx} \right] dx.$$

On simplifying the above equation, we get the following stationary conditions

$$1 + \frac{\alpha}{t} \lambda(t) - \lambda'(t) = 0, \quad \frac{\alpha \left(x \lambda'(x) - \lambda(x) \right)}{x^2} - \lambda''(x) = 0, \quad \lambda(t) = 0.$$

Solving the above expression for λ, we obtained

$$\lambda(x) = \begin{cases} x \log\left(\frac{x}{t} \right), & \alpha = 1 \\ \frac{x \left(x^{\alpha-1} - t^{\alpha-1} \right)}{(\alpha - 1) t^{\alpha-1}}, & \alpha \neq 1 \end{cases} \tag{2.1}$$

3 Construction of homotopy variational iteration method (HVIM)

In this part, we propose HVIM for solving the LEEs. Using the Lagrange multiplier λ, we have the following variational iteration formula

$$V_{k+1}(t) = V_k(t) + c_0 \int_0^t \lambda(x) \left[\frac{d^2 V_k(x)}{dx^2} + \frac{\alpha}{x} \frac{dV_k(x)}{dx} + f(x, V_k) \right] dx, \tag{3.1}$$

where c_0 is a control parameter.

To obtain the series solution of (1.1), we couple the concept of VIM with HAM. Using the homotopy analysis method, we get the following general zero-order deformation equation for (3.1) (see appendix in [12])

$$H(t,q,v) := (1-q)[v_0 - V] = qh\int_0^t \lambda(x)\left[\frac{d^2V(x)}{dx^2} + \frac{\alpha}{x}\frac{dV(x)}{dx} + f(x,V(x))\right]dx, \tag{3.2}$$

where $q \in [0,1]$ is an embedding parameter, $h = -c_0$ is a convergence control parameter and v_0 is the initial guess satisfying (1.2). We have from (3.2)

$$\begin{aligned} H(t,0,V) &= V - v_0, \\ H(t,1,V) &= h\int_0^t \lambda(x)\left[\frac{d^2V(x)}{dx^2} + \frac{\alpha}{x}\frac{dV(x)}{dx} + f(x,V(x))\right]dx. \end{aligned}$$

Therefore $V(t,q)$ changes from $v_0(t)$ to the best approximate solution of (3.1) as q varies from 0 to 1. Taylor series expansion of v(t,q) w.r.t. parameter q

$$V(t,q) = v_0 + \sum_{m=1}^{\infty} v_m q^m, \tag{3.3}$$

where

$$v_m = \frac{1}{m!}\frac{\partial^m v(t,q)}{\partial q^m}\Big|_{q=0}. \tag{3.4}$$

If $h \neq 0$ is chosen properly then the series (3.3) will be convergent at $q = 1$

$$V(t,1) = V(t) = \sum_{m=0}^{\infty} v_m, \tag{3.5}$$

which will be the solution of (1.1)

Defining the vector $\vec{v}_m = \{v_0, v_1, \ldots, v_m\}$ and differentiating (3.2) m times w.r.t. parameter q, dividing it by $m!$ and setting subsequently $q = 0$ then the mth-order deformation equation is obtained

$$v_m = \eta v_{m-1} - hR_m(t, \vec{v}_{m-1}), \tag{3.6}$$

where

$$\eta = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{3.7}$$

and

$$\begin{aligned}
R_m(t,\vec{v}_{m-1}) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\int_0^t \lambda(x)\left[\frac{d^2V(x,q)}{dx^2}+\frac{\alpha}{x}\frac{dV(x,q)}{dx}+f(x,q,V)\right]dx \\
&= \int_0^t \lambda(x)\left[\frac{d^2v_{m-1}(x)}{dx^2}+\frac{\alpha}{x}\frac{dv_{m-1}(x)}{dx}+\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}f\left(\sum_{i=0}^{\infty}v_iq^i\right)|_{q=0}\right]dx \\
&= \int_0^t \lambda(x)\left[\frac{d^2v_{m-1}(x)}{dx^2}+\frac{\alpha}{x}\frac{dv_{m-1}(x)}{dx}+P_{m-1}\right]dx, \qquad (3.8)
\end{aligned}$$

where

$$P_{m-1}=\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}f\left(\sum_{i=0}^{\infty}v_iq^i\right)|_{q=0}\ . \qquad (3.9)$$

Therefore choosing the initial guess v_0 satisfying (1.2), $v_m, m \geq 1$ are successively obtained. Hence the kth-order approximate solution of (1.1) can be obtained by

$$V_k(t)=\sum_{m=0}^{k}v_m. \qquad (3.10)$$

4 Convergence analysis

Theorem 4.1 Suppose the Lipschitz condition $\mid f(t,V(t))-f(t,V^*(t))\mid\leq M\mid V(t)-V^*(t)\mid$, where $f(t,V(t))$ is nonlinear function, such that there exist $K\in(0,1)$ then the series $\sum_{m=0}^{k}v_m$ obtained from (3.10) is convergent in Banach space $X=(C[0,1],||V||)$ with defined norm

$$||V||=\max_{t\in[0,1]}\mid V(t)\mid,\quad V\in X$$

Proof From (3.6)-(3.10), we have

$$\begin{aligned}
V_k &= \sum_{m=0}^{k}v_m=\sum_{m=0}^{k}v_{m-1}-h\sum_{m=0}^{k}\int_0^t\lambda(x)\left[\frac{d^2v_{m-1}(x)}{dx^2}+\frac{\alpha}{x}\frac{dv_{m-1}(x)}{dx}+P_{m-1}\right]dx \\
&= V_{k-1}-h\int_0^t\lambda(x)\left[\frac{d^2V_{k-1}(x)}{dx^2}+\frac{\alpha}{x}\frac{dV_{k-1}(x)}{dx}+\sum_{m=0}^{k}P_{m-1}\right]dx.
\end{aligned}$$

For $n > p$ and for all $n, p \in N$,

$$\begin{aligned}
\|V_k - V_p\| &= \max_{t\in[0,1]} \Big| (V_{k-1} - V_{p-1}) - \\
&\quad h\int_0^t \lambda(x)\left[\frac{d^2V_{k-1}(x)}{dx^2} - \frac{d^2V_{p-1}(x)}{dx^2} + \frac{\alpha}{x}\left(\frac{dV_{k-1}(x)}{dx} - \frac{dV_{p-1}(x)}{dx}\right) + \sum_{m=0}^{k} P_{m-1} - \sum_{m=0}^{p} P_{m-1}\right] dx \Big| \\
&\leq \|V_{k-1} - V_{p-1}\| + \\
&\quad \max_{t\in[0,1]} \left| h\int_0^t \lambda(x)\left[\frac{d^2V_{k-1}(x)}{dx^2} - \frac{d^2V_{p-1}(x)}{dx^2} + \frac{\alpha}{x}\left(\frac{dV_{k-1}(x)}{dx} - \frac{dV_{p-1}(x)}{dx}\right) + \sum_{m=0}^{k} P_{m-1} - \sum_{m=0}^{p} P_{m-1}\right] dx \right|.
\end{aligned}$$

Using the relation $\sum_{m=0}^{k} P_m \leq f(V_k)$ [9]

$$\begin{aligned}
\|V_k - V_p\| &\leq \|V_{k-1} - V_{p-1}\| + \\
&\quad | h | \max_{t\in[0,1]} \int_0^t \left| \lambda(x)\left[\frac{d^2V_{k-1}(x)}{dx^2} - \frac{d^2V_{p-1}(x)}{dx^2} + \frac{\alpha}{x}\left(\frac{dV_{k-1}(x)}{dx} - \frac{dV_{p-1}(x)}{dx}\right) + (f(V_k) - f(V_p))\right] dx \right|.
\end{aligned} \tag{4.1}$$

As $\frac{d^2}{d\xi^2}$ and $\frac{d}{d\xi}$ are continuous linear operators in X, so there exists the real numbers β_1 and β_2 such that

$$\left\| \frac{d^2V_k}{dx^2} - \frac{d^2V_{k-1}}{dx^2} \right\| \leq \beta_1 \|V_k - V_{k-1}\|, \tag{4.2}$$

$$\left\| \frac{dV_k}{dx} - \frac{dV_{k-1}}{dx} \right\| \leq \beta_2 \|V_k - V_{k-1}\|, \tag{4.3}$$

$$\gamma_1 = \max_{t\in[0,1]} \int_0^t |\lambda(x)| dx, \quad and \quad \gamma_2 = \max_{t\in[0,1]} \int_0^t \left| \frac{\lambda(x)}{x} \right| dx. \tag{4.4}$$

From (4.1)-(4.4) and using the Lipschitz condition, we have

$$\begin{aligned}
\|V_k - V_p\| &\leq \|V_{k-1} - V_{p-1}\| + | h | \{\gamma_1\beta_1\|V_{k-1} - V_{p-1}\| + | \alpha | \gamma_2\beta_2\|V_{k-1} - V_{p-1}\| + \gamma_1 M\|V_{k-1} - V_{p-1}\|\} \\
&= (1 + | h | \{\gamma_1\beta_1 + | \alpha | \gamma_2\beta_2 + \gamma_1 M\}) \|V_{k-1} - V_{p-1}\| \\
&= K\|V_{k-1} - Z_{p-1}\|,
\end{aligned}$$

where $K = 1 + | h | \{\gamma_1\beta_1 + | \alpha | \gamma_2\beta_2 + \gamma_1 M\}$.

Setting $k = p + 1$, we get

$$\|V_{p+1} - V_p\| \leq K\|V_p - V_{p-1}\| \leq K^2\|V_{p-1} - V_{p-2}\| \leq ... \leq K^p\|V_1 - V_0\|.$$

Using triangular inequality for all $k, p \in N$ with $k > p$

$$\begin{aligned}
\|V_k - V_p\| &\leq \|V_k - X_{k-1}\| + \|V_{k-1} - V_{k-2}\| + \|V_{k-2} - V_{k-3}\| + ... + \|V_{p+1} - V_p\| \\
&\leq K^p \left(1 + K + K^2 + ... + K^{k-p-1}\right) \|V_1 - V_0\| \\
&= K^s \left(\frac{1 - K^{n-s}}{1 - K}\right) \|v_1\|.
\end{aligned}$$

Since $K \in (0,1)$, therefore

$$\|V_k - V_p\| \leq \left(\frac{K^p}{1-K}\right)\|v_1\|. \tag{4.5}$$

Taking $k, p \to \infty$, we get $V_k \to V_p$. Hence, $< V_k >$ converges to the analytic solution.

5 Numerical results

In this section, we include eight application problems of the Lane-Emden type equation with various initial and boundary conditions and compare the obtained numerical results by using the proposed homotopy variational iteration method with those obtained by advanced Adomian decomposition method (AADM) [20, 21] and homotopy perturbation method (HPM) [12] to illustrate the applicability and accuracy of the homotopy variational iteration method.

Example 1. Consider the following nonlinear LEE that arises in equilibrium isothermal gas sphere

$$\frac{d^2V(t)}{dt^2} + \frac{2}{t}\frac{d}{dt}V(t) = -V^5, \tag{5.1}$$

with the initial and boundary conditions

$$V(0) = 1, \quad V'(0) = 0, \quad V(1) = \sqrt{\frac{3}{4}}.$$

The Exact solution is

$$V(t) = \sqrt{\frac{3}{x+t^2}}.$$

Here $\alpha = 2$, so LEE in (5.1) is spherical.

On applying the proposed HVIM, we get the following $(m+1)^{th}$ iterative scheme for (5.1)

$$V_{k+1}(t) \quad = \quad V_k - h\int_0^t \frac{x\,(x-t)}{t}\left[\frac{d^2V_k(x)}{dx^2} + \frac{\alpha}{x}\frac{dV_k(x)}{dx} + \sum_{m=0}^{k} P_m\right]dx, \tag{5.2}$$

Where P_m can be obtain from (3.9).

Applying the proposed HVIM, with the initial guess $V_0 = v_0 = 1$ and $h = -1$, the 4th-order approximate solution for (5.1) can be obtained as

$$V_4(t) \quad = \quad 1 - 0.166667t^2 + 0.0416667t^4 - 0.0115741t^6 + 0.00337577t^8$$

Table 1: Comparison of numerical results of Example 1

t	$V(t)$	$V_4(t)$	AADM [21]	HPM [12]	R_4	r_4 [21]	r_4 [12]
0.1	0.9983374885	0.9983374885	0.9819992464	0.9901138001	$1.01030E-13$	$1.63382E-02$	$8.22369E-03$
0.2	0.9933992678	0.9933992679	0.9774514890	0.9856630325	$1.02452E-10$	$1.59478E-02$	$7.73624E-03$
0.3	0.9853292782	0.9853292840	0.9700154210	0.9783541835	$5.82008E-09$	$1.53139E-02$	$6.97509E-03$
0.4	0.9743547037	0.9743548049	0.9599063336	0.9683460216	$1.01246E-07$	$1.44484E-02$	$6.00868E-03$
0.5	0.9607689228	0.9607698417	0.9474256347	0.9558512822	$9.18870E-07$	$1.33433E-02$	$4.91764E-03$
0.6	0.9449111825	0.9449167000	0.9329608489	0.9411276222	$5.51748E-06$	$1.19503E-02$	$3.78356E-03$
0.7	0.9271455408	0.9271704283	0.9169856171	0.9244666981	$2.48875E-05$	$1.01599E-02$	$2.67884E-03$
0.8	0.9078412990	0.9079322864	0.9000596969	0.9061819006	$9.09874E-05$	$7.78160E-03$	$1.65940E-03$
0.9	0.8873565094	0.8876397215	0.8828289622	0.8865953896	$2.83212E-04$	$4.52755E-03$	$7.61120E-04$
1.0	0.8660254038	0.8668016975	0.8660254038	0.8660252039	$7.76294E-04$	$1.11022E-16$	$1.99841E-07$

In Table 1, we compare the approximate solutions and corresponding absolute errors obtained by HVIM with the existing AADM [21], HPM [12] and available Exact solution. Figure 1 displays the comparison of numerical solutions and corresponding absolute errors obtained by HVIM with the existing methods [12,21]. We conclude that the proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

Example 2. Consider the following nonlinear LEE that arises in thermal explosion in cylindrical vessel

$$\frac{d^2V(t)}{dt^2} + \frac{1}{t}\frac{d}{dt}V(t) = -ce^{V(t)}, \tag{5.3}$$

with the initial and boundary conditions

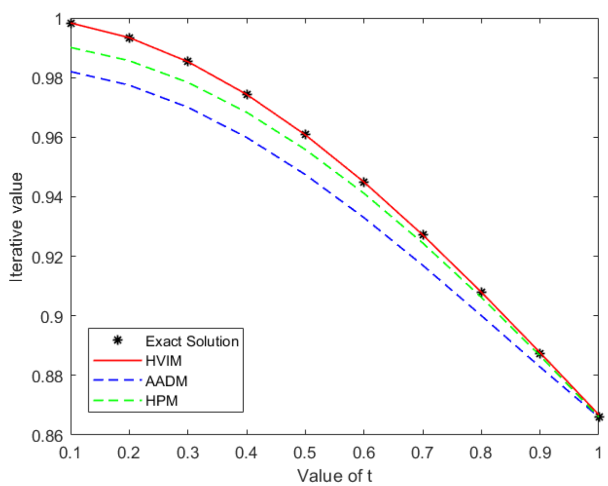
$$V(0) = 2\,\ln 2(2-2\sqrt{3}), \quad V'(0) = 0, \quad V(1) = 0.$$

The Exact solution is

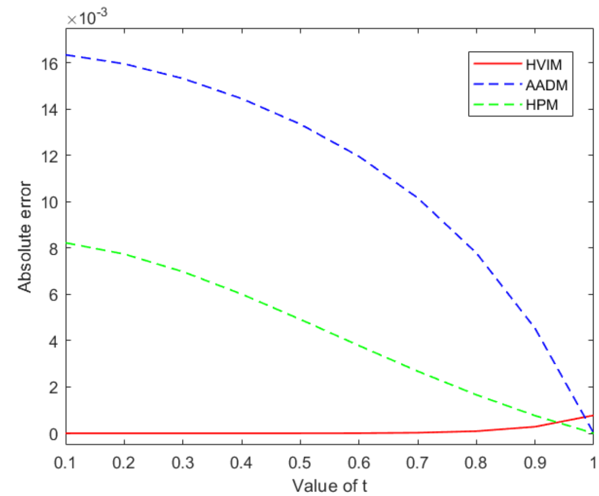
$$V(t) = 2\,\ln\left(\frac{1+c_1}{1+c_1t^2}\right), \text{ where } c_1 = \frac{(8-2c) + -\sqrt{(8-2c)^2 - 4c^2}}{2c}$$

Here $\alpha = 1$, so LEE in (5.3) is cylindrical.

On applying the proposed HVIM for $c = 1$ with the initial guess $V_0 = v_0 = 2\,\ln 2(2-2\sqrt{3})$ and $h = -1$, the



(a) HVIM Vs AADM Vs HPM



(b) Absolute error

Figure 1: Graphical comparison for example 1

4th-order approximate solution for (5.3) can be obtained as

$$V_4(t) = 0.316694 - 0.343146t^2 + 0.0294373t^4 - 0.00336709t^6 + 0.000433276t^8$$

In Table 2, we compare the approximate solutions and corresponding absolute errors obtained by HVIM with the existing AADM [21], HPM [12] and available Exact solution. Figure 2 displays the comparison of numerical solutions and corresponding absolute errors obtained by HVIM with the existing methods [12,21]. We conclude that the proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

Example 3. Consider the following nonlinear singular boundary value problem of LEE

$$\frac{d^2V(t)}{dt^2} + \frac{2}{t}\frac{d}{dt}V(t) = \frac{c_1 V(t)}{c_2 + V(t)} \tag{5.4}$$

with the boundary conditions

$$V'(0) = 0, \quad 5V(1) + V'(1) = 5.$$

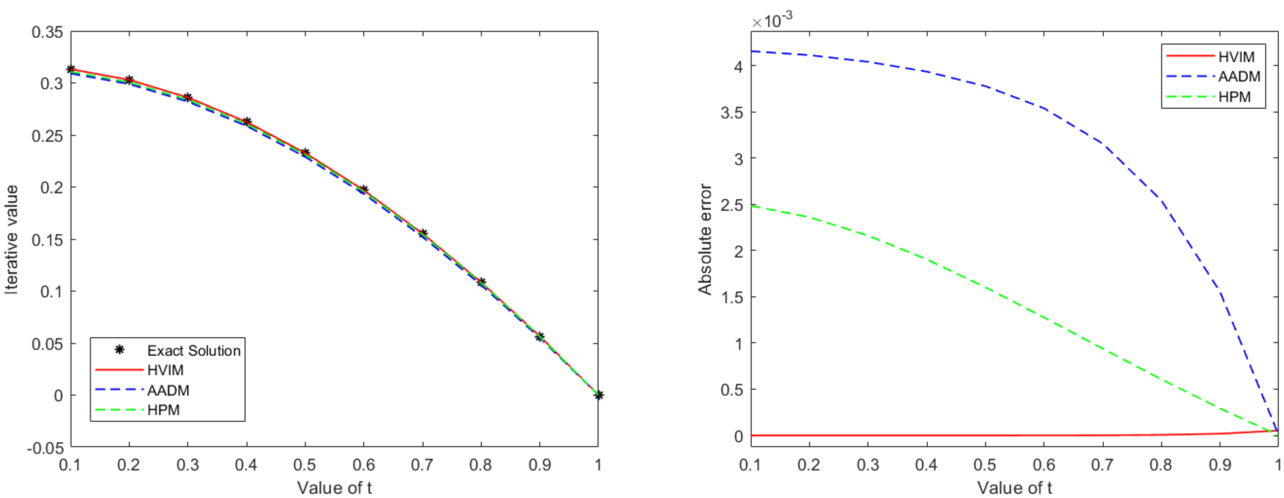
where $c_1 = 0.76129$ and $c_2 = 0.03119$ Here $\alpha = 2$, so LEE in (5.4) is spherical.

On applying the proposed HVIM with the initial guess $V_0 = v_0 = 1$ and $h = -1$, the 4th-order approximate solution for (5.4) can be obtained as

$$V_4(t) = 1. + 0.123044t^2 + 0.000137378t^4 - 7.73281588853816 \times 10^{-6}t^6 + 5.307578546053464 \times 10^{-7}t^8$$

Table 2: Comparison of numerical results of Example 2

t	$V(t)$	$V_4(t)$	AADM [21]	HPM [12]	R_4	r_4 [21]	r_4 [12]
0.1	0.3132658505	0.3132658505	0.3091110525	0.3107822715	$5.88418E-15$	$4.15480E-03$	$2.48358E-03$
0.2	0.3030154228	0.3030154228	0.2989032977	0.3006563961	$6.05554E-12$	$4.11213E-03$	$2.35903E-03$
0.3	0.2860472653	0.2860472657	0.2820071442	0.2838860900	$3.46709E-10$	$4.04012E-03$	$2.16118E-03$
0.4	0.2625311275	0.2625311336	0.2585977491	0.2606277580	$6.09658E-09$	$3.93338E-03$	$1.90337E-03$
0.5	0.2326967839	0.2326968399	0.2289203320	0.2310948050	$5.60745E-08$	$3.77645E-03$	$1.60198E-03$
0.6	0.1968268057	0.1968271477	0.1932901754	0.1955522821	$3.42017E-07$	$3.53663E-03$	$1.27452E-03$
0.7	0.1552481067	0.1552496768	0.1520926245	0.1543102970	$1.57011E-06$	$3.15548E-03$	$9.3781E-04$
0.8	0.1083227634	0.1083286149	0.1057830873	0.1077164120	$5.85148E-06$	$2.53968E-03$	$6.06351E-04$
0.9	0.0564386025	0.0564571922	0.0548870346	0.0561472994	$1.85898E-05$	$1.55157E-03$	$2.91303E-04$
1.0	0.0000000000	0.0000520553	$-2.082E-17$	$-2.594E-08$	$5.20553E-05$	$2.08167E-17$	$2.59375E-08$



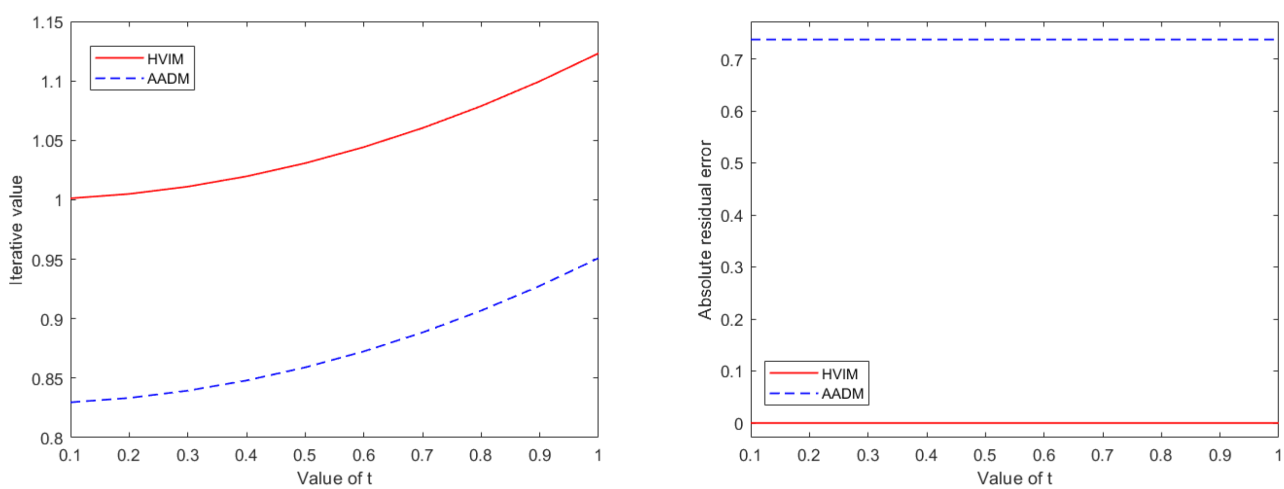
(a) HVIM Vs AADM Vs HPM

(b) Absolute error

Figure 2: Graphical comparison for example 2

Table 3: Numerical comparison of Example 3

t	$V_4(t)$	AADM [21]	R_4	r_4 [21]
0.1	1.0012304530	0.8296798519	$4.47975E-14$	$7.37113E-01$
0.2	1.0049219764	0.8333484897	$1.14257E-11$	$7.37113E-01$
0.3	1.0110750606	0.8394636718	$2.91129E-10$	$7.37113E-01$
0.4	1.0196905138	0.8480265769	$2.88457E-09$	$7.37113E-01$
0.5	1.0307694490	0.8590388551	$1.70170E-08$	$7.37113E-01$
0.6	1.0443132659	0.8725026281	$7.22638E-08$	$7.37113E-01$
0.7	1.0603236294	0.8884204889	$2.44447E-07$	$7.37113E-01$
0.8	1.0788024451	0.9067955021	$6.99766E-07$	$7.37113E-01$
0.9	1.0997518334	0.9276312038	$1.76274E-06$	$7.37113E-01$
1.0	1.1231741029	0.9509316014	$4.01325E-06$	$7.37113E-01$



(a) HVIM Vs AADM (b) Absolute residual error

Figure 3: Graphical comparison for example 3

In Table 3, we compare the approximate solutions and corresponding absolute residual errors obtained by HVIM with the existing AADM [20]. Figure 3 displays the comparison of numerical solutions and corresponding absolute residual errors obtained by HVIM with the AADM [20]. We conclude that the proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

Example 4. Consider the following nonlinear initial value problem of LEE arises in astrophysics

$$\frac{d^2V(t)}{dt^2} + \frac{2}{t}\frac{d}{dt}V(t) = -e^{V(x)} \tag{5.5}$$

with the initial conditions

$$V(0) = 0, \quad V'(0) = 0.$$

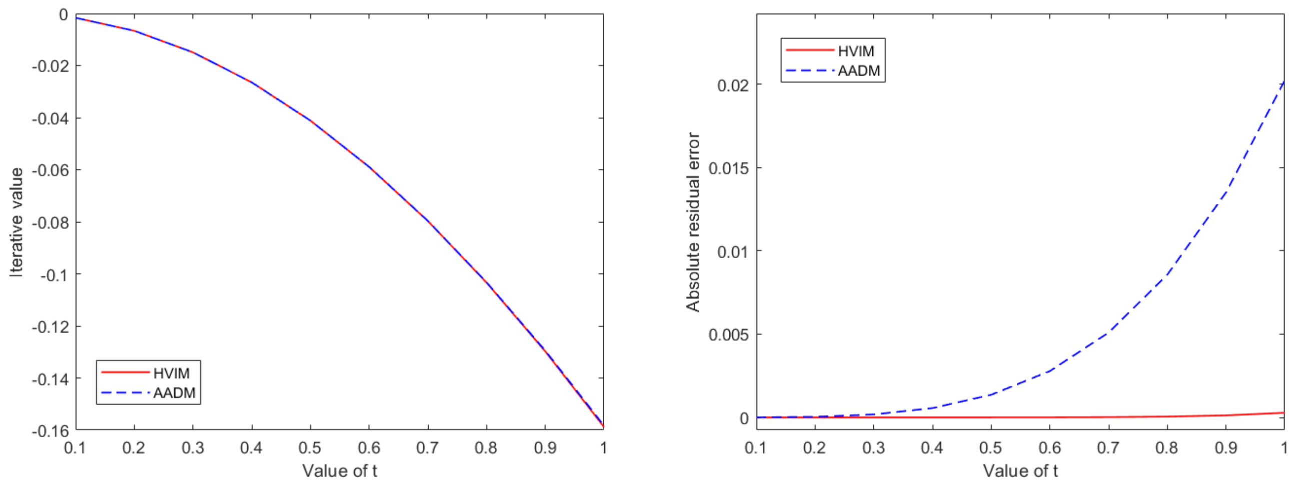
On applying the proposed HVIM with the initial guess $V_0 = v_0 = 0$ and $h = -1$, the 4th-order approximate solution for (5.5) can be obtained as

$$V_4(t) \quad = \quad -0.166667t^2 + 0.00833333t^4 - 0.000529101t^6 + 0.0000373555t^8$$

Table 4: Numerical comparison of Example 4

t	$V_4(t)$	AADM [20]	R_4	r_4 [20]
0.1	−0.0016658339	−0.0016658333	$3.07854E-12$	$2.22006E-06$
0.2	−0.0066533671	−0.0066533333	$7.85679E-10$	$3.54178E-05$
0.3	−0.0149328833	−0.0149325000	$2.00348E-08$	$1.78437E-04$
0.4	−0.0264554760	−0.0264533333	$1.98721E-07$	$5.60158E-04$
0.5	−0.0411539546	−0.0411458333	$1.17389E-06$	$1.35583E-03$
0.6	−0.0589440583	−0.0589200000	$4.99298E-06$	$2.78219E-03$
0.7	−0.0797259280	−0.0796658333	$1.69206E-05$	$5.09154E-03$
0.8	−0.1033857667	−0.1032533333	$4.85356E-05$	$8.56513E-03$
0.9	−0.1297976054	−0.1295325000	$1.22531E-04$	$1.35060E-02$
1.0	−0.1588250784	−0.1583333333	$2.79618E-04$	$2.02319E-02$

In Table 4, we compare the approximate solutions and corresponding absolute residual errors obtained by HVIM with the existing AADM [20]. Figure 4 displays the comparison of numerical solutions and corresponding absolute residual errors obtained by HVIM with the AADM [20]. We conclude that the



(a) HVIM Vs AADM

(b) Absolute residual error

Figure 4: Graphical comparison for example 4

proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

Example 5. Consider the following nonlinear trigonometric problem of LEE

$$\frac{d^2V(t)}{dt^2} + \frac{2}{t}\frac{d}{dt}V(t) = -\mathrm{Sin}(V(t)) \tag{5.6}$$

with the initial conditions

$$V(0) = 1, \quad V'(0) = 0.$$

On applying the proposed HVIM with the initial guess $V_0 = v_0 = 1$ and $h = -1$, the 4th-order approximate solution for (5.6) can be obtained as

$$V_4(t) \quad = \quad 1 - 0.140245t^2 + 0.00378874t^4 + 0.000148292t^6 - 0.0000107728t^8$$

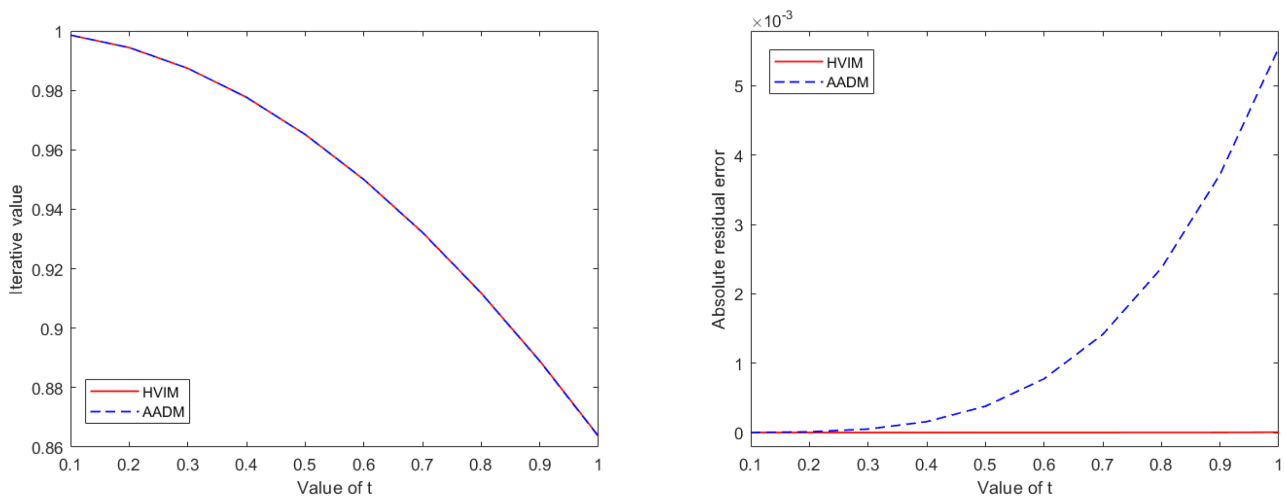
In Table 5, we compare the approximate solutions and corresponding absolute residual errors obtained by HVIM with the existing AADM [20]. Figure 5 displays the comparison of numerical solutions and corresponding absolute residual errors obtained by HVIM with the AADM [20]. We conclude that the proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

Example 6. Consider the following nonlinear trigonometric problem of LEE

$$\frac{d^2V(t)}{dt^2} + \frac{2}{t}\frac{d}{dt}V(t) = -\mathrm{Cos}(V(t)) \tag{5.7}$$

Table 5: Numerical comparison of Example 5

t	$V_4(t)$	AADM [20]	R_4	r_4 [20]
0.1	0.9985979274	0.9985979272	$9.82547E-15$	$6.22130E-07$
0.2	0.9943962649	0.9943962554	$2.75274E-12$	$9.92073E-06$
0.3	0.9874087314	0.9874086240	$8.26175E-11$	$4.99427E-05$
0.4	0.9776583658	0.9776577655	$9.93206E-10$	$1.56603E-04$
0.5	0.9651777802	0.9651755052	$7.19872E-09$	$3.78449E-04$
0.6	0.9500094993	0.9500027615	$3.76175E-08$	$7.74951E-04$
0.7	0.9322063712	0.9321895459	$1.55867E-07$	$1.41434E-03$
0.8	0.9118320290	0.9117949626	$5.42366E-07$	$2.37100E-03$
0.9	0.8889613799	0.8888872089	$1.64595E-06$	$3.72255E-03$
1.0	0.8636810942	0.8635435751	$4.47304E-06$	$5.54644E-03$



(a) HVIM Vs AADM (b) Absolute residual error

Figure 5: Graphical comparison for example 5

with the initial conditions

$$V(0) = 1 + \frac{\pi}{2}, \quad V'(0) = 0.$$

On applying the proposed HVIM with the initial guess $V_0 = v_0 = 1 + \frac{\pi}{2}$ and $h = -1$, the 4th-order approximate solution for (5.7) can be obtained as

$$V_4(t) = 2.5708 + 0.140245t^2 + 0.00378874t^4 - 0.000148292t^6 - 0.0000107728t^8$$

Table 6: Numerical comparison of Example 6

t	$V_4(t)$	AADM [20]	R_4	r_4 [20]
0.1	2.5721991572	2.5721991573	$8.93730E-15$	$6.23521E-07$
0.2	2.5764121858	2.5764121953	$1.99557E-12$	$1.00098E-05$
0.3	2.5834489715	2.5834490804	$3.89438E-11$	$5.09567E-05$
0.4	2.5933319303	2.5933325448	$2.17682E-10$	$1.62300E-04$
0.5	2.6060920549	2.6060944140	$2.33805E-11$	$4.00182E-04$
0.6	2.6217685068	2.6217756065	$7.09392E-09$	$8.39837E-04$
0.7	2.6404080661	2.6404261335	$5.29620E-08$	$1.57792E-03$
0.8	2.6620644183	2.6621050994	$2.51166E-07$	$2.73540E-03$
0.9	2.6867972559	2.6868807016	$9.29641E-07$	$4.46098E-03$
1.0	2.7146711656	2.7148302302	$2.90866E-06$	$6.93514E-03$

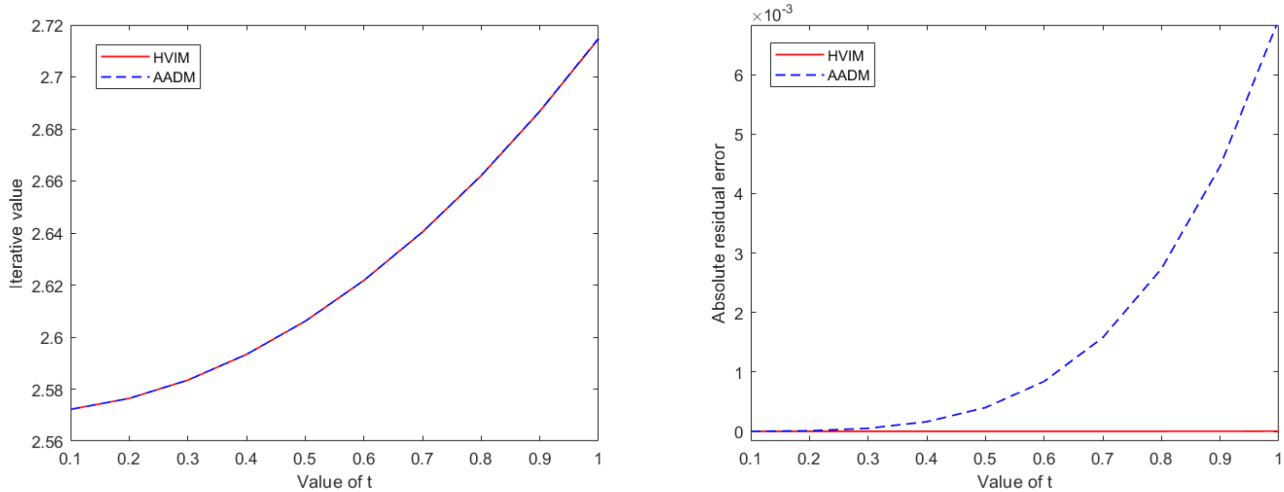
In Table 6, we compare the approximate solutions and corresponding absolute residual errors obtained by HVIM with the existing AADM [20]. Figure 6 displays the comparison of numerical solutions and corresponding absolute residual errors obtained by HVIM with the AADM [20]. We conclude that the proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

Example 7. Consider the following nonlinear hyperbolic problem of LEE

$$\frac{d^2V(t)}{dt^2} + \frac{2}{t}\frac{d}{dt}V(t) = -\text{Sinh}(V(t)) \tag{5.8}$$

with the initial conditions

$$V(0) = 1, \quad V'(0) = 0.$$



(a) HVIM Vs AADM (b) Absolute residual error

Figure 6: Graphical comparison for example 6

On applying the proposed HVIM with the initial guess $V_0 = v_0 = 1$ and $h = -1$, the 4th-order approximate solution for (5.8) can be obtained as

$$V_4(t) \quad = \quad 1 - 0.195867t^2 + 0.0151119t^4 - 0.00109194t^6 + 0.000098555t^8$$

In Table 7, we compare the approximate solutions and corresponding absolute residual errors obtained by HVIM with the existing AADM [20]. Figure 7 displays the comparison of numerical solutions and corresponding absolute residual errors obtained by HVIM with the AADM [20]. We conclude that the proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

Example 8. Consider the following nonlinear hyperbolic problem of LEE

$$\frac{d^2V(t)}{dt^2} + \frac{2}{t}\frac{d}{dt}V(t) = -\text{Cosh}(V(t)) \tag{5.9}$$

with the initial conditions

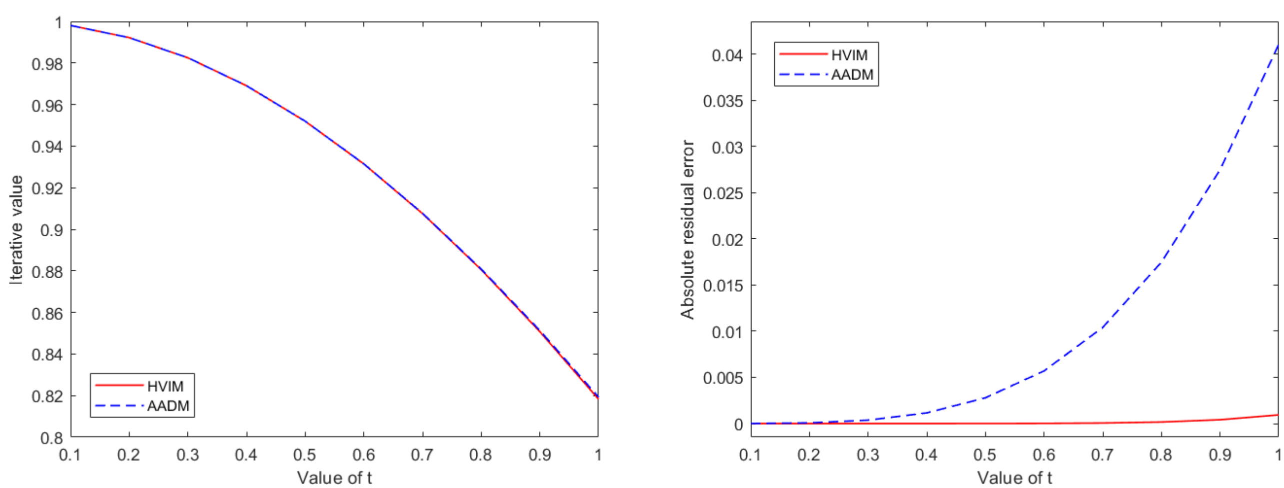
$$V(0) = 1, \quad V'(0) = 0.$$

On applying the proposed HVIM with the initial guess $V_0 = v_0 = 1$ and $h = -1$, the 4th-order approximate solution for (5.9) can be obtained as

$$V_4(t) \quad = \quad 1 - 0.25718t^2 + 0.0151119t^4 - 0.00163787t^6 + 0.000156302t^8$$

Table 7: Numerical comparison of Example 7

t	$V_4(t)$	AADM [20]	R_4	r_4 [20]
0.1	0.9980428414	0.9980428425	$1.05561E-11$	$4.58075E-06$
0.2	0.9921894348	0.9921895044	$2.69214E-09$	$7.30338E-05$
0.3	0.9824935991	0.9824943887	$6.85614E-08$	$3.67576E-04$
0.4	0.9690437586	0.9690481667	$6.78844E-07$	$1.15231E-03$
0.5	0.9519611019	0.9519777785	$4.00117E-06$	$2.78429E-03$
0.6	0.9313971428	0.9314464330	$1.69733E-05$	$5.70182E-03$
0.7	0.9075308232	0.9076536075	$5.73456E-05$	$1.04108E-02$
0.8	0.8805653368	0.8808350478	$1.63936E-04$	$1.74699E-02$
0.9	0.8507248911	0.8512627685	$4.12348E-04$	$2.74757E-02$
1.0	0.8182516669	0.8192450528	$9.37302E-04$	$4.10482E-02$

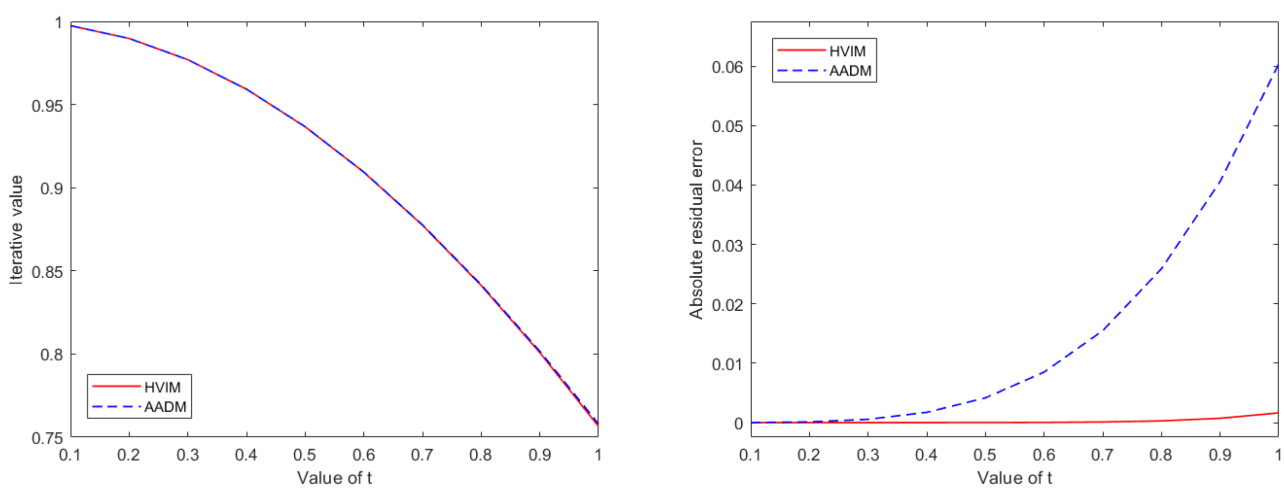


(a) HVIM Vs AADM (b) Absolute residual error

Figure 7: Graphical comparison for example 7

Table 8: Numerical comparison of Example 8

t	$V_4(t)$	AADM [20]	R_4	r_4 [20]
0.1	0.9974297085	0.9974297101	$1.87563E-11$	$6.86973E-06$
0.2	0.9897368704	0.9897369748	$4.78081E-09$	$1.09470E-04$
0.3	0.9769750133	0.9769761970	$1.21639E-07$	$5.50470E-04$
0.4	0.9592314419	0.9592380482	$1.20279E-06$	$1.72350E-03$
0.5	0.9366244873	0.9366494684	$7.07739E-06$	$4.15761E-03$
0.6	0.9092998754	0.9093736665	$2.99615E-05$	$8.49689E-03$
0.7	0.8774264367	0.8776101198	$1.00985E-04$	$1.54763E-02$
0.8	0.8411914399	0.8415945740	$2.87901E-04$	$2.58950E-02$
0.9	0.8007958965	0.8015990439	$7.21930E-04$	$4.05892E-02$
1.0	0.7564502464	0.7579318126	$1.63541E-03$	$6.04043E-02$



(a) HVIM Vs AADM

(b) Absolute residual error

Figure 8: Graphical comparison for example 8

In Table 8, we compare the approximate solutions and corresponding absolute residual errors obtained by HVIM with the existing AADM [20]. Figure 8 displays the comparison of numerical solutions and corresponding absolute residual errors obtained by HVIM with the AADM [20]. We conclude that the proposed HVIM has greater efficiency, and we reach good accuracy within a few iterations.

6 Conclusion

In this research, we introduced an effective numerical algorithm called the homotopy variational iteration method using the variational iteration method combined with the homotopy analysis method to determine the numerical solution of Lane-Emden type equations. The convergence study is addressed under general conditions. Eight problems of different kinds of LEEs with strong nonlinearities are included and solved by means of the proposed HVIM to test the efficiency. The method is compared with the exact solution and existing methods [12, 20, 21]. Unlike other methods, the proposed algorithm does not require discretization or perturbation and can be applied easily and accurately. The proposed HVIM can solve highly nonlinear LEEs with less computational work and computation time and converge to the solution in a few iterations.

Statements and Declarations

Competing Interests: The authors declare that there are no known competitors with regards to this publication.

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