An Efficient Technique Via the $\mathbb{J}$-Transform Decomposition Method: Theoretical Analysis with Applications

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Abstract:

The nonlinear Klein–Gordon equation, which describes nonlinear wave interaction and arises from the study of quantum field theory, is one of the most notable differential equations. In this research, we solve the equation using a novel approach. It is used in many areas of mathematics, such as conformal mapping theory, physics, and algebraic geometry. In the current work, the $\mathbb{J}-$transform Adomian decomposition method ($\mathbb{JADM}$) is applied to provide exact solutions for a variety of nonlinear partial differential equations (PDES). We provide comprehensive proofs for novel theorems related to the $\mathbb{J}-$transform methodology. This method is based on the $\mathbb{J}-$transform method ($\mathbb{JTM}$) and the Adomian decomposition method ($\mathbb{ADM}$). The theoretical analysis of the $\mathbb{JADM}$ is investigated and computed using easily obtained terms for some differential equations. Our results are compared with exact solutions obtained by other methods that can be found in the literature. The paper describes the important aspects of the $\mathbb{JADM}$. The $\mathbb{JADM}$ has demonstrated a high degree of efficiency, accuracy, and adaptability to a wide range of differential equations, both linear and nonlinear. Mathematica was used for much of the symbolic and numerical calculations Ω.

Keywords: Klein-Gordon Differential Equation, $\mathbb{J}-$Transform Method, Adomian Decomposition Method, Schrödinger Equation, Fixed Point Theory.

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1 Introduction

When solving differential and integral equations with initial and boundary value conditions, integral transformations play a crucial role. One of the most popular methods for handling differential and integral problems is the integral transform method [1–11]. The Laplace transform is also the technique that is used the most frequently in the literature [12]. In 1993, Watugala proposed the Laplace-Carson transform for the first time [13]. Another name for it is the $p-$multiplied version of the conventional Laplace transform. It is closely related to the Sumudu transform, which was employed to address controlled engineering problems. The N-transformation, sometimes referred to as the natural transform, was initially presented in 2008 and is comparable to the Sumudu integral transform and the Laplace-Carson transform. The N-transform can solve the unsteady fluid flow through a flat wall problem by changing the variables, which provide both Laplace and Sumudu integral transforms [5–8].

Most linear and nonlinear techniques found in a variety of scientific domains, such as population models, fluid mechanics, solid state physics, plasma physics, and chemical kinetics, can be modeled using differential equations. As a result, it is still very difficult to obtain exact or approximate solutions to linear and nonlinear differential equations in applied mathematics and physics, which demands developing new techniques [14–20]. Many powerful mathematical techniques, such as the Adomian decomposition method ($\mathbb{ADM}$) [14–18], natural Adomian decomposition method ($\mathbb{NADM}$) [21], reduced differential transform method ($\mathbb{RDTM}$) [22–25], and homotopy perturbation method ($\mathbb{HPM}$) [26], have been proposed to obtain exact or approximate analytical solutions of those existing phenomena.

For the most part, nonlinear phenomena may be modeled using partial differential equations (PDEs) of integer order [21]. Analytical and numerical investigations are necessary for addressing the complex problem of dynamically studying PDEs. Notably, integral transformations are the most efficient and successful methods for determining the exact and approximate analytical solutions of PDEs. Interestingly, and this is significant, integral transforms do not contain perturbations or long-lasting polynomials.

A novel method for solving both linear and nonlinear differential equations is the $\mathbb{J}-$transform Adomian decomposition method ($\mathbb{JADM}$), as stated in [27]. It has been motivated and inspired by the ongoing studies in this area. Furthermore, Shehu Maitama and Wei dong Zhao developed accurate solutions to partial differential equations (PDEs) by applying the $\mathbb{JTM}$. Recall that the study's exact solutions were incorrect, and you can easily confirm this by applying the initial conditions to the exact solutions.

The Korteweg-deVries (KdV) equation, which describes a long wave moving through a canal, was developed through research on shallow water. There is a single wave solution to the KdV equation. The single wave solution of the nonlinear Schrödinger equation is typically observed. A solitary or soliton wave equation is one in which the propagation speed and wave equation amplitude are totally unrelated. One of the most basic model equations in fluid mechanics is Burger's equation. It consistently illustrates how diffusion and convection processes are related.

This work is arranged as follows: The definitions and significant features of the $\mathbb{J}-$transform and the Adomian decomposition technique are provided in section 2 along with some background on the theory of integral transform. The $\mathbb{J}-$transform theories are examined in full with proofs in Section 3. In section 4, we present the theoretical analysis of the $\mathbb{JADM}$ for nonlinear PDEs. Exact solutions to nonlinear partial differential equations are provided in Section 5. Section 6 focuses on providing exact solutions to nonlinear partial differential equations. Finally, in section 7, we present the conclusion of our work.

2 The $\mathbb{J}-$Transform Adomian Decomposition Method

The concept of $\mathbb{J}-$transform method was introduced by Shehu Maitama and Wei dong Zhao[27]. We provide some background information regarding the nature of the $\mathbb{J}-$transform method in this section.

Definition 2.1 *Suppose that $\theta(s)$ is a piece-wise continuous function over $\mathbb{R}$ and $K, p > 0$. Suppose that $\mathcal{A} = \left\{\theta(s) : |\theta(s)| < Ke^{c\,s}\chi_{(0,\infty)}(s)\right\}$, where $\chi_{(0,\infty)}(s)$ is the characteristic function and its given by:*

$$\chi_{(0,\infty)}(s) = \begin{cases} 1 & s \in (0,\infty), \\ 0 & s \notin (0,\infty). \end{cases}$$

So, $|\theta(s)| \leq Ke^{c\,s}$ for $s \longrightarrow \infty$ i.e. given any $\theta(s) \in \mathcal{A}$, where $r, v > 0$ we have:

$$\left|\int_0^\infty e^{-r\,s}\theta(sv)ds\right| \leq K\int_0^\infty e^{-r\,s}e^{c|s\,v|}ds$$
$$= K\int_0^\infty e^{(cv-r)s}ds.$$

If $c\,v - r < 0$, the above is convergent. Therefore, the order of $\theta(s)$ is exponential.

The $\mathbb{J}-$transformation is then provided as follows:

$$\mathbb{J}\left(\theta(s)\right) = \Theta(r,v) = v\int_0^\infty e^{\frac{-r\,s}{v}}\,\theta(s)ds,\; s, v > 0, \tag{2.1}$$

where $\mathbb{J}$ is the $\mathbb{J}-$transformation of $\theta(s)$ and r, v are the $\mathbb{J}-$transformation variables.

So, Eq. (2.1) can be written as,

$$\mathbb{J}\left(\theta(s)\right) = \Theta(r,v) = v^2\int_0^\infty e^{-r\,s}\,\theta(v\,s)\,ds,\; r, v \in (0,\infty). \tag{2.2}$$

It is crucial to understand the inverse property of the $\mathbb{J}$-transform before we demonstrate its applications. First, we present the following two crucial theorems.

Theorem 2.1 [28]. *Assume that the function $\theta(w)$ is analytic on a region which includes γ and its interior, and that γ is a simple closed curve. Assumed to be in a counterclockwise orientation is γ. Next, for every w_0 within γ*

$$\theta(w_0) = \frac{1}{2\pi i}\int_\gamma \frac{\Theta(w)}{w - w_0}dw.$$

Theorem 2.2 [28]. *Provided that γ is a simple closed, positively oriented contour and θ is analytic on $\mathbb{C}$ except for a few points $w_1, w_2, ..., w_n$ inside it, then*

$$\oint_\gamma \theta(w)\,dw = 2\pi i\sum_{k=1}^{n} Res_{\,\theta}\left(w_k\right).$$

Theorem 2.1 leads to Theorem 2.2, which is invaluable in computing the real integral by using the proper contour in $\mathbb{C}$, see [28].

Definition 2.2 *(Inverse $\mathbb{J}$-transform)* [27]. *Suppose $\mathbb{J}^{-1}$ is referred to as the inverse $\mathbb{J}-$transform of $\Theta(r,v)$, where $\Theta(r,v)$ is the $\mathbb{J}-$transform of the function $\theta(s)$. Then,*

$$\mathbb{J}^{-1}[\Theta(r,v)] = \theta(s), \text{ for } s \geqslant 0.$$

Equivalently, based on Theorem 2.1 and Theorem 2.2, the complex inverse $\mathbb{J}-$transform is defined as:

$$\begin{aligned}\theta(s) &= \lim_{\beta\to\infty}\frac{1}{2\pi i}\int_{\alpha-i\beta}^{\alpha+i\beta}\frac{1}{w^2}e^{\left(\frac{rs}{w}\right)}\Theta(r,w)dr\\ &= \sum \text{ residues of } \frac{1}{w^2}e^{\left(\frac{rs}{w}\right)}\Theta(r,w) \text{ at the poles of } \Theta(r,w).\end{aligned}$$

Significant Properties:

Some fundamental properties of $\mathbb{J}$-transforms are as follows; for a list of these properties, see [27]:

1. $\mathbb{J}\left[C\right] = \frac{C\,v^2}{r}$.

2. $\mathbb{J}\left[\zeta^m\right] = \frac{m!v^{m+2}}{r^{m+1}}$, where $m \geq 0$.

3. $\mathbb{J}\left[e^{c\zeta}\right] = \frac{v^2}{(r-cv)}$.

4. Suppose the function $\theta\left(s,z\right)$ has a $\mathbb{J}$-transformation if $\theta^{(k)}\left(s,z\right)$ is its kth derivative. Then,

$$\mathbb{J}\left[\theta^{(k)}\left(s,z\right)\right] = \frac{r^k}{v^k}\Theta\left(x,r,v\right) - \sum_{j=0}^{k-1}\frac{r^{k-(j+1)}}{v^{k-(j+2)}}\theta^{(j)}\left(s,0\right).$$

Adomian Polynomials Evaluations:

Now let's introduce the Adomian polynomials, which are a useful tool for efficiently decomposing the complex nonlinear component into smaller, more manageable components that can be integrable as a Taylor series. The following is a representation of the unknown function ϕ, as shown in [14]:

$$\phi = \sum_{m=0}^{\infty}\phi_m, \tag{2.3}$$

Thus it is necessary to build a recursive relation in order to identify the components ϕ_m , $m \geq 0$. The following formula can be used to define $F(\phi)$ as an infinite series, also referred to as Adomian polynomials A_m, when working with nonlinear terms:

$$F(\phi) = \sum_{m=0}^{\infty}A_m(\phi_0,\,\phi_1,....,\phi_m), \tag{2.4}$$

where the A_m of the nonlinear term $F(\phi)$ can be computed using the following formula from [17]:

$$A_{_m} = \frac{1}{m!}\frac{d^m}{d\mu^m}\left[F\left(\sum_{j=0}^{m}\mu^j\phi_j\right)\right]_{\mu=0}, m = 0,1,2,... \tag{2.5}$$

The generic formula for Eq. (2.5) can then be expressed as follows:

Let the nonlinear function be represented by $F(\phi)$. Using the Adomian polynomial definition and Eq. (2.5), the following can be obtained:

$$\begin{aligned} A_0 &= F(\phi_0), \\ A_1 &= \phi_1 F'(\phi_0), \\ A_2 &= \phi_2 F'(\phi_0) + \tfrac{1}{2!}\phi_1^2 F''(\phi_0). \end{aligned} \tag{2.6}$$

Lastly, a similar process can be used to construct the other terms. Two important observations are provided by the polynomials previously given in Eq. (2.6). A_0 depends exclusively on ϕ_0, A_1 depends solely on ϕ_0 and ϕ_1, A_2 depends solely on ϕ_0, ϕ_1 and ϕ_2, etc.

Additionally, by substituting Eq. (2.6) for Eq. (2.4), one can produce:

$$\begin{aligned} F(\phi) &= A_0 + A_1 + A_2 + \cdots \\ &= F(\phi_0) + (\phi_1 + \phi_2 + \phi_3 + \cdots)F'(\phi_0) \\ &+ \frac{1}{2!}(\phi_1^2 + 2\phi_1\phi_2 + 2\phi_1\phi_3 + \phi_2^2 + \cdots)F''(\phi_0) \\ &+ \frac{1}{3!}(\phi_1^3 + 3\phi_1^2\phi_2 + 3\phi_1^2\phi_3 + 6\phi_1\phi_2\phi_3 + \cdots)F'''(\phi_0) + \cdots \\ &= F(\phi_0) + (\phi - \phi_0)F'(\phi_0) + \frac{1}{2!}(\phi - \phi_0)^2 F''(\phi_0) + \cdots \end{aligned}$$

3 $\mathbb{J}$-Transform Theories Derivation

The new and comprehensive proofs of several theorems pertaining to the $\mathbb{J}$-transformation will be examined in this part. Additionally, we will use these to exactly solve a few PDEs under suitable initial conditions.

Theorem 3.1 *Let $\theta(s) = e^{\alpha s} \in \mathcal{A}$, where $\alpha \in \mathbb{R}$. Then its $\mathbb{J}-$transform is given by*

$$\mathbb{J}[e^{\alpha s}] = \frac{w^2}{r - \alpha w}. \tag{3.1}$$

Proof: *By employing the definition of $\mathbb{J}-$transform, we arrive at:*

$$\begin{aligned} \mathbb{J}[e^{\alpha s}] &= w \int_0^\infty e^{\frac{-rs}{w}} e^{\alpha s} ds \\ &= w \int_0^\infty e^{-(\frac{r}{w} - \alpha)s} ds. \end{aligned} \tag{3.2}$$

Equation (3.2) can be used to integrate by substitution to get:

Let

$$v = \left(\frac{r}{w} - \alpha\right) s \Rightarrow dv \ = \left(\frac{r}{w} - \alpha\right) ds \Rightarrow ds = \frac{w}{r - \alpha w} dv.$$

So,

$$\begin{aligned}
\mathbb{J}[e^{\alpha s}] = \ & w \int_0^\infty e^{-v} \frac{w}{r - \alpha w} dv \\
= \ & \frac{w^2}{r - \alpha w} \left[\int_0^\infty e^{-v} dv \right] \\
= \ & \frac{w^2}{r - \alpha w} \left[\lim_{n \to \infty} (-e^{-v}) \Big|_0^n \right] \\
= \ & \frac{w^2}{r - \alpha w} [1] \\
= \ & \frac{w^2}{r - \alpha w}.
\end{aligned}$$

■

Theorem 3.2 *Given $\theta(s) = \frac{s^k}{k!}$, its $\mathbb{J}-$transform may be found as*

$$\mathbb{J}\left[\frac{s^k}{k!}\right] = \frac{w^{k+2}}{r^{k+1}} \quad k = 0, 1, 2, 3, \cdots \tag{3.3}$$

Proof: *By using the $\mathbb{J}-$transform definition, we obtain:*

$$\begin{aligned}
\mathbb{J}\left[\frac{s^k}{k!}\right] &= w \int_0^\infty e^{\frac{-rs}{w}} \frac{s^k}{k!} ds \\
&= \frac{w}{k!} \int_0^\infty e^{\frac{-rs}{w}} s^k ds.
\end{aligned} \tag{3.4}$$

Using Eq. (3.4)'s and integration by parts, we get:

$$\begin{aligned}
v = s^k &\longrightarrow dv = k s^{k-1}, \\
dz = e^{\frac{-rs}{w}} &\longrightarrow z = -\frac{w}{r} e^{\frac{-rs}{w}}.
\end{aligned}$$

So,

$$\mathbb{J}\left[\frac{s^k}{k!}\right] = \frac{w}{k!}\left[\frac{-ws^k}{r}e^{-\frac{rs}{w}}\bigg|_0^\infty + \int_0^\infty \frac{kws^{k-1}}{r}e^{-\frac{rs}{w}}ds\right]$$

$$= \frac{w}{k!}\int_0^\infty \frac{kws^{k-1}}{r}e^{\frac{-rs}{w}}ds$$

$$= \frac{w}{r}\left[w\int_0^\infty e^{\frac{-rs}{w}}\frac{s^{k-1}}{(k-1)!}ds\right]$$

$$= \frac{w}{r}\mathbb{J}\left[\frac{s^{k-1}}{(k-1)!}\right].$$

Hence,

$$\mathbb{J}\left[\frac{s^k}{k!}\right] = \frac{w}{r}\mathbb{J}\left[\frac{s^{k-1}}{(k-1)!}\right].$$

Using induction now, we observe that,

$$\textit{For } k=0 \Rightarrow \mathbb{J}\left[\frac{s^0}{0!}\right] = \mathbb{J}[1] = \frac{w^2}{r}.$$

$$\textit{For } k=1 \Rightarrow \mathbb{J}\left[\frac{s}{1!}\right] = \frac{w}{r}\mathbb{J}\left[\frac{s^0}{0!}\right] = \frac{w}{r}\mathbb{J}[1] = \frac{w^3}{r^2}.$$

$$\textit{For } k=2 \Rightarrow \mathbb{J}\left[\frac{s^2}{2!}\right] = \frac{w}{r}\mathbb{J}\left[\frac{s^1}{1!}\right] = \frac{w}{r}\mathbb{J}[s] = \frac{w^4}{r^3}.$$

$$\textit{For } k=3 \Rightarrow \mathbb{J}\left[\frac{s^3}{3!}\right] = \frac{w}{r}\mathbb{J}\left[\frac{s^2}{2!}\right] = \frac{w}{r}\frac{w^4}{r^3} = \frac{w^5}{r^4}.$$

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We keep doing this to obtain:

$$\mathbb{J}\left[\frac{s^k}{k!}\right] = \frac{w^{k+2}}{r^{k+1}} \quad, k = 0, 1, 2, 3, \cdots$$

■

Theorem 3.3 *Consider the following:* $\theta(s) = \frac{\sinh \alpha s}{\alpha} \in \mathcal{A}$. *Then, its* $\mathbb{J}-$*transform is obtained as*

$$\mathbb{J}\left[\frac{\sinh \alpha s}{\alpha}\right] = \frac{w^3}{r^2 - \alpha^2 w^2}. \tag{3.5}$$

Proof: *Utilizing the* $\mathbb{J}-$*transform definition, we arrive at*

$$\begin{aligned}
\mathbb{J}\left[\frac{\sinh \alpha s}{\alpha}\right] &= w^2 \int_0^\infty e^{-rs} \frac{\sinh(\alpha s w)}{\alpha}\, ds \\
&= \frac{w^2}{\alpha} \int_0^\infty e^{-rs} \sinh(\alpha s w)\, ds \\
&= \frac{w^2}{\alpha} \int_0^\infty e^{-rs} \frac{e^{\alpha s w} - e^{-\alpha s w}}{2}\, ds \\
&= \frac{w^2}{2\alpha} \int_0^\infty [e^{-(r-\alpha w)s} - e^{-(r+\alpha w)s}]\, ds \\
&= \frac{w^2}{2\alpha} \left[-\frac{e^{-(r-\alpha w)s}}{-(r-\alpha w)} - \frac{e^{-(r+\alpha w)}s}{-(r+\alpha w)} \Bigg|_0^\infty \right] \\
&= \frac{w^2}{2\alpha} \left[\frac{1}{r-\alpha w} - \frac{1}{r+\alpha w} \right] \\
&= \frac{w^2}{2\alpha} \left[\frac{2\alpha w}{r^2 - \alpha^2 w^2} \right].
\end{aligned}$$

Hence,

$$\mathbb{J}\left[\frac{\sinh \alpha s}{\alpha}\right] = \frac{w^3}{r^2 - \alpha^2 w^2}.$$

■

Theorem 3.4 *Assume* $\theta(s) = \cosh(\alpha s) \in \mathcal{A}$. *Subsequently its* $\mathbb{J}-$*transform is obtained as:*

$$\mathbb{J}[\cosh \alpha s] = \frac{w^2 r}{r^2 - w^2 \alpha^2}. \tag{3.6}$$

Proof: *With the help of the* $\mathbb{J}-$*transform definition, we derive:*

$$
\begin{aligned}
\mathbb{J}[\cosh \alpha s] =& \; w^2 \int_0^\infty e^{-rs} \cosh(\alpha s w) \; ds \\
=& \; w^2 \int_0^\infty e^{-rs} \left[\frac{e^{\alpha s w} + e^{-(\alpha s w)}}{2} \right] \; ds \\
=& \; \frac{w^2}{2} \int_0^\infty \left[e^{-(r-\alpha w)s} + e^{-(r+\alpha w)s} \right] \; ds \\
=& \; \frac{w^2}{2} \left[\frac{e^{-(r-\alpha w)s}}{-(r-\alpha w)} + \frac{e^{-(r+\alpha w)s}}{-(r+\alpha w)} \Big|_0^\infty \right] \\
=& \; \frac{w^2}{2} \left[\frac{1}{r-\alpha w} + \frac{1}{r+\alpha w} \right] \\
=& \; \frac{w^2}{2} \left[\frac{2r}{r^2 - \alpha^2 w^2} \right].
\end{aligned}
$$

Hence,

$$
\mathbb{J}[\cosh \alpha s] = \frac{w^2 r}{r^2 - \alpha^2 w^2}.
$$

■

Theorem 3.5 *Given* $Ci(s) = \int_s^\infty \frac{\cos(w)}{w} dw$ *is the cosine integral function. Then, the following is its* $\mathbb{J}-$*transform:*

$$
\mathbb{J}[Ci(s)] = \frac{-w^2}{2r} \ln \left(\frac{w^2}{r^2 + w^2} \right). \tag{3.7}
$$

Proof: *First,*

$$
Ci(s) = \int_s^\infty \frac{\cos(w)}{w} dw. \tag{3.8}
$$

Let

$$
\begin{aligned}
& w = sv \Rightarrow dw = s \; dv \\
& w = s \longrightarrow v = 1 \\
& w = \infty \longrightarrow v = \infty.
\end{aligned}
$$

Eq. (3.8) then becomes into,

$$
\begin{aligned}
Ci(s) =& \int_1^\infty \frac{\cos(sv)}{sv} s dv \\
=& \int_1^\infty \frac{\cos(sv)}{v} dv.
\end{aligned} \tag{3.9}
$$

The following can be obtained by applying the $\mathbb{J}-$*transform to both sides of Eq. (3.9):*

$$
\begin{aligned}
\mathbb{J}[Ci(s)] &= \mathbb{J}\left[\int_1^\infty \frac{\cos(sv)}{v}dv\right] \\
&= w\int_0^\infty e^{-rs}\left[\int_1^\infty \frac{\cos(svw)}{v}dv\right]ds.
\end{aligned} \tag{3.10}
$$

By altering the integration order in Eq. (3.10), we get:

$$
\begin{aligned}
\mathbb{J}[Ci(s)] &= \int_1^\infty \frac{1}{v}\left[w\int_0^\infty e^{-rs}\cos(svw)ds\right]dv \\
&= \int_1^\infty \frac{1}{v}\mathbb{J}\left[\cos(sv)\right]dv \\
&= \int_1^\infty \frac{1}{v}\frac{w^2 s}{r^2+v^2w^2}dv \\
&= w^2 r\int_1^\infty \frac{1}{v(r^2+(vw)^2)}dv.
\end{aligned}
$$

Using partial fraction, the following can be determined:

$$\mathbb{J}[Ci(s)] = w^2 r \int_1^\infty \frac{1}{r^2}\left[\frac{1}{v} - \frac{1}{2}\frac{2w^2 v}{r^2 + (wv)^2}\right] dv$$

$$= \frac{w^2 r}{r^2}\int_1^\infty \frac{1}{v} - \frac{1}{2}\frac{2w^2 v}{r^2 + (wv)^2} dv$$

$$= \frac{w^2}{r}\left[\ln(v) - \frac{1}{2}\ln(r^2 + (wv)^2)\Big|_1^\infty\right]$$

$$= \frac{w^2}{r}\left[\ln(v) - \ln\sqrt{(r^2 + (wv)^2)}\Big|_1^\infty\right]$$

$$= \frac{w^2}{r}\left[\ln\frac{v}{\sqrt{(r^2 + (wv)^2)}}\Big|_1^\infty\right]$$

$$= \frac{w^2}{r}\left[\ln\frac{1}{\sqrt{(\frac{r^2}{v^2} + w^2)}}\Big|_1^\infty\right]$$

$$= \frac{w^2}{r}\left[\ln\frac{1}{w} - \ln\frac{1}{\sqrt{r^2 + w^2}}\right]$$

$$= \frac{w^2}{r}\ln\frac{\left(r^2 + w^2\right)^{\frac{1}{2}}}{w}$$

$$= \frac{w^2}{r}\ln\left(\frac{r^2 + w^2}{w^2}\right)^{\frac{1}{2}}$$

$$= \frac{w^2}{r}\ln\left(\frac{w^2}{r^2 + w^2}\right)^{\frac{-1}{2}}.$$

Hence,

$$\mathbb{J}[Ci(s)] = -\frac{w2}{2r}\ln\left(\frac{w^2}{r^2 + w^2}\right).$$

■

4 Analysis of the $\mathbb{JADM}$ for Nonlinear ODEs

We are going to talk about the theoretical analysis, including the error estimate, the convergence and uniqueness theorems, and the proposed method. Examining a nonlinear partial differential equation with the following structure:

$$\theta_z(s,z) = F(\theta(s,z)) + \theta(s,z). \tag{4.1}$$

Accompanied along the I.C.

$$\theta(s,0) = \alpha(s). \tag{4.2}$$

Where $F(\theta(s,z)) = \theta_{zz}(s,z) + \theta^2(s,z)$ are the nonlinear source.

One can employ the $\mathbb{J}$-transformation along with property 4 in Eq. (4.1) to get:

$$\Theta\left(r,w\right) = \tfrac{w^2}{r^2}\alpha(s) + \tfrac{w}{r^2}\mathbb{J}\left[\theta(s,z) + F(\theta(s,z))\right] \ . \tag{4.3}$$

Applying the inverse $\mathbb{J}$-transformation of Eq. (4.3), we get:

$$\theta(s,z) = \gamma(s,z) + \mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[\theta(s,z) + F(\theta(s,z))\right]\right].$$

Note $\gamma(s,z)$ describes the conditions and the non-homogeneous source.

Suppose we have a solution as follows:

$$\theta(s,z) = \sum_{j=0}^{\infty}\theta_j(s,z). \tag{4.4}$$

The nonlinear term $F(\theta(s,z)) = \sum_{i=0}^{\infty} C_i$, where the C_i's are the Adomian polynomials.

Using Eq. (4.4), Eq. (4.3) can be rewritten as follows:

$$\sum_{j=0}^{\infty}\theta_j(s,z) = \gamma(s,z) + \mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[\sum_{j=0}^{\infty}C_j + \sum_{j=0}^{\infty}\theta_j\right]\right]. \tag{4.5}$$

By looking at Eq. (4.5), one can observe that $\gamma(s,z)$ represents the source term along with initial conditions.

So, one can achieve the recursive formula as:

$$\theta_{j+1}(s,z) = \mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[C_j + \theta_j\right]\right],\ j \geq 0. \tag{4.6}$$

In this case, our intended solution is:

$$\theta(s,z) = \sum_{j=0}^{\infty}\theta_j(s,z). \tag{4.7}$$

Theorem 4.1 ***(Uniqueness Theorem)****. Suppose* $0 < \delta < 1$ *and* $\delta = (K_1 + K_2)\, z$*. Then Eq. (4.1) will have a unique solution.*

Proof: Given the norm $\|.\|$ and $\Xi = (C[\Omega], \|.\|)$ as the Banach space of all continuous functions on $\Omega = [c, d]$, define $\Upsilon : \Xi \to \Xi$, where

$$\theta_{j+1}(s,z) = \gamma(s,z) + \mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[M\left(\theta_j(s)\right) + L\left(\theta_j(s,z)\right)\right]\right].$$

Given $L[\theta(s,z)] = \theta(s,z)$, $M[\theta(s,z)] = F(\theta(s,z))$ with $\left|M(\theta) - M(\hat{\theta})\right| < K_1\left|\theta - \hat{\theta}\right|$ and $\left|L(\theta) - L(\hat{\theta})\right| < K_2\left|\theta - \hat{\theta}\right|$, where K_1, K_2 are constants related to Lipschitz and $\theta, \hat{\theta}$ are distinct solutions for Eq. (4.1).

$$\begin{aligned}
\left\|\Upsilon(\theta) - \Upsilon(\hat{\theta})\right\| &= \max_{z\in\Omega}\left|\begin{array}{l} \mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[L(\theta) + M(\theta)\right]\right] \\ \\ -\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[L(\hat{\theta}) + M(\hat{\theta})\right]\right] \end{array}\right| \\
&= \max_{z\in\Omega}\left|\begin{array}{l} \mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[L(\theta) - L(\hat{\theta})\right]\right] \\ \\ +\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[M(\theta) - M(\hat{\theta})\right]\right] \end{array}\right| \\
&\le \max_{z\in\Omega}\left[K_1\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[\left|\theta - \hat{\theta}\right|\right]\right] + K_2\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[\left|\theta - \hat{\theta}\right|\right]\right]\right] \\
&\le \max_{z\in\Omega}\left(K_1 + K_2\right)\left[\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[\left|\theta - \hat{\theta}\right|\right]\right]\right] \\
&\le \left(K_1 + K_2\right)\left[\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[\left\|\theta(s,z) - \hat{\theta}(s,z)\right\|\right]\right]\right] \\
&= \left\|\theta - \hat{\theta}\right\|\left(K_1 + K_2\right) z.
\end{aligned}$$

The Banach fixed-point theorem for contraction states that there is a unique solution to Eq. (4.1) since $0 < \delta < 1$ implies that Υ is contraction mapping.

Hence, Theorem 4.1 proof is now complete. ■

Theorem 4.2 ***(Convergence Theorem)*** *The solution for Eq. (4.1) with* $0 < \delta < 1$ *and* $|\theta_1| < \infty$ *will eventually converge when the* $\mathbb{JADM}$ *applied.*

Proof: Assume the k^{th} partial sum is q_k i.e. $q_k = \sum_{j=0}^{k} \theta_j(s)$. We will demonstrate that in the Banach space Ξ, $\{q_k\}$ is a Cauchy sequence. Assume the Adomian polynomials mentioned in [29], which is new format of $M(q_k) = \tilde{C}_k + \sum_{j=0}^{k-1} \tilde{C}_j$. Let q_i and q_k be any distinct sums with $k \geq i$. Thus,

$$\begin{aligned}
\|q_k - q_i\| &= \max_{z\in\Omega} |q_k - q_i| \\
&= \max_{z\in\Omega} \left| \sum_{j=i+1}^{k} \hat{\theta}_i(s,z) \right|, \ k = 1, 2, ... \\
&\leq \max_{z\in\Omega} \left| \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[L \left(\sum_{j=i+1}^{k} \theta_{j-1}(s,z) \right) \right] \right] + \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[\sum_{j=i+1}^{k} C_{j-1}(s,z) \right] \right] \right| \\
&= \max_{z\in\Omega} \left| \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[L \left(\sum_{j=i}^{k-1} \theta_i(s,z) \right) \right] \right] + \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[\sum_{j=i}^{k-1} C_j(s,z) \right] \right] \right| \\
&\leq \max_{z\in\Omega} \left| \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[L(q_{k-1}) - L(q_{i-1}) \right] \right] + \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[M(q_{k-1}) - M(q_{i-1}) \right] \right] \right| \\
&\leq K_1 \max_{z\in\Omega} \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[|q_{k-1} - q_{i-1}| \right] \right] + K_2 \max_{z\in\Omega} \mathbb{J}^{-1} \left[\tfrac{w}{r^2} \mathbb{J} \left[|q_{k-1} - q_{i-1}| \right] \right] \\
&= (K_1 + K_2)\, z \, \|q_{k-1} - q_{i-1}\| .
\end{aligned}$$

Thus, $\|q_k - q_i\| \leq \delta \|q_{k-1} - q_{i-1}\|$. Choose $k = i + 1$, then:

$$\|q_{i+1} - q_i\| \leq \delta \|q_i - q_{i-1}\| \leq \delta^2 \|q_{i-1} - q_{i-2}\| \leq ... \leq \delta^i \|q_1 - q_0\| .$$

Similarly, using the triangle inequality:

$$\begin{aligned}
\|q_k - q_i\| &\leq \|q_{i+1} - q_i\| + \|q_{i+2} - q_{i+1}\| + ... + \|q_k - q_{k-1}\| \\
&\leq \left[\delta^i + \delta^{i+1} + ... + \delta^{m-1} \right] \|q_1 - q_0\| \\
&\leq \delta^i \left[\tfrac{1-\delta^{k-i}}{1-\delta} \right] \|\theta_1\| .
\end{aligned}$$

But, $0 < \delta < 1$, then $1 - \delta^{k-i} < 1$. Thus,

$$\|q_k - q_i\| \leq \frac{\delta^i}{1-\delta} \max_{z\in\Omega} |\theta_1| . \tag{4.8}$$

Since $\theta(s,z)$ is bounded, then $|\theta_1| < \infty$. So, as $i \to \infty$, then $\|q_k - q_i\| \to 0$. Thus, the sequence $\{q_k\}$ is a Cauchy in Ξ. Therefore, $\theta(s,z) = \sum_{j=0}^{\infty} \theta_j(s,z)$ converges.

Hence, Theorem 5.2 proof is now complete. ■

Theorem 4.3 ***(Error Estimate)*** *It is expected that the largest absolute truncation error of the series solution in equations (4.8) to (4.1) will be*

$$\max_{s\in\Omega} \left| \theta(s,z) - \sum_{k=0}^{i} \theta_k(s,z) \right| \leq \frac{\delta^i}{1-\delta} \|\theta_1(s,z)\| .$$

Proof: Using both Theorem 4.2 and Eq. (4.8) one can conclude: $\|q_k - q_i\| \leq \frac{\delta^i}{1-\delta} \max_{s\in\Omega} |\theta_1|$. So as $k \to \infty$, we have $q_k \to \theta(s,z)$. Then, $\|\theta(s,z) - q_i\| \leq \frac{\delta^i}{1-\delta} \max_{s\in\Omega} |\theta_1(s,z)|$.

Thus, the absolute truncation error reaches its maximum in Ω as:

$$\max_{z\in\Omega} \left| \theta(s,z) - \sum_{k=0}^{i} \theta_k(s,z) \right| \leq \frac{\delta^i}{1-\delta} \max_{z\in\Omega} |\theta_1(s,z)| = \frac{\delta^i}{1-\delta} \|\theta_1(s,z)\| .$$

Hence, Theorem 4.3 proof is now complete. ■

Remark:

The $\mathbb{JADM}$ for nonlinear PDE convergence analysis was successfully applied in the current work. This is very important to the research community because of its importance and the fact that it shows how the technique converges and the solution is unique.

5 Applications of $\mathbb{JADM}$

We now demonstrate the use of the $\mathbb{J}-$transform to the solution of many nonlinear ordinary differential equations. A nonlinear differential equation's nonlinear terms can all be easily handled with the aid of the Adomian polynomials.

Assume the nonlinear partial differential equation exists:

$$L(\theta(s,z)) + R(\theta(s,z)) + F(\theta(s,z)) = g(s,z). \tag{5.1}$$

And its I.C:

$$\theta(s,0) = \gamma(s), \tag{5.2}$$

where the nonhomogeneous term is $g(s,z)$, the nonlinear term is $F(\theta(s,z))$, L represents the operator of the largest derivative, and R represents the remaining differential operator.

After that, $\mathbb{J}-$transform and property 4 are applied to Eq. (5.1) to yield:

$$\frac{r^2\Theta(x,r,w)}{w} - w\,\gamma(s) + \mathbb{J}\left[R(\theta(s,z))\right] + \mathbb{J}\left[F(\theta(s,z))\right] = \mathbb{J}\left[g(s,z)\right]. \tag{5.3}$$

From Eq. (5.2) and Eq. (5.3) together, the following is true:

$$\Theta(x,r,w)=\frac{\gamma(s)w^2}{r^2}+\frac{w}{r^2}\mathbb{J}\left[g(s,z)\right]-\frac{w}{r^2}\mathbb{J}\left[R(\theta(s,z))+F(\theta(s,z))\right]. \tag{5.4}$$

Using the inverse $\mathbb{J}$-transform on Eq. (5.4) to get:

$$\theta(s,z)=G(s,z)-\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[R(\theta(s,z))+F(\theta(s,z))\right]\right], \tag{5.5}$$

where $G(s,z)$ represents both the initial condition and the nonhomogeneous part.

Assume an infinite series solution of $\theta(s,z)$ of the form:

$$\theta(s,z)=\sum_{j=0}^{\infty}\theta_j(s,z). \tag{5.6}$$

We rewrite Eq. (5.5) as follows using Eq. (5.6):

$$\sum_{j=0}^{\infty}\theta_j(s,z)=G(s,z)-\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[R\sum_{j=0}^{\infty}\theta_j(s,z)+\sum_{j=0}^{\infty}A_j(s,z)\right]\right], \tag{5.7}$$

The Adomian polynomials are the $A_j(s,z)'s$ in the nonlinear term $F(\theta(s,z))=\sum_{j=0}^{\infty}A_j(s,z)$.

When the two sides of equation (5.7) are compared, $\theta_0(s,z)=G(s,z)$ is the outcome.

$$\begin{aligned}
\theta_0(s,z) &= G(s,z),\\
\theta_1(s,z) &= -\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[R\theta_0(s,z)+A_0(s,z)\right]\right],\\
\theta_2(s,z) &= -\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[R\theta_1(s,z)+A_1(s,z)\right]\right],\\
\theta_3(s,z) &= -\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[R\theta_2(s,z)+A_2(s,z)\right]\right].
\end{aligned}$$

Next, one can generate the following general relation:

$$\theta_{j+1}(s,z)=-\mathbb{J}^{-1}\left[\frac{w}{r^2}\mathbb{J}\left[R\theta_j(s,z)+A_j(s,z)\right]\right], \quad j\geq 0. \tag{5.8}$$

The expected exact solution's final expression is as follows:

$$\theta(s,z)=\sum_{j=0}^{\infty}\theta_j(s,z). \tag{5.9}$$

Worked Examples:

We use the $\mathbb{JADM}$ to illustrate the performance of our recently designed system in a number of nonlinear PDE situations.

Example 5.1 *Assume Burger's equation, which has the following form:*

$$\theta_z + \theta\theta_s = \theta_{ss}. \tag{5.10}$$

Accompanied by its condition:

$$\theta(s,0) = 4\tan(2s). \tag{5.11}$$

Using $\mathbb{J}$-transformation on Eq. (5.10), we get:

$$\frac{r\Theta(x,r,w)}{w} - w\theta(s,0) + \mathbb{J}\left[\theta\theta_s\right] = \mathbb{J}\left[\theta_{ss}\right]. \tag{5.12}$$

Substitute Eq. (5.11) in Eq. (5.11) to produce:

$$\theta(x,r,w) = \frac{4w^2\tan(2s)}{r} + \frac{w}{r}\mathbb{J}\left[\theta_{ss} - \theta\theta_s\right]. \tag{5.13}$$

Employing the $\mathbb{J}$-inverse transformation of Eq. (5.13) to obtain:

$$\theta(s,z) = 4\tan(2s) + \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[\theta_{ss} - \theta\theta_s\right]\right]. \tag{5.14}$$

Suppose our intended solution $\theta(s,z)$ is of the form:

$$\theta(s,z) = \sum_{j=0}^{\infty}\theta_j(s,z). \tag{5.15}$$

Putting Eq. (5.15) in place of Eq. (5.14) results in:

$$\sum_{j=0}^{\infty}\theta_j(s,z) = 4\ tan\,(2s) + \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[\sum_{j=0}^{\infty}(\theta_j)_{ss} - \sum_{j=0}^{\infty}A_j\right]\right], \tag{5.16}$$

where the Adomian polynomials $A_j(s,z)$ stand in for the nonlinear term $\theta(s)\theta'(s)$.

Thus,

$$\begin{aligned}
\theta_0(s,z) &= 4\tan(2s),\\
\theta_1(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_0)_{ss} - A_0\right]\right],\\
\theta_2(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_1)_{ss} - A_1\right]\right].
\end{aligned}$$

Finally,

$$\theta_{j+1}(s,z) = \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_j)_{ss} - A_j\right]\right], \quad \forall j \geq 0. \tag{5.17}$$

Then using Eq. (5.17) we can arrive at:

$$\begin{aligned}\theta_1(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_0)_{ss} - A_0\right]\right]\\ &= \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_0)_{ss} - \theta_0(\theta_0)_s\right]\right]\\ &= \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[32\,sec^2(2s)\tan(2s) - 32\,sec^2(2s)\tan(2s)\right]\right]\\ &= 0.\end{aligned}$$

$$\begin{aligned}\theta_2(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_1)_{ss} - A_1\right]\right]\\ &= \mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_1)_{ss} - (\theta_1\theta_{0s} + \theta_0(\theta_1)_s)\right]\right]\\ &= 0.\end{aligned}$$

Eventually, we can conclude:

$$\theta_{j+1}(s,z) = 0, \qquad \forall j \geq 1. \tag{5.18}$$

Consequently, the following provides the solution to the nonlinear Burgers equation:

$$\begin{aligned}\theta(s,z) &= \sum_{j=0}^{\infty}\theta_j(s,z)\\ &= \theta(s,z) + \theta_1(s,z) + \theta_2(s,z) + \cdots\\ &= 4\tan(2s) + 0 + 0 + \cdots\\ &= 4\tan(2s).\end{aligned}$$

Therefore, the following provides exact solution to Burger's equation:

$$\theta(s,z) = 4\tan(2s).$$

As a result, our exact solution and the one found in the literature agree when employing the $\mathbb{JADM}$.

Example 5.2 *Examine the nonlinear Schrödinger equation of the following form:*

$$i\theta_z + \theta_{ss} - 2|\theta|^2\theta = 0. \tag{5.19}$$

Accompanied by its condition:

$$\theta(s,0) = e^{is}. \tag{5.20}$$

By multiplying Eq. (5.19) by i, we obtain:

$$\begin{aligned} -\theta_z + i\theta_{ss} - 2i|\theta|^2\theta = 0, \\ \theta_z = i\theta_{ss} - 2i|\theta|^2\theta. \end{aligned} \tag{5.21}$$

Using $\mathbb{J}$-transformation on Eq. (5.21), we obtain:

$$\frac{r\Theta(x,r,w)}{w} - w\theta(s,0) = i\mathbb{J}\left[\theta_{ss}\right] - i2\mathbb{J}\left[|\theta|^2\theta\right]. \tag{5.22}$$

Substitute Eq. (5.21) in Eq. (5.20) to produce:

$$\Theta(x,r,w) = \frac{w^2}{r}e^{is} + i\frac{w}{r}\mathbb{J}\left[\theta_{ss} - 2|\theta|^2\theta\right]. \tag{5.23}$$

Employing the $\mathbb{J}$-inverse transformation of Eq. (5.23) to obtain:

$$\theta(s,z) = e^{is} + i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[\theta_{ss} - 2|\theta|^2\theta\right]\right]. \tag{5.24}$$

Assume our anticipated solutions for $\theta(s,z)$ in the form:

$$\theta(s,z) = \sum_{j=0}^{\infty}\theta_j(s,z). \tag{5.25}$$

Substituting Eq. (5.25) into Eq. (5.24) will result in:

$$\sum_{j=0}^{\infty}\theta_j(s,z) = e^{is} + i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[\sum_{j=0}^{\infty}(\theta_j)_{ss} - 2\sum_{j=0}^{\infty}A_j\right]\right], \tag{5.26}$$

where the nonlinear term $|\theta|^2\theta$ is represented by the Adomian polynomial A_j.

We continue in a similar manner to obtain:

$$\begin{aligned} \theta_0(s,z) &= e^{is}, \\ \theta_1(s,z) &= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_0)_{ss} - 2A_0\right]\right], \\ \theta_2(s,z) &= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_1)_{ss} - 2A_1\right]\right], \\ \theta_3(s,z) &= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_2)_{ss} - 2A_2\right]\right]. \end{aligned}$$

Finally,

$$\theta_{j+1}(s,z) = i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_j)_{ss} - 2A_n\right]\right], \quad \forall j \geq 0. \tag{5.27}$$

Then using Eq. (5.27) we can arrive at:

$$
\begin{aligned}
\theta_1(s,z) &= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_0)_{ss}-2A_0\right]\right] \\
&= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_0)_{ss}-2v_0^2\bar{\theta_0}\right]\right] \\
&= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[-e^{is}-2e^{2is}e^{-is}\right]\right] \\
&= -3ie^{is}\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[1\right]\right] \\
&= -3ie^{is}\mathbb{J}^{-1}\left[\frac{w^3}{r^2}\right] \\
&= -3ize^{is}.
\end{aligned}
$$

$$
\begin{aligned}
\theta_2(s,z) &= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_1)_{ss}-2A_1\right]\right] \\
&= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[(\theta_1)_{ss}-2\left(2\theta_0\theta_1 th\bar{e}ta_0+\theta_0^2\bar{\theta_1}\right)\right]\right] \\
&= i\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[3ize^{is}-2\left(-6ize^{is}+3ize^{is}\right)\right]\right] \\
&= -9e^{is}\mathbb{J}^{-1}\left[\frac{w}{r}\mathbb{J}\left[z\right]\right] \\
&= -9e^{is}\mathbb{J}^{-1}\left[\frac{w^4}{r^3}\right] \\
&= \frac{1}{2!}\left(3iz\right)^2 e^{is}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\theta(s,z) &= \sum_{j=0}^{\infty}\theta_j(s,z) \\
&= \theta_0(s,z)+\theta_1(s,z)+\theta_2(s,z)\theta_3(s,z)+\cdots \\
&= e^{is}-3ize^{is}+\frac{1}{2!}\left(3iz\right)^2 e^{is}-\frac{1}{3!}\left(3iz\right)^3 e^{is}+\cdots \\
&= e^{is}\left(1-(3iz)+\frac{1}{2!}\left(3iz\right)^2-\frac{1}{3!}\left(3iz\right)^3+\cdots\right) \\
&= e^{i(s-3z)}.
\end{aligned}
$$

Hence, the exact solution $\theta(s,z)$ is given by:

$$\theta(s,z) = e^{i(s-3z)}.$$

As a result, our exact solution and the one found in the literature agree when employing the $\mathbb{JADM}$.

Example 5.3 *Examine the homogeneous Korteweg-deVries (KdV) equation that is as follows:*

$$\theta_z(s,z) - 6\theta(s,z)\theta_s(s,z) + \theta_{sss}(s,z) = 0. \tag{5.28}$$

Accompanied by its conditions:

$$\theta(s,0) = \frac{s-4}{18}. \tag{5.29}$$

Using $\mathbb{J}-$transformation on Eq. (5.28), we obtain:

$$\frac{r\Theta(x,r,w)}{w} - w\theta(s,0) - \mathbb{J}\left[6\theta\theta_s\right] + \mathbb{J}\left[\theta_{sss}\right] = 0. \tag{5.30}$$

Substitute in Eq. (5.29) using Eq. (5.30) to produce:

$$\Theta(x,r,w) = \frac{w^2(s-4)}{18r} - \frac{w}{r}\left[\ \mathbb{J}\left[6\theta\theta_s - \theta_{sss}\right]\right]. \tag{5.31}$$

Below, we employ the $\mathbb{J}$-inverse transformation of Eq. (5.31) for our purposes:

$$\theta(s,z) = \frac{(s-4)}{18} - \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6\theta\theta_s - \theta_{sss}\right]\right]\right]. \tag{5.32}$$

Assume the following is how we want our solutions to look: $v(x,t)$ of the form:

$$\theta(s,z) = \sum_{j=0}^{\infty}\theta_j(s,z). \tag{5.33}$$

Putting Eq. (5.33) in place of Eq. (5.32) results in:

$$\sum_{j=0}^{\infty}\theta_j(s,z) = \frac{(s-4)}{18} - \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6\sum_{j=0}^{\infty}A_j - \sum_{j=0}^{\infty}(\theta_j)_{sss}\right]\right]\right]. \tag{5.34}$$

The nonlinear terms' Adomian polynomials $\theta\theta_s$ is denoted by A_j.

Then, using Eq. (5.34) we can arrive at:

$$\begin{aligned}
\theta_0(s,z) &= \frac{(s-4)}{18},\\
\theta_1(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6A_0 - (\theta_0)_{sss}\right]\right]\right],\\
\theta_2(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6A_1 - (v\theta_1)_{sss}\right]\right]\right],\\
\theta_3(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6A_2 - (\theta_2)_{sss}\right]\right]\right].
\end{aligned}$$

Thus,

$$\theta_{j+1}(s,z) = \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6A_j - (\theta_j)_{sss}\right]\right]\right], \qquad \forall j \geq 0. \tag{5.35}$$

From Eq. (5.35), we obtain:

$$\begin{aligned}
\theta_1(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6A_0 - (\theta_0)_{sss}\right]\right]\right] \\
&= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6\theta_0\theta_{0s} - (v_0)_{sss}\right]\right]\right] \\
&= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[\frac{(s-4)}{54} - 0\right]\right]\right] \\
&= \frac{(s-4)}{54}\mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[1\right]\right]\right] \\
&= \frac{(s-4)z}{54}. \\
\theta_2(s,z) &= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6A_1 - (\theta_1)_{sss}\right]\right]\right] \\
&= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[6\left(\theta_1(\theta_0)_s + (\theta_1)_s\theta_0\right) - (\theta_1)_{sss}\right]\right]\right] \\
&= \mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[12\left(\frac{(s-4)z}{972} + \frac{(s-4)z}{972}\right)\right]\right]\right] \\
&= \frac{(s-4)}{81}\mathbb{J}^{-1}\left[\frac{w}{r}\left[\mathbb{J}\left[z\right]\right]\right] \\
&= \frac{(s-4)z^2}{162}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\theta(s,z) &= \sum_{j=0}^{\infty}\theta_j(s,z) \\
&= \theta_0(s,z) + \theta_1(s,z) + \theta_3(s,z) + \cdots \\
&= \frac{(s-4)}{18} + \frac{(s-4)z}{54} + \frac{(s-4)z^2}{162} + \cdots \\
&= \frac{(s-4)}{18}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right) \\
&= \frac{s-4}{6(3-z)}.
\end{aligned}$$

Therefore, the precise answer to Eq. (5.28) is:

$$\theta(s,z)=\frac{s-4}{6(3-z)}.$$

As a result, our exact solution and the one found in the literature agree when employing the $\mathbb{JADM}$.

Example 5.4 *The following nonlinear Klein-Gordon equation given below needs to be investigated.*

$$\theta_{zz}-\theta_{ss}+\theta^2=6s^3z-6sz^3+s^6z^6. \tag{5.36}$$

Accompanied by its conditions:

$$\theta(s,0)=\theta_z(s,0)=0. \tag{5.37}$$

Using $\mathbb{J}$*-transformation on Eq. (5.36), we obtain:*

$$\frac{r^2\Theta(x,r,w)}{w^2}-r\theta(s,0)-w\theta_z(s,0)-\mathbb{J}\left[\theta_{ss}\right]+\mathbb{J}\left[\theta^2\right]=\frac{6s^3w^3}{r^2}-\frac{36sw^5}{r^4}+\frac{720s^6w^8}{r^7}. \tag{5.38}$$

Substitute Eq. (5.37) into Eq. (5.37) to produce:

$$\Theta(x,r,w)=\frac{6s^3w^5}{r^4}-\frac{36rw^7}{r^6}+\frac{720s^6w^{10}}{r^9}+\frac{w^2}{r^2}\left[\mathbb{J}\left[\theta_{ss}-\theta^2\right]\right]. \tag{5.39}$$

Employing the $\mathbb{J}$*-inverse transformation on Eq. (5.39) to get:*

$$\theta(s,z)=s^3z^3-\frac{3sz^5}{10}+\frac{s^6z^8}{56}+\mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[\theta_{ss}-v^2\right]\right]\right]. \tag{5.40}$$

Assume the following is how we want our solutions to look: $\theta(s,z)$ *of the form:*

$$\theta(s,z)=\sum_{j=0}^{\infty}\theta_j(s,z). \tag{5.41}$$

Putting Eq. (5.40) in place of Eq. (5.41) results in:

$$\sum_{j=0}^{\infty}\theta_j(s,z)=s^3z^3-\frac{3sz^5}{10}+\frac{s^6z^8}{56}+\mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[\sum_{j=0}^{\infty}(\theta_j)_{ss}-\sum_{j=0}^{\infty}A_j\right]\right]\right], \tag{5.42}$$

where the nonlinear term θ^2 *is represented by the Adomian polynomial* $A_j(s)$.

We continue in a similar manner to obtain:

$$\begin{aligned}
\theta_0(s,z) &= s^3z^3-\frac{3sz^5}{10}+\frac{s^6z^8}{56},\\
\theta_1(s,z) &= \mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[(\theta_0)_{ss}-A_0\right]\right]\right],\\
\theta_2(s,z) &= \mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[(\theta_1)_{ss}-A_1\right]\right]\right],\\
\theta_3(s,z) &= \mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[(\theta_2)_{ss}-A_2\right]\right]\right].
\end{aligned}$$

Finally,

$$\theta_{j+1}(s,z) = \mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[(\theta_j)_{ss} - A_n\right]\right]\right], \qquad \forall j \geq 0. \tag{5.43}$$

Then using Eq. (5.41) we can arrive at:

$$\begin{aligned}
\theta_1(s,z) &= \mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[(\theta_0)_{ss} - A_0\right]\right]\right] \\
&= \mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[(\theta_0)_{ss} - \theta_0^2\right]\right]\right] \\
&= \mathbb{J}^{-1}\left[\frac{w^2}{r^2}\left[\mathbb{J}\left[\left(6sz^3 + \frac{15s^4z^8}{28}\right) - \left(s^3z^3 - \frac{3sz^5}{10} + \frac{s^6z^8}{56}\right)^2\right]\right]\right] \\
&= \mathbb{J}^{-1}\left[\left(\frac{36sw^7}{r^6} - \frac{720s^6w^{10}}{r^9} + \cdots\right)\right] \\
&= \frac{3sz^5}{10} - \frac{s^6z^8}{56} + \cdots.
\end{aligned}$$

Ultimately, one can observe that the remaining terms continue to satisfy the equation by eliminating any of the noise terms in $\theta_0(s,z)$ that also occur in $\theta_1(s,z)$. This leads to an exact solution that takes the following form:

$$\theta(s,z) = s^3z^3.$$

As a result, our precise solution and the one found in the literature agree when employing the $\mathbb{JADM}$.

Remark: *The term pairs with the same sign but opposite sign are called noise terms.*

6 Conclusion

One of the most intriguing differential equations is the nonlinear Klein–Gordon equation rises from the study of quantum field theory and is used to describe nonlinear wave interaction, which we solve in this paper using a unique strategy. It has applications in algebraic geometry, physics, and the theory of conformal mapping, among other branches of mathematics. Using the $\mathbb{JADM}$, Klein-Gordon equation problem has been effectively resolved. We also provided exact solutions for Schrodinger equation and Burger's equation. The results show that the convergence rate of the $\mathbb{JADM}$ is faster than other methods reported in the literature. The relevance of $\mathbb{JADM}$ was proved by its application in applied science and engineering fields. Additionally, we implemented the suggested technique to other situations, demonstrating its efficacy and adaptability. According to the aforementioned study, the $\mathbb{JADM}$ can also be used to precisely solve other non-linear ODEs and PDEs, such as systems of ODEs and PDEs, which are commonly encountered in science and engineering. So, from the $\mathbb{JADM}$ solutions to various scenarios, a fuller understanding of the real-world applications represented by these modeling challenges will become clear.

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Compliance with ethical standards

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Consent to participate Participants are informed that if they have any objections or issues about the way the research is or has been done, they can get in touch with the University of Vermont Ethics Officer.

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