ANALYSIS OF FRACTIONAL ORDER SCHRÖDINGER EQUATION WITH SINGULAR AND NON-SINGULAR KERNEL DERIVATIVES VIA NOVEL HYBRID SCHEME

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Abstract In the present study, a novel semi-analytic scheme is proposed to obtain exact and approximate series solutions for the time fractional linear and non-linear Schrödinger equation. This hybrid scheme employs the general bivariate transform followed by the homotopy perturbation method to formulate the recurrence relation. The recurrence relation leads to a system of linear differential equations that associates with the desired components of the series solution. To characterize the considered model with memory effects, the fractional temporal order is considered in the Caputo, Caputo-Fabrizio, and Atangana-Baleanu in Caputo senses. The adapted scheme appears efficient and competent in identifying a diverse collection of trigonometric, wave, and soliton solutions with the availability of initial data. Configurational variations in the governing phenomena with alterations in the fractional order are addressed through graphical illustrations. The potential of the developed regime is affirmed through the uniqueness and convergence analysis of the acquired results. Numerical results are found to be in accordance with existing results in terms of absolute error norms. The main highlight of the proposed scheme is its efficacy and simplicity in constructing a series solution that rapidly converges to the exact solution.

Keywords Fractional derivative, Partial differential equations, Integral transform, Homotopy perturbation method.

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1. Introduction

The concept of calculus of fractional operators was developed to find an answer to the question that prompted L' Hospital in 1695 to ask Leibnitz about the scenario of n = 1/2 for derivative of order n i.e. $\frac{d^n f}{d\zeta^n}$. An essential role of the fractional operators is that it furnishes the real-world models with an arbitrary order of differentiation and integrals for characterizing memory and hereditary effects. These are increasingly employed in the disciplines of applied sciences to decode the non-local physical appeal of complex systems [2, 7, 9, 12, 24, 56]. As a result of the usefulness of the fractional calculus in the mathematical analysis its practical and theoretical aspects are emerged as an enthralling topic in recent decades. For instance, mathematical modeling of viscoelastic materials frequently makes use of fractional derivatives [50]. A differential equation of diffusion that does not use integers can be used to describe anomalous diffusion phenomena in non-homogeneous media [14]. Fractance, an electrical circuit with non-integer order impedance that possesses both resistance and capacitance qualities, is another example of an element with fractional order [36]. Additionally, it has been demonstrated that fractional order calculus can be used to more accurately simulate the dynamic process of heat conduction [22]. It has been demonstrated in biology that biological cell membranes exhibit fractal order conductance and are classified as non-integer order systems [21, 26]. Some financial systems are known to exhibit fractional order dynamics in economics [37]. Due to the wide range of multidisciplinary applications, the piqued attention of researchers glorified the literature with innumerable definitions of differentiation and integration of arbitrary order. The capacity to choose the most advantageous definition increased the ability of mathematical simulation to adapt the given facts. Some of the renowned definitions are Riemann-Liouville derivative, Caputo-derivative, Caputo-Fabrizio derivative, Atangana Baleanu derivative, Grunwald-Letnikov derivative and Riesz derivative etc [15, 19, 33, 44, 45]. Despite the abundance of literature on fractional calculus, the optimal definition of a fractional derivative is still lacking. Researchers have reported the setbacks of prevailing non-integer derivatives in several aspects. R-L derivative have singular kernel and results in variable function even for derivative of constant function. All existing fractional derivatives just inherited the linearity property from the classical derivative. All fractional derivatives do not follow the product rule, quotient rule, chain rule, Rolle's theorem and mean value theorem. Most of the fractional derivatives except Caputo derivative do not cope up with zero for constant function. After discarding the R-L derivative the Caputo derivative is considered as most preferable operator by incorporating the limitations of R-L derivative. Since, mathematical model within the framework of Caputo derivative possess the classical initial conditions, the behaviour of complex evolution processes is widely investigated with aid of the Caputo derivative and its extensions.

Experimental studies regarding wave-particle duality witnessed the wave motion of particles in microscopic system instead of obeying Newtonian laws of motions governed in macroscopic systems. To specify the quantitative relation between the behaviour of particle and its associated wave function, prompted by de Broglie postulate, Schrödinger's theory developed as a generalization of Newton's theory of the motion of particles in macroscopic systems. The mathematical model evolved from Schrödinger's theory entitled as Schrödinger equation (SG) have emerged as fundamental narrator of the aspects of the particle for its specific potential energy

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 $\nu(\zeta, \tau)$, through the behaviour of associated wave function $\psi(\zeta, \tau)$. Potential energy is usually a function of ζ only and possibly can be of τ also. However, for the constant value potential energy, SG equation describes the phenomena of free particle. The presence of imaginary unit *i* at a fundamental level in SG equation led to the relation of first power of total energy to the second power of momentum. Furthermore, it associates the wave function to two real functions simultaneously, its real part and imaginary part. This is in contrast to a wave function from classical mechanics. Taking into account, the uncertainty principal of microscopic world knowns as quantum mechanics SG equation expresses the particle-wave relation in terms of probability density, first stated in 1926 by Max Born [18] as :

$$\mathcal{P}(\zeta,\tau) = |\psi(\zeta,\tau)|^2$$

This quantity indicates the probability of finding the particle close to the coordinate ζ at time τ , in per unit length of the ζ -axis. With development of non-linear science, the non-linear SG equation established as a multidisciplinary model interpreting non-linear wave evolution phenomena frequently arise in quantum mechanics, chemical kinetics, optics, hydrodynamics, bio-genetics, plasma physics, and so forth [13,16,20,34,54,55]. The integrable cubic non-linear SG equation has special interest in revolutionized communication system fabricated with optical fibers [17]. It exhibits the basic features of optical wave propagation in Kerr media but leaves certain finer details such as spin which is a purely quantum phenomena [35]. For being general, non-linear SG equations incorporates a variety of non-linearities including, octic-septic-quintic-cubic non-linearity, Kerr non-linearity, logarithmic non-linearity, dual-power law non-linearity etc [8, 30, 43, 51–53].

The Feynman path integrals is generated by quantum dynamics with Brownian-like attributes. However, commonly known as $\psi(\tau) \sim \tau^{1/2}$, square root law of governing Brownian motion regarding change in location over time has not been spotted for many complex quanta and classical physics [38–40]. To analyze complex quantum dynamic processes, broader evolution law $\psi(\tau) \sim \tau^{1/\varpi}$ has been confined with general scaling $1/\varpi$; $0 < \varpi \leq 2$, known as Levy flights. As a fundamental result of extending the Feynman path integrals to Levy-like quantum mechanical path, Nick Lasin discovered eminent fractional Schrödinger equation interpreting the mechanics of fractional quantum [38–40]. To extract the dynamical information of variants of fractional SG equation, various analytical and numerical studies have been organized with different preeminent techniques [3–6, 10, 27, 29, 31, 32, 42, 47].

In present study, to characterize the SG equation with the memory effect, the fractional temporal operator has been adapted with three terminologies such as Caputo(C), Caputo-Fabrizio (CF) and Atangana Baleanu in Caputo sense (ABC) in the following form:

$$iC^{\rho}_{\tau}\psi(\zeta,\tau) = \beta\psi_{\zeta\zeta}(\zeta,\tau) + \nu(\zeta)\psi(\zeta,\tau) + \lambda\psi(\zeta,\tau)|\psi(\zeta,\tau)|^2.$$
(1.1)

Initially,

$$\psi(\zeta, 0) = \phi(\zeta), \tag{1.2}$$

where $i = \sqrt{-1}, \psi(\zeta, \tau)$ is complex-valued function while ζ and τ are spatial and temporal variables, respectively. The parameter ρ decribes the order of fractional temporal derivative. β stands for coefficient of group velocity dispersion (GVD) and λ represents the coefficient of non-linearity. The term $\psi(\zeta, \tau)|\psi(\zeta, \tau)|^2$ represents Kerr-law non-linearity which implies direct proportional dependence of refractive index of light on its intensity. $\nu(\zeta)$ is the trapping potential relies on the description of loss and dispersion management schemes. $\phi(\zeta)$ is the known initial function. The paper comprises seven sections. Section 1 is about general introduction of the problem. Section 2 consists of fundamental terminologies of fractional derivatives and general bivariate (GB) transform. Section 3 gives brief description of general bivariate homotopy perturbation method (GB-HPM). Sufficient conditions for uniqueness and convergence of the solution are presented in Section 4. Several numerical examples are discussed in Section 5. Section 6 briefs the results and

thorough observations. Finally, conclusions are given in Section 7.

2. Basic Definitions

This section covers basic concepts and definitions of Caputo, CF, ABC fractional derivatives and the GB-transform that will be used in this work.

Definition 2.1. [44] In Caputo's terminology the fractional operator over function $\varphi(\tau)$ is defined as:

$${}_{0}^{C}C_{\tau}^{\rho}(\varphi(\tau)) = \begin{cases} \frac{1}{\Gamma(n-\rho)} \int_{0}^{\tau} \frac{\varphi^{n}(\xi)}{(\tau-\xi)^{\rho+1-n}} d\xi, & n-1 < \rho < n, \\ \frac{d^{n}}{d\tau^{n}}\varphi(\tau), & \rho = n. \end{cases}$$
(2.1)

By replacing the singular power law kernel and multiplier of intergral in Caputo's fractional derivative fractional operators with non-singular kernels are defined as:

Definition 2.2. [12, 19] In CF's terminology the fractional operator of order ρ over function $\varphi(\tau)$ is defined as:

$${}_{0}^{CF}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right) = \begin{cases} \frac{B(\rho)}{1-\rho} \int_{0}^{\tau} \exp\left(-\frac{\rho(\tau-\xi)}{1-\rho}\right) \varphi'\left(\xi\right) \, d\xi, & 0 < \rho < 1, \\ \frac{d}{d\tau}\varphi(\tau), & \rho = 1, \end{cases}$$
(2.2)

where $B(\rho)$ is the normalization function with B(0) = B(1) = 1.

Definition 2.3. [2,15] In ABC's terminology the fractional operator of order ρ over function $\varphi(\tau)$ is defined as:

$${}^{ABC}_{0}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right) = \begin{cases} \frac{B(\rho)}{1-\rho} \int_{0}^{\tau} E_{\rho}\left(-\frac{\rho\left(\tau-\xi\right)^{\rho}}{1-\rho}\right)\varphi'\left(\xi\right) d\xi, & 0 < \rho < 1, \\ \frac{d}{d\tau}\varphi(\tau), & \rho = 1, \end{cases}$$
(2.3)

where $B(\rho)$ is the normalization function and $E_{\rho}(.)$ is the Mittag-Leffler function.

In present study, $B(\rho) = 1$ is considered for simplicity.

These terminologies of fractional calculus have been widely explored to trace the non-local behaviour of numerous physical systems. In search of efficient tools, integral transforms have been emerged as competent tools to construct the solution of fractional models. However, efficiency of these transforms is totally problem oriented. The present study directs to the capabilities of recently introduced GB transform [11] for fractional models framed with different Caputo's type fractional derivatives. GB transform reduces to ARA [49] and Formable transform [48] for specific values of its accounted parameters.

Definition 2.4. [11] The GB transform of function $\varphi(\tau)$ is given by:

$$\mathscr{A}_{m}\left(\varphi\left(\tau\right)\right) = \mathscr{P}_{m}\left(s,\gamma\right) = \frac{s}{\gamma^{m}} \int_{0}^{\infty} \tau^{m-1} \exp\left(-\frac{s}{\gamma}\tau\right) \varphi\left(\tau\right) d\tau, \qquad (2.4)$$

where m is order, s and γ are transformation variable.

Definition 2.5. [11] If $\mathscr{P}_m(s,\gamma)$ is the GB transform of function $\varphi(\tau)$, then the inverse of $\mathscr{P}_m(s,\gamma)$ is the $\varphi(\tau)$ is such that

$$\mathscr{A}_{m}^{-1}\left(\mathscr{P}_{m}\left(s,\gamma\right)\right) = \varphi\left(\tau\right).$$

$$(2.5)$$

Lemma 2.1. [11] If $\mathscr{A}_{m}(\varphi_{1}(\tau)) = \mathscr{P}_{m}(s,\gamma)$ and $\mathscr{A}_{m}(\varphi_{2}(\tau)) = \mathscr{Q}_{m}(s,\gamma)$ then

$$\mathscr{A}_{m}\left(\lambda_{1}\varphi_{1}\left(\tau\right)+\lambda_{2}\varphi_{2}\left(\tau\right)\right)=\lambda_{1}\mathscr{P}_{m}\left(s,\gamma\right)+\lambda_{2}\mathscr{Q}_{m}\left(s,\gamma\right),$$
(2.6)

where λ_1 and λ_2 are constants.

Lemma 2.2. [11] (Shifting in n-domain)

$$\mathscr{A}_{m}\left(\tau^{n}\varphi\left(\tau\right)\right) = \gamma^{n}\mathscr{A}_{m+n}\left(\varphi\left(\tau\right)\right), \qquad (2.7)$$

where $m + n - 1 \ge 0$.

Theorem 2.1. The GB transform for fractional operator defined in Caputo's sense is given by:

$$\mathscr{A}_{m}\left[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{1}{\Gamma\left(n-\rho\right)}\sum_{r=1}^{m}\binom{m-1}{r-1}\Gamma\left(n+m-\rho-r\right)\frac{\gamma^{n-\rho}}{s^{n+m-\rho-r}}\mathscr{A}_{r}\left[\varphi^{n}\left(\tau\right)\right]$$
(2.8)

where $n-1 < \rho \leq n$.

Proof. By using definition of GB transform and Caputo's derivative one can have

$$\mathscr{A}_{m}\left[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{s}{\gamma^{m}}\int_{0}^{\infty}\tau^{m-1}\exp\left(-\frac{s}{\gamma}\tau\right)\left(\frac{1}{\Gamma\left(n-\rho\right)}\int_{0}^{\tau}\frac{\varphi^{n}\left(\xi\right)}{\left(\tau-\xi\right)^{\rho-n+1}}d\xi\right)d\tau.$$
(2.9)

Altering the order of integration in Eq.(2.9), gives

$$\mathscr{A}_{m}\left[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{s}{\gamma^{m}\Gamma\left(n-\rho\right)} \int_{0}^{\infty} \varphi^{n}\left(\xi\right) \int_{\xi}^{\infty} \frac{\tau^{m-1}}{\left(\tau-\xi\right)^{\rho-n+1}} \exp\left(-\frac{s}{\gamma}\tau\right) d\tau \, d\xi.$$
(2.10)

Letting $\lambda = \tau - \xi$ in Eq.(2.10) leads to

$$\mathscr{A}_{m}\left[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{s}{\gamma^{m}\Gamma\left(n-\rho\right)} \int_{0}^{\infty} \varphi^{n}\left(\xi\right) \exp\left(-\frac{s}{\gamma}\xi\right) \int_{0}^{\infty} \exp\left(-\frac{s}{\gamma}\lambda\right) \left(\lambda+\xi\right)^{m-1} \lambda^{n-\rho-1} d\lambda \, d\xi. \quad (2.11)$$

Using the binomial formula, Eq.(2.11) gives

$$\mathscr{A}_{m}\left[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{s}{\gamma^{m}\Gamma\left(n-\rho\right)}\int_{0}^{\infty}\varphi^{n}\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\int_{0}^{\infty}\exp\left(-\frac{s}{\gamma}\lambda\right)\sum_{r=1}^{m}\binom{m-1}{r-1}$$

$$\lambda^{n+m-\rho-r-1}\xi^{r-1}d\lambda \,d\xi. \tag{2.12}$$

Using the definition of GB transform of order one, Eq.(2.12) implies

$$\mathscr{A}_{m}\left[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{1}{\gamma^{m-1}\Gamma\left(n-\rho\right)} \int_{0}^{\infty} \varphi^{n}(\xi) \exp\left(-\frac{s}{\gamma}\xi\right) \mathscr{A}_{1}\left[\sum_{r=1}^{m} \binom{m-1}{r-1} \lambda^{n+m-\rho-r-1}\right] \xi^{r-1} d\xi, \qquad (2.13)$$

$$=\frac{1}{\Gamma\left(n-\rho\right)}\sum_{r=1}^{m}\binom{m-1}{r-1}\Gamma\left(n+m-\rho-r\right)\frac{\gamma^{n-\rho-r}}{s^{n+m-\rho-r-1}}\int_{0}^{\infty}\varphi^{n}\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\xi^{r-1}d\xi$$
(2.14)

Using the definition of GB transform of order r, Eq.(2.14) implies

$$\mathscr{A}_{m}\left[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{1}{\Gamma\left(n-\rho\right)}\sum_{r=1}^{m}\binom{m-1}{r-1}\Gamma\left(n+m-\rho-r\right)\frac{\gamma^{n-\rho}}{s^{n+m-\rho-r}}\mathscr{A}_{r}\left[\varphi^{n}\left(\tau\right)\right].$$

Remark 2.1. (Special Cases of Eq. (2.8))

i. For m=1:

$$\mathscr{A}_1[{}_0^C C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right)] = \frac{\gamma^{n-\rho}}{s^{n-\rho}} \mathscr{A}_1[\varphi^n(\tau)], \quad n-1 < \rho \le n.$$

ii. For m=2:

$$\mathscr{A}_{2}[{}_{0}^{C}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)] = (n-\rho)\frac{\gamma^{n-\rho}}{s^{n-\rho+1}}\mathscr{A}_{1}[\varphi^{n}(\tau)] + \frac{\gamma^{n-\rho}}{s^{n-\rho}}\mathscr{A}_{2}[\varphi^{n}(\tau)], \quad n-1 < \rho \leq n$$

For $\gamma = 1$, the resulting equations of GB transform for Caputo's operator are in alignment with existing equations of ARA transform for Caputo's operator [46].

Theorem 2.2. The GB transform for fractional operator defined in Caputo-Fabrizio's sense is given by

$$\mathscr{A}_{m} \begin{bmatrix} {}_{0}^{CF}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right) \end{bmatrix} = B\left(\rho\right) \sum_{r=1}^{m} \binom{m-1}{r-1} \frac{(m-r)!}{(1-\rho)} \frac{\gamma}{\left(s + \frac{\rho}{1-\rho}\gamma\right)^{m-r+1}} \mathscr{A}_{r}\left[\varphi'\left(\tau\right)\right],$$

$$(2.15)$$

where $0 < \rho \leq 1$.

Proof. By using definition of GB transform and Caputo-Fabrizio's derivative one can have

$$\mathscr{A}_{m} \begin{bmatrix} {}^{CF}_{0}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right) \end{bmatrix} = \frac{B\left(\rho\right)s}{\left(1-\rho\right)\gamma^{m}} \int_{0}^{\infty} \tau^{m-1} \exp\left(-\frac{s}{\gamma}\tau\right) \left[\int_{0}^{\tau} \varphi'\left(\xi\right) \exp\left(-\frac{\rho\left(\tau-\xi\right)}{1-\rho}\right) d\xi\right] d\tau.$$

$$(2.16)$$

Altering the order of integration in Eq.(2.16), gives

$$\mathscr{A}_{m}\left[{}_{0}^{CF}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{B\left(\rho\right)s}{\left(1-\rho\right)\gamma^{m}} \int_{0}^{\infty}\varphi'\left(\xi\right)\int_{\xi}^{\infty}\exp\left(-\frac{s}{\gamma}\tau\right)\exp\left(-\frac{\rho\left(\tau-\xi\right)}{1-\rho}\right)\tau^{m-1}d\tau\,d\xi$$
(2.17)

Letting $\lambda = \tau - \xi$ in Eq.(2.17) leads to

$$\mathscr{A}_{m} \begin{bmatrix} {}^{CF}_{0} C^{\rho}_{\tau} \left(\varphi\left(\tau\right)\right) \end{bmatrix} = \frac{B\left(\rho\right) s}{\left(1-\rho\right) \gamma^{m}} \int_{0}^{\infty} \varphi'\left(\xi\right) \int_{0}^{\infty} \exp\left(-\frac{s}{\gamma} \left(\lambda+\xi\right)\right) \exp\left(-\frac{\rho\lambda}{1-\rho}\right) \left(\lambda+\xi\right)^{m-1} d\lambda \, d\xi.$$
(2.18)

Using the binomial formula, Eq.(2.18) gives

$$\mathcal{A}_{m} \begin{bmatrix} {}_{0}^{CF}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right) \end{bmatrix} = \frac{B\left(\rho\right)s}{\left(1-\rho\right)\gamma^{m}} \int_{0}^{\infty}\varphi'\left(\xi\right) \int_{0}^{\infty}\exp\left(-\frac{s}{\gamma}\left(\lambda+\xi\right)\right)\exp\left(-\frac{\rho\lambda}{1-\rho}\right)$$
$$\sum_{r=1}^{m} \binom{m-1}{r-1}\lambda^{m-r}\xi^{r-1}d\lambda\,d\xi.$$
(2.19)

From the definition of GB transform of order one, one can get

$$\mathcal{A}_{m} \begin{bmatrix} {}^{CF}_{0}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right) \end{bmatrix} = \frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}} \int_{0}^{\infty} \varphi'\left(\xi\right) \exp\left(-\frac{s}{\gamma}\xi\right) \mathcal{A}_{1} \left[\sum_{r=1}^{m} \binom{m-1}{r-1}\right) \\ \exp\left(-\frac{\rho\lambda}{1-\rho}\right) \lambda^{m-r} \left[\xi^{r-1}d\xi\right] \\ = \frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}} \sum_{r=1}^{m} \binom{m-1}{r-1} \left(m-r\right)! \frac{s\gamma^{m-r}}{\left(s+\frac{\rho}{1-\rho}\gamma\right)^{m-r+1}} \int_{0}^{\infty} \varphi'\left(\xi\right) \exp\left(-\frac{s}{\gamma}\xi\right) \xi^{r-1}d\xi.$$

$$(2.20)$$

Using the definition of GB transform of order r, Eq.(2.20) implies

$$\mathscr{A}_{m}\left[{}_{0}^{CF}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = B\left(\rho\right)\sum_{r=1}^{m}\binom{m-1}{r-1}\frac{(m-r)!}{1-\rho}\frac{\gamma}{\left(s+\frac{\rho}{1-\rho}\gamma\right)^{m-r+1}}\mathscr{A}_{r}\left[\varphi'\left(\tau\right)\right].$$

Remark 2.2. (Special Cases of Eq. (2.15))

i. For m = 1:

$$\mathscr{A}_{1} \begin{bmatrix} C^{F} C^{\rho}_{\tau} \left(\varphi \left(\tau \right) \right) \end{bmatrix} = \frac{B\left(\rho \right)}{1 - \rho + \rho \frac{\gamma}{s}} \frac{\gamma}{s} \mathscr{A}_{1} [\varphi'(\tau)], \quad 0 < \rho \leq 1.$$

ii. For m = 2:

$$\mathscr{A}_{2} \begin{bmatrix} CF \\ 0 \end{bmatrix} C_{\tau}^{\rho} \left(\varphi\left(\tau\right)\right) \end{bmatrix} = \frac{B\left(\rho\right) \gamma(1-\rho)}{[s(1-\rho)+\rho\gamma]^{2}} \mathscr{A}_{1} [\varphi'(\tau)] + \frac{B\left(\rho\right) \gamma}{s(1-\rho)+\rho\gamma} \mathscr{A}_{2} [\varphi'(\tau)], \quad 0 < \rho \le 1.$$

Theorem 2.3. The GB transform for fractional operator defined in Atangana Baleanu in Caputo sense is given by

$$\mathscr{A}_{m}\left[{}_{0}^{ABC}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{B\left(\rho\right)}{\left(1-\rho\right)}\sum_{r=1}^{m}\binom{m-1}{r-1}\frac{1}{s^{m-r}}\sum_{k=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^{k}\frac{1}{\Gamma\left(\rho k+1\right)}\left(\rho k+m-r\right)!$$
$$\frac{\gamma^{\rho k+1}}{s^{\rho k+1}}\mathscr{A}_{r}\left[\varphi'\left(\tau\right)\right],\qquad(2.21)$$

where $0 < \rho \leq 1$.

Proof. By using definition of GB transform and Atangana Baleanu in Caputo sense derivative one can have

$$\mathscr{A}_{m}\left[{}_{0}^{ABC}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{sB\left(\rho\right)}{\left(1-\rho\right)\gamma^{m}}\int_{0}^{\infty}\tau^{m-1}\exp\left(-\frac{s}{\gamma}\tau\right)\left[\int_{0}^{\tau}\varphi'\left(\xi\right)E_{\rho}\left(-\frac{\rho\left(\tau-\xi\right)^{\rho}}{1-\rho}\right)d\xi\right]d\tau.$$

$$(2.22)$$

Altering the order of integration in Eq.(2.22), gives

$$\mathscr{A}_{m}\left[{}_{0}^{ABC}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right)\right] = \frac{sB\left(\rho\right)}{\left(1-\rho\right)\gamma^{m}}\int_{0}^{\infty}\varphi'\left(\xi\right)\int_{\xi}^{\infty}\exp\left(-\frac{s}{\gamma}\tau\right)E_{\rho}\left(-\frac{\rho\left(\tau-\xi\right)^{\rho}}{1-\rho}\right)\tau^{m-1}d\tau\,d\xi.$$
(2.23)

Letting $\lambda = \tau - \xi$ in Eq.(2.23) leads to

$$\mathscr{A}_{m} \begin{bmatrix} ABC \\ 0 \end{bmatrix} = \frac{sB(\rho)}{(1-\rho)\gamma^{m}} \int_{0}^{\infty} \varphi'(\xi) \int_{0}^{\infty} \exp\left(-\frac{s}{\gamma}(\lambda+\xi)\right) E_{\rho}\left(-\frac{\rho\lambda^{\rho}}{1-\rho}\right) \\ (\lambda+\xi)^{m-1} d\lambda d\xi.$$
(2.24)

Using the binomial formula, Eq.(2.24) gives

$$\mathscr{A}_{m}\left[{}_{0}^{ABC}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right)\right] = \frac{sB\left(\rho\right)}{\left(1-\rho\right)\gamma^{m}} \int_{0}^{\infty}\varphi'\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\left[\int_{0}^{\infty}\exp\left(-\frac{s}{\gamma}\lambda\right)E_{\rho}\left(-\frac{\rho\lambda^{\rho}}{1-\rho}\right)\right]$$
$$\sum_{r=1}^{m}\binom{m-1}{r-1}\lambda^{m-r}d\lambda\left]\xi^{r-1}d\xi.$$
(2.25)

From the definition of GB transform of order one, one can get

$$\mathscr{A}_{m}\left[{}_{0}^{ABC}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right)\right] = \frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}} \int_{0}^{\infty}\varphi'\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\mathscr{A}_{1}\left[\sum_{r=1}^{m}\binom{m-1}{r-1}E_{\rho}\left(-\frac{\rho\lambda^{\rho}}{1-\rho}\right)\right]$$
$$\lambda^{m-r}\xi^{r-1}d\xi.$$
 (2.26)

By using series expansion of $E_{\rho}(.)$ [2,45], one gets

$$\mathscr{A}_{m} \begin{bmatrix} {}^{ABC}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right) \end{bmatrix} = \frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}} \sum_{r=1}^{m} \binom{m-1}{r-1} \frac{1}{s^{m-r}} \sum_{k=0}^{\infty} \left(-\frac{\rho}{1-\rho}\right)^{k} \frac{1}{\Gamma\left(\rho k+1\right)} \left(\rho k+m\right) - r \left(\frac{\gamma^{\rho k+m-r}}{s^{\rho k}} \int_{0}^{\infty} \varphi'\left(\xi\right) \exp\left(-\frac{s}{\gamma}\xi\right) \xi^{r-1} d\xi.$$

$$(2.27)$$

Using the definition of GB transform of order r, Eq.(2.27) implies

$$\mathscr{A}_{m}\left[{}_{0}^{ABC}C_{\tau}^{\rho}\left(\varphi\left(\tau\right)\right)\right] = \frac{B\left(\rho\right)}{\left(1-\rho\right)}\sum_{r=1}^{m} \binom{m-1}{r-1}\frac{1}{s^{m-r}}\sum_{k=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^{k}\frac{1}{\Gamma\left(\rho k+1\right)}\left(\rho k+m-r\right)!}\frac{\gamma^{\rho k+1}}{s^{\rho k+1}}\mathscr{A}_{r}\left[\varphi'\left(\tau\right)\right].$$

Remark 2.3. (Special Case of Eq. (2.21) for m = 1)

$$\begin{aligned} \mathscr{A}_{1} \begin{bmatrix} {}^{ABC}C^{\rho}_{\tau}\left(\varphi\left(\tau\right)\right) \end{bmatrix} &= \frac{B(\rho)}{(1-\rho)} \sum_{k=0}^{\infty} (-1)^{k} \left[\frac{\rho}{1-\rho} \left(\frac{\gamma}{s}\right)^{\rho} \right]^{k} \frac{\gamma}{s} \mathscr{A}_{1}[\varphi'(\tau)], \\ &= \frac{B(\rho)}{(1-\rho)} \left[1 + \frac{\rho}{1-\rho} \left(\frac{\gamma}{s}\right)^{\rho} \right]^{-1} \frac{\gamma}{s} \mathscr{A}_{1}[\varphi'(\tau)], \\ &= \frac{B(\rho)}{\left[1-\rho+\rho \left(\frac{\gamma}{s}\right)^{\rho} \right]} \frac{\gamma}{s} \mathscr{A}_{1}[\varphi'(\tau)], \quad 0 < \rho \le 1. \end{aligned}$$

3. Methodology

In this section, the fundamental operation of proposed semi-analytic procedure derived from amalgamation of GB transform and HPM [1,28] is presented to solve time fractional non-linear partial differential equations. Consider the generic non-linear non-homogeneous PDE with temporal fractional operator as:

$$C^{\rho}_{\tau}(\psi(\zeta,\tau)) = \Lambda(\psi(\zeta,\tau)) + \Theta(\psi(\zeta,\tau)) + f(\zeta,\tau), \qquad (3.1)$$

with the initial condition

$$\psi(\zeta, 0) = \phi(\zeta), \tag{3.2}$$

where C_{τ}^{ρ} is the fractional temporal operator. Λ and Θ specify the linear and nonlinear operators, respectively. $f(\zeta, \tau)$ is the source term and $\phi(\zeta)$ is the known initial function.

By employing the GB transform of order one on Eq.(3.1) with consideration of fractional derivatives of Caputo type, yields

$$\frac{1}{r(s,\gamma,\rho)} \left[\mathscr{A}_1 \left[\psi(\zeta,\tau) \right] - \phi(\zeta) \right] = \mathscr{A}_1 \left[M(\zeta,\tau) \right], \tag{3.3}$$

where,

$$M(\zeta,\tau) = \Lambda(\psi(\zeta,\tau)) + \Theta(\psi(\zeta,\tau)) + f(\zeta,\tau), \qquad (3.4)$$

and the function $r(s, \gamma, \rho)$ emerges in the following forms:

i. (**GB-HPM**_C) For Caputo's terminology:

$$r(s,\gamma,\rho) = \left(\frac{\gamma}{s}\right)^{\rho}.$$
(3.5)

ii. $(\mathbf{GB-HPM}_{CF})$ For CF's terminology:

$$r(s,\gamma,\rho) = 1 - \rho + \rho\left(\frac{\gamma}{s}\right).$$
(3.6)

iii. $(\mathbf{GB-HPM}_{ABC})$ For ABC's terminology:

$$r(s,\gamma,\rho) = \frac{1-\rho+\rho\left(\frac{\gamma}{s}\right)^{\rho}}{B(\rho)}.$$
(3.7)

By operating inverse of GB transform, Eq.(3.3) can be re-written as

$$\psi(\zeta,\tau) = \phi(\zeta) + \mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[M(\zeta,\tau)]].$$
(3.8)

Embedding the homotopy parameter i.e., $q \in [0,1]$ constructs the perturbation equation as

$$\psi(\zeta,\tau) = \phi(\zeta) + q\mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[M(\zeta,\tau)]].$$
(3.9)

The solution of Eq.(3.1) is presumed in the form of series expanded with respect to the small parameter q as

$$\psi(\zeta,\tau) = \sum_{l=0}^{\infty} q^l \psi_l(\zeta,\tau).$$
(3.10)

Decompose the non-linear operator $\Theta(\psi(\zeta, \tau))$ as

$$\Theta(\psi(\zeta,\tau)) = \sum_{l=0}^{\infty} \mathscr{H}_l, \qquad (3.11)$$

where the \mathscr{H}_l represent the He's polynomials [25]:

$$\mathscr{H}_{l}(\psi_{0},\psi_{1},\ldots,\psi_{l}) = \frac{1}{l!} \frac{\partial^{l}}{\partial q^{l}} \left[\Theta\left(\sum_{j=0}^{\infty} q^{j} \psi_{j}(\zeta,\tau)\right) \right]_{q=0}, \quad l = 0, 1, 2 \dots$$
(3.12)

Expression (3.12) can be expanded as:

$$\mathcal{H}_{0} = \Theta(\psi_{0}),$$

$$\mathcal{H}_{1} = \psi_{1}\Theta'(\psi_{0}),$$

$$\mathcal{H}_{2} = \psi_{2}\Theta'(\psi_{0}) + \frac{1}{2!}\psi_{1}^{2}\Theta''(\psi_{0}), \dots$$
(3.13)

By inserting the Eq.(3.10) and Eq.(3.11) into Eq.(3.9), one obtains the following system of linear equations corresponding to the components of desired series solution as:

$$\psi_{0}(\zeta,\tau) = \phi(\zeta) + \mathscr{A}_{1}^{-1}[r(s,\gamma,\rho)\mathscr{A}_{1}[f(\zeta,\tau)]],$$

$$\psi_{1}(\zeta,\tau) = \mathscr{A}_{1}^{-1}[r(s,\gamma,\rho)\mathscr{A}_{1}[\Lambda(\psi_{0}(\zeta,\tau)) + \mathscr{H}_{0}]],$$

$$\vdots$$

$$\psi_{l+1}(\zeta,\tau) = \mathscr{A}_{1}^{-1}[r(s,\gamma,\rho)\mathscr{A}_{1}[\Lambda(\psi_{l}(\zeta,\tau)) + \mathscr{H}_{l}]], \ l = 0, 1, 2, \dots$$
(3.14)

By substituting calculated components from system (3.14) into Eq.(3.10) with $q \to 1$ gives the solution of Eq.(3.1) as

$$\psi(\zeta,\tau) = \psi_0(\zeta,\tau) + \psi_1(\zeta,\tau) + \psi_2(\zeta,\tau) + \psi_3(\zeta,\tau) + \dots,$$
(3.15)

where truncated series of n components termed as n^{th} order approximate solution and closed form represents analytic solution.

4. Analysis of GB-HPM

Theorem 4.1. [2, 56] Let $\mathfrak{B} = (\mathcal{C}[\Omega], \|.\|)$ be the Banach space that incorporates $\|\psi\| = \max_{(\zeta,\tau)\in\Omega} |\psi(\zeta,\tau)|$, for all continuous functions defined on Ω . A and Θ (linear and non-linear operators in GB-HPM scheme) are Lipschitz operators, that is

$$|\Lambda[\psi] - \Lambda[\psi^*]| < \kappa_1 |\psi - \psi^*|, \qquad (4.1)$$

and

$$|\Theta[\psi] - \Theta[\psi^*]| < \kappa_2 |\psi - \psi^*|, \qquad (4.2)$$

where κ_1 and κ_2 are Lipschitz constants. Then GB-HPM solution is convergent if $\exists \kappa \in (0,1)$ such that

i.

$$\kappa = (\kappa_1 + \kappa_2) \frac{\tau^{\rho}}{\Gamma(\rho + 1)} \text{ for } GB\text{-}HPM_C \text{ solution.}$$
(4.3)

ii.

$$\kappa = (\kappa_1 + \kappa_2)(1 - \rho + \rho\tau) \text{ for GB-HPM}_{CF} \text{ solution.}$$
(4.4)

iii.

$$\kappa = (\kappa_1 + \kappa_2) \left(1 - \rho + \rho \frac{\tau^{\rho}}{\Gamma(\rho + 1)} \right) \text{ for } GB\text{-}HPM_{ABC} \text{ solution.}$$
(4.5)

Proof. Let $\Psi_m = \sum_{r=0}^m \psi_r$. To show that Ψ_m is a Cauchy sequence in \mathfrak{B} . Consider,

$$\begin{split} \|\Psi_{m} - \Psi_{n}\| &= \max_{(\zeta,\tau)\in\Omega} |\Psi_{m} - \Psi_{n}| = \max_{(\zeta,\tau)\in\Omega} \Big| \sum_{r=n+1}^{m} \psi_{r}(\zeta,\tau) \Big|, \ n = 1, 2, 3..., \\ &\leq \max_{(\zeta,\tau)\in\Omega} \Big| \mathscr{A}_{1}^{-1} \left\{ r(s,\gamma,\rho) \mathscr{A}_{1} \left[\sum_{r=n+1}^{m} \left(\Lambda(\psi_{r-1}(\zeta,\tau)) + \Theta(\psi_{r-1}(\zeta,\tau)) \right) \right] \right\} \Big|, \\ &\leq \max_{(\zeta,\tau)\in\Omega} \Big| \mathscr{A}_{1}^{-1} \left\{ r(s,\gamma,\rho) \mathscr{A}_{1} \left[\sum_{r=n}^{m-1} \left(\Lambda(\psi_{r}(\zeta,\tau)) + \Theta(\psi_{r}(\zeta,\tau)) \right) \right] \right\} \Big|, \\ &\leq \max_{(\zeta,\tau)\in\Omega} \Big| \mathscr{A}_{1}^{-1} \left\{ r(s,\gamma,\rho) \mathscr{A}_{1} \left[\Lambda(\psi_{m-1}) - \Lambda(\psi_{n-1}) + \Theta(\psi_{m-1}) - \Theta(\psi_{n-1}) \right] \right\} \Big|, \\ &\leq (4.6) \end{split}$$

i. **GB-HPM**_C solution: Using Eq.(3.5), Eq.(4.6) drives

$$\leq (\kappa_1 + \kappa_2) \frac{\tau^{\rho}}{\Gamma(\rho+1)} \parallel \psi_{m-1} - \psi_{n-1} \parallel$$

Let m = n + 1, then

$$\| \psi_{n+1} - \psi_n \| \le \kappa \| \psi_n - \psi_{n-1} \| \le \kappa^2 \| \psi_{n-1} - \psi_{n-2} \| \le \dots \le \kappa^n \| \psi_1 - \psi_0 \|,$$

where $\kappa = (\kappa_1 + \kappa_2) \frac{\tau^{\nu}}{\Gamma(\rho+1)}$. In the same manner,

$$\|\psi_{m-1} - \psi_{n-1}\| \le \|\psi_{n+1} - \psi_n\| + \|\psi_{n+2} - \psi_{n+1}\| + \ldots + \|\psi_m - \psi_{m-1}\|,$$

$$\leq \left(\kappa^{n} + \kappa^{n+1} + \ldots + \kappa^{m-1}\right) \parallel \psi_{1} - \psi_{0} \parallel,$$
$$\leq \kappa^{n} \left(\frac{1 - \kappa^{m-n}}{1 - \kappa}\right) \parallel \psi_{1} \parallel.$$

As $0 < \kappa < 1$ implies $1 - \kappa^{m-n} < 1$. Therefore

$$\|\Psi_m - \Psi_n\| \le \frac{\kappa^n}{1-\kappa} \|\psi_1\|$$

Since $\| \psi_1 \| < \infty$. $\| \Psi_m - \Psi_n \| \to_0$ when $n \to \infty$. Hence, Ψ_m is a Cauchy sequence in \mathfrak{V} which is sufficient to prove the convergence of series solution. ii. **GB-HPM**_{CF} solution: Using Eq.(3.6), Eq.(4.6) gives

$$\leq (\kappa_1 + \kappa_2) (1 - \rho + \rho \tau) \| \psi_{m-1} - \psi_{n-1} \|.$$

Further proof is analogous to the (i) part. iii.**GB-HPM**_{ABC} solution: Using Eq.(3.7), Eq.(4.6) gives

$$\leq (\kappa_1 + \kappa_2) \left(1 - \rho + \rho \frac{\tau^{\rho}}{\Gamma(\rho+1)} \right) \parallel \psi_{m-1} - \psi_{n-1} \parallel.$$

Further proof is analogous to the (i) part.

Theorem 4.2. [2, 56] Presume the conditions of Theorem 4.1 then the solution obtained with the aid of GB-HPM solution is unique whenever $0 < \kappa < 1$, where

i.

$$\kappa = (\kappa_1 + \kappa_2) \frac{\tau^{\rho}}{\Gamma(\rho + 1)} \quad for \ GB-HPM_C solution.$$
(4.7)

ii.

$$\kappa = (\kappa_1 + \kappa_2) (1 - \rho + \rho \tau) \text{ for GB-HPM}_{CF} \text{ solution.}$$
(4.8)

iii.

$$\kappa = (\kappa_1 + \kappa_2) \left(1 - \rho + \rho \frac{\tau^{\rho}}{\Gamma(\rho+1)} \right) \quad \text{for GB-HPM}_{ABC} \text{ solution.}$$
(4.9)

Proof. Eq. (3.8) gives

$$\psi\left(\zeta,\tau\right) = \phi(\zeta) + \mathscr{A}_{1}^{-1}\left[r\left(s,\gamma,\rho\right)\mathscr{A}_{1}\left[\Lambda\left(\psi\left(\zeta,\tau\right.\right)\right) + \Theta\left(\psi\left(\zeta,\tau\right)\right)\right]\right].$$

If possible, let ψ and ψ^* be the two distinct function values. Then

$$\| \psi - \psi^* \| = \max |\mathscr{A}_1^{-1} [r(s, \gamma, \rho) \mathscr{A}_1 [\Lambda(\psi(\zeta, \tau)) + \Theta(\psi(\zeta, \tau))]] | - \mathscr{A}_1^{-1} [r(s, \gamma, \rho) \mathscr{A}_1 [\Lambda(\psi^*(\zeta, \tau)) + \Theta(\psi^*(\zeta, \tau))]] |, \leq \max |\mathscr{A}_1^{-1} [r(s, \gamma, \rho) \mathscr{A}_1 [\kappa_1 | \psi - \psi^*|]] | + \max |\mathscr{A}_1^{-1} [r(s, \gamma, \rho) \mathscr{A}_1 [\kappa_2 | \psi - \psi^*|]] |, \leq (\kappa_1 + \kappa_2) \mathscr{A}_1^{-1} [r(s, \gamma, \rho) \mathscr{A}_1 \| \psi - \psi^* \|],$$

$$(4.10)$$

i. (**GB-HPM**_C) For Caputo's terminology Eq.(4.10) reduces to

$$= (\kappa_1 + \kappa_2) \mathscr{A}_1^{-1} \left[\left(\frac{\gamma}{s} \right)^{\rho} \mathscr{A}_1 \parallel \psi - \psi^* \parallel \right],$$

$$\| \psi - \psi^* \| \le (\kappa_1 + \kappa_2) \frac{\tau^{\rho}}{\Gamma(\rho + 1)} \| \psi - \psi^* \|,$$

(1 - \kappa) \| \psi - \psi^* \| \le 0. (4.11)

Since $0 < \kappa = (\kappa_1 + \kappa_2) \frac{\tau^{\rho}}{\Gamma(\rho+1)} < 1$, then $\parallel \psi - \psi^* \parallel = 0$ which implies $\psi = \psi^*$. This proves the uniqueness of GB-HPM_C solution.

ii. (GB-HPM $_{CF}$) For CF's terminology:

Using $r(s, \gamma, \rho) = 1 - \rho + \rho\left(\frac{\gamma}{s}\right)$ uniqueness condition for GB-HPM_{CF} solution can be derived as part (i).

iii. (**GB-HPM**_{ABC}) For ABC's terminology: Using $r(s, \gamma, \rho) = \frac{1-\rho+\rho(\frac{\gamma}{s})^{\rho}}{B(\rho)}$ uniqueness condition for GB-HPM_{ABC} solution can

5. Numerical Experiments

This section provides the demonstration of developed method with error analysis for considered examples.

Example 5.1. Consider the time fractional linear SG equation as follows:

$$C^{\rho}_{\tau}\psi + i\psi_{\zeta\zeta} = 0, \quad 0 < \rho \le 1, \tag{5.1}$$

with initial condition

$$\psi\left(\zeta,0\right) = e^{3i\zeta}.\tag{5.2}$$

The exact solution for Eq.(5.1) is established as $\psi(\zeta, \tau) = e^{3i(\zeta+3\tau)}$ for $\rho = 1$. Proceeding as developed methodology with choice of $\Lambda(\psi(\zeta, \tau)) = -i\psi_{\zeta\zeta}$ and $\Theta(\psi(\zeta, \tau)) = 0$ yields the recurrent connection in the following form

$$\psi_0 = \psi(\zeta, 0),$$

$$\psi_{l+1}(\zeta, \tau) = -i\mathscr{A}_1^{-1}[r(s, \gamma, \rho)\mathscr{A}_1[(\psi_l)_{\zeta\zeta}]], l = 0, 1, 2, \dots$$
(5.3)

i. **GB-HPM**_C solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.5) for Caputo's terminology, the solution of the above system of successive iterative components can be written in the form

$$\psi_0(\zeta,\tau) = e^{3i\zeta},$$

$$\psi_1(\zeta,\tau) = \frac{9i\tau^{\rho}}{\Gamma(1+\rho)}e^{3i\zeta},$$

$$\psi_2(\zeta,\tau) = \frac{(9i\tau^{\rho})^2}{\Gamma(1+2\rho)}e^{3i\zeta},$$

$$\psi_3(\zeta,\tau) = \frac{(9i\tau^{\rho})^3}{\Gamma(1+3\rho)}e^{3i\zeta}, \dots$$

Consequently, the series of the $GB-HPM_C$ solution is represented as

$$\psi^{c}(\zeta,\tau) = e^{3i\zeta} + \frac{9i\tau^{\rho}}{\Gamma(1+\rho)}e^{3i\zeta} + \frac{(9i\tau^{\rho})^{2}}{\Gamma(1+2\rho)}e^{3i\zeta} + \frac{(9i\tau^{\rho})^{3}}{\Gamma(1+3\rho)}e^{3i\zeta} + \dots , \quad (5.4)$$
$$= e^{3i\zeta}E_{n}(9it^{\rho}). \quad (5.5)$$

where $E_n(.)$ is Mittag-Leffler function.

ii. **GB-HPM**_{CF} solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.6) for CF's terminology, the solution of the above system of successive iterative components can be written in the form

$$\begin{split} \psi_{0}\left(\zeta,\tau\right) &= e^{3i\zeta},\\ \psi_{1}\left(\zeta,\tau\right) &= 9ie^{3i\zeta}\left[\rho\left(\tau-1\right)+1\right],\\ \psi_{2}\left(\zeta,\tau\right) &= \frac{\left(9i\right)^{2}}{2!}e^{3i\zeta}\left[\rho^{2}\left(\tau^{2}-4\tau+2\right)+4\rho\left(\tau-1\right)+2\right],\\ \psi_{3}\left(\zeta,\tau\right) &= \frac{\left(9i\right)^{3}}{3!}e^{3i\zeta}\left[\rho^{3}\left(\tau^{3}-9\tau^{2}+18\tau-6\right)+\rho^{2}\left(9\tau^{2}-36\tau+18\right)+18\rho\left(\tau-1\right)+6\right],\dots \end{split}$$

Consequently, the series of the GB-HPM $_{CF}$ solution is represented as

$$\psi^{CF}(\zeta,\tau) = e^{3i\zeta} \left\{ 1 + 9i\left[\rho\left(\tau-1\right)+1\right] + \frac{(9i)^2}{2!} \left[\rho^2\left(\tau^2-4\tau+2\right)+4\rho\left(\tau-1\right)+2\right] + \frac{(9i)^3}{3!} \left[\rho^3\left(\tau^3-9\tau^2+18\tau-6\right)+\rho^2\left(9\tau^2-36\tau+18\right)+18\rho\left(\tau-1\right)+6\right], \dots \right\}.$$
(5.6)

iii. **GB-HPM**_{ABC} solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.7) for ABC's terminology, the solution of the above system of successive iterative components can be written in the form

$$\begin{split} \psi_{0}\left(\zeta,\tau\right) &= e^{3i\zeta},\\ \psi_{1}\left(\zeta,\tau\right) &= \frac{9ie^{3i\zeta}}{\Gamma\left(\rho+1\right)} \left[\rho\tau^{\rho} - \left(\rho-1\right)\Gamma\left(\rho+1\right)\right],\\ \psi_{2}\left(\zeta,\tau\right) &= e^{3i\zeta} \left(\frac{162\rho^{2}\tau^{\rho}}{\Gamma\left(\rho+1\right)} - \frac{162\rho\tau^{\rho}}{\Gamma\left(\rho+1\right)} - \frac{81\rho^{2}\tau^{2\rho}}{\Gamma\left(2\rho+1\right)} - 81 - 81\rho^{2} + 162\rho\right),\dots \,. \end{split}$$

Consequently, the series of the GB-HPM $_{ABC}$ solution is represented as

$$\psi^{ABC}(\zeta,\tau) = e^{3i\zeta} \left\{ 1 + \frac{9ie^{3i\zeta}}{\Gamma(\rho+1)} \left[\rho \tau^{\rho} - (\rho-1) \Gamma(\rho+1) \right] + \left(\frac{162\rho^2 \tau^{\rho}}{\Gamma(\rho+1)} - \frac{162\rho \tau^{\rho}}{\Gamma(\rho+1)} - \frac{81\rho^2 \tau^{2\rho}}{\Gamma(2\rho+1)} - 81 - 81\rho^2 + 162\rho \right) + \dots \right\}.$$
(5.7)

By considering $\rho = 1$, the GB-HPM_{ABC}, GB-HPM_{CF} and GB-HPM_{ABC} solutions become

$$\psi(\zeta,\tau) = e^{3i\zeta} \sum_{n=0}^{\infty} \frac{(9i\tau)^n}{n!} = e^{3i(\zeta+3\tau)}.$$
(5.8)

which represents the exact solution for the linear SG equation (5.1) with classical derivative.

Example 5.2. Consider $\beta = -\frac{1}{2}$, $\nu(\zeta) = 0$ and $\lambda = -1$ then time fractional non-linear SG equation (1.1) becomes

$$iC_{\tau}^{\rho}\psi + \frac{1}{2}\psi_{\zeta\zeta} + |\psi|^2\psi = 0, \quad 0 < \rho \le 1,$$
(5.9)

with initial condition

$$\psi\left(\zeta,0\right) = e^{i\zeta}.\tag{5.10}$$

The exact solution for Eq.(5.9) is established as $\psi(\zeta, \tau) = e^{i(\zeta + \tau/2)}$ in the case of $\rho = 1$. Proceeding as developed methodology with choice of $\Lambda(\psi(\zeta, \tau)) = i\frac{1}{2}\psi_{\zeta\zeta}$ and $\Theta(\psi(\zeta, \tau)) = i|\psi|^2\psi$ yields the recurrent connection in the following form

$$\psi_{0} = \psi\left(\zeta, 0\right),$$

$$\psi_{l+1}\left(\zeta, \tau\right) = i\mathscr{A}_{1}^{-1}\left[r\left(s, \gamma, \rho\right)\mathscr{A}_{1}\left[\left(\frac{1}{2}\psi_{l}\right)_{\zeta\zeta} + \mathscr{H}_{l}\right]\right], \ l = 0, 1, 2, \dots$$
(5.11)

where non-linear term written as $\Theta(\psi(\zeta, \tau)) = i |\psi|^2 \psi = i\psi^2 \bar{\psi}$ is represented in terms of He's polynomials (3.12) in following form:

$$\mathscr{H}_{0} = i\psi_{0}^{2}\bar{\psi}_{0}, \mathscr{H}_{1} = i\psi_{0}\left(\psi_{0}\bar{\psi}_{1} + 2\bar{\psi}_{0}\psi_{1}\right), \mathscr{H}_{2} = i[2\psi_{0}\left(\bar{\psi}_{0}\psi_{2} + \psi_{1}\bar{\psi}_{1}\right) + \psi_{1}^{2}\psi_{0} + \psi_{0}^{2}\bar{\psi}_{2}], \dots$$
(5.12)

i. **GB-HPM**_C solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.5) for Caputo's terminology the series solution of 3^{rd} -order approximation is obtained as following:

$$\begin{split} \psi_0\left(\zeta,\tau\right) &= e^{i\zeta},\\ \psi_1\left(\zeta,\tau\right) &= \frac{i}{2} \frac{\tau^{\rho}}{\Gamma\left(\rho+1\right)} e^{i\zeta},\\ \psi_2\left(\zeta,\tau\right) &= -\frac{1}{4} \frac{\tau^{2\rho}}{\Gamma\left(2\rho+1\right)} e^{i\zeta},\\ \psi_3\left(\zeta,\tau\right) &= -\frac{i}{8} \frac{\tau^{3\rho}}{\Gamma\left(3\rho+1\right)} \left(5 - 2\frac{\Gamma\left(2\rho+1\right)}{\left(\Gamma\left(\rho+1\right)^2\right)}\right) e^{i\zeta},\dots \end{split}$$

Similarly, one can get more components for different values of l in Eq.(5.11). Then the GB-HPM_C solution can be written in following series

$$\psi^{C}(\zeta,\tau) = e^{i\zeta} \left[1 + \frac{i}{2} \frac{\tau^{\rho}}{\Gamma(\rho+1)} - \frac{1}{4} \frac{\tau^{2\rho}}{\Gamma(2\rho+1)} - \frac{i}{8} \frac{\tau^{3\rho}}{\Gamma(3\rho+1)} \left(5 - 2\frac{\Gamma(2\rho+1)}{(\Gamma(\rho+1)^{2}} \right) + \dots \right]$$
(5.13)

ii. **GB-HPM**_{CF} solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.6) for CF's terminology the series solution of 3^{rd} -order approximation is obtained as following:

$$\begin{split} \psi_0\left(\zeta,\tau\right) &= e^{i\zeta},\\ \psi_1\left(\zeta,\tau\right) &= ie^{i\zeta}\left(\rho\left(\tau-1\right)+1\right),\\ \psi_2\left(\zeta,\tau\right) &= -\frac{i}{2}e^{i\zeta}\left[\rho^2(\tau^2-4\tau+2)+4\rho\left(\tau-1\right)+2\right],\\ \psi_3\left(\zeta,\tau\right) &= -\frac{i}{6}e^{i\zeta}\left[\rho^3\left(\tau^3-21\tau^2+54\tau+18\right)+\rho^2\left(2\tau^2-108\tau+54\right)+54\rho\left(\tau-1\right)+18\right],\dots \end{split}$$

Similarly, one can get more components for different values of l in Eq.(5.11). Then the GB-HPM_{CF} solution can be written in following series

$$\psi^{CF}(\zeta,\tau) = e^{i\zeta} \left\{ 1 + i\left[\rho\left(\tau-1\right)+1\right] - \frac{i}{2}\left[\rho^{2}\left(\tau^{2}-4\tau+2\right)+4\rho\left(\tau-1\right)+2\right] - \frac{i}{6}e^{i\zeta}\left[\rho^{3}\left(\tau^{3}-21\tau^{2}+54\tau+18\right)+\rho^{2}\left(2\tau^{2}-108\tau+54\right)+54\rho\left(\tau-1\right)+18\tau+18\right] + \dots \right\}.$$
(5.14)

iii. **GB-HPM**_{ABC} solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.7) for ABC's terminology the approximate series solution is obtained as following:

$$\begin{split} \psi_{0}(\zeta,\tau) &= e^{i\zeta}, \\ \psi_{1}(\zeta,\tau) &= \frac{ie^{i\zeta}}{2\Gamma(\rho+1)} \left[\rho \tau^{\rho} - (\rho-1) \Gamma(\rho+1) \right], \\ \psi_{2}(\zeta,\tau) &= e^{i\zeta} \left(-\frac{1}{4} - \frac{\rho^{2}}{4} + \frac{\rho}{2} + \frac{\rho^{2} \tau^{\rho}}{2\Gamma(\rho+1)} - \frac{\rho \tau^{\rho}}{2\Gamma(\rho+1)} - \frac{\rho^{2} \tau^{2\rho}}{4\Gamma(2\rho+1)} \right), \dots \end{split}$$

Similarly, one can get more components for different values of l in Eq.(5.11). Then the GB-HPM_{ABC} solution can be written in following series

$$\psi^{ABC}(\zeta,\tau) = e^{i\zeta} \left\{ 1 + \frac{ie^{i\zeta}}{2\Gamma(\rho+1)} \left[\rho\tau^{\rho} - (\rho-1)\Gamma(\rho+1) \right] + \left(-\frac{1}{4} - \frac{\rho^2}{4} + \frac{\rho}{2} + \frac{\rho^2\tau^{\rho}}{2\Gamma(\rho+1)} - \frac{\rho\tau^{\rho}}{2\Gamma(\rho+1)} - \frac{\rho^2\tau^{2\rho}}{4\Gamma(2\rho+1)} \right) + \dots \right\}.$$
(5.15)

As $\rho \rightarrow 1$, the GB-HPM_C, GB-HPM_{CF} and GB-HPM_{ABC} solutions become

$$\psi\left(\zeta,\tau\right) = e^{i\zeta} \sum_{n=0}^{\infty} \frac{\left(\frac{i\tau}{2}\right)^n}{n!} = e^{i\left(\zeta+\frac{\tau}{2}\right)}.$$
(5.16)

which represents the exact solution for the non-linear SG equation (5.9) with classical derivatives.

Example 5.3. Consider $\beta = -\frac{1}{2}$, $\nu(\zeta) = -\cos^2(\zeta)$ and $\lambda = 1$ then time fractional non-linear SG equation (1.1) becomes

$$iC_{\tau}^{\rho}\psi + \frac{1}{2}\psi_{\zeta\zeta} - \cos^2(\zeta)\psi - |\psi|^2\psi = 0, \quad 0 < \rho \le 1,$$
(5.17)

with initial condition

$$\psi\left(\zeta,0\right) = \sin(\zeta).\tag{5.18}$$

The exact solution for Eq.(5.17) is given as $\psi(\zeta, \tau) = e^{-3i\tau/2} sin(\zeta)$ when $\rho = 1$.

Proceeding as developed methodology with choice of $\Lambda(\psi(\zeta, \tau)) = i\left(\frac{1}{2}\psi_{\zeta\zeta} - \cos^2(\zeta)\psi\right)$ and $\Theta(\psi(\zeta, \tau)) = -i|\psi|^2\psi$ yields the recurrent connection in the following form

$$\psi_0 = \psi\left(\zeta, 0\right),$$

$$\psi_{l+1}(\zeta,\tau) = i\mathscr{A}_1^{-1} \left[r\left(s,\gamma,\rho\right) \mathscr{A}_1 \left[\left(\frac{1}{2}\psi_l\right)_{\zeta\zeta} - \cos^2\left(\zeta\right)\psi_l - \mathscr{H}_l \right] \right], \ l = 0, 1, 2 \dots$$

$$(5.19)$$

where in order to deal with existing non-linearity $\Theta(\psi(\zeta, \tau)) = -i |\psi|^2 \psi = -i \psi^2 \bar{\psi}$, He's polynomials (3.12) are used in following form:

$$\mathscr{H}_{0} = -i\psi_{0}^{2}\bar{\psi}_{0}, \mathscr{H}_{1} = -i\psi_{0}\left(\psi_{0}\bar{\psi}_{1} + 2\bar{\psi}_{0}\psi_{1}\right), \mathscr{H}_{2} = -i[2\psi_{0}\left(\bar{\psi}_{0}\psi_{2} + \psi_{1}\bar{\psi}_{1}\right) + \psi_{1}^{2}\psi_{0} + \psi_{0}^{2}\bar{\psi}_{2}], \dots$$
(5.20)

i. **GB-HPM**_C solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.5) for Caputo's terminology the series solution of 3^{rd} -order approximation is obtained as following:

$$\begin{split} \psi_{0}(\zeta,\tau) &= \sin(\zeta), \\ \psi_{1}(\zeta,\tau) &= -\frac{3i\tau^{\rho}\sin(\zeta)}{2\Gamma(\rho+1)}, \\ \psi_{2}(\zeta,\tau) &= -\frac{9\tau^{2\rho}\sin(\zeta)}{4\Gamma(2\rho+1)}, \\ \psi_{3}(\zeta,\tau) &= \frac{9i\tau^{3\rho}sin(\zeta)[4(\Gamma(\rho+1))^{2}sin^{2}(\zeta) - 2\Gamma(2\rho+1)sin^{2}(\zeta) + 3(\Gamma(\rho+1))^{2}]}{8(\Gamma(\rho+1))^{2}\Gamma(3\rho+1)}, \dots \end{split}$$

Similarly, one can get more components for different values of l in Eq.(5.19). Then the GB-HPM_C solution can be written in following series

$$\psi^{C}(\zeta,\tau) = \sin(\zeta) \left\{ 1 - \frac{3i\tau^{\rho}}{2\Gamma(\rho+1)} - \frac{9\tau^{2\rho}}{4\Gamma(2\rho+1)} - \frac{9\tau^{2\rho}}{4\Gamma(2\rho+1)} \frac{9i\tau^{3\rho}[4(\Gamma(\rho+1))^{2}sin^{2}(\zeta) - 2\Gamma(2\rho+1)sin^{2}(\zeta) + 3(\Gamma(\rho+1))^{2}]}{8(\Gamma(\rho+1))^{2}\Gamma(3\rho+1)}, \dots \right\}.$$
(5.21)

ii. **GB-HPM**_{CF} solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.6) for CF's terminology the series solution of 3^{rd} -order approximation is obtained as following:

$$\psi_{0}(\zeta,\tau) = \sin(\zeta),$$

$$\psi_{1}(\zeta,\tau) = -\frac{3i\sin(\zeta)}{2} \left[\rho(\tau-1)+1\right],$$

$$\psi_{2}(\zeta,\tau) = -\frac{9\sin(\zeta)}{8} \left[\rho^{2}(\tau^{2}-4\tau+2)+4\rho(\tau-1)+2\right],\dots$$

Similarly, one can get more components for different values of l in Eq.(5.19). Then the GB-HPM_{CF} solution can be written in following series

$$\psi^{CF}(\zeta,\tau) = \sin(\zeta) \left\{ 1 - \frac{3i}{2} [\rho(\tau-1)+1] - \frac{9}{8} [\rho^2(\tau^2 - 4\tau + 2) + 4\rho(\tau-1) + 2] + \dots \right\}.$$
(5.22)

iii. **GB-HPM**_{ABC} solution: Using explicit form of $r(s, \gamma, \rho)$ described in Eq.(3.7) for ABC's terminology the approximate series solution is obtained as following:

$$\begin{split} \psi_{0}\left(\zeta,\tau\right) &= \sin(\zeta), \\ \psi_{1}\left(\zeta,\tau\right) &= -\frac{3i\sin\left(\zeta\right)}{2\Gamma\left(\rho+1\right)}\left(\rho\tau^{\rho} - \left(\rho-1\right)\Gamma\left(\rho+1\right)\right), \\ \psi_{2}\left(\zeta,\tau\right) &= -\frac{9\sin\left(\zeta\right)}{4\Gamma\left(\rho+1\right)\Gamma\left(2\rho+1\right)} \left(\Gamma\left(\rho+1\right)\Gamma\left(2\rho+1\right) - 2\rho\Gamma\left(\rho+1\right)\Gamma\left(2\rho+1\right) \\ &+ 2\rho\tau^{\rho}\Gamma\left(2\rho+1\right) + \rho^{2}\Gamma\left(\rho+1\right)\Gamma\left(2\rho+1\right) - 2\rho^{2}\tau^{\rho}\Gamma\left(2\rho+1\right) + \rho^{2}\tau^{2\rho}\Gamma\left(\rho+1\right)\right), \dots \,. \end{split}$$

Similarly, one can get more components for different values of l in Eq.(5.19). Then the GB-HPM_{ABC} solution can be written in following series

$$\psi^{ABC}(\zeta,\tau) = \sin(\zeta) \Biggl\{ 1 - \frac{3i\sin(\zeta)}{2\Gamma(\rho+1)} \left(\rho\tau^{\rho} - (\rho-1)\Gamma(\rho+1)\right) - \frac{9\sin(\zeta)}{4\Gamma(\rho+1)\Gamma(2\rho+1)} \\ \left(\Gamma(\rho+1)\Gamma(2\rho+1) - 2\rho\Gamma(\rho+1)\Gamma(2\rho+1) + 2\rho\tau^{\rho}\Gamma(2\rho+1) + \rho^{2}\tau^{2}\Gamma(\rho+1)\right) + \rho^{2}\Gamma(\rho+1)\Gamma(2\rho+1) - 2\rho^{2}\tau^{\rho}\Gamma(2\rho+1) + \rho^{2}\tau^{2}\Gamma(\rho+1)\right) + \dots \Biggr\}.$$
(5.23)

Considering $\rho=1$, the GB-HPM $_{C}$, GB-HPM $_{CF}$ and GB-HPM $_{ABC}$ solutions gives

$$\psi\left(\zeta,\tau\right) = \sin(\zeta) \sum_{n=0}^{\infty} \frac{\left(\frac{-3i\tau}{2}\right)^n}{n!} = \sin(\zeta) e^{-\frac{3i\tau}{2}}.$$
(5.24)

which represents the exact solution for the non-linear SG equation (5.17) with classical derivative.

6. Results and Discussions

The advantageous feature about this scheme is that, unlike the variables separation strategy and most of numerical schemes, which requires the use of both initial and boundary conditions, it solves for an analytic solution by using only the initial condition. In Example 1 the GB-HPM solution is found in accordance with existing solutions obtained using Sumudu transform iterative method (STIM) [31] and differential transform method (DTM) [23] for Caputo's derivative. In Figure 1, two-dimensional graphs are provided to compare approximate real and imaginary part of the GB-HPM solution for Caputo, CF and ABC terminologies when $\rho = 0.5, 0.7, 0.9$ and 1 at $\tau = 0.5$ for Example 5.1. Resulting in waves with higher amplitude displays the impact of variation in fractional order. Table 1. describes the performance of present scheme in terms of absolute error for approximate solution of Example 5.1. It demonstrates the efficiency of suggested method with maximum error of order 10^{-5} .

Figure 2 depicts the graphical presentation of solutions obtained for distinct fractional order values ($\rho = 0.5, 0.7, 0.9$ and 1) of Example 5.2 with Caputo, CF and ABC operators in computational domain $-5 \leq \zeta \leq 5$ at $\tau = 0.5$. Figure 2 demonstrates that each of the subfigures behave almost in identical manner, are in comparable nature and have good precision alignment. For Example 5.2, the absolute errors of present scheme are compared to the existing results of modified generalized Mittag-Leffler function method (MGMLFM) [10] which verify the superiority of present scheme for non-linear differential equations. Table 3 exhibits the fast convergence of approximate series solutions to exact solution with evaluation of more components of series solutions.

The wave form behaviour of numerical results obtained for Example 5.3 with Caputo, CF and ABC derivatives at different values of fractional order ($\rho = 0.5, 0.7, 0.9$ and 1) is presented in Figure 3 for limited spatial domain of $-5 \le \zeta \le 5$ at $\tau = 0.5$. In Table 4, the tabulated results of Example 5.3 again indicates the better performance of present scheme in comparison of MGMLFM [10] results. Table 5 ensures that the required precision can be attained through the assessment of additional components of series solutions.

In present study it is observed that however the CF and ABC operators with nonsingular kernels are considered extensions of Caputo operator but due to their high sensitivity with respect to the small change in fractional order, Caputo operator still holds its legacy. GB-HPM_C solutions are found more consistent with respect to the small variations in ρ . For $\rho = 1$ GB-HPM solution for Caputo, CF and ABC operators coincides and results in a rapidly convergent series solution. Finding the closed form of a series solution allows one to obtain the desired exact solution for considered problem.

7. Conclusion

In this study, approximate semi-analytic solutions for time fractional linear and non-linear Schrödinger equations are investigated using three types of fractional derivatives: Caputo, Caputo-Fabrizio (CF), and Atangana-Baleanu-Caputo (ABC). These solutions are obtained through the recently proposed GB transform followed by the HPM. The study establishes the application of the GB transform to the considered fractional derivatives and demonstrates the GB-HPM procedure for general time fractional non-linear equations.

Compared to the traditional HPM scheme, the suggested GB-HPM scheme streamlines the sequence estimation process by avoiding the need to compute fractional derivatives and fractional integrals in the recursive mechanism. Numerical results of the GB-HPM solutions are displayed in figures and tables for different values of the parameter ρ across all three fractional terminologies. Graphical simulations help in understanding the impact of Caputo, CF, and ABC operators on the complex internal structure of wave propagation.

A comparative error analysis of the GB-HPM solution with previously published results reflects the supremacy and consistency of the proposed scheme. The numerical results demonstrate high accuracy, even when only a few components of the series solution are considered. The straightforward GB-HPM scheme is effective in resolving non-linear fractional differential equations and has enormous potential for identifying hidden mechanisms in dynamic mathematical systems.

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ζ	au	$\operatorname{GB-HPM}_C$	$\operatorname{GB-HPM}_{CF}$	$\operatorname{GB-HPM}_{ABC}$	Exact
-5	0.1	4.7123×10^{-16}	4.7123×10^{-16}	4.7123×10^{-16}	1.0000
	0.3	$2.2250{\times}10^{-11}$	$2.2250{\times}10^{-11}$	$2.2250{\times}10^{-11}$	1.0000
	0.6	$4.5730{\times}10^{-5}$	$4.5730{ imes}10^{-5}$	$4.5730{ imes}10^{-5}$	1.0000
-1	0.1	1.1102×10^{-16}	1.1102×10^{-16}	1.1102×10^{-16}	1.0000
	0.3	$2.2250{\times}10^{-11}$	$2.2250{\times}10^{-11}$	$2.2250{\times}10^{-11}$	1.0000
	0.6	$4.5730{\times}10^{-5}$	$4.5730{ imes}10^{-5}$	$4.5730{ imes}10^{-5}$	1.0000
1	0.1	1.1102×10^{-16}	1.1102×10^{-16}	1.1102×10^{-16}	1.0000
	0.3	2.2249×10^{-11}	2.2249×10^{-11}	2.2249×10^{-11}	1.0000
	0.6	$4.5730{\times}10^{-5}$	$4.5730{ imes}10^{-5}$	$4.5730{ imes}10^{-5}$	1.0000
5	0.1	2.9894×10^{-16}	2.9894×10^{-16}	2.9894×10^{-16}	1.0000
	0.3	$2.2250{\times}10^{-11}$	$2.2250{\times}10^{-11}$	$2.2250{\times}10^{-11}$	1.0000
	0.6	$4.5730{ imes}10^{-5}$	$4.5730{ imes}10^{-5}$	$4.5730{ imes}10^{-5}$	1.0000

Table 1. Computed absolute errors for GB-HPM $_{20^{th}-order}$ solutions for Example 5.1 when $\rho{=}1$

Table 2. Comparison of the absolute errors of obtained solutions using GB-HPM_{3rd-order} and modified generalized Mittag-Leffler function method (MGMLFM) for Example 5.2 when $\rho = 1$

ζ	au	$\operatorname{GB-HPM}_C$	$GB-HPM_{CF}$	$GB-HPM_{ABC}$	MGMLFM [10]
-5	0.01	2.6042×10^{-11}	2.6042×10^{-11}	$2.6042{\times}10^{-11}$	8.3335×10^{-8}
	0.03	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.2506{\times}10^{-6}$
	0.06	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	1.8020×10^{-5}
-1	0.01	$2.6042{\times}10^{-11}$	$2.6042{\times}10^{-11}$	$2.6042{\times}10^{-11}$	8.3335×10^{-8}
	0.03	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.2506{\times}10^{-6}$
	0.06	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	1.8020×10^{-5}
1	0.01	$2.6042{\times}10^{-11}$	2.6042×10^{-11}	$2.6042{\times}10^{-11}$	8.3335×10^{-8}
	0.03	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.2506{\times}10^{-6}$
	0.06	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	1.8020×10^{-5}
5	0.01	$2.6042{\times}10^{-11}$	2.6042×10^{-11}	$2.6042{\times}10^{-11}$	8.3335×10^{-8}
	0.03	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.1094{ imes}10^{-9}$	$2.2506{\times}10^{-6}$
	0.06	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	$3.3750{ imes}10^{-8}$	1.8020×10^{-5}

Table 3. Comparison of the absolute errors of obtained GB-HPM solutions for Example 5.2 when $\rho = 1$ and $\zeta = 5$.

	3^{rd} -order solutions			5^{th} -order solutions		
au	$\operatorname{GB-HPM}_C$	$\operatorname{GB-HPM}_{CF}$	$GB-HPM_{ABC}$	$\operatorname{GB-HPM}_C$	$\operatorname{GB-HPM}_{CF}$	$\mathrm{GB}\text{-}\mathrm{HPM}_{ABC}$
0.1	2.6041×10^{-7}	2.6041×10^{-7}	2.6041×10^{-7}	2.1701×10^{-11}	2.1701×10^{-11}	2.1701×10^{-11}
0.2	4.1661×10^{-6}	$4.1661{\times}10^{-6}$	4.1661×10^{-6}	1.3888×10^{-9}	1.3888×10^{-9}	1.3888×10^{-9}
0.3	2.1087×10^{-5}	2.1087×10^{-5}	2.1087×10^{-5}	1.5818×10^{-8}	1.5818×10^{-8}	$1.5818{ imes}10^{-8}$
0.4	6.6631×10^{-5}	6.6631×10^{-5}	6.6631×10^{-5}	8.8862×10^{-8}	8.8862×10^{-8}	8.8862×10^{-8}
0.5	$1.6262{ imes}10^{-4}$	$1.6262{ imes}10^{-4}$	1.6262×10^{-4}	3.3892×10^{-7}	$3.3892{ imes}10^{-7}$	$3.3892{ imes}10^{-7}$
0.6	$3.3710{ imes}10^{-4}$	$3.3710{ imes}10^{-4}$	3.3710×10^{-4}	1.0118×10^{-6}	1.0118×10^{-6}	1.0118×10^{-6}
0.7	6.2424×10^{-4}	6.2424×10^{-4}	6.2424×10^{-4}	2.5508×10^{-6}	$2.5508{\times}10^{-6}$	$2.5508{\times}10^{-6}$
0.8	1.0644×10^{-3}	1.0644×10^{-3}	1.0644×10^{-3}	5.6819×10^{-6}	$5.6819 imes 10^{-6}$	$5.6819{ imes}10^{-6}$
0.9	1.7040×10^{-3}	1.7040×10^{-3}	1.7040×10^{-3}	1.1515×10^{-5}	1.1515×10^{-5}	$1.1515{\times}10^{-5}$
1	2.5955×10^{-3}	$2.5955{\times}10^{-3}$	2.5955×10^{-3}	2.1660×10^{-5}	$2.1660{\times}10^{-5}$	$2.1660{\times}10^{-5}$

Table 4. Comparison of the absolute errors of obtained solutions using GB-HPM₃ $_{rd-order}$ and modified generalized Mittag-Leffler function method (MGMLFM) for Example 5.3 when $\rho = 1$

ζ	au	$\operatorname{GB-HPM}_C$	$\operatorname{GB-HPM}_{CF}$	$\operatorname{GB-HPM}_{ABC}$	$\mathrm{MGMLFM}\ [10]$
-5	0.01	$2.0227{ imes}10^{-9}$	$2.0227{ imes}10^{-9}$	$2.0227{ imes}10^{-9}$	6.6173×10^{-7}
	0.03	1.6384×10^{-7}	1.6384×10^{-7}	1.6384×10^{-7}	1.7956×10^{-5}
	0.06	2.6212×10^{-6}	2.6212×10^{-6}	$2.6212{ imes}10^{-6}$	1.4613×10^{-4}
-1	0.01	1.7750×10^{-9}	1.7750×10^{-9}	1.7750×10^{-9}	4.4717×10^{-7}
	0.03	1.4377×10^{-7}	1.4377×10^{-7}	1.4377×10^{-7}	1.2139×10^{-5}
	0.06	2.3001×10^{-6}	2.3001×10^{-6}	$2.3001{\times}10^{-6}$	9.8906×10^{-5}
1	0.01	1.7750×10^{-9}	1.7750×10^{-9}	1.7750×10^{-9}	4.4717×10^{-7}
	0.03	1.4377×10^{-7}	1.4377×10^{-7}	1.4377×10^{-7}	1.1395×10^{-5}
	0.06	$2.3001{\times}10^{-6}$	$2.3001{\times}10^{-6}$	$2.3001{\times}10^{-6}$	9.8906×10^{-5}
5	0.01	2.0227×10^{-9}	2.0227×10^{-9}	$2.0227{ imes}10^{-9}$	6.6173×10^{-7}
	0.03	$1.6384{ imes}10^{-7}$	$1.6384{ imes}10^{-7}$	$1.6384{ imes}10^{-7}$	1.7956×10^{-5}
	0.06	2.6212×10^{-6}	2.6212×10^{-6}	2.6212×10^{-6}	1.4613×10^{-4}

	3	rd -order solutio	ns	5^{th} -order solutions		
au	$\operatorname{GB-HPM}_C$	$\operatorname{GB-HPM}_{CF}$	$\operatorname{GB-HPM}_{ABC}$	$\operatorname{GB-HPM}_C$	$\operatorname{GB-HPM}_{CF}$	$\mathrm{GB}\text{-}\mathrm{HPM}_{ABC}$
0.1	2.0221×10^{-5}	2.0221×10^{-5}	2.0221×10^{-5}	1.5168×10^{-8}	1.5168×10^{-8}	$1.5168{ imes}10^{-8}$
0.2	3.2325×10^{-4}	3.2325×10^{-4}	3.2325×10^{-4}	9.7024×10^{-7}	9.7024×10^{-7}	9.7024×10^{-7}
0.3	1.6340×10^{-3}	1.6340×10^{-3}	1.6340×10^{-3}	1.1042×10^{-5}	1.1042×10^{-5}	1.1042×10^{-5}
0.4	5.1534×10^{-3}	5.1534×10^{-3}	5.1534×10^{-3}	6.1967×10^{-5}	$6.1967{ imes}10^{-5}$	$6.1967{ imes}10^{-5}$
0.5	$1.2548{ imes}10^{-2}$	1.2548×10^{-2}	1.2548×10^{-2}	2.3602×10^{-4}	2.3602×10^{-4}	$2.3602{ imes}10^{-4}$
0.6	$2.5933{ imes}10^{-2}$	$2.5933{ imes}10^{-2}$	2.5933×10^{-2}	7.0343×10^{-4}	7.0343×10^{-4}	7.0343×10^{-4}
0.7	$4.7858{ imes}10^{-2}$	4.7858×10^{-2}	4.7858×10^{-2}	1.7698×10^{-3}	1.7698×10^{-3}	$1.7698{ imes}10^{-3}$
0.8	8.1279×10^{-2}	8.1279×10^{-2}	8.1279×10^{-2}	3.9334×10^{-3}	3.9334×10^{-3}	3.9334×10^{-3}
0.9	$1.2953{ imes}10^{-1}$	1.2953×10^{-1}	1.2953×10^{-1}	7.9510×10^{-3}	$7.9510 imes 10^{-3}$	7.9510×10^{-3}
1	$1.9632{ imes}10^{-1}$	1.9632×10^{-1}	1.9632×10^{-1}	1.4913×10^{-2}	1.4913×10^{-2}	1.4913×10^{-2}

Table 5. Comparison of the absolute errors of obtained GB-HPM solutions for Example 5.3 when $\rho = 1$ and $\zeta = 5$.



Figure 1. Waveform behaviour of real (Re) and imaginary (Im) part of GB-HPM solution for Caputo, CF and ABC operators at $\tau = 0.5$.



Figure 2. Waveform behaviour of real (Re) and imaginary (Im) part of GB-HPM solution for Caputo, CF and ABC operators at $\tau = 0.5$.



Figure 3. Waveform behaviour of real (Re) and imaginary (Im) part of GB-HPM solution for Caputo, CF and ABC operators at $\tau = 0.5$.