Positive solutions of discrete boundary value problem for a second-order nonlinear difference equation with singular ϕ-Laplacian

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Abstract

We establish the nonexistence, existence and multiplicity of positive solutions of the following discrete boundary value problem for a second-order nonlinear difference equation with singular ϕ-Laplacian

$$\begin{cases} -\nabla(k^{N-1}\phi(\triangle v_k)) = \lambda N k^{N-1}\left(\dfrac{f'(\varphi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} - f(\varphi^{-1}(v_k))H(\varphi^{-1}(v_k),k)\right), k\in[2,n-1]_{\mathbb{Z}}, \\ |\triangle v_k|<1, \\ \triangle v_1 = 0 = v_n, \end{cases}$$

where $\phi(s)=\frac{s}{\sqrt{1-s^2}}$, $\phi:(-1,1)\to\mathbb{R}$ is an increasing homeomorphism with $\phi(0)=0$, λ is a positive parameter, $\triangle$ is the forward difference operator defined by $\triangle v_k = v_{k+1}-v_k$, ∇ is the backward difference operator defined by $\nabla v_k = v_k - v_{k-1}$, $f\in C^\infty(I)$ and $f>0$, I is an open interval in $\mathbb{R}$, $\varphi(s)=\int_0^s \frac{dt}{f(t)}$, φ^{-1} is the inverse function of φ, $H: I\times[2,n-1]_{\mathbb{Z}}\to\mathbb{R}$ is a continuous function, $[2,n-1]_{\mathbb{Z}} := 2,3,\ldots,n-1$, and the integer $n\geq 4$. By using the method of upper and lower solutions, topological degree theory and Szulkin's critical point theory for convex, lower semicontinous perturbations of C^1-functionals, we determine the interval of parameter λ in which the above problem has zero, one, two positive solutions according to sublinear at zero.

Keywords: positive solutions, discrete boundary value problem, singular ϕ-Laplacian, Lower and upper solutions, Szulkin's critical point theory.

MR(2020) 34B18, 35A01, 35J93, 39A27

1 Introduction

Let $I\subseteq\mathbb{R}$ be an open interval in $\mathbb{R}$ with the metric $-dt^2$. Denote by $\mathcal{M}$ the $(N+1)$-dimensional product manifold $I\times\mathbb{R}^N$ with $N\geq 1$ endowed with the Lorentzian metric $g=-dt^2+f^2(t)dx^2$, where $f\in C^\infty(I)$, $f>0$, is called the scale factor or warping function. Clearly, $\mathcal{M}$ is a Lorentzian warped product with base $(I,-dt^2)$, fiber $(\mathbb{R}^N, dx^2)$ and warping function f, we refer it as a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. In cosmology, the FLRW spacetime describes a spatially homogeneous and isotropic universe, and plays an important role in the study of the

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exact solution of the field equations of Einstein's general theory of relativity. In this case, the warping function $f(t)$ is also interpreted as the radius of the universe at time t, and the sign of its derivative indicates whether the universe is expanding or contracting at time t in [1, 15, 17, 28, 32]. In particular, when $f(t) \equiv 1$ it is the Minkowski spacetime. For more details, see [14] and the references therein.

Given $f \in C^\infty(I)$, $f > 0$, for each $u \in C^\infty(\Omega)$, where Ω is a domain of $\mathbb{R}^N$, such that $u(\Omega) \subseteq I$, we can consider its graph $M = \{(x, u(x)) : x \in \Omega\}$ in the FLRW spacetime $\mathcal{M}$. The graph M is spacelike whenever $|\mathrm{grad}u| < f(u)$ in Ω, where $\mathrm{grad}u$ is the gradient of u in $\mathbb{R}^N$, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. In this case, the unit timelike normal vector field in the same time orientation of ∂_t is given by

$$A = \frac{f(u)}{\sqrt{f(u) - |\mathrm{grad}u|^2}} \left(\frac{1}{f^2(u)} \mathrm{grad}u + \partial_t \right),$$

and the corresponding mean curvature associated to A, is defined by

$$\frac{1}{N} \left\{ \mathrm{div} \left(\frac{\mathrm{grad}u}{f(u)\sqrt{f^2(u) - |\mathrm{grad}u|^2}} \right) + \frac{f'(u)}{\sqrt{f^2(u) - |\mathrm{grad}u|^2}} \left(N + \frac{|\mathrm{grad}u|^2}{f^2(u)} \right) \right\} := Q,$$

where div denotes the divergence operator of $\mathbb{R}^N$, $f'(u) := f' \circ u$. Since $|\mathrm{grad}u| < f(u)$ in Ω, then Q is a quasilinear elliptic operator. The question we are interested in is the existence of spacelike graphs with a prescribed mean curvature function in the FLRW spacetime $\mathcal{M}$. The general problem of the curvature prescription is, given a function $H : I \times \mathbb{R}^N \to \mathbb{R}$, to obtain solutions of the quasilinear elliptic equation

$$Q(u) = H(u, x), \qquad |\mathrm{grad}u| < f(u), \quad x \in \Omega.$$

This equation is also known as the prescribed mean curvature spacelike equation in FLRW spacetime. If H is a constant, it is called the prescribed constant mean curvature spacelike equation. If $H = 0$, it is also called the maximal spacelike graph equation.

In recent years, the solution of the boundary value problem for the prescribed mean curvature spacelike equation in Minkowski spacetime has been widely concerned by many scholars, whose attention is mainly on the positive solution. Readers may refer to the literature [2-4, 8, 10-13, 16, 18-20, 23-26] and its citations. In particular, based on the bifurcation method, Dai, Ma, Xu, Coelho et al. obtain the existence and multiplicity of positive solutions in [10, 12, 13, 27]. For radially symmetric solutions on the sphere, the results of existence and multiple solutions have been established in [3, 4]. By using the method of lower and upper solutions and topological degree, the nonexistence, existence and multiple solutions of positive solutions for Dirichlet system problems are studied in [18, 19]. In addition, some scholars have focused on discrete forms of these problems [5, 6, 9, 21, 22, 27]. Based on the method of lower and upper solutions, topological degree and variational method, Chen, Ma, Liang obtained the nonexistence, existence and multiple solutions of the positive solution of discrete robin boundary value problem in [9]. The solvability of Dirichlet problem is obtained by combining shooting method with Euler's method in [21].

Compared with the study in Minkowski spacetime, the results of the boundary value problem for the prescribed mean curvature spacelike equation in FLRW spacetime are relatively few. Only in recent years, Mawhin and Torres [28, 30] studied the existence of solutions to Neumann boundary value problems by using Leray-Schauder degree theory. Bereanu, de la Fuente, Romero and Torres [1, 17] prove that all solutions of Dirichlet boundary value problem are radially symmetric by using Schauder fixed point theorem and give sufficient conditions for the existence of positive solutions. Dai, Romero, Torres [15] use the global bifurcation theory to study the nonexistence, existence and multiple solutions of the positive solutions of Dirichlet boundary value problems on the sphere. Bereanu and Torres [7] use the critical point theory for strongly indefinite functionals to study the infinite number of solutions of the Neumann boundary value problem when certain assumptions are satified. Xu and Ma [31] obtained the existence of the solution of the discrete Neumann boundary value problem based on the Brouwer degree. Xu and Ma [32] proved the existence of solutions to the corresponding differential and difference problems for Dirichlet boundary value problems, and that the solutions of the discrete problems converge to the solutions of the continuous problems.

In this paper, we study the nonexistence, existence and multiplicity of positive solutions of the following discrete boundary value problem for a second-order nonlinear difference equation with singular ϕ-Laplacian

$$\begin{cases} -\nabla(k^{N-1}\phi(\triangle v_k)) = \lambda N k^{N-1}\left(\dfrac{f'(\varphi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} - f(\varphi^{-1}(v_k))H(\varphi^{-1}(v_k),k)\right), \quad k\in[2,n-1]_{\mathbb{Z}}, \\ |\triangle v_k|<1, \\ \triangle v_1 = 0 = v_n, \end{cases} \tag{1.1}$$

where $\phi(s)=\frac{s}{\sqrt{1-s^2}}$, $\phi:(-1,1)\to\mathbb{R}$ is an increasing homeomorphism with $\phi(0)=0$, λ is a positive parameter, $\triangle$ is the forward difference operator defined by $\triangle v_k = v_{k+1}-v_k$, ∇ is the backward difference operator defined by $\nabla v_k = v_k - v_{k-1}$, $f\in C^\infty(I)$ and $f>0$, I is an open interval in $\mathbb{R}$, $\varphi(s)=\int_0^s \frac{dt}{f(t)}$, φ^{-1} is the inverse function of φ, $H: I\times[2,n-1]_{\mathbb{Z}}\to\mathbb{R}$ is a continuous function, $[2,n-1]_{\mathbb{Z}} := 2,3,\ldots,n-1$, and the integer $n\geq 4$.

This study is mainly motivated by the numerical approximation of radially symmetric spacelike solutions of the nonlinear Dirichlet problem for the prescribed mean curvature spacelike equation in FLRW spacetime

$$\begin{cases} \operatorname{div}\Big(\dfrac{\operatorname{grad}u}{f(u)\sqrt{f^2(u)-|\operatorname{grad}u|^2}}\Big) + \dfrac{f'(u)}{\sqrt{f^2(u)-|\operatorname{grad}u|^2}}\Big(N+\dfrac{|\operatorname{grad}u|^2}{f^2(u)}\Big) = NH(u,|x|), \quad x\in B, \\ |\operatorname{grad}u| < f(u), \quad x\in B, \\ u=0, \quad x\in\partial B, \end{cases}$$

where $B=\{x\in\mathbb{R}^N : |x|<R\}$, R is a positive number, $f\in C^\infty(I)$, $f>0$ and $H: I\times[0,+\infty)\to\mathbb{R}$ is the prescribed mean curvature function.

Because (1.1) shows that $\|\triangle v\|_\infty < 1$, we can deduce $\|v\|_\infty = \left\|\sum_{i=k}^{n-1}\triangle v_i\right\|_\infty < n-2$, where

$\|v\|_\infty := \max_{k\in[2,n-1]_{\mathbb{Z}}} |v_k|$, this implies the image of nonnegative $v=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$ is in $[0,n-2]$. Therefore, when discussing (1.1), we always assume $\varphi^{-1}([0,n-2])\subset I$, which is equivalent to

$$I_fR := \left[0, \int_0^{n-2} f(\varphi^{-1}(s))ds\right] \subset I.$$

The main result is as follows.

Theorem 1.1. Assume that $I_fR\subset I$ and $f'(t)>0$, $H(t,k)<\frac{f'}{f}(t)$ for all $k\in[2,n-1]_{\mathbb{Z}}$ and $t\in I_fR\setminus\{0\}$, and assume also that

$$(A_{fH}) \qquad \begin{cases} \lim\limits_{t\to0^+} \frac{Nf'(t)}{\varphi(t)} = f_0, \\ \lim\limits_{t\to0^+} \frac{Nf(t)H(t,k)}{\varphi(t)} = H_0, \\ f_0 - H_0 = 0. \end{cases}$$

Then there is a $\Lambda > \frac{MN^2}{(n+1)^{N+2}}$ such that problem (1.1) has zero, at least one or at least two positive solutions when $\lambda\in(0,\Lambda),\ \lambda=\Lambda,\ \lambda>\Lambda$.

The rest of this paper is arranged as follows. In Section 2, we consider a more general case than (1.1), give its fixed point reformulation, prove that all possible solutions have a prior bound, and compute the corresponding topological degrees. In Section 3, we present a lower and upper solution result for problem (1.1) with $\lambda=1$. In Section 4, we deal with a mixed boundary value problem, involving a more general nonlinearity than that in (1.1). Using the Szulkin's critical point theory we prove the existence of positive solutions. Finally in Section 5 we give proof of the main results.

2 Fixed point, a priori bound

The following assumptions are used throughout the paper.

If $v\in\mathbb{R}^p$ $(p\geq1)$, where p is an integer, then we define $\|v\|_\infty := \max_{k\in[1,p]_{\mathbb{Z}}} |v_k|$. For every $i,j\in\mathbb{N}$ with $i>j$, we set $\sum_{k=i}^{j} v_k = 0$. If $\alpha,\beta\in\mathbb{R}^p$, we write $\alpha\leq\beta$ (respectively, $\alpha<\beta$) if $\alpha_k\leq\beta_k$ (respectively, $\alpha_k<\beta_k$) for all $k\in[1,p]_{\mathbb{Z}}$. Next we introduce a closed subpace $V^{n-2}=\{v\in\mathbb{R}^n : \triangle v_1 = 0 = v_n\}$ with the orientation of $\mathbb{R}^n$ and the norm $\|v\|_\infty := \max_{k\in[2,n-1]_{\mathbb{Z}}} |v_k|$, whose elements can be associated with the coordinates $(v_2,v_3,\cdots,v_{n-1})$ and correspond to the elements of $\mathbb{R}^n$ of the form $(v_2,v_2,\cdots,v_{n-1},0)$.

Let $v=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$. And then we define

$$\triangle v = (\triangle v_1,\cdots,\triangle v_{n-1}) \in \mathbb{R}^{n-1},$$

where $\triangle v_k = v_{k+1}-v_k$, $k\in[1,n-1]_{\mathbb{Z}}$, in addition, if there is $\|\triangle v\|_\infty := \max_{k\in[1,n-1]_{\mathbb{Z}}} |\triangle v_k| < 1$,

$$\nabla\left(k^{N-1}\phi(\triangle v)\right) = \left(\nabla(2^{N-1}\phi(\triangle v_2)),\cdots,\nabla\left((n-1)^{N-1}\phi(\triangle v_{n-1})\right)\right) \in \mathbb{R}^{n-2}$$

is defined, where

$$\nabla\left(k^{N-1}\phi(\triangle v_k)\right) = k^{N-1}\phi(\triangle v_k) - (k-1)^{N-1}\phi(\triangle v_{k-1}), \qquad k \in [2, n-1]_{\mathbb{Z}}.$$

If $v^0 \in V^{n-2}$, then $B(v^0, \rho) := \{v \in V^{n-2} : \|v - v^0\|_\infty < \rho\}$ $(\rho > 0)$ is defined, and we simply refer to $B(0, \rho)$ as B_ρ.

Moreover, to prove the simplicity of the procedure, if $v = (v_1, \cdots, v_p) \in \mathbb{R}^p$ $(p \geq 1)$, where p is an integer, then we sometimes express it as v without causing ambiguity. It also satisfies the assumption that if $v = (0, \cdots, 0) \in V^{n-2}$, then there is always $H(0, k) = \frac{f'}{f}(0)$, where $k \in [2, n-1]_{\mathbb{Z}}$.

Next we consider the problem

$$\begin{cases} \nabla(k^{N-1}\phi(\triangle v_k)) + k^{N-1}g(k, v_k, \triangle v_k) = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \\ |\triangle v_k| < 1, \\ \triangle v_1 = 0 = v_n, \end{cases} \tag{2.1}$$

where $N \geq 1$, and we also assume that

(A_ϕ) $\phi : (-1, 1) \to \mathbb{R}$ is an odd, increasing homeomorphism with $\phi(0) = 0$;

(A_g) $g : [2, n-1]_{\mathbb{Z}} \times [0, +\infty) \times (-1, 1) \to [0, +\infty)$ is continuous and $g(k, v_k, \triangle v_k) > 0$ for all $(k, v_k, \triangle v_k) \in [2, n-1]_{\mathbb{Z}} \times (0, \infty) \times (-1, 1)$.

Let $\sigma(k) = 1/k^{N-1}$, we define the operator

$$S : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}, \quad Sv_k = \sigma(k)\sum_{i=2}^{k} i^{N-1}v_i, \quad k \in [2, n-1]_{\mathbb{Z}},$$

$$K : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}, \quad Kv_k = \sum_{i=k}^{n-1} v_i, \quad k \in [2, n-1]_{\mathbb{Z}},$$

clearly, $K \circ \phi^{-1} \circ S : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is continuous. Furthermore, given a function $h = (h_2, \cdots, h_{n-1})$, the discrete problem

$$\nabla(k^{N-1}\phi(\triangle v_k)) + k^{N-1}h_k = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \quad |\triangle v_k| < 1, \quad \triangle v_1 = 0 = v_n$$

has a unique solution $v \in V^{n-2}$ and

$$v_k = K \circ \phi^{-1} \circ S \circ h_k, \quad k \in [2, n-1]_{\mathbb{Z}}.$$

Let N_g be the Nemytskii operator associated with g,

$$N_g : \mathbb{R}^n \to \mathbb{R}^{n-2}, \quad N_g(v) = (g(2, v_2, \triangle v_2), \cdots, g(n-1, v_{n-1}, \triangle v_{n-1})).$$

So problem (2.1) has the following fixed point reformulation.

Lemma 2.1. $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ is the solution of (2.1) if and only if the continuous operator

$$\mathcal{N}_g : V^{n-2} \to V^{n-2}, \quad \mathcal{N}_g = K \circ \phi^{-1} \circ S \circ N_g$$

has a fixed point, and in addition the fixed point of $\mathcal{N}_g$ satisfies

$$\|\triangle v\|_\infty < 1, \quad \|v\|_\infty < n-2 \tag{2.2}$$

and

$$d_B[I-\mathcal{N}_g, B_\rho, 0] = 1, \quad \text{for all } \rho \geq n-2.$$

Proof. Since the range of ϕ^{-1} is $(-1,1)$, the inequality (2.2) holds. Next, consider the compact homotopy

$$\mathcal{H} : [0,1]\times V^{n-2} \to V^{n-2}, \quad \mathcal{H}(\tau,\cdot) = \tau\mathcal{N}_g(\cdot),$$

and

$$\mathcal{H}([0,1]\times V^{n-2}) \subset B_{n-2}.$$

Then, from the invariance under homotopy of the Brouwer degree it follows that

$$\begin{aligned} d_B[I-\mathcal{H}(0,\cdot), B_\rho, 0] &= d_B[I-\mathcal{H}(1,\cdot), B_\rho, 0] \\ &= d_B[I-\mathcal{N}_g, B_\rho, 0] \\ &= d_B[I, B_\rho, 0] = 1, \end{aligned}$$

for all $\rho \geq n-2$. □

Lemma 2.2. Suppose that conditions (A_ϕ) and (A_g). Let $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ be a nontrivial solution of (2.1), then $v_k > 0$, $k \in [1, n-1]_\mathbb{Z}$. Moreover, v_k is strictly decreasing on $k \in [2,n]_\mathbb{Z}$.
Proof. From (2.1), we know that

$$\triangle v_k = -\phi^{-1}\left(\frac{1}{k^{N-1}}\sum_{i=2}^{k} i^{N-1} g(i, v_i, \triangle v_i)\right),$$

and $\triangle v_k \leq 0$ by using the assumption condition (A_g). Because of $v_n = 0$, we can derive $v_k \geq 0$, for all $k \in [1,n]_\mathbb{Z}$. Since v is a nontrivial solution, v_k is not identically zero, and then from (2.1) we can deduce $\triangle v_k < 0$ and $v_k > 0$, for all $k \in [2, n-1]_\mathbb{Z}$. Finally, using condition $\triangle v_1 = 0 = v_n$, we can know that $v_k > 0$, $k \in [1, n-1]_\mathbb{Z}$ and v_k is strictly decreasing on $k \in [2,n]_\mathbb{Z}$. □

In the next lemma we assume that g is sublinear with respect to ϕ at zero.

Lemma 2.3. Assume that conditions (A_ϕ), (A_g),

$$\lim_{s\to 0^+} \frac{g(k,s,\triangle s)}{\phi(s)} = 0 \quad \text{uniformly for } k\times\triangle s \in [2, n-1]_\mathbb{Z} \times (-1,1) \tag{2.3}$$

and

$$\liminf_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} > 0 \quad \text{for all } \sigma > 0. \tag{2.4}$$

Then there exists $\rho_0 > 0$ such that

$$d_B[I-\mathcal{N}_g, B_\rho, 0] = 1, \quad \text{for all } 0 < \rho \leq \rho_0,$$

where $\mathcal{N}_g$ is the fixed point operator associated to (2.1).

Proof. From (2.4) we know that there exists $\varepsilon > 0$ such that

$$\frac{n^N \varepsilon}{N} \leq \liminf_{s \to 0} \frac{\phi(s/(n-2))}{\phi(s)}. \tag{2.5}$$

From (2.3) we know that $s_\varepsilon > 0$ makes

$$g(k, s, \triangle s) \leq \varepsilon \phi(s), \quad \text{for } k \times s \times \triangle s \in [2, n-1]_{\mathbb{Z}} \times [0, s_\varepsilon] \times (-1, 1). \tag{2.6}$$

In the following we consider the compact homotopy

$$\mathcal{H} : [0,1] \times V^{n-2} \to V^{n-2}, \quad \mathcal{H}(\tau, v) = \tau \mathcal{N}_g(v),$$

where $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$.

We claim that there exists $\rho_0 > 0$ such that

$$v \neq \mathcal{H}(\tau, v), \quad \text{for } (\tau, v) \in [0,1] \times (\overline{B}_{\rho_0} \setminus \{0\}). \tag{2.7}$$

In fact, suppose there exists

$$v^m = \tau_m \mathcal{N}_g(v^m), \quad \tau_m \in [0,1],$$

where $v^m = (v_1^m, \cdots, v_n^m)$, $v^m \in V^{n-2} \setminus \{0\}$, $m \in \mathbb{N}$ and $\|v^m\|_\infty \to 0$. According to lemma 2.2, we obtain that v_k^m is strictly decreasing function with respect to k, and $v_k^m > 0$ on $k \in [1, n-1]_{\mathbb{Z}}$.

Suppose we have that $\|v^m\|_\infty \leq s_\varepsilon$ for all $m \in \mathbb{N}$, and we know that

$$g(k, v_k^m, \triangle v_k^m) \leq \varepsilon \phi(\|v^m\|_\infty), \quad \text{for } m \in \mathbb{N}, \quad k \in [2, n-1]_{\mathbb{Z}}.$$

from (2.6). So for any $m \in \mathbb{N}$, we have that

$$\begin{aligned} \|v^m\|_\infty &\leq \sum_{j=k}^{n-1} \phi^{-1} \left[\frac{1}{j^{N-1}} \sum_{i=2}^{j} i^{N-1} g(i, v_i^m, \triangle v_i^m) \right] \\ &\leq \sum_{j=k}^{n-1} \phi^{-1} \left(\frac{(j+1)^N}{N \cdot j^{N-1}} \cdot \varepsilon \phi(\|v^m\|_\infty) \right) \\ &< (n-2) \phi^{-1} \left(\frac{n^N}{N} \cdot \varepsilon \phi(\|v^m\|_\infty) \right), \end{aligned}$$

consequently,

$$\frac{\phi\left(\frac{\|v^m\|_\infty}{n-2}\right)}{\phi(\|v^m\|_\infty)} < \frac{n^N \varepsilon}{N}.$$

Since $\|v^m\|_\infty \to 0$, this contradicts (2.5). Thus, the homotopy invariance of Brouwer degree shows that for any $\rho \in (0, \rho_0]$ there is

$$d_B[I - \mathcal{H}(1, \cdot), B_\rho, 0] = d_B[I - \mathcal{H}(0, \cdot), B_\rho, 0] = 1.$$

Then the lemma is proved. $\square$

Lemma 2.4. Let $k^0 \in (0,1)$ be given. Then

$$|\triangle v_s| \leq 1 - k^0, \ \forall v \in \mathcal{A}, \ \forall s \in [2, n-1]_{\mathbb{Z}},$$

where

$$\mathcal{A} := \left\{ v \in V^{n-2} \mid -1 < \triangle v_k < 0 \text{ for } k \in [2, n-1]_{\mathbb{Z}}, \|v\|_\infty \leq 1 - k^0 \right\}.$$

Proof. Let $a = 1 - k^0$, $I = [2, n-1]_{\mathbb{Z}}$. Then

$$0 < a < 1.$$

Since $\triangle v_k < 0$ on $[2, n-1]_{\mathbb{Z}}$. If there exists $s \in I$ such that $|\triangle v_s| > 1 - k^0 = a$, then $\triangle v_s > a$ or $\triangle v_s < -a$. If $\triangle v_s < -a$, which is $\triangle v_s = v_{s+1} - v_s < -a$. Thus, we have that $-a > \triangle v_s = v_{s+1} - v_s \geq v_n - v_s \geq -v_s$, $s \in [2, n-1]_{\mathbb{Z}}$, which means $v_s > a = 1 - k^0$. This is a contradiction. Analogously, we can get a contradiction for the other case. □

3 lower and upper solutions

Next in this section, we develop the method of lower and upper solutions for the mixed boundary value problem

$$\begin{cases} \nabla(k^{N-1}\phi(\triangle v_k)) + Nk^{N-1}\left(\dfrac{f'(\varphi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} - f(\varphi^{-1}(v_k))H(\varphi^{-1}(v_k), k)\right) = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \\ |\triangle v_k| < 1, \\ \triangle v_1 = 0 = v_n. \end{cases} \tag{3.1}$$

Let's say $F : [2, n-1]_{\mathbb{Z}} \times \varphi(I) \times (-1, 1) \to \mathbb{R}$ is defined by

$$F(r, s, t) = \frac{Nf'(\varphi^{-1}(s))}{\sqrt{1-t^2}} - Nf(\varphi^{-1}(s))H(\varphi^{-1}(s), r).$$

Moreover we define the Nemytskii operator associated with F as

$$N_F : \mathbb{R}^n \to \mathbb{R}^{n-2}, \quad N_F(v) = (F(2, v_2, \triangle v_2), \cdots, F(n-1, v_{n-1}, \triangle v_{n-1})).$$

Note that if $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ is the solution of (3.1) if and only if $v \in V^{n-2}$ and v is a fixed point of the continuous operator

$$\mathcal{N}_F : V^{n-2} \to V^{n-2}, \qquad \mathcal{N}_F = K \circ \phi^{-1} \circ S \circ N_F.$$

Definition 3.1. A function $\alpha = (\alpha_1, \cdots, \alpha_n)$ is called a lower solution of (3.1) if $\|\triangle \alpha\|_\infty < 1$, $I_f R \subset I$ and

$$\begin{cases} -\nabla(k^{N-1}\phi(\triangle \alpha_k)) \leq Nk^{N-1}\left(\dfrac{f'(\varphi^{-1}(\alpha_k))}{\sqrt{1-(\triangle \alpha_k)^2}} - f(\varphi^{-1}(\alpha_k))H(\varphi^{-1}(\alpha_k), k)\right), \ k \in [2, n-1]_{\mathbb{Z}}, \\ \qquad \triangle \alpha_1 = 0, \ \alpha_n \leq 0. \end{cases}$$

A function $\beta = (\beta_1, \cdots, \beta_n)$ is called an upper solution of (3.1) if $\|\triangle\beta\|_\infty < 1$, $I_f R \subset I$ and

$$\begin{cases} -\nabla(k^{N-1}\phi(\triangle\beta_k)) \geq Nk^{N-1}\left(\dfrac{f'(\varphi^{-1}(\beta_k))}{\sqrt{1-(\triangle\beta_k)^2}} - f(\varphi^{-1}(\beta_k))H(\varphi^{-1}(\beta_k), k)\right), \ k \in [2, n-1]_{\mathbb{Z}}, \\ \qquad\qquad \triangle\beta_1 = 0, \ \beta_n \geq 0. \end{cases}$$

Such a lower or an upper solution is called strict if the above inequalities are strict.

Theorem 3.1. Assume that $I_f R \subset I$ and $f'(t) > 0$, $H(t,k) < \frac{f'}{f}(t)$ for all $k \in [2, n-1]_{\mathbb{Z}}$ and $t \in I_f R \setminus \{0\}$. If (3.1) has a lower solution $\alpha = (\alpha_1, \cdots, \alpha_n)$ and an upper solution $\beta = (\beta_1, \cdots, \beta_n)$ such that $\alpha_k \leq \beta_k$ for all $k \in [1, n]_{\mathbb{Z}}$, then (3.1) has at least one solution $v = (v_1, \cdots, v_n)$ such that $\alpha_k \leq v_k \leq \beta_k$, where $k \in [1, n]_{\mathbb{Z}}$.

Proof. Let $\gamma : [2, n-1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(k, v_k) = \begin{cases} \alpha_k, & v_k < \alpha_k, \\ v_k, & \alpha_k \leq v_k \leq \beta_k, \\ \beta_k, & v_k > \beta_k. \end{cases}$$

We consider the modified problem

$$\begin{cases} \nabla(k^{N-1}\phi(\triangle v_k)) + k^{N-1}\Big(\dfrac{Nf'(\varphi^{-1}(\gamma(k, v_k)))}{\sqrt{1-(\triangle v_k)^2}} \\ \qquad - NH(\varphi^{-1}(\gamma(k, v_k)), k) f(\varphi^{-1}(\gamma(k, v_k))) - v_k + \gamma(k, v_k)\Big) = 0, \qquad k \in [2, n-1]_{\mathbb{Z}}, \\ |\triangle v_k| < 1, \\ \triangle v_1 = 0 = v_n. \end{cases} \tag{3.2}$$

We claim that (3.2) has at least one solution. Denote

$$F(r, s, t) = \frac{Nf'(\varphi^{-1}(s))}{\sqrt{1-t^2}} - Nf(\varphi^{-1}(s))H(\varphi^{-1}(s), r).$$

In fact, (3.2) is equivalent to the fixed point problem $v = \mathcal{N}_{\widetilde{F}}(v)$, where $\widetilde{F} = F(k, \gamma(k, v_k), \triangle v_k) - v_k + \gamma(k, v_k)$, $k \in [2, n-1]_{\mathbb{Z}}$. Since $\|\triangle v\|_\infty = \|\phi^{-1} \circ S \circ N_{\widetilde{F}}(v)\|_\infty < 1$, we can have $\|v\|_\infty = \|K(\triangle v)\|_\infty$, which leads to $\|v\|_\infty < n-2$. Using Schauder fixed point theorem, we can deduce that there exists $v \in V^{n-2}$ such that $v = \mathcal{N}_{\widetilde{F}}(v)$.

Below we will prove that if v is a solution of (3.2), then there is $\alpha_k \leq v_k \leq \beta_k$ on $k \in [1, n]_{\mathbb{Z}}$.

Suppose by contradiction that there is some $i \in [1, n]_{\mathbb{Z}}$ such that

$$\max_{j \in [1,n]_{\mathbb{Z}}} [\alpha_j - v_j] = \alpha_i - v_i > 0.$$

If $i \in [2, n-1]_{\mathbb{Z}}$, there is $\triangle\alpha_i \leq \triangle v_i$ and $\triangle v_{i-1} \leq \triangle\alpha_{i-1}$. Since ϕ is an increasing homeomorphism, there is

$$\nabla(i^{N-1}\phi(\triangle\alpha_i)) \leq \nabla(i^{N-1}\phi(\triangle v_i)).$$

From $f'(t) > 0$ and α is a lower solution of (3.1), we know that

$$\begin{aligned}\nabla(i^{N-1}\phi(\triangle\alpha_i)) &\le \nabla(i^{N-1}\phi(\triangle v_i))\\ &= i^{N-1}\left(-\frac{Nf'(\varphi^{-1}(\alpha_i))}{\sqrt{1-(\triangle v_i)^2}} + NH(\varphi^{-1}(\alpha_i),i)f(\varphi^{-1}(\alpha_i)) + v_i - \alpha_i\right)\\ &< i^{N-1}\left(-\frac{Nf'(\varphi^{-1}(\alpha_i))}{\sqrt{1-(\triangle v_i)^2}} + NH(\varphi^{-1}(\alpha_i),i)f(\varphi^{-1}(\alpha_i))\right)\\ &\le i^{N-1}\left(-\frac{Nf'(\varphi^{-1}(\alpha_i))}{\sqrt{1-(\triangle \alpha_i)^2}} + NH(\varphi^{-1}(\alpha_i),i)f(\varphi^{-1}(\alpha_i))\right)\\ &\le \nabla(i^{N-1}\phi(\triangle\alpha_i)),\end{aligned}$$

but this is a contradiction.

If $i = n$, then $\alpha_n - v_n > 0$, and it follows from $v_n = 0$ and definition of lower solutions that this is also a contradiction.

If $i = 1$, then using $\triangle\alpha_1 = 0 = \triangle v_1$ we know that $\alpha_1 = \alpha_2$ and $v_1 = v_2$, which means that $i = 2$ is also the maximum point. The proof process is the same as $i \in [2, n-1]_{\mathbb{Z}}$, which is still a contradiction. So for $k \in [1, n]_{\mathbb{Z}}$, we have that $\alpha_k \le v_k$. Analogously, for $k \in [1, n]_{\mathbb{Z}}$, we have that $v_k \le \beta_k$. The proof of the theorem is completed. □

Theorem 3.2. Assume that (3.1) has a lower solution $\alpha = (\alpha_1, \cdots, \alpha_n)$ and an upper solution $\beta = (\beta_1, \cdots, \beta_n)$ such that $\alpha_k \le \beta_k$ for all $k \in [1, n]_{\mathbb{Z}}$. Let $\Omega_{\alpha,\beta} := \{v \in V^{n-2} : \alpha \le v \le \beta\}$. Assume also that (3.1) has an unique solution v^0 in $\Omega_{\alpha,\beta}$ and there exists $\rho_0 > 0$ such that $\overline{B}(v^0, \rho_0) \subset \Omega_{\alpha,\beta}$. Then

$$d_B[I - \mathcal{N}_g, B(v^0, \rho), 0] = 1, \quad \text{for } 0 < \rho \le \rho_0,$$

where $\mathcal{N}_g$ is the fixed point operator associated to (3.1).

Proof. Let $\mathcal{N}_\gamma$ be the fixed point operator associated to the modified problem (3.2). According to theorem 3.1, any fixed point v of $\mathcal{N}_\gamma$ is contained in $\Omega_{\alpha,\beta}$ and v is also a fixed point of $\mathcal{N}_g$. Thus, v^0 is the unique fixed point of $\mathcal{N}_\gamma$. Now,

$$d_B[I - \mathcal{N}_\gamma, B(v^0, \rho), 0] = 1, \qquad \text{for all } \rho > 0$$

is obtained from lemma 2.1 and the excision property of the Brouwer degree. The result follows from the fact $\mathcal{N}_\gamma(v) = \mathcal{N}_g(v)$ for $v \in \overline{B}(v^0, \rho_0)$. □

4 Variational solutions

First consider the mixed boundary value problem

$$\nabla(k^{N-1}\phi(\triangle v_k)) + k^{N-1}g(k, v_k, \triangle v_k) = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \quad |\triangle v_k| < 1, \quad \triangle v_1 = 0 = v_n, \qquad (4.1)$$

and satisfy the following assumptions

$(A_{g'})$ $g : [2, n-1]_{\mathbb{Z}} \times \mathbb{R} \times (-1, 1) \to \mathbb{R}$ is a continuous function;

(A_Φ) $\Phi : [-1,1] \to \mathbb{R}$ is continuous, of class C^1 on $(-1,1)$, $\Phi(0) = 0$ and $\phi := \Phi' : (-1,1) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$.

Define a convex set

$$K := \{u \in \mathbb{R}^n : \|\triangle u\|_\infty \le 1\},$$

where $u = (u_1, \cdots, u_n)$, and obviously the convex set K is closed in $\mathbb{R}^n$. This means that

$$K_0 := \{u \in K : u_n = 0\}$$

is also a convex, closed subset of $\mathbb{R}^n$. Since there is

$$\|u\|_\infty \le n-1 \quad \text{for all } u \in K_0, \tag{4.2}$$

we know that K_0 is bounded in $\mathbb{R}^n$.

Next, we introduce the functional $\Psi : \mathbb{R}^n \to (-\infty, +\infty]$, defined by

$$\Psi(u) = \begin{cases} \sum_{k=2}^{n-1} k^{N-1}\Phi(\triangle u_k), & \text{if } u \in K_0, \\ +\infty, & \text{if } u \in \mathbb{R}^n \setminus K_0 \end{cases}$$

is proper, convex and lower semicontinuous. Obviously Ψ is bounded on K_0.

We define $G : [2, n-1]_{\mathbb{Z}} \times \mathbb{R} \times (-1,1) \to \mathbb{R}$ by

$$G(k,s,t) = -\int_0^s k^{N-1} g(k,\xi,t)d\xi \quad \text{for all } (k,s,t) \in [2,n-1]_{\mathbb{Z}} \times \mathbb{R} \times (-1,1),$$

and

$$\mathcal{G}(u) = \sum_{k=2}^{n-1} G(k, u_k, \triangle u_k) \quad \text{for } u \in \mathbb{R}^n.$$

It's obvious that $\mathcal{G} \in C^1(\mathbb{R}^n, \mathbb{R})$. Thus the energy functional $I = \Psi + \mathcal{G}$ has the structure required by Szulkin's critical point theory [29]. Accordingly, a function $v \in \mathbb{R}^n$ is a critical point of I if $v \in K_0$ and

$$\Psi(u) - \Psi(v) + \langle \mathcal{G}'(v), u - v\rangle \ge 0 \quad \text{for all } u \in \mathbb{R}^n,$$

or, equivalently

$$\sum_{k=2}^{n-1} k^{N-1}(\Phi(\triangle u_k) - \Phi(\triangle v_k) - g(k, v_k, \triangle v_k)(u_k - v_k)) \ge 0, \quad \text{for all } u \in K_0. \tag{4.3}$$

Lemma 4.1. Assuming that (A_Φ). Then for every $h \in \mathbb{R}^{n-2}$ and $h = (h_2, \cdots, h_{n-1})$, the problem

$$\nabla(k^{N-1}\phi(\triangle v_k)) + k^{N-1}h_k = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \quad \triangle v_1 = 0 = v_n \tag{4.4}$$

has a unique solution $v(h)$, which is also the unique solution in K_0 of the variational inequality

$$\sum_{k=2}^{n-1} k^{N-1}(\Phi(\triangle u_k) - \Phi(\triangle v_k) - h_k(u_k - v_k)) \ge 0, \quad \text{for all } u \in K_0, \tag{4.5}$$

and the unique minimum over K_0 of the strictly convex functional $\mathcal{J} : K_0 \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \sum_{k=2}^{n-1} k^{N-1}(\Phi(\triangle u_k) - h_k u_k), \quad \text{for all } u \in K_0.$$

Proof. Clearly (4.4) has a unique solution

$$v_k(h) = K \circ \phi^{-1} \circ S \circ h_k.$$

Set $v := v(h)$ is a solution of (4.4). Next, taking $u \in K_0$, multiplying (4.4) by $u_k - v_k$ and summing over $[2, n-1]_{\mathbb{Z}}$, we can derive

$$\sum_{k=2}^{n-1} k^{N-1}(\phi(\triangle v_k)(\triangle u_k - \triangle v_k) - h_k(u_k - v_k)) = 0, \quad \text{for all } u \in K_0.$$

And then from the convexity inequality

$$\Phi(\triangle u_k) - \Phi(\triangle v_k) \geq \phi(\triangle v_k)(\triangle u_k - \triangle v_k),$$

we deduce that (4.5) holds.

In fact, it is easy to conclude that $v \in K_0$ is a solution of (4.5) if and only if it is a minimum of $\mathcal{J}$ on K_0. Moreover, since Φ is strictly convex, we know that $\mathcal{J}$ is also strictly convex. This means that the uniqueness of the minimum of $\mathcal{J}$ on K_0. □

Lemma 4.2. Assumes that the conditions (A_Φ) and $(A_{g'})$ are satisfied. Then every critical point of I is a solution of (4.1). In addition, (4.1) has a solution that is a minimum point of I on $\mathbb{R}^n$.

Proof. Let $v \in K_0$ be the critical point of I. Then v solves the variational inequality (4.5) with $h_k = (k, v_k, \triangle v_k)$ (see (4.3)), whereas from lemma 4.1, it is known that v is the solution of (4.1). From the definition of I, we know that $I(u) = +\infty$ on $u \in \mathbb{R}^n \setminus K_0$, thus

$$\inf_{\mathbb{R}^n} I := \inf_{K_0} I := c_0.$$

According to (4.2) we know that I is bounded on K_0. Let $\{v^m\} \subset K_0$ be such that $I(v^m) \to c_0$. Using the compactness of K_0 that we have assumed, we know that there exists $v \in K_0$ such that there is $v^m \to v$ in $\mathbb{R}^n$. It follows that $\mathcal{G}(v^m) \to \mathcal{G}(v)$ and $\Psi(v) \leq \liminf_{k\to\infty} \Psi(v^m)$. Therefore $I(v) \leq c_0$ and v is a minimum of I on $\mathbb{R}^n$. It can be seen from [29] that v is a critical point of I and hence a solution of (4.1). □

Now consider the special example of $g(k, s, t) = \left(\frac{Nf'(\varphi^{-1}(s))}{\sqrt{1-t^2}} - Nf(\varphi^{-1}(s))H(\varphi^{-1}(s), k)\right)$ in (4.1).

$$\begin{cases} \nabla\left(k^{N-1}\dfrac{\triangle v_k}{\sqrt{1-(\triangle v_k)^2}}\right) + k^{N-1}\left(\dfrac{Nf'(\varphi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} - Nf(\varphi^{-1}(v_k))H(\varphi^{-1}(v_k), k)\right) = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \\ |\triangle v_k| < 1, \\ \triangle v_1 = 0 = v_n, \end{cases} \tag{4.6}$$

satisfy the following assumption

(A_s) $g : [2, n-1]_{\mathbb{Z}} \times (0, +\infty) \times (-1, 1) \to \mathbb{R}$ is a continuous function such that $g(k, 0, t) = 0$ and $g(k, s, t) > 0$ for all $s > 0$.

According to lemma 2.2, the solution of (4.6) is nonnegative. we use

$$\widetilde{g}(k, s, t) = \begin{cases} g(k, s, t), & s \geq 0, \\ 0, & s < 0 \end{cases}$$

instead of $g(k, s, t)$ in (4.6). In order to simplify notation, the modified function $\widetilde{g}(k, s, t)$ is still represented by $g(k, s, t)$. Accordingly, $\Phi(s) = 1 - \sqrt{1 - s^2}$ $(s \in [-1, 1])$ and

$$G(k, s, t) = -\int_0^s k^{N-1} g(k, \xi, t) d\xi,$$

the energy functional $I : \mathbb{R}^n \to (-\infty, +\infty]$ associated to (4.6) will be

$$I(u) = \sum_{k=2}^{n-1} k^{N-1} \left(1 - \sqrt{1 - (\triangle u_k)^2} - \int_0^{u_k} g(k, \xi, t) d\xi \right)$$

and $I = +\infty$ on $\mathbb{R}^n \setminus K_0$.

Theorem 4.1. Suppose that (A_s) holds. Let

$$\inf_{K_0} I < 0.$$

Then problem (4.6) has at least one solution v such that $v_k > 0$ on $k \in [2, n-1]_{\mathbb{Z}}$ and v_k is strictly decreasing with respect to $k \in [2, n]_{\mathbb{Z}}$.

Proof. By lemma 4.2 and $I(0) = 0$, we obtain that (4.6) has a nontrivial nonnegative solution v. By lemma 2.2, we can easily conclude that v_k s strictly decreasing with respect to $k \in [2, n]_{\mathbb{Z}}$. □

Corollary 4.1. Suppose that condition (A_s) is satisfied. Then for sufficiently large $\lambda > 0$ the problem

$$\begin{cases} -\nabla(k^{N-1}\phi(\triangle v_k)) = \lambda N k^{N-1} \left(\dfrac{f'(\varphi^{-1}(v_k))}{\sqrt{1 - (\triangle v_k)^2}} - f(\varphi^{-1}(v_k)) H(\varphi^{-1}(v_k), k) \right), k \in [2, n-1]_{\mathbb{Z}}, \\ |\triangle v_k| < 1, \\ \triangle v_1 = 0 = v_n \end{cases}$$

has at least one solution $v \in \mathbb{R}^n$ such that $v_k > 0$ on $k \in [2, n-1]_{\mathbb{Z}}$ and v_k is strictly decreasing with respect to $k \in [2, n]_{\mathbb{Z}}$.

5 Proof of main result

Proof of Theorem 1.1. Let's say

$$S_j := \{\lambda > 0 : \text{ (1.1) has at least } j \text{ positive solutions}\}, \quad (j = 1, 2).$$

1. *The existence of* Λ.

Let $\lambda > 0$, $v = (v_1, \dots, v_n)$ is a positive solution of (1.1). Firstly, using hypothesis (A_{fH}), we have that for any $\varepsilon_0 > 0$, there exists δ_1, such that $|\varphi^{-1}(v_k) - 0| < \delta_1$, there can be $\left|\frac{Nf'(\varphi^{-1}(v_k))}{v_k} - f_0\right| < \varepsilon_0$. For the above ε_0, there exists δ_2, such that there exists $|\varphi^{-1}(v_k) - 0| < \delta_2$, there implies that $\left|\frac{Nf(\varphi^{-1}(v_k))H(\varphi^{-1}(v_k),k)}{v_k} - H_0\right| < \varepsilon_0$.

Secondly, using lemma 2.4, let $k^0 = a_0$, a_0 is the constant that satisfies the definition, then there is $I = [2, n-1]_{\mathbb{Z}}$. Hence, $\|v\|_\infty \le 1 - a_0$, $|\triangle v_s| \le 1 - a_0$ for all $s \in I$.

To sum (1.1) from 2 to k, we obtain that

$$\begin{aligned}
-k^{N-1}\phi(\triangle v_k) &= \lambda \sum_{i=2}^{k} i^{N-1} \left(\frac{Nf'(\varphi^{-1}(v_i))}{\sqrt{1-(\triangle v_i)^2}} - Nf(\varphi^{-1}(v_i))H(\varphi^{-1}(v_i), i) \right) \\
&< \lambda \sum_{i=2}^{k} i^{N-1} \left(\frac{f_0 v_i}{\sqrt{1-(\triangle v_i)^2}} - H_0 v_i \right) \\
&\le \lambda \sum_{i=2}^{k} i^{N-1} f_0 v_i \left(\frac{1}{\sqrt{1-(1-a_0)^2}} - 1 \right) \\
&< \lambda \sum_{i=2}^{k} i^{N-1} f_0 \,(n-2) \left(\frac{1}{\sqrt{1-(1-a_0)^2}} - 1 \right) \\
&\le \lambda M_0 (n-2) \frac{(k+1)^N}{N},
\end{aligned}$$

where $M_0 = f_0 \left(\frac{1}{\sqrt{1-(1-a_0)^2}} - 1 \right)$.

Since v_k is strictly decreasing on $k \in [2, n]_{\mathbb{Z}}$, there is

$$-\triangle v_k \le -\frac{\triangle v_k}{\sqrt{1-(\triangle v_k)^2}} < \frac{\lambda M_0 (n-2)(k+1)^N}{N \cdot k^{N-1}} \tag{5.1}$$

for each $k \in [2, n-1]_{\mathbb{Z}}$.

Summing (5.1) from 2 to $n-1$, we have that

$$v_2 < \frac{\lambda M_0 (n-2)}{N} \cdot \sum_{i=2}^{n-1} \frac{(i+1)^N}{i^{N-1}} < \frac{\lambda M_0 (n-2)(n+1)^{N+1}}{N^2} < \frac{\lambda M_0 (n+1)^{N+2}}{N^2}. \tag{5.2}$$

Next, using $v_2 > 0$, we obtain that

$$\lambda > \frac{MN^2}{(n+1)^{N+2}},$$

where $M = v_2 / M_0$.

We know from corollary 4.1 that the problem (1.1) has at least one positive solution for sufficiently large $\lambda > 0$. In particular, $S_1 \ne \emptyset$. We can define

$$\Lambda = \Lambda(n) := \inf S_1.$$

Clearly, we have that $\Lambda \ge \frac{MN^2}{(n+1)^{N+2}}$. We will prove $\Lambda \in S_1$.

Let $\lambda_m \to \Lambda$, $v^m = (v_1^m, \dots, v_n^m) > 0$, $\lambda_m \times v^m \in S_1 \times V^{n-2}$, and

$$v_k^m = K \circ \phi^{-1} \circ S \circ \left(\lambda_m \left(\frac{Nf'(\varphi^{-1}(v_k^m))}{\sqrt{1-(\triangle v_k^m)^2}} - Nf(\varphi^{-1}(v_k^m))H(\varphi^{-1}(v_k^m), k) \right) \right).$$

From (2.2) and the Arzelà-Ascoli theorem we know that there exists $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, after taking a subsequence such that $\{v^m\} \to v$ in $\mathbb{R}^n$. We have that $v_k > 0$ and

$$v_k = K \circ \phi^{-1} \circ S \circ \left(\Lambda \left(\frac{N f'(\varphi^{-1}(v_k))}{\sqrt{1 - (\triangle v_k)^2}} - N f(\varphi^{-1}(v_k)) H(\varphi^{-1}(v_k), k) \right) \right).$$

It can be seen from (5.2) that there is a constant $c_1 > 0$ which makes $v_2^m > c_1$, for all $m \in \mathbb{N}$. This means that $v_2 \geq c_1$. By lemma 2.2, we get $v_k > 0$ on $k \in [1, n-1]_{\mathbb{Z}}$. Hence, $\Lambda \in S_1$. Obviously, $\Lambda > \frac{MN^2}{(n+1)^{N+2}}$.

Next, let $\lambda_0 > \Lambda$, where λ_0 is arbitrary. We will show $\lambda_0 \in S_1$. Suppose v^1 is a positive solution of (1.1) with $\lambda = \Lambda$. It is easy to know that v^1 is a lower solution to problem (1.1) when $\lambda = \lambda_0$. Construct the upper solution, let $H > 0$, the integer $\widetilde{n} > n$, while considering the problem

$$\nabla \left(k^{N-1} \frac{\triangle v_k}{\sqrt{1 - (\triangle v_k)^2}} \right) + k^{N-1} H = 0, \quad k \in [2, \widetilde{n} - 1]_{\mathbb{Z}}, \ \ |\triangle v_k| < 1, \ \ \triangle v_1 = 0 = v_{\widetilde{n}}. \tag{5.3}$$

By calculation we can get

$$v_k = \sum_{j=k}^{\widetilde{n}-1} \frac{\frac{1}{j^{N-1}} \sum\limits_{i=2}^{j} i^{N-1} H}{\sqrt{1 + \left(\frac{1}{j^{N-1}} \sum\limits_{i=2}^{j} i^{N-1} H \right)^2}}.$$

For fixed $\lambda_2 > \lambda_0$, let v^2 is the solution of problem (5.3) corresponding to $H = \lambda_2 M_0 (\widetilde{n} - 2)$. By $v_n^2 > 0$ and

$$\lambda_0 \left(\frac{N f'(\varphi^{-1}(v_k^2))}{\sqrt{1 - (\triangle v_k^2)^2}} - N f(\varphi^{-1}(v_k^2)) H(\varphi^{-1}(v_k^2), k) \right) \leq \lambda_2 M_0 (\widetilde{n} - 2), \quad \text{for all } k \in [2, n-1]_{\mathbb{Z}}.$$

We can see that v^2 is an upper solution of problem (1.1) when $\lambda = \lambda_0$, then

$$v_n^2 = \sum_{j=n}^{\widetilde{n}-1} \frac{\frac{1}{j^{N-1}} \sum\limits_{i=2}^{j} i^{N-1} H}{\sqrt{1 + \left(\frac{1}{j^{N-1}} \sum\limits_{i=2}^{j} i^{N-1} H \right)^2}}.$$

Hence there exists $\widetilde{n}$ such that $v_2^1 < v_n^2$. Consider that v_k^1, v_k^2 is strictly decreasing, then there is $v_k^1 < v_k^2$ for all $k \in [1, n]_{\mathbb{Z}}$. It follows from theorem 2.1 that $\lambda_0 \in S_1$. Hence, $S_1 = [\Lambda, \infty)$.

2.*Multiplicity.*

Let $\lambda_0 > \Lambda$. We will show that $\lambda_0 \in S_2$. Let v^1, v^2 be constructed as above. Let v^0 be a solution of (1.1) with $\lambda = \lambda_0$ such that $v^1 \leq v^0 \leq v^2$, where $v^0 \in \Omega_{v^1, v^2} := \{v \in V^{n-2} : v^1 \leq v \leq v^2\}$.

First, we claim that there exists $\varepsilon > 0$ such that $\overline{B}(v^0, \varepsilon) \subset \Omega_{v^1, v^2}$. For all $k \in [2, n-1]_{\mathbb{Z}}$, there is

$$v_k^2 = \sum_{j=k}^{\widetilde{n}-1} \phi^{-1} \left(\frac{1}{j^{N-1}} \sum_{i=2}^{j} \lambda_2 i^{N-1} M_0 (\widetilde{n} - 2) \right).$$

Consequently,

$$
\begin{aligned}
v_k^2 &> \sum_{j=k}^{n-1} \phi^{-1}\left(\frac{1}{j^{N-1}}\sum_{i=2}^{j}\lambda_2 i^{N-1}\left(\frac{Nf'(\varphi^{-1}(v_i^2))}{\sqrt{1-(\triangle v_i^2)^2}} - Nf(\varphi^{-1}(v_i^2))H(\varphi^{-1}(v_i^2),i)\right)\right)\\
&\geq \sum_{j=k}^{n-1} \phi^{-1}\left(\frac{1}{j^{N-1}}\sum_{i=2}^{j}\lambda_0 i^{N-1}\left(\frac{Nf'(\varphi^{-1}(v_i^0))}{\sqrt{1-(\triangle v_i^0)^2}} - Nf(\varphi^{-1}(v_i^0))H(\varphi^{-1}(v_i^0),i)\right)\right)\\
&= v_k^0.
\end{aligned}
$$

Therefore, there exists $\varepsilon_2 > 0$ such that $v^2 \geq v$ for all $v \in \overline{B}(v^0, \varepsilon_2)$. Similarly on $[1,m]_{\mathbb{Z}}$ there is $v^1 < v^0$ for some $m \in [2, n-1]_{\mathbb{Z}}$. Therefore $\varepsilon_3 > 0$ can be found such that

$$
v \in V^{n-2}, \text{ and } \|v - v^0\|_\infty \leq \varepsilon_3 \Rightarrow v \geq v^1 \quad \text{on } k \in [1,m]_{\mathbb{Z}}. \tag{5.4}
$$

Obviously if $m = n-1$, our claim holds. Otherwise, $\triangle v_k^0 < \triangle v_k^1$ on $k \in [m, n-1]_{\mathbb{Z}}$ is obtained from

$$
-\triangle v_k^0 = \phi^{-1} \circ S \circ \lambda_0 \left(\frac{Nf'(\varphi^{-1}(v_k^0))}{\sqrt{1-(\triangle v_k^0)^2}} - Nf(\varphi^{-1}(v_k^0))H(\varphi^{-1}(v_k^0),k)\right)
$$

and

$$
-\triangle v_k^1 = \phi^{-1} \circ S \circ \Lambda \left(\frac{Nf'(\varphi^{-1}(v_k^1))}{\sqrt{1-(\triangle v_k^1)^2}} - Nf(\varphi^{-1}(v_k^1))H(\varphi^{-1}(v_k^1),k)\right).
$$

So there exists $\varepsilon_1 \in (0, \varepsilon_3)$ such that $\triangle v_k < \triangle v_k^1$ on $k \in [m, n-1]_{\mathbb{Z}}$, where $v \in B(v^0, \varepsilon_1)$. Then using $v_n^0 = 0 = v_n$, we know that $v > v^1$ on $k \in [m, n-1]_{\mathbb{Z}}$ for all $v \in \overline{B}(v^0, \varepsilon_1)$. From (5.4) we can see that our claim is valid when $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$. If the second solution of (1.1) is contained in Ω_{v^1,v^2}, and this solution is nontrivial, then the proof of the multiplicity is completed.

If not, using theorem 3.2 we have that

$$
d_B[I - \mathcal{N}_{\lambda_0}, B(v^0, \rho), 0] = 1, \quad \text{for all } 0 < \rho \leq \varepsilon,
$$

where $\mathcal{N}_{\lambda_0}$ is the fixed point operator associated to (1.1) with $\lambda = \lambda_0$. Moreover, according to lemma 2.1, we obtain that

$$
d_B[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1, \quad \text{for all } \rho \geq n-2.
$$

From lemma 2.3 we have that

$$
d_B[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1, \quad \text{for all } \rho \text{ sufficiently small.}
$$

When ρ_1, ρ_2 is sufficiently small and $\rho_3 \geq n-2$ such that $\overline{B}(v^0, \rho_1) \cap \overline{B}_{\rho_2} = \emptyset$, $\overline{B}(v^0, \rho_1) \cup \overline{B}_{\rho_2} \subset B_{\rho_3}$. Thus, from the additivity-excision property of the Brouwer degree it follows that

$$
d_B[I - \mathcal{N}_{\lambda_0}, B_{\rho_3} \setminus [\overline{B}(v^0, \rho_1) \bigcup \overline{B}_{\rho_2}], 0] = -1,
$$

which, together with the existence property of the Brouwer degree, imply that $\mathcal{N}_{\lambda_0}$ has a fixed point $\widetilde{v}^0 \in B_{\rho_3} \setminus [\overline{B}(v^0, \rho_1) \bigcup \overline{B}_{\rho_2}]$. So we conclude that (1.1) has a second positive solution, and the proof is completed. □

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