On Banach's fixed point theorem in perturbed metric spaces

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Abstract

The measurement of the distance between two points is always tainted by errors. The causes of such errors are varied. For instance, the imperfection in the adjustment of instruments affects the accuracy of measurements. These errors are generally "small", however their accumulations can become significant. Motivated by this fact, in this paper, we introduce the notion of perturbed metric spaces and establish an interesting generalization of Banach's fixed point theorem in such spaces.

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1 Introduction

The most existence results in nonlinear analysis have been established making use of Banach's fixed point theorem [2]. For instance, the existence of local solutions to evolution equations, the existence of solutions to integral equations and the existence of solutions to matrix equations, see e.g. [7,8,17,20]. The literature includes several extensions and generalizations of Banach's fixed point theorem. We can classify such extensions or generalizations in two categories: the first one is concerned with the study of new classes of mappings satisfying generalized contractions, see e.g. [3, 5, 9, 11, 14–16, 18, 21]; the second one is concerned with study of contraction mappings, where X is equipped with a generalized metric, see e.g. [1, 4, 6, 10, 12, 13, 19].

Banach's fixed point theorem asserts that, if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, that is,

$$d(Tu, Tv) \le \lambda d(u, v), \quad u, v \in X \tag{1.1}$$

for some constant $\lambda \in (0, 1)$, then T admits one and only one fixed point. On the other hand, due to the possible errors made in the measurement of the distance between two points, a natural question arises. Namely, if instead (1.1), one has

$$D(Tu, Tv) \le \lambda D(u, v), \quad u, v \in X,$$

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where $D(u,v)$ is the experimental measurement of $d(u,v)$, what we can say in this case? Namely, the Banach fixed point result will be affected by the experimental measurements? Notice that D is not necessarily a metric on X. In this paper, we study the above question by introducing the concept of perturbed metric spaces. Next, an interesting generalization of Banach's fixed point theorem is obtained in the setting of perturbed metric spaces.

The rest of the paper is organized as follows. In Section 2, we introduce the notion of perturbed metric spaces and some topological concepts related to such spaces. Our main result and its proof are given in Section 3. An interesting example is also provided to illustrate our obtained result.

2 Perturbed metric spaces

Throughout this paper, X denotes an arbitrary non-empty set. The set of nonnegative integers is denoted by $\mathbb{N}$.

We introduce below the notion of a perturbed metric sapce.

Definition 2.1. Let $D, P : X \times X \to [0,\infty)$ be two given mappings. We say that D is a perturbed metric on X with respect to P, if

$$\begin{aligned} D - P : X \times X &\to \mathbb{R} \\ (x,y) &\mapsto D(x,y) - P(x,y) \end{aligned}$$

is a metric on X, i.e., for all $x, y, z \in X$,

(i) $(D-P)(x,y) \geq 0$;

(ii) $(D-P)(x,y) = 0$ if and only if $x = y$;

(iii) $(D-P)(x,y) = (D-P)(y,x)$;

(iv) $(D-P)(x,y) \leq (D-P)(x,z) + (D-P)(z,y)$.

We call P a perturbed mapping, $d = D - P$ an exact metric and (X, D, P) a perturbed metric space.

Notice that a perturbed metric on X is not necessarily a metric on X. Some examples are provided below to illustrate this fact.

Example 2.2. Let $D : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ be the mapping defined by

$$D(x,y) = |x-y| + x^2y^4, \quad x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping $P : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ given by

$$P(x,y) = x^2y^4, \quad x, y \in \mathbb{R}.$$

In this case, the exact metric is the mapping $d : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ defined by

$$d(x,y) = |x-y|, \quad x, y \in \mathbb{R}.$$

Remark that D is not a metric on X. This can be easily seen observing that $D(1,1) = 1 \neq 0$.

Example 2.3. Let $D : C([0,1]) \times C([0,1]) \to [0,\infty)$ be the mapping defined by

$$D(f,g) = \int_0^1 |f(t) - g(t)|\,dt + (f(0) - g(0))^2, \quad f,g \in C([0,1]),$$

where $C([0,1)) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous on } [0,1]\}$. Then D is a perturbed metric on $C([0,1])$ with respect to the perturbed mapping $P : C([0,1]) \times C([0,1]) \to [0,\infty)$ given by

$$P(f,g) = (f(0) - g(0))^2, \quad f,g \in C([0,1]).$$

In this case, the exact metric is the mapping $d : C([0,1]) \times C([0,1]) \to [0,\infty)$ defined by

$$d(f,g) = \int_0^1 |f(t) - g(t)|\,dt, \quad f,g \in C([0,1]).$$

Remark that D is symmetric and $D(f,g) = 0$ if and only if $f = g$. However, D is not a metric on $C([0,1])$. Namely, consider three constant functions

$$f_1 \equiv C_1, \quad f_2 \equiv C_2, \quad f_3 \equiv C_3.$$

Then

$$D(f_1,f_3) = |C_1 - C_3| + (C_1 - C_3)^2,$$
$$D(f_1,f_2) = |C_1 - C_2| + (C_1 - C_2)^2$$

and

$$D(f_2,f_3) = |C_2 - C_3| + (C_2 - C_3)^2.$$

In particular, for $(C_1,C_2,C_3) = \left(0,\frac{1}{2},1\right)$, we get

$$D(f_1,f_3) = 2, \quad D(f_1,f_2) = \frac{3}{4}, \quad D(f_2,f_3) = \frac{3}{4},$$

which yields

$$D(f_1,f_3) > D(f_1,f_2) + D(f_2,f_3).$$

This shows that the triangle inequality is violated by D.

Example 2.4. Let $D : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ be the mapping defined by

$$D(n,m) = (n-m)^2, \quad n,m \in \mathbb{N}. \tag{2.1}$$

Then D is a perturbed metric on $\mathbb{N}$, where the perturbed mapping $P : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ is given by

$$P(n,m) = (n-m)^2 - |n-m|, \quad n,m \in \mathbb{N}, \tag{2.2}$$

and the exact metric $d : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ is given by

$$d(n,m) = |n-m|, \quad n,m \in \mathbb{N}. \tag{2.3}$$

Remark that D is not a metric on $\mathbb{N}$ but it is a b-metric on $\mathbb{N}$, see e.g. [6] for more details about b-metric spaces.

In the following example, we construct a perturbed metric with respect to different perturbed mappings.

Example 2.5. Let $X = 2\mathbb{N}$ be the set of nonnegative even integers, that is,

$$X = \{0, 2, \cdots, 2k, \cdots\}.$$

Let $D : 2\mathbb{N} \times 2\mathbb{N} \to [0, \infty)$ be the b-metric on $2\mathbb{N}$ defined by (2.1). From Example (2.4), D is a perturbed metric on $2\mathbb{N}$ with respect to the perturbed mapping $P : 2\mathbb{N} \times 2\mathbb{N} \to [0, \infty)$ given by (2.2), and the exact metric $d : 2\mathbb{N} \times 2\mathbb{N} \to [0, \infty)$ is given by (2.3). Consider now the mapping $Q : 2\mathbb{N} \times 2\mathbb{N} \to [0, \infty)$ defined by

$$Q(n, m) = (n - m)^2 - 2|n - m|, \quad n, m \in 2\mathbb{N}.$$

Then D is also a perturbed metric on $2\mathbb{N}$ with respect to the perturbed mapping Q, and the exact metric is $2d$.

We provide below some elementary properties of perturbed metric spaces.

Proposition 2.6. Let $D, P, Q : X \times X \to [0, \infty)$ be three given mappings and $\alpha > 0$.

(i) If (X, D, P) and (X, D, Q) are two perturbed metric spaces, then $\left(X, D, \frac{P+Q}{2}\right)$ is a perturbed metric space.

(ii) If (X, D, P) is a perturbed metric space, then $(X, \alpha D, \alpha P)$ is a perturbed metric space.

Proof. (i) Since $D - P$ and $D - Q$ are two metrics on X, then

$$\frac{1}{2}\left[(D - P) + (D - Q)\right] = D - \frac{P + Q}{2}$$

is a metric on X, which proves (i).
(ii) Since $D - P$ is a metric on X and $\alpha > 0$, then

$$\alpha(D - P) = \alpha D - \alpha P$$

is a metric on X, which proves (ii). □

We now introduce some topological concepts in perturbed metric spaces.

Definition 2.7. Let (X, D, P) be a perturbed metric space, $\{z_n\}$ a sequence in X and $T : X \to X$.

(i) We say that $\{z_n\}$ is a perturbed convergent sequence in (X, D, P), if $\{z_n\}$ is a convergent sequence in the metric space (X, d), where d is the exact metric (i.e., $d = D - P$).

(ii) We say that $\{z_n\}$ is a perturbed Cauchy sequence in (X, D, P), if $\{z_n\}$ is a Cauchy sequence in the metric space (X, d).

(iii) We say that (X, D, P) is a complete perturbed metric space, if (X, d) is a complete metric space, or, equivalently, if every perturbed Cauchy sequence in (X, D, P) is a perturbed convergent sequence in (X, D, P).

(iv) We say that T is a perturbed continuous mapping, if T is continuous with respect to the exact metric d.

3 A generalization of Banach's fixed point theorem

In this section, we extend the Banach fixed point theorem from standard metric spaces to perturbed metric spaces.

Theorem 3.1. *Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ be a given mapping. Assume that*

(i) *T is a perturbed continuous mapping;*

(ii) *There exists $\lambda \in (0, 1)$ such that*

$$D(Tu, Tv) \leq \lambda D(u, v) \tag{3.1}$$

for all $u, v \in X$.

Then T admits one and only one fixed point.

Proof. Let $z_0 \in X$ be fixed. Consider the Picard sequence $\{z_n\} \subset X$ defined by

$$z_{n+1} = Tz_n, \quad n \in \mathbb{N}.$$

Taking $(u, v) = (z_0, z_1)$ in (3.1), we obtain

$$D(Tz_0, Tz_1) \leq \lambda D(z_0, z_1),$$

thai is,

$$D(z_1, z_2) \leq \lambda D(z_0, z_1). \tag{3.2}$$

Similarly, taking $(u, v) = (z_1, z_2)$ in (3.1), we obtain

$$D(z_2, z_3) \leq \lambda D(z_1, z_2),$$

which implies by (3.2) that

$$D(z_2, z_3) \leq \lambda^2 D(z_0, z_1).$$

Continuing in the same way, by induction, we get

$$D(z_n, z_{n+1}) \leq \lambda^n \tau, \quad n \in \mathbb{N}, \tag{3.3}$$

where $\tau = D(z_0, z_1)$. Let $d = D - P$ be the exact metric. From (3.3), we deduce that

$$d(z_n, z_{n+1}) + P(z_n, z_{n+1}) \leq \lambda^n \tau, \quad n \in \mathbb{N}.$$

Since $d(z_n, z_{n+1}) \leq d(z_n, z_{n+1}) + P(z_n, z_{n+1})$, it holds that

$$d(z_n, z_{n+1}) \leq \lambda^n \tau, \quad n \in \mathbb{N}.$$

Following a standard argument, the above inequality implies that $\{z_n\}$ is a Cauchy sequence in the metric space (X, d), that is, $\{z_n\}$ is a perturbed Cauchy sequence in the perturbed metric space (X, D, P). By the completeness of the perturbed metric space (X, D, P), we deduce that there exists $z^* \in X$ such that

$$\lim_{n\to\infty} d(z_n, z^*) = 0. \tag{3.4}$$

We now show that z^* is a fixed point of T. Since T is a perturbed continuous mapping, then (3.4) yields
$$\lim_{n\to\infty} d(Tz_n, Tz^*) = 0,$$
that is,
$$\lim_{n\to\infty} d(z_{n+1}, Tz^*) = 0. \tag{3.5}$$
Since $d = D - P$ is a metric on X, by the uniqueness of the limit, we get $z^* = Tz^*$, that is, z^* is a fixed point of T.

We now show that T admits a unique fixed point. We argue by contradiction supposing that $u, v \in X$ are two distinct fixed points of T. By (3.1), we have
$$D(u, v) = D(Tu, Tv) \leq \lambda D(u, v),$$
which yields
$$d(u, v) + P(u, v) \leq \lambda(d(u, v) + P(u, v)).$$
Since $u \neq v$, then $d(u, v) + P(u, v) \neq 0$, and the above inequality yields $\lambda \geq 1$, which contradicts the condition $\lambda \in (0, 1)$. Consequently, z^* is the unique fixed point of T. This completes the proof of Theorem 3.1. □

We now show that Theorem 3.1 includes Banach's fixed point theorem.

Corollary 3.2 (Banach's fixed point theorem)**.** Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Assume that there exists $\lambda \in (0, 1)$ such that
$$d(Tu, Tv) \leq \lambda d(u, v) \tag{3.6}$$
for all $u, v \in X$. Then T admits one and only one fixed point.

Proof. Let $D = d$ and $P \equiv 0$ ($P(u, v) = 0$ for all $u, v \in X$). Then (X, D, P) is a perturbed metric space. Furthermore, by (3.6), T is continuous with respect to the exact metric d, and (3.1) holds. Then Theorem 3.1 applies. □

We provide below an interesting example to illustrate Theorem 3.1.

Example 3.3. Let $X = \{A_1, A_2, A_3, A_4\} \subset \mathbb{R}^3$, where A_i, $i = 1, 2, 3, 4$, are the vertices of a regular tetrahedron (see Figure 1) with
$$\|A_i - A_j\| = 1, \quad i \neq j.$$
Here, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^3$. We consider the mapping $T : X \to X$ defined by
$$TA_1 = A_1,\ TA_2 = A_3,\ TA_3 = A_4,\ TA_4 = A_1.$$
We introduce the mapping $P : X \times X \to [0, \infty)$ defined by
$$\begin{aligned}
P(A_1, A_2) &= P(A_2, A_1) = 4,\\
P(A_1, A_3) &= P(A_3, A_1) = 3,\\
P(A_1, A_4) &= P(A_4, A_1) = 2,\\
P(A_2, A_3) &= P(A_3, A_2) = 4,\\
P(A_2, A_4) &= P(A_4, A_2) = 9,\\
P(A_3, A_4) &= P(A_4, A_3) = 3,\\
P(A_i, A_i) &= 0,\ i = 1, 2, 3, 4.
\end{aligned}$$

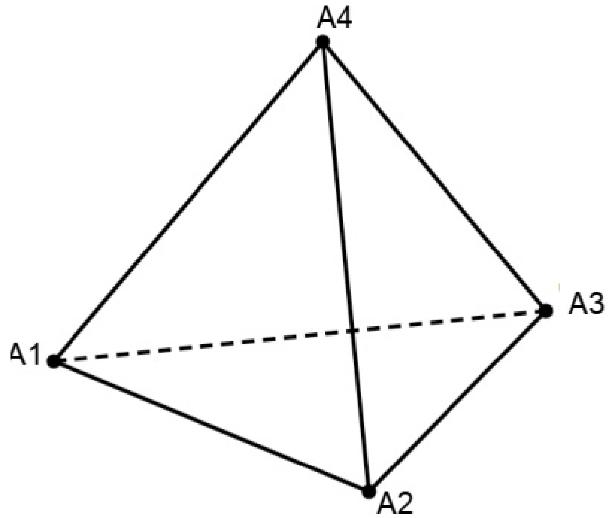

Figure 1: The set X

We also consider the mapping $D : X \times X \to [0,\infty)$ defined by

$$D(A_i, A_j) = \|A_i - A_j\| + P(A_i, A_j), \quad i, j \in \{1, 2, 3, 4\}.$$

Observe that (X, D, P) is a perturbed metric space. In this case, the exact metric is the discrete metric $d : X \times X \to [0,\infty)$ defined by

$$d(A_i, A_j) = \|A_i - A_j\|, \quad i, j \in \{1, 2, 3, 4\},$$

that is,

$$d(A_i, A_j) = \begin{cases} 1 & \text{if} \quad i \neq j, \\ 0 & \text{if} \quad i = j. \end{cases} \tag{3.7}$$

Remark also that D is not a metric on X. Namely, we have

$$\begin{aligned} D(A_2, A_4) &= 1 + P(A_2, A_4) = 1 + 9 = 10, \\ D(A_2, A_3) &= 1 + P(A_2, A_3) = 1 + 4 = 5, \\ D(A_3, A_4) &= 1 + P(A_3, A_4) = 1 + 3 = 4, \end{aligned}$$

which shows that $D(A_2, A_4) > D(A_2, A_3) + D(A_3, A_4)$.

We now show that the mapping T satisfies conditions (i) and (ii) of Theorem 3.1. It is clear that T is a perturbed continuous mapping, that is, (i) is satisfied. On the other hand, (3.1) is equivalent to

$$\|TA_i - TA_j\| + P(TA_i, TA_j) \leq \lambda \left(\|A_i - A_j\| + P(A_i, A_j)\right), \quad i, j \in \{1, 2, 3, 4\}. \tag{3.8}$$

Observe that, if $i = j$, by the definition of the perturbed mapping P, we have

$$\|TA_i - TA_i\| + P(TA_i, TA_i) = 0,$$

which shows that (3.8) is satisfied for all $\lambda > 0$. Assume now that $i \neq j$. Table 1 provides the values of $\|TA_i - TA_j\| + P(TA_i, TA_j)$ and $\|A_i - A_j\| + P(A_i, A_j)$. From Table 1, we

(i,j)	$\|TA_i - TA_j\| + P(TA_i, TA_j)$	$\|A_i - A_j\| + P(A_i, A_j)$
$(1,2)$	4	5
$(1,3)$	3	4
$(1,4)$	0	3
$(2,3)$	4	5
$(2,4)$	4	10
$(3,4)$	3	4

Table 1: The values of $\|TA_i - TA_j\| + P(TA_i, TA_j)$ & $\|A_i - A_j\| + P(A_i, A_j)$

deduce that

$$\max_{1 \leq i < j \leq 4} \frac{\|TA_i - TA_j\| + P(TA_i, TA_j)}{\|A_i - A_j\| + P(A_i, A_j)} \leq \frac{4}{5},$$

Then, by symmetry (notice that P is a symmetric mapping), we deduce that (3.8) holds for all $\lambda \in \left[\frac{4}{5}, 1\right)$, which shows that condition (ii) of Theorem 3.1 is satisfied.

Observe that the only fixed point of T is the point A_1, which confirms the result provided by Theorem 3.1.

We point out that the Banach fixed point theorem (see Corollary 3.2) is not applicable in the metric space (X, d), where d is the exact metric defined by (3.7). This can be easily seen observing for instance that

$$\frac{d(TA_1, TA_2)}{d(A_1, A_2)} = \frac{d(A_1, A_3)}{d(A_1, A_2)} = 1.$$

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