Inertial Extrapolation Scheme for System of Variational Inclusions Using Generalized Yosida and Cayley Operators

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Abstract

The aim of this work is to study a system of variational inclusions involving generalized Yosida and Cayley operators through inertial extrapolation scheme in real Banach space. To obtain faster convergence of the sequences generated by algorithm, we use one inertial extrapolation scheme, although we have established some more iterative schemes. To achieve our goal, we prove an important Lemma ensuring the convergence of sum of two sequences. We provide a numerical example.

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1 Introduction

Variational inclusions are application oriented and can be treated as mathematical version of many problems of day-to-day life, such as economics, physics, engineering and space sciences, etc.. The system of variational inclusions extends the concept of variational inequalities. These systems have applications across various fields such as mathematical analysis, biological sciences, elasticity, image processing, biomedical sciences, and optimization. Furthermore, investigating variational inclusion systems provides novel methods for tackling analytical problems. For more literature on system of variational inclusions, one can see [1–10] and references therein.

Yosida approximation operator are of great importance due to their applications. In the study of wave equations, heat equations and heat flow, etc., one can found the clear applications of Yosida approximation operators. For more details, we refer to [11–13].

The Cayley transform is a mapping that connects skew-symmetric matrices to special orthogonal matrices and is utilized in real, complex, and quaternionic analysis. In the context of Hilbert spaces, it serves as a mapping between linear operators. Considering the real projective line, the Cayley transform permutes the elements $1, 0, -1, \infty$ in sequence and maps the positive real numbers to the interval $[-1, 1]$. Consequently, the Legendre polynomials can be applied to functions on the positive real numbers using the Cayley transform, resulting in Legendre rational functions.

On the Riemann sphere, the Cayley transform is given by

$$f(z) = \frac{z - i}{z + i}.$$

This transform maps the points $\{\infty, 1, -1\}$ to $\{1, -i, i\}$. As a Mbius transformation, it permutes generalized circles in the complex plane, mapping the real line onto the unit circle. For more details, see [14–19].

Various iterative algorithms appeared in the literature using proximal operators, resolvent operators, projection operators as well as sub-differential operator. In order to obtain faster convergence of the sequences generated by the considered algorithm, we have to choose such a scheme which expedite the speed of convergence. Several authors have used inertial extrapolation scheme using inertial extrapolation term $\gamma(u_n - u_{n-1})$, where γ is the extrapolation factor which accelerates the convergence rate of the method. While dealing with heavy ball method, Polyak [20], introduced inertial-type algorithm. There are two steps in the inertial-type algorithm, through these two steps consecutive iterations are gained by using former two terms, for reference see [21–23].

In view of the above mentioned facts, in this paper, we study a system of variational inclusions involving generalized Yosida and Cayley operators using inertial extrapolation scheme in real Banach space. Simultaneously, we have developed some more iterative schemes for our problem. The existence of a solution and the convergence of the sequences produced by our scheme are demonstrated. We construct a numerical example.

2 Preliminary tools and Hypothesis

Let $\widehat{\mathcal{X}}$ be a real Banach space with its topological dual $\widehat{\mathcal{X}}^*$. We denote the norm on $\widehat{\mathcal{X}}$ by $\|\cdot\|$ and duality pairing by $\langle \cdot, \cdot \rangle$ between $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{X}}^*$. The class of all non-empty subsets of $\widehat{\mathcal{X}}$ are denoted by $2^{\widehat{\mathcal{X}}}$.

Definition 2.1. *The normalized duality mapping* $J : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}^*$ *is defined by*

$$J(\widehat{\mathrm{p}}) = \{\widehat{\mathrm{q}} \in \widehat{\mathcal{X}}^* : \langle \widehat{\mathrm{p}}, \widehat{\mathrm{q}} \rangle = \|\widehat{\mathrm{p}}\|^2,\ \|\widehat{\mathrm{p}}\| = \|\widehat{\mathrm{q}}\|\},\ \textit{for all}\ \widehat{\mathrm{p}} \in \widehat{\mathcal{X}}.$$

Definition 2.2. *The operator* $\widetilde{\mathcal{A}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *is said to be*

(i) accretive, if

$$\langle \widetilde{\mathcal{A}}(\widehat{\mathrm{p}}) - \widetilde{\mathcal{A}}(\widehat{\mathrm{q}}), J(\widehat{\mathrm{p}} - \widehat{\mathrm{q}}) \rangle \geq 0,\ \textit{for all}\ \widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}},$$

(ii) strongly accretive, if

$$\langle \widetilde{\mathcal{A}}(\widehat{\mathrm{p}}) - \widetilde{\mathcal{A}}(\widehat{\mathrm{q}}), J(\widehat{\mathrm{p}} - \widehat{\mathrm{q}}) \rangle \geq r_1 \|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|^2,\ \textit{for all}\ \widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}},$$

where $r_1 > 0$ *is a constant,*

(iii) Lipschitz continuous, if

$$\|\widetilde{\mathcal{A}}(\widehat{\mathrm{p}}) - \widetilde{\mathcal{A}}(\widehat{\mathrm{q}})\| \leq \lambda_{\widetilde{\mathcal{A}}} \|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|,\ \textit{for all}\ \widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}},$$

where $\lambda_{\widetilde{\mathcal{A}}} > 0$ *is a constant.*

Definition 2.3. *A multi-valued mapping* $\mathcal{M} : \widehat{\mathcal{X}} \to 2^{\widehat{\mathcal{X}}}$ *is said to be accretive, if for all* $\widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}}$

$$\langle u - v, J(\widehat{\mathrm{p}} - \widehat{\mathrm{q}}) \rangle \geq 0,\ \textit{for all}\ u \in \mathcal{M}(\widehat{\mathrm{p}}),\ v \in \mathcal{M}(\widehat{\mathrm{q}}).$$

Definition 2.4. *[24, 25] Let* $\widetilde{\mathcal{A}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be a mapping. A multi-valued mapping* $\mathcal{M} : \widehat{\mathcal{X}} \to 2^{\widehat{\mathcal{X}}}$ *is said to be* $\widetilde{\mathcal{A}}$*-accretive if* $\mathcal{M}$ *is accretive and*

$$[\widetilde{\mathcal{A}} + \rho\mathcal{M}](\widehat{\mathcal{X}}) = \widehat{\mathcal{X}},\ \textit{where}\ \rho > 0\ \textit{is a constant}.$$

Definition 2.5. *[24] Let* $\widetilde{\mathcal{A}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be a mapping and* $\mathcal{M} : \widehat{\mathcal{X}} \to 2^{\widehat{\mathcal{X}}}$ *be A-accretive multi-valued mapping. The generalized resolvent operator* $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *associated with* $\widetilde{\mathcal{A}}$*, is defined as:*

$$R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) = [\widetilde{\mathcal{A}} + \rho\mathcal{M}]^{-1}(\mathrm{p}),\quad \textit{for all}\ \widehat{\mathrm{p}} \in \widehat{\mathcal{X}}\ \textit{and}\ \rho > 0\ \textit{is a constant}.$$

Theorem 2.1. *[26] Let* $\widetilde{\mathcal{A}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be strongly accretive operator with constant* r_1 *and* $\mathcal{M} : \widehat{\mathcal{X}} \to 2^{\widehat{\mathcal{X}}}$ *be* $\widetilde{\mathcal{A}}$*-accretive multi-valued mapping. Then*

$$\left\| R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) - R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{q}}) \right\| \leq \frac{1}{r_1} \|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|,\ \textit{for all}\ \widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}}.$$

That is, the generalized resolvent operator $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$ *is Lipschitz continuous.*

Definition 2.6. *[26] Let* $\widetilde{\mathcal{B}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be a mapping and* $R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *is the generalized resolvent operator associated with* $\widetilde{\mathcal{B}}$*. The generalized Cayley operator* $C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *is defined as*

$$C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{p}}) = \left[2R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} - \widetilde{\mathcal{B}}\right](\widehat{\mathrm{p}}),\ \textit{for all}\ \widehat{\mathrm{p}} \in \widehat{\mathcal{X}}\ \textit{and}\ \gamma > 0\ \textit{is a constant}.$$

Proposition 2.1. *[26, 27] The generalized Cayley operator $C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ is Lipschitz continuous with constant λ_C, that is*

$$\left\| C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{p}}) - C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}}) \right\| \leq \lambda_C \|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|, \ \textit{for all } \widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}},$$

where $\lambda_C = \dfrac{2 + \lambda_{\widetilde{\mathcal{B}}} r_2}{r_2}$ *and the generalized resolvent operator* $R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *is* $\dfrac{1}{r_2}$*-Lipschitz continuous.*

Definition 2.7. *[28] The generalized Yosida approximation operator $Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ is defined as*

$$Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) = \frac{1}{\rho}\left[\widetilde{\mathcal{A}} - R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\right](\widehat{\mathrm{p}}), \ \textit{for all } \widehat{\mathrm{p}} \in \widehat{\mathcal{X}} \textit{ and } \rho > 0 \textit{ is a constant.}$$

Proposition 2.2. *[28] The generalized Yosida approximation operator $Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ is Lipschitz continuous with constant λ_Y, that is*

$$\left\| Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) - Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{q}}) \right\| \leq \lambda_Y \|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|, \ \textit{for all } \widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}},$$

where $\lambda_Y = \dfrac{\lambda_{\widetilde{\mathcal{A}}} r_1 + 1}{\rho r_1}$ *and the generalized resolvent operator* $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *is* $\dfrac{1}{r_1}$*-Lipschitz continuous.*

Lemma 2.1. *Let $\{s_n\}$ and $\{t_n\}$ be sequences of non-negative real numbers such that*

$$\begin{aligned} s_{n+1} &\leq (1 - a_n)s_n + a_n\hat{\alpha_n} + \xi_n \\ \text{and} \quad \mathrm{t}_{n+1} &\leq (1 - a_n)t_n + a_n\hat{\beta_n} + \delta_n, \quad \textit{for all } n \geq 1, \end{aligned}$$

where

(i) $\{a_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} a_n = \infty$ *or equivalently* $\Pi_{n=1}^{\infty}(1 - a_n) = 0$,

(ii) $\limsup(\hat{\alpha_n} + \hat{\beta_n}) \leq 0$,

(iii) $\xi_n \geq 0$, $\delta_n \geq 0$, $\sum_{n=1}^{\infty} \xi_n < \infty$, $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then

$$s_n + t_n \to 0, \quad \textit{as} \quad n \to \infty.$$

Proof. For any $\epsilon > 0$, let N be an integer such that

$$\alpha_n < \frac{\epsilon}{2}, \ \beta_n < \frac{\epsilon}{2}, \ \sum_{n=N}^{\infty} \xi_n < \frac{\epsilon}{2} \textit{ and } \sum_{n=N}^{\infty} \delta_n < \frac{\epsilon}{2}, n \geq N.$$

Given,

$$s_{n+1} \leq (1 - a_n)s_n + a_n\hat{\alpha_n} + \xi_n, \tag{2.1}$$

so, we have

$$\begin{aligned}
s_1 &\leq (1-a_0)s_0 + a_0\hat{\alpha_0} + \xi_0,\\
s_2 &\leq (1-a_1)s_1 + a_1\hat{\alpha_1} + \xi_1,\\
&\vdots\\
s_N &\leq (1-a_{N-1})s_{N-1} + a_{N-1}\hat{\alpha}_{N-1} + \xi_{N-1},\\
&\vdots\\
s_n &\leq (1-a_{n-1})s_{n-1} + a_{n-1}\hat{\alpha}_{n-1} + \xi_{n-1}.
\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$s_{n+1} \leq (1-a_n)\left\{(1-a_{n-1})s_{n-1} + a_{n-1}\hat{\alpha}_{n-1} + \xi_{n-1}\right\} + a_n\hat{\alpha_n} + \xi_n. \tag{2.3}$$

Rearranging the terms and using all the previous inequalities with (2.3), we have

$$s_{n+1} \leq \left(\Pi_{k=N}^{n}(1-a_k)\right) s_N + \left(1-\Pi_{k=N}^{n}(1-a_k)\right)\frac{\epsilon}{2} + \sum_{k=N}^{n}\xi_k. \tag{2.4}$$

Similarly, we can write

$$t_{n+1} \leq \left(\Pi_{k=N}^{n}(1-a_k)\right) t_N + \left(1-\Pi_{k=N}^{n}(1-a_k)\right)\frac{\epsilon}{2} + \sum_{k=N}^{n}\delta_k. \tag{2.5}$$

Adding (2.4) and (2.5), we obtain

$$(s_{n+1}+t_{n+1}) \leq \left(\Pi_{k=N}^{n}(1-a_k)\right)(s_N+t_N) + \left(1-\Pi_{k=N}^{n}(1-a_k)\right)\epsilon + \sum_{k=N}^{n}\xi_k + \sum_{k=N}^{n}\delta_k. \tag{2.6}$$

Using conditions (ii) and (iii), we obtain

$$s_n + t_n \to 0, \quad as \quad n \to \infty.$$

□

3 Phrasing of Problem and Iterative Schemes

Let $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ be single-valued mappings. Suppose that $\mathcal{M}, \mathcal{N} : \widehat{\mathcal{X}} \to CB(\widehat{\mathcal{X}})$ are multi-valued mappings, $Y_{\widetilde{\mathcal{A}},\rho}^{\mathcal{M}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ and $C_{\widetilde{\mathcal{B}},\gamma}^{\mathcal{N}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ are generalized Yosida approximation operator and generalized Cayley operator, respectively. We will study the following system of variational inclusions involving generalized Yosida and Cayley operators.

Find $\widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}}$ such that

$$\begin{aligned}
0 &\in Y_{\widetilde{\mathcal{A}},\rho}^{\mathcal{M}}(\widehat{\mathrm{p}}) + \mathcal{M}(\widehat{\mathrm{q}})\\
0 &\in C_{\widetilde{\mathcal{B}},\gamma}^{\mathcal{N}}(\widehat{\mathrm{q}}) + \mathcal{N}(\widehat{\mathrm{p}}).
\end{aligned} \tag{3.1}$$

If $Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) = 0 = C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}})$, then the problem (3.1) reduces to the system of variational inclusions, that is, find $\widehat{\mathrm{p}}, \widehat{\mathrm{q}}, \in \widehat{\mathcal{X}}$ such that

$$\begin{aligned} &0 \in \mathcal{M}(\widehat{\mathrm{q}}) \\ &0 \in \mathcal{N}(\widehat{\mathrm{p}}). \end{aligned} \tag{3.2}$$

One can obtain many previously studied systems of variational inclusions from system (3.1).

The fixed point formulation of system (3.1) is given below.

Lemma 3.1. *The system of variational inclusions involving generalized Yosida and Cayley operators* (3.1) *has a solution* $\widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}}$ *if and only if the following system of fixed point equations is satisfied:*

$$\widehat{\mathrm{q}} = R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}) - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}})\right], \tag{3.3}$$

$$\widehat{\mathrm{p}} = R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\left[\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}) - \gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}})\right]. \tag{3.4}$$

Proof. Proof is easy and hence omitted. □

Using Lemma 3.1, we suggest the following iterative scheme for solving system (3.1).

Iterative Scheme 3.1. *For any* $\widehat{\mathrm{p}}_0, \widehat{\mathrm{q}}_0 \in \widehat{\mathcal{X}}$*, compute sequences* $\{\widehat{\mathrm{p}}_n\}$ *and* $\{\widehat{\mathrm{q}}_n\}$ *by the following scheme:*

$$\widehat{\mathrm{q}}_{n+1} = (1-\alpha_n)\widehat{\mathrm{q}}_n + \alpha_n R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[A(\widehat{\mathrm{q}}_n) - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}_n)\right], \tag{3.5}$$

$$\widehat{\mathrm{p}}_{n+1} = (1-\beta_n)\widehat{\mathrm{p}}_n + \beta_n R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\left[\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}_n) - \gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}}_n)\right], \tag{3.6}$$

where $n = 0, 1, 2, \cdots,\ \alpha_n, \beta_n \in [0,1], \rho > 0$ *and* $\gamma > 0$ *are constants.*

Equations (3.3) *and* (3.4) *can be rewritten as*

$$\widehat{\mathrm{q}} = R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}) + \widetilde{\mathcal{A}}(\widehat{\mathrm{q}})}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}})\right], \tag{3.7}$$

$$\text{and} \quad \widehat{\mathrm{p}} = R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\left[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}) + \widetilde{\mathcal{B}}(\widehat{\mathrm{p}})}{2} - \gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}})\right]. \tag{3.8}$$

Based on (3.7) *and* (3.8)*, we suggest the following iterative scheme to solve the system* (3.1).

Iterative Scheme 3.2. *For any* $\widehat{\mathrm{p}}_0, \widehat{\mathrm{q}}_0 \in \widehat{\mathcal{X}}$*, compute the sequences* $\{\widehat{\mathrm{p}}_{n+1}\}$ *and* $\{\widehat{\mathrm{q}}_{n+1}\}$ *by the recurrence relations:*

$$\widehat{\mathrm{q}}_{n+1} = (1-\alpha_n)\widehat{\mathrm{q}}_n + \alpha_n R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}_n) + \widetilde{\mathcal{A}}(\widehat{\mathrm{q}}_{n+1})}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}_{n+1})\right], \tag{3.9}$$

$$\widehat{\mathrm{p}}_{n+1} = (1-\beta_n)\widehat{\mathrm{p}}_n + \beta_n R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\left[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}_n) + \widetilde{\mathcal{B}}(\widehat{\mathrm{p}}_{n+1})}{2} - \gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}}_{n+1})\right]. \tag{3.10}$$

Where $n = 0, 1, 2, \cdots,\ \alpha_n, \beta_n \in [0,1], \rho > 0$ *and* $\gamma > 0$ *are constants.*

We established the following inertial extrapolation scheme.

Iterative Scheme 3.3. *For any* $\widehat{\mathrm{p}}_0, \widehat{\mathrm{q}}_0 \in \widehat{\mathcal{X}}$, *compute the sequences* $\{\widehat{\mathrm{p}}_{n+1}\}$ *and* $\{\widehat{\mathrm{q}}_{n+1}\}$ *by the recurrence relations:*

$$\begin{aligned}
\widehat{w}_n &= \widehat{\mathrm{q}}_n + \gamma'_n(\widehat{\mathrm{p}}_n - \widehat{\mathrm{p}}_{n-1}), && (3.11)\\
\widehat{\mathrm{q}}_{n+1} &= (1-\alpha_n)\widehat{\mathrm{q}}_n + \alpha_n R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}_n) + \widetilde{\mathcal{A}}(\widehat{w}_n)}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{w}_n)\right], && (3.12)\\
\widehat{v}_n &= \widehat{\mathrm{p}}_n + \gamma''_n(\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}_{n-1}), && (3.13)\\
\widehat{\mathrm{p}}_{n+1} &= (1-\beta_n)\widehat{\mathrm{p}}_n + \beta_n R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\left[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}_n) + \widetilde{\mathcal{B}}(\widehat{v}_n)}{2} - \gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{v}_n)\right]. && (3.14)
\end{aligned}$$

Where $\alpha_n, \beta_n, \gamma'_n, \gamma''_n \in [0,1]$, γ'_n *and* γ''_n *are the extrapolating terms for* $n \geq 1, \rho > 0$ *and* $\gamma > 0$ *are constants.*

4 Existence and Convergence Results

Existence and convergence results for the system (3.1) discussed below.

Theorem 4.1. *Let* $\widehat{\mathcal{X}}$ *be a real Banach space. Let* $\widetilde{\mathcal{A}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be the single-valued mappings such that* $\widetilde{\mathcal{A}}$ *is* $\lambda_{\widetilde{\mathcal{A}}}$*-Lipschitz continuous, strongly accretive with constant* r_1; $\widetilde{\mathcal{B}}$ *is* $\lambda_{\widetilde{\mathcal{B}}}$*-Lipschitz continuous and strongly accretive with constant* r_2. *Let* $\mathcal{M}, \mathcal{N} : \widehat{\mathcal{X}} \to 2^{\widehat{\mathcal{X}}}$ *be the multi-valued mappings such that* $\mathcal{M}$ *is* $\widetilde{\mathcal{A}}$*-accretive and* $\mathcal{N}$ *is* $\widetilde{\mathcal{B}}$*-accretive. Let* $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}, R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be the generalized resolvent operators such that* $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$ *is* $\frac{1}{r_1}$*-Lipschitz continuous and* $R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}$ *is* $\frac{1}{r_2}$*-Lipschitz continuous. Let* $Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be the generalized Yosida approximation operator and* $C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ *be the generalized Cayley operator such that* $Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$ *is* λ_Y*-Lipschitz continuous and* $C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}$ *is* λ_C*-Lipschitz continuous. Suppose that the following conditions are satisfied for* $\alpha_n, \beta_n, \gamma'_n, \gamma''_n \in [0,1]$, *for all* $n \geq 1$ *such that*

$$\frac{r_1 + \lambda_{\widetilde{\mathcal{A}}}}{r_1} > 1, \quad \frac{r_2 + \lambda_{\widetilde{\mathcal{B}}}}{r_2} > 1, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} \beta_n = \infty. \tag{4.1}$$

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty} \gamma'_n\Big[\xi(\theta_1)\|\widehat{\mathrm{p}}_n - \widehat{\mathrm{p}}_{n-1}\|\Big] < \infty\\
&\sum_{n=1}^{\infty} \gamma''_n\Big[\xi(\theta_2)\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}_{n-1}\|\Big] < \infty
\end{aligned}\right\}. \tag{4.2}$$

$$\limsup[\alpha_n \theta \rho \lambda_Y] \leq 0, \quad \limsup[\beta_n \theta' \gamma \lambda_C] \leq 0 \quad \}, \tag{4.3}$$

where $\theta = \frac{1}{r_1}$, $\theta' = \frac{1}{r_2}$, $\lambda_Y = \frac{\lambda_{\widetilde{\mathcal{A}}} r_1 + 1}{\rho r_1}$, $\lambda_C = \frac{2 + \lambda_{\widetilde{\mathcal{B}}} r_2}{r_2}$, *all the constants are positive and* γ'_n, γ''_n *are the extrapolating terms.*

Then, the sequences $\{\widehat{\mathrm{p}}_n\}$ *and* $\{\widehat{\mathrm{q}}_n\}$ *produced by scheme 3.1 converge strongly to the solution of system* (3.1).

Proof. Let $\widehat{\mathrm{p}}, \widehat{\mathrm{q}} \in \widehat{\mathcal{X}}$ be the solution of system of variational inclusions involving generalized Yosida and Cayley operators (3.1). Using (3.7) and (3.8), we have

$$\begin{aligned} \widehat{\mathrm{q}}^* &= (1-\alpha_n)\widehat{\mathrm{q}}^* + \alpha_n R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)+\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}^*)\right], \quad (4.4) \\ \widehat{\mathrm{p}}^* &= (1-\beta_n)\widehat{\mathrm{p}}^* + \beta_n R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\left[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)+\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)}{2} - \gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}}^*)\right], \quad (4.5) \end{aligned}$$

where $\alpha_n, \beta_n \in [0,1]$, for all $n \geq 1$. Using (3.12), (4.4) and (3.11) and Lipschitz continuity of the generalized resolvent operator $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$, we evaluate

$$\begin{aligned} \|\widehat{\mathrm{q}}_{n+1} - \widehat{\mathrm{q}}^*\| &= \Bigg\| \left\{(1-\alpha_n)\widehat{\mathrm{q}}_n + \alpha_n R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}_n)+\widetilde{\mathcal{A}}(\widehat{w}_n)}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{w}_n)\right]\right\} \\ &\quad - \left\{(1-\alpha_n)\widehat{\mathrm{q}}^* + \alpha_n R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)+\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}^*)\right]\right\} \Bigg\| \\ &\leq (1-\alpha_n)\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| + \alpha_n\theta \Bigg\| \left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}_n)+\widetilde{\mathcal{A}}(\widehat{w}_n)}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{w}_n)\right] \\ &\quad - \left[\frac{\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)+\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)}{2} - \rho Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}^*)\right] \Bigg\| \\ &\leq (1-\alpha_n)\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| + \frac{\alpha_n\theta}{2}\|\widetilde{\mathcal{A}}(\widehat{\mathrm{q}}_n) - \widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)\| \\ &\quad + \frac{\alpha_n\theta}{2}\|\widetilde{\mathcal{A}}(\widehat{w}_n) - \widetilde{\mathcal{A}}(\widehat{\mathrm{q}}^*)\| + \alpha_n\theta\rho \left\| Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{w}_n) - Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}^*)\right\|. \quad (4.6) \end{aligned}$$

Using the Lipschitz continuity of the mapping $\widetilde{\mathcal{A}}$ and generalized Yosdia approximation operator $Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$, from (4.6), we obtain

$$\begin{aligned} \|\widehat{\mathrm{q}}_{n+1} - \widehat{\mathrm{q}}^*\| &\leq (1-\alpha_n)\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| + \frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| \\ &\quad + \frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}\|\widehat{w}_n - \widehat{\mathrm{q}}^*\| + \alpha_n\theta\rho\lambda_Y \|\widehat{w}_n - \widehat{\mathrm{p}}^*\|. \quad (4.7) \end{aligned}$$

Applying (3.11), we can write

$$\begin{aligned} \|\widehat{w}_n - \widehat{\mathrm{q}}^*\| &= \|\widehat{\mathrm{q}}_n + \gamma_n'(\widehat{\mathrm{p}}_n - \widehat{\mathrm{p}}_{n-1}) - \widehat{\mathrm{q}}^*\| \\ &\leq \|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| + \gamma_n'\|\widehat{\mathrm{p}}_n - \widehat{\mathrm{p}}_{n-1}\|, \quad (4.8) \end{aligned}$$

and

$$\|\widehat{w}_n - \widehat{\mathrm{p}}^*\| \leq \|\widehat{\mathrm{q}}_n - \widehat{\mathrm{p}}^*\| + \gamma_n'\|\widehat{\mathrm{p}}_n - \widehat{\mathrm{p}}_{n-1}\|. \quad (4.9)$$

Making use of (4.8) and (4.9), (4.7) becomes

$$\begin{aligned} \|\widehat{\mathrm{q}}_{n+1} - \widehat{\mathrm{q}}^*\| &\leq (1-\alpha_n)\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| + \frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| + \frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}\Big[\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{q}}^*\| \\ &\quad + \gamma_n'\|\widehat{\mathrm{p}}_n - \widehat{\mathrm{p}}_{n-1}\|\Big] + \alpha_n\theta\rho\lambda_Y\Big[\|\widehat{\mathrm{q}}_n - \widehat{\mathrm{p}}^*\| + \gamma_n'\|\widehat{\mathrm{p}}_n - \widehat{\mathrm{p}}_{n-1}\|\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq \Big[(1-\alpha_n)+\frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}+\frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}\Big]\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}^*\| \\
&\quad +\frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}\gamma_n'\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}_{n-1}\|+\alpha_n\theta\rho\lambda_Y\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{p}}^*\| \\
&\qquad +\alpha_n\theta\rho\lambda_Y\gamma_n'\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}_{n-1}\| \\
&\leq \Big[(1-\alpha_n)+\alpha_n\theta\ \lambda_{\widetilde{\mathcal{A}}}\Big]\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}^*\| \\
&\quad +\Big[\Big(\frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}+\alpha_n\theta\rho\lambda_Y\Big)\gamma_n'\Big]\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}_{n-1}\| \\
&\qquad +\alpha_n\theta\rho\lambda_Y\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{p}}^*\|. \qquad (4.10)
\end{aligned}
$$

Using (3.14), (4.5) and using the Lipschitz continuity of generalized resolvent operator $R^M_{\widetilde{\mathcal{B}},\gamma}$, we evaluate

$$
\begin{aligned}
\|\widehat{\mathrm{p}}_{n+1}-\widehat{\mathrm{p}}^*\| &= \Big\|\Big\{(1-\beta_n)\widehat{\mathrm{p}}_n+\beta_n R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\Big[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}_n)+\widetilde{\mathcal{B}}(\widehat{v}_n)}{2}-\gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{v}_n)\Big]\Big\} \\
&\quad -\Big\{(1-\beta_n)\widehat{\mathrm{p}}^*+\beta_n R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}\Big[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)+\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)}{2}-\gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}}^*)\Big]\Big\}\Big\| \\
&\leq (1-\beta_n)\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|+\beta_n\theta'\Big\|\Big[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}_n)+\widetilde{\mathcal{B}}(\widehat{v}_n)}{2}-\gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{v}_n)\Big] \\
&\qquad -\Big[\frac{\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)+\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)}{2}-\gamma C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}}^*)\Big]\Big\| \\
&\leq (1-\beta_n)\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|+\frac{\beta_n\theta'}{2}\|\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}_n)-\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)\| \\
&\quad +\frac{\beta_n\theta'}{2}\|\widetilde{\mathcal{B}}(\widehat{v}_n)-\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}^*)\|+\beta_n\theta'\gamma\Big\|C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{v}_n)-C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}}^*)\Big\|. \qquad (4.11)
\end{aligned}
$$

Using the Lipschitz continuity of the mapping $\widetilde{\mathcal{B}}$ and generalized Cayley operator $C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}$, from (4.11), we obtain

$$
\begin{aligned}
\|\widehat{\mathrm{p}}_{n+1}-\widehat{\mathrm{p}}^*\| &\leq (1-\beta_n)\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|+\frac{\beta_n\theta'}{2}\lambda_{\widetilde{\mathcal{B}}}\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\| \\
&\quad +\frac{\beta_n\theta'}{2}\lambda_{\widetilde{\mathcal{B}}}\|\widehat{v}_n-\widehat{\mathrm{p}}^*\|+\beta_n\theta'\gamma\lambda_C\|\widehat{v}_n-\widehat{\mathrm{q}}^*\|. \qquad (4.12)
\end{aligned}
$$

Applying (3.13), we can write

$$
\begin{aligned}
\|\widehat{v}_n-\widehat{\mathrm{p}}^*\| &= \|\widehat{\mathrm{p}}_n+\gamma_n''(\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1})-\widehat{\mathrm{p}}^*\| \\
&\leq \|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|+\gamma_n''\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|, \qquad (4.13)
\end{aligned}
$$

and

$$
\|\widehat{v}_n-\widehat{\mathrm{q}}^*\|\leq\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{q}}^*\|+\gamma_n''\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|. \qquad (4.14)
$$

Making use of (4.13) and (4.14), (4.12) becomes

$$
\|\widehat{\mathrm{p}}_{n+1}-\widehat{\mathrm{p}}^*\| \leq (1-\beta_n)\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|+\frac{\beta_n\theta'}{2}\lambda_{\widetilde{\mathcal{B}}}\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|+\frac{\beta_n\theta'}{2}\lambda_{\widetilde{\mathcal{B}}}\Big[\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|
$$

$$
\begin{aligned}
&+\gamma_n''\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|\Big]+\beta_n\theta'\gamma\lambda_C\Big[\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{q}}^*\|+\gamma_n''\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|\Big]\\
\leq\ &\Big[(1-\beta_n)+\beta_n\theta'\lambda_{\widetilde{\mathcal{B}}}\Big]\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|\\
&+\Big[\left(\frac{\beta_n\theta'}{2}\lambda_{\widetilde{\mathcal{B}}}+\beta_n\theta'\gamma\lambda_C\right)\gamma_n''\Big]\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|\\
&\qquad+\beta_n\theta'\gamma\lambda_C\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{q}}^*\|. \qquad (4.15)
\end{aligned}
$$

Adding (4.10) and (4.15), we obtain

$$
\begin{aligned}
\|\widehat{\mathrm{q}}_{n+1}-\widehat{\mathrm{q}}^*\|+\|\widehat{\mathrm{p}}_{n+1}-\widehat{\mathrm{p}}^*\| \leq\ &\Big[(1-\alpha_n)+\alpha_n\theta\ \lambda_{\widetilde{\mathcal{B}}}\Big]\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}^*\|\\
&+\Big[\left(\frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{B}}}+\alpha_n\theta\rho\lambda_Y\right)\gamma_n'\Big]\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}_{n-1}\|\\
&\qquad+\alpha_n\theta\rho\lambda_Y\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{p}}^*\|\\
&+\Big[(1-\beta_n)+\beta_n\theta'\lambda_B\Big]\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|\\
&+\Big[\left(\frac{\beta_n\theta'}{2}\lambda_{\widetilde{\mathcal{B}}}+\beta_n\theta'\gamma\lambda_C\right)\gamma_n''\Big]\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|\\
&\qquad+\beta_n\theta'\gamma\lambda_C\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{q}}^*\|\\
=\ &\Big[(1-\alpha_n(1-\theta\lambda_{\widetilde{\mathcal{B}}})\Big]\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}^*\|\\
&+\Big[(1-\beta_n(1-\theta'\lambda_{\widetilde{\mathcal{B}}})\Big]\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|\\
&+\xi(\theta_1)\gamma_n'\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}_{n-1}\|+\xi(\theta_2)\gamma_n''\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|\\
&+\alpha_n\theta\rho\lambda_Y\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{p}}^*\|+\beta_n\theta'\gamma\lambda_C\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{q}}^*\|\\
=\ &\xi(\hat{\theta})[\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}^*\|+\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}^*\|]\\
&+\xi(\theta_1)\gamma_n'\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}_{n-1}\|+\xi(\theta_2)\gamma_n''\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|\\
&+\alpha_n\theta\rho\lambda_Y\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{p}}^*\|+\beta_n\theta'\gamma\lambda_C\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{q}}^*\|,
\end{aligned}
\qquad (4.16)
$$

where

$$
\xi(\hat{\theta})=\mathrm{map}\{[1-\alpha_n(1-\theta\lambda_{\widetilde{\mathcal{A}}})],[1-\beta_n(1-\theta'\lambda_{\widetilde{\mathcal{B}}})]\},
$$
$$
\xi(\theta_1)=\frac{\alpha_n\theta}{2}\lambda_{\widetilde{\mathcal{A}}}+\alpha_n\theta\rho\lambda_Y,\qquad \xi(\theta_2)=\frac{\beta_n\theta}{2}\lambda_{\widetilde{\mathcal{B}}}+\beta_n\theta'\gamma\lambda_C,
$$

By condition (4.1),

$$
1-\theta\lambda_{\widetilde{\mathcal{A}}}<1,\quad 1-\theta'\lambda_{\widetilde{\mathcal{B}}}<1,\quad \sum_{n=1}^{\infty}\alpha_n=\infty\ \ \textit{and}\ \ \sum_{n=1}^{\infty}\beta_n=\infty.
$$

By condition (4.2),

$$
\left.\begin{aligned}
&\sum_{n=1}^{\infty}\gamma_n'\Big[\xi(\theta_1)\|\widehat{\mathrm{p}}_n-\widehat{\mathrm{p}}_{n-1}\|\Big]<\infty\\
&\sum_{n=1}^{\infty}\gamma_n''\Big[\xi(\theta_2)\|\widehat{\mathrm{q}}_n-\widehat{\mathrm{q}}_{n-1}\|\Big]<\infty
\end{aligned}\right\}.
$$

Also, applying condition (4.3), we have

$$\limsup[\alpha_n\theta\rho\lambda_Y] \leq 0, \quad \limsup[\beta_n\theta'\gamma\lambda_C] \leq 0.$$

Applying Lemma 2.1, we conclude that $\widehat{\mathrm{p}}_n \to \widehat{\mathrm{p}}$ and $\widehat{\mathrm{q}}_n \to \widehat{\mathrm{q}}$, as $n \to \infty$. This completes the proof. $\square$

The following numerical example is constructed showing that all the conditions of Theorem 4.1 are satisfied. We also establish the convergence graph and computational table for illustration.

Example 4.1. *Let $\widehat{\mathcal{X}} = \mathbb{R}$ with usual inner product and norm. Let $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ be the mappings such that $\widetilde{\mathcal{A}}(\widehat{\mathrm{p}}) = (\frac{3\widehat{\mathrm{p}}}{2})$ and $\widetilde{\mathcal{B}}(\widehat{\mathrm{p}}) = (\frac{7\widehat{\mathrm{p}}}{6})$ and the multi-valued mappings $\mathcal{M}, \mathcal{N} : \widehat{\mathcal{X}} \to CB(\widehat{\mathcal{X}})$ defined by $\mathcal{M}(\widehat{\mathrm{p}}) = \{\frac{2\widehat{\mathrm{p}}}{5}\}$ and $\mathcal{N}(\widehat{\mathrm{p}}) = \{\frac{3\widehat{\mathrm{p}}}{7}\}$.*

(i) *$\widetilde{\mathcal{A}}$ is $\lambda_{\widetilde{\mathcal{A}}}$-Lipschitz and r_1-strongly accretive*

$$\begin{aligned}\|\widetilde{\mathcal{A}}(\widehat{\mathrm{p}}) - \widetilde{\mathcal{A}}(\widehat{\mathrm{q}})\| &= \|\frac{3\widehat{\mathrm{p}}}{2} - \frac{3\widehat{\mathrm{q}}}{2}\| \\ &= \frac{3}{2}\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\| \\ &\leq 2\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|,\end{aligned}$$

that is, $\widetilde{\mathcal{A}}$ is $\lambda_{\widetilde{\mathcal{A}}} = 2$-Lipschitz continuous.
$\widetilde{\mathcal{A}}$ is r_1-strongly accretive.

$$\begin{aligned}\langle \widetilde{\mathcal{A}}(\widehat{\mathrm{p}}) - \widetilde{\mathcal{A}}(\widehat{\mathrm{q}}), \widehat{\mathrm{p}} - \widehat{\mathrm{q}}\rangle &= \langle \frac{3\widehat{\mathrm{p}}}{2} - \frac{3\widehat{\mathrm{q}}}{2}, \widehat{\mathrm{p}} - \widehat{\mathrm{q}}\rangle \\ &= \frac{3}{2}\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|^2 \geq \frac{2}{3}\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|^2,\end{aligned}$$

that is, $\widetilde{\mathcal{A}}$ is $r_1 = \frac{2}{3}$ strongly accretive.

(ii) *Similarly, one can prove that $\widetilde{\mathcal{B}}$ is $\lambda_{\widetilde{\mathcal{B}}} = \frac{8}{6}$-Lipschitz continuous and $r_2 = \frac{3}{4}$-strongly accretive.*

(iii) *For $\rho = \gamma = 1$, it is easy to show that $\mathcal{M}$ is $\widetilde{\mathcal{A}}$-accretive mapping and $\mathcal{N}$ is $\widetilde{\mathcal{B}}$-accretive mapping.*

(iv) *For $\rho = \gamma = 1$, we calculate*

$$R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) = [\widetilde{\mathcal{A}} + \rho\mathcal{M}]^{-1}(\widehat{\mathrm{p}}) = \left(\frac{10\mathrm{p}}{19}\right),$$

$$R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{p}}) = [\widetilde{\mathcal{B}} + \gamma\mathcal{N}]^{-1}(\widehat{\mathrm{p}}) = \left(\frac{42\widehat{\mathrm{p}}}{67}\right).$$

The Lipschitz continuity of $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$ and $R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}$ is calculated below:

$$\|R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) - R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{q}})\| = \|\frac{10\widehat{\mathrm{p}}}{19} - \frac{10\widehat{\mathrm{q}}}{19}\|$$

$$\leq \frac{3}{2}\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|.$$

Similarly,

$$\|R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{p}}) - R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}})\| = \|\frac{42\widehat{\mathrm{p}}}{67} - \frac{42\widehat{\mathrm{q}}}{67}\|$$
$$\leq \frac{4}{3}\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|,$$

that is, $R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$ *is* $\frac{1}{(2/3)}$*-Lipschitz continuous and* $R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}$ *is* $\frac{1}{(3/4)}$*-Lipschitz continuous.*

(*v*) *The generalized Yosida approximation operator and generalized Cayley operator are calculated as:*

$$Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) = \frac{1}{\rho}\left[\widetilde{\mathcal{A}} - R^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}\right](\widehat{\mathrm{p}}) = \left(\frac{37\widehat{\mathrm{p}}}{38}\right),$$
$$C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{p}}) = \left[2R^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma} - \widetilde{\mathcal{B}}\right](\widehat{\mathrm{p}}) = \left(\frac{35\widehat{\mathrm{p}}}{402}\right).$$

Also,
$$\|Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{p}}) - Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}(\widehat{\mathrm{q}})\| = \left\|\frac{37\widehat{\mathrm{p}}}{38} - \frac{37\widehat{\mathrm{q}}}{38}\right\|$$
$$\leq \frac{75}{76}\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|,$$

that is, $Y^{\mathcal{M}}_{\widetilde{\mathcal{A}},\rho}$ *is* $\lambda_Y = \frac{75}{76}$*-Lipschitz continuous. And*

$$\|C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{p}}) - C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}(\widehat{\mathrm{q}})\| = \left\|\frac{35\widehat{\mathrm{p}}}{402} - \frac{35\widehat{\mathrm{q}}}{402}\right\|$$
$$\leq \frac{13}{134}\|\widehat{\mathrm{p}} - \widehat{\mathrm{q}}\|,$$

that is, $C^{\mathcal{N}}_{\widetilde{\mathcal{B}},\gamma}$ *is* $\lambda_C = \frac{13}{134}$*-Lipschitz continuous.*

(*vi*) *For* $\alpha_n = 1 - \frac{1}{n}$, $\beta_n = 1 - \frac{1}{n+2}$, *we compute the sequence* $\widehat{\mathrm{p}}_n$ *and* $\widehat{\mathrm{q}}_n$ *by the Iterative scheme 3.1, in the following way:*

$$\widehat{\mathrm{q}}_{n+1} = \left(\frac{1}{n}\right)\widehat{\mathrm{q}}_n + \frac{10}{19}\left(\frac{n-1}{n}\right)\left(\frac{3\widehat{\mathrm{q}}_n}{2} - \frac{37\widehat{\mathrm{p}}_n}{38}\right),$$
$$\widehat{\mathrm{p}}_{n+1} = \left(\frac{1}{n+2}\right)\widehat{\mathrm{p}}_n + \frac{42(n+1)}{67(n+2)}\left(\frac{7\widehat{\mathrm{p}}_n}{6} - \frac{35\widehat{\mathrm{q}}_n}{402}\right).$$

(*vii*) *It is easy to check that condition* (4.1), (4.2) *and* (4.3) *of Theorem 4.1 are satisfied. Thus, system* (3.1) *admits a solution.*

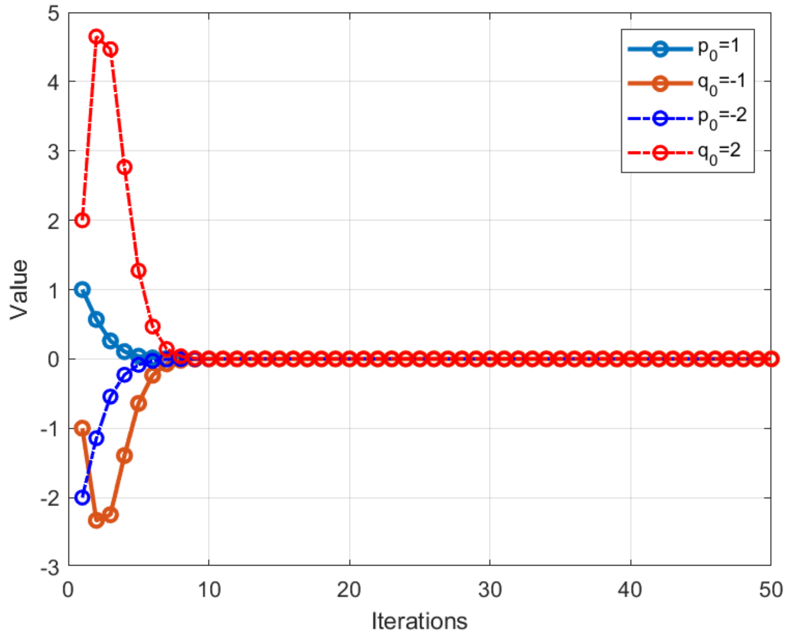

Figure 1: Convergence Graph for different initial values of $\widehat{\mathrm{p}}$ and $\widehat{\mathrm{q}}$

No. of Iteration	For $\widehat{\mathrm{p}}_0 = 1$ $\widehat{\mathrm{p}}_n$	For $\widehat{\mathrm{q}}_0 = -1.$ $\widehat{\mathrm{q}}_n$	For $\widehat{\mathrm{p}}_0 = -2$ $\widehat{\mathrm{p}}_n$	For $\widehat{\mathrm{q}}_0 = 2$ $\widehat{\mathrm{q}}_n$
n=1	1	-1	-2	2
n=2	0.56867229	-2.3285738	-1.1438657	4.6506265
n=3	0.25954374	-2.2475433	-0.5439893	4.4687547
n=4	0.10520368	-1.39417701	-0.229106768	2.7665369
n=5	0.03907625	-0.64115197	-0.085211068	1.2730781
n=10	6.3088058e-05	-0.0008671193	-0.0001278830	0.0085015
n=15	7.8936310e-09	-7.57314208e-08	-1.58118674e-08	1.5119973e-07
n=20	1.5590855e-13	-1.1832846e-12	-3.116451e-13	2.363213e-12
n=25	8.029934e-19	-5.2264155e-18	-1.60430115e-18	1.0439134e-17
n=30	1.447128e-24	-8.479071e-24	-2.8908184e-24	1.693656e-23
n=35	1.1019388e-30	-5.990450e-30	-2.2011664e-30	1.1965819e-29
n=40	4.0400406e-37	-2.07791406e-36	-8.07003053e-37	4.1506175e-36
n=45	7.848087e-44	-3.8693066e-43	-1.567657e-43	7.7289247e-43
n=50	8.6912112e-51	-4.1449943e-50	-1.736069e-50	8.27961e-50

Table 1: Computational Table showing the output for different initial values of $\widehat{\mathrm{p}}$ and $\widehat{\mathrm{q}}$

5 Conclusions

Due to applications of Yosida approximation operator and Cayley operator in contemporary science, this paper is centered on solving a system of variational inclusions involving the generalized Yosida and the Cayley operators in real Banach space. The solution to our problem has been achieved by developing an inertial extrapolation scheme, although several other schemes have also been developed. It is well-known that the inertial extrapolation scheme provides a faster rate of convergence.

We remark that scientists of other discipline may use our results for practical and applications purposes.

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