Improved convergence theorem for the general modulus-based matrix splitting method*

Ya-Li Liu[†], Shi-Liang Wu[‡], Cui-Xia Li[§]

[†]*School of Information Engineering, College of Arts and Sciences·Kunming, Kunming, Yunnan, 650221, P.R. China*

[‡,§]*School of Mathematics, Yunnan Normal University, Kunming, Yunnan, 650500, P.R. China*

[‡]*Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Yunnan Normal University, Kunming, Yunnan, 650500, P.R. China*

Abstract

In this note, based on the published work by Li [A general modulus-based matrix splitting method for linear complementarity problems of H-matrices, Appl. Math. Lett. 26 (2013) 1159-1164], we further study the convergence property of the general modulus-based matrix splitting (GMMS) method for linear complementarity problems. A new sufficient condition of the GMMS method is obtained, which is weaker than the result in the above work.

Keywords: Linear complementarity problems; GMMS method; convergence

AMS classification: 65F10

1 Introduction

The linear complementarity problems (denoted by LCP(q, A)) is that we need to find that $z \in \mathbb{R}^n$ satisfies

$$w := Az + q \geq 0,\ \ z \geq 0 \text{ and } z^T w = 0, \tag{1.1}$$

*This research was supported by National Natural Science Foundation of China (No.11961082), Yunnan Key Laboratory of Modern Analytical Mathematics and Applications (202302AN360007), Cross-integration Innovation team of modern Applied Mathematics and Life Sciences in Yunnan Province, China (202405AS350003).

[†]liuyali80818@126.com

[‡]slwuynnu@126.com

[§]Corresponding author:lixiatkynu@126.com

where $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are given in [1,2]. Li in [3] presented a general modulus-based matrix splitting (GMMS) method for solving the LCP(q, A) with matrix A being an H-matrix. Essentially, the idea of the GMMS method is to introduce two positive diagonal matrices for the equivalent absolute value equation of the LCP(q, A). Concretely, using

$$z = \Omega_1(|x| + x) \text{ and } w = \Omega_2(|x| - x), \tag{1.2}$$

where $|\cdot|$ denotes the absolute value, the LCP(q, A) can be equivalently transformed into the following absolute value equation

$$(\Omega_2 + M_{\Omega_1})x = N_{\Omega_1}x + (\Omega_2 - A\Omega_1)|x| - q, \tag{1.3}$$

where $A\Omega_1 = M_{\Omega_1} - N_{\Omega_1}$ with $\det(M_{\Omega_1}) \neq 0$ is a matrix splitting of matrix $A\Omega_1$, Ω_1 and Ω_2 are two positive diagonal matrices.

Based on (1.3), the GMMS method works below.

Method 1.1 [3] Let Ω_1 and Ω_2 be given positive diagonal matrices. Then for any initial vector $x^{(0)} \in \mathbb{R}^n$, calculate $x^{(k+1)}$ by

$$(\Omega_2 + M_{\Omega_1})x^{(k+1)} = N_{\Omega_1}x^{(k)} + (\Omega_2 - A\Omega_1)|x^{(k)}| - q, \text{ for } k = 0, 1, 2, \ldots, \tag{1.4}$$

where $A\Omega_1 = M_{\Omega_1} - N_{\Omega_1}$ is a matrix splitting of matrix $A\Omega_1$. Then set

$$z^{(k+1)} = \Omega_1(|x^{(k+1)}| + x^{(k+1)})$$

until the iteration sequence $\{z^{(k)}\}_{k=1}^{+\infty} \subset \mathbb{R}^n$ is convergent.

Let $A = M - N$. For $\gamma > 0$, if we set

$$\Omega_1 = \frac{1}{\gamma}I, \Omega_2 = \frac{1}{\gamma}\Omega, M_{\Omega_1} = \frac{1}{\gamma}M, N_{\Omega_1} = \frac{1}{\gamma}N,$$

then the GMMS method reduces to the modulus-based matrix splitting (MMS) method, see Method 3.1 in [4].

For later discussion, some necessary concepts, notations and lemmas are reminded. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. It is called as a Z-matrix if $a_{ij} \leq 0$ for $i \neq j$; a nonsingular M-matrix if A is a Z-matrix and $A^{-1} \geq 0$; an H-matrix if its comparison matrix $\langle A\rangle = (\langle a\rangle_{ij}) \in \mathbb{R}^{n\times n}$ ($\langle a\rangle_{ii} = |a_{ii}|$ and $\langle a\rangle_{ij} = -|a_{ij}|$ for $i \neq j$) is a nonsingular M-matrix; a strictly diagonally dominant (SDD) (by rows) matrix if

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, i = 1, 2, \ldots, n,$$

see [6]. In addition, an H-matrix with positive diagonal is called an H_+-matrix in [4]. If $A \leq B$ with A being an M-matrix and B being a Z-matrix, then B is an M-matrix [6]. The matrix splitting, $A = M - N$, of $A \in \mathbb{R}^{n\times n}$ is called as an H-splitting if $\langle M\rangle - |N|$ is a nonsingular M-matrix with $|N| = (|n_{ij}|)$. $\rho(A)$, $\|A\|_\infty$ and D_A denote the spectral radius, the infinite norm and the diagonal part of the matrix A, respectively. It is well known that LCP(q, A) has a unique solution if A is an H_+-matrix.

Lemma 1.1 [5] *Let $A \in \mathbb{R}^{n\times n}$ with $A \geq 0$. If there exists $u \in \mathbb{R}^n$ with $u > 0$ such that $Au < u$, then $\rho(A) < 1$.*

Lemma 1.2 [7] *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix, and $B = D_A - A$. Then $|A^{-1}| \leq \langle A\rangle^{-1}$ and $\rho(|D_A|^{-1}|B|) < 1$.*

For the convergence of the GMMS method, Li in [3] gave the following main result.

Theorem 1.1 [3] *Let $A\Omega_1 = M_{\Omega_1} - N_{\Omega_1}$ be an H-splitting of the H_+-matrix A, and Ω_1 and Ω_2 be two positive diagonal matrices. If*

$$\Omega_2 e > D_A\Omega_1 e - V^{-1}(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve \tag{1.5}$$

for any positive diagonal matrix V such that $(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)V$ is an SDD matrix, where $e = (1, 1, \ldots, 1)^T$, then Method 1.1 is convergent for any initial guess $x^{(0)} \in \mathbb{R}^n$.

The purpose of this paper is to establish a new sufficient condition for convergence of the GMMS method, which is superior to those previously published works in [3, 4, 9].

2 Main result

To give our main result, we first present Lemma 2.1.

Lemma 2.1 *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix, and $A = M - N$ be its an H-splitting. Then $\langle M\rangle - |N| \leq \langle A\rangle$.*

Proof. By the simple computations, we have

$$a_{ii} = m_{ii} - n_{ii} = |m_{ii} - n_{ii}| \geq |m_{ii}| - |n_{ii}|$$

and

$$-|a_{ij}| = -|m_{ij} - n_{ij}| \geq -|m_{ij}| - |n_{ij}|,$$

where $A = (a_{ij}), M = (m_{ij})$ and $N = (n_{ij})$. Hence, the result of Lemma 2.1 is valid. □

Next, for the GMMS method, we give the following main result, see Theorem 2.1.

Theorem 2.1 *Let A, Ω_1 and Ω_2 be defined in Theorem 1.1, and $A\Omega_1 = M_{\Omega_1} - N_{\Omega_1}$ be an H-splitting of $A\Omega_1$. If*

$$\Omega_2 e > D_A\Omega_1 e - \frac{1}{2}V^{-1}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve \tag{2.1}$$

for any positive diagonal matrix V such that $(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)V$ is an SDD matrix, where $e = (1, 1, \ldots, 1)^T$, then Method 1.1 is convergent for any initial guess $x^{(0)} \in \mathbb{R}^n$.

Proof. Assume that (z^*, w^*) is a solution of the LCP(q, A). Using Eq. (1.2), we get that $x^* = \frac{1}{2}(\Omega_1^{-1}z^* - \Omega_2^{-1}w^*)$ and $|x^*| = \frac{1}{2}(\Omega_1^{-1}z^* + \Omega_2^{-1}w^*)$, which meets

$$(\Omega_2 + M_{\Omega_1})x^* = N_{\Omega_1}x^* + (\Omega_2 - A\Omega_1)|x^*| - q. \tag{2.2}$$

Combining (1.4) with (2.2), we obtain

$$(\Omega_2 + M_{\Omega_1})(x^{(k+1)} - x^*) = N_{\Omega_1}(x^{(k)} - x^*) + (\Omega_2 - A\Omega_1)(|x^{(k)}| - |x^*|). \tag{2.3}$$

Since $A\Omega_1 = M_{\Omega_1} - N_{\Omega_1}$ is an H-splitting of $A\Omega_1$, $\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|$ is an M-matrix. Clearly, we have

$$\langle M_{\Omega_1}\rangle - |N_{\Omega_1}| \le \langle M_{\Omega_1}\rangle,$$

which implies that matrix $\langle M_{\Omega_1}\rangle$ is an M-matrix. Further, we can obtain that $\Omega_2 + M_{\Omega_1}$ is an H_+-matrix. Based on Lemma 1.2, we have

$$|(\Omega_2 + M_{\Omega_1})^{-1}| \le \langle \Omega_2 + M_{\Omega_1}\rangle^{-1} = (\Omega_2 + \langle M_{\Omega_1}\rangle)^{-1}.$$

From (2.3), we have

$$\begin{aligned} |x^{(k+1)} - x^*| &= |(\Omega_2 + M_{\Omega_1})^{-1}||N_{\Omega_1}(x^{(k)} - x^*) + (\Omega_2 - A\Omega_1)(|x^{(k)}| - |x^*|)| \\ &\le T|x^{(k)} - x^*|, \end{aligned} \tag{2.4}$$

where

$$T = (\Omega_2 + \langle M_{\Omega_1}\rangle)^{-1}(|N_{\Omega_1}| + |\Omega_2 - A\Omega_1|).$$

Obviously, the GMMS method is convergent for $\rho(T) < 1$.

Next, we consider two cases: $\Omega_2 e \ge D_A\Omega_1 e$ and

$$D_A\Omega_1 e - \frac{1}{2}V^{-1}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve < \Omega_2 e < D_A\Omega_1 e.$$

Case (I): Since $\Omega_2 e \ge D_A\Omega_1 e$, we know that $\Omega_2 \ge D_A\Omega_1$. In this case, we have

$$\Omega_2 - |\Omega_2 - A\Omega_1| = \langle A\Omega_1\rangle$$

and

$$\begin{aligned} T &= (\Omega_2 + \langle M_{\Omega_1}\rangle)^{-1}(\Omega_2 + \langle M_{\Omega_1}\rangle - \Omega_2 - \langle M_{\Omega_1}\rangle + |N_{\Omega_1}| + |\Omega_2 - A\Omega_1|) \\ &= I - (\Omega_2 + \langle M_{\Omega_1}\rangle)^{-1}(\Omega_2 + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}| - |\Omega_2 - A\Omega_1|) \\ &= I - (\Omega_2 + \langle M_{\Omega_1}\rangle)^{-1}(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}| + \langle A\Omega_1\rangle) \\ &\le I - 2(\Omega_2 + \langle M_{\Omega_1}\rangle)^{-1}(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|). \end{aligned}$$

Noticing that $\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|$ is an M-matrix, there exists a positive vector u so that

$$(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)u > 0.$$

Therefore,

$$Tu \le (I - 2(\Omega_2 + \langle M_{\Omega_1}\rangle)^{-1}(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|))u < u.$$

It follows that $\rho(T) < 1$ from Lemma 1.1.

Case (II): Since $\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|$ is an M-matrix, there exists a positive diagonal matrix V such that $(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)V$ is an SDD matrix. Then from the equivalent statement M_{35} of the nonsingular M-matrix in [Page 137, 6], we have

$$\langle M_{\Omega_1}\rangle Ve \ge (\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve > 0.$$

So,

$$\langle M_{\Omega_1}\rangle Ve + \Omega_2 Ve > (|N_{\Omega_1}| + \Omega_2)Ve > 0.$$

Moreover, we can get that the interval $(D_A\Omega_1 e - \frac{1}{2}V^{-1}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve, D_A\Omega_1 e)$ is nonempty from Lemma 2.1.

Since

$$D_A\Omega_1 e - \frac{1}{2}V^{-1}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve < \Omega_2 e,$$

we obtain

$$\begin{aligned}
&2VD_A\Omega_1 e - (\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve < 2V\Omega_2 e\\
\Leftrightarrow\ &[2D_A\Omega_1 - (\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)]Ve < 2\Omega_2 Ve\\
\Leftrightarrow\ &(2D_A\Omega_1 - \langle A\Omega_1\rangle - \langle M_{\Omega_1}\rangle + |N_{\Omega_1}|)Ve < 2\Omega_2 Ve\\
\Leftrightarrow\ &(|A\Omega_1| - \langle M_{\Omega_1}\rangle + |N_{\Omega_1}|)Ve < 2\Omega_2 Ve\\
\Leftrightarrow\ &(2\Omega_2 + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}| - |A\Omega_1|)Ve > 0.
\end{aligned}$$

In addition, when $\Omega_2 e < D_A\Omega_1 e$, we have

$$|\Omega_2 - A\Omega_1| = |A\Omega_1| - \Omega_2 \ge 0.$$

Let

$$\bar{M} = \Omega_2 + \langle M_{\Omega_1}\rangle, \bar{N} = |N_{\Omega_1}| + |\Omega_2 - A\Omega_1|.$$

Then $T = \bar{M}^{-1}\bar{N}$ and

$$\begin{aligned}
(\bar{M} - \bar{N})Ve &= (\Omega_2 + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}| - |\Omega_2 - A\Omega_1|)Ve\\
&= (2\Omega_2 + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}| - |A\Omega_1|)Ve\\
&> 0.
\end{aligned}$$

By using Theorem 3 in [8], we have

$$\begin{aligned}
\rho(T) =& \rho(V^{-1}TV)\\
\le& \|V^{-1}TV\|_\infty\\
=& \|((\Omega_2 + \langle M_{\Omega_1}\rangle)V)^{-1}(|N_{\Omega_1}| + |\Omega_2 - A\Omega_1|)V\|_\infty\\
\le& \max_{1\le i\le n} \frac{((|N_{\Omega_1}| + |\Omega_2 - A\Omega_1|)Ve)_i}{((\Omega_2 + \langle M_{\Omega_1}\rangle)Ve)_i}\\
<& 1.
\end{aligned}$$

Combining Case (I) with Case (II), the result of Theorem 2.1 is valid. □

Comparing the condition (2.1) with the condition (1.5), the former is weaker than the latter. In fact, by simple computation,

$$\begin{aligned}
&D_A\Omega_1 e - V^{-1}(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve - [D_A\Omega_1 e - \frac{1}{2}V^{-1}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve]\\
=& - V^{-1}(\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve + \frac{1}{2}V^{-1}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|)Ve\\
=&\frac{1}{2}V^{-1}(\langle A\Omega_1\rangle - (\langle M_{\Omega_1}\rangle - |N_{\Omega_1}|))Ve \geq 0.
\end{aligned}$$

This implies that the convergence condition (1.4) of Theorem 1.1 in [3] is improved.

When $\Omega_2 = \Omega$ and $\Omega_1 = I$, Corollary 2.1 is obtained.

Corollary 2.1 *Let $A = M - N$ be an H-splitting of the H_+-matrix A. If*

$$\Omega e > D_A e - \frac{1}{2}V^{-1}(\langle A\rangle + \langle M\rangle - |N|)Ve \tag{2.5}$$

for any positive diagonal matrix V such that $(\langle M\rangle - |N|)V$ is an SDD matrix, where $e = (1, 1, \ldots, 1)^T$, then the MMS method (see Method 3.1 in [4]*) is convergent for any initial guess $x^{(0)} \in \mathbb{R}^n$.*

Further, it is easy to find that the condition (2.5) in Corollary 2.1 is also weaker than those in Theorem 4.3 in [4] and Theorems 3.1 in [9].

It's important to note that the matrix V involved in Theorem 2.1 and Corollary 2.1 may be not easily available, including Theorem 1.1 as well. Whereas, when the H_+-matrix A is an SDD matrix, the matrix V can be easily chosen. That is to say, for this situation, we can choose $V = I$, and then make use of the condition (2.1) to judge the convergence of the GMMS method and the condition (2.5) to judge the convergence of the MMS method.

Finally, we use two simple examples to compare Theorem 1.1 [3] with Theorem 2.1.

Example 2.1 To compare Theorem 1.1 [3] with Theorem 2.1, we set $V = \Omega_1 = I$ and

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Clearly, $A\Omega_1 = A$. Taking

$$M_{\Omega_1} = \begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix}, N_{\Omega_1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\langle A\Omega_1\rangle = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \langle M_{\Omega_1}\rangle = \begin{bmatrix} 4 & 0 \\ -2 & 4 \end{bmatrix}, |N_{\Omega_1}| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

So,

$$\langle M_{\Omega_1}\rangle - |N_{\Omega_1}| = \begin{bmatrix} 4 & -1 \\ -3 & 4 \end{bmatrix} \text{ and } \frac{1}{2}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|) = \begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}.$$

By Theorem 1.1 [3], the choice of matrix Ω_2 satisfies $\Omega_2 > 3I$ to ensure the convergence of the GMMS method. Whereas, using Theorem 2.1, the choice of matrix Ω_2 only need to satisfy $\Omega_2 > 2I$ to ensure the convergence of the GMMS method. This shows that Theorem 2.1 is weaker than Theorem 1.1 [3]. Further, for the interval $(2I, 3I]$, if we take $\Omega_2 = 3I$, then $\rho(T) = 0.4918 < 1$. This shows that the GMMS method is convergent. But, in this case, Theorem 1.1 [3] does not satisfy to judge the convergence of the GMMS method.

Example 2.2 Let $V = \Omega_1 = I$ and $A = \text{tridiag}(1,4,1) \in \mathbb{R}^{n\times n}$ with $n \geq 3$. Taking $M_{\Omega_1} = \text{tridiag}(2,4,0)$, $N_{\Omega_1} = \text{tridiag}(1,0,-1)$. Then, by the simple calculations, we have

$$\langle M_{\Omega_1}\rangle - |N_{\Omega_1}| = \text{tridiag}(-3,4,-1), \frac{1}{2}(\langle A\Omega_1\rangle + \langle M_{\Omega_1}\rangle - |N_{\Omega_1}|) = \text{tridiag}(-2,4,-1).$$

For $n \geq 3$, by Theorem 1.1 [3], the choice of matrix Ω_2 satisfies $\Omega_2 > 4I$ to ensure the convergence of the GMMS method. Whereas, using Theorem 2.1, the choice of matrix Ω_2 only need to satisfy $\Omega_2 > 3I$ to ensure the convergence of the GMMS method.

Next, we consider $\Omega_2 = 4I$ for $A = M_{\Omega_1} - N_{\Omega_1}$ with $M_{\Omega_1} = \text{tridiag}(2,4,0)$ and $N_{\Omega_1} = \text{tridiag}(1,0,-1)$. In our computations, the starting vector is zero, the relative residual error (denoted by 'RES'), which is defined by

$$\text{RES}(x^{(k)}) = \|\min(Az^{(k)} + p, z^{(k)})\|_2.$$

All the test results are run on an Intel@ Celeron@ G4900, where the CPU 3.10GHz and the memory is 8.00 GB, and the language is MATLAB 7.0.

n	300	600	900
$\rho(T)$	0.6475	0.6557	0.6591
IT	16	16	16
CPU	0.0156	0.0781	0.1719
RES	3.9499e-7	5.6384e-7	6.9269e-7

Table 1: Numerical results of GMMS with $\Omega_2 = 4I$ and $\Omega_1 = I$.

Table 1 lists some numerical results of the GMMS method with $q = -Az^*$ and $z^* = (1,2,\ldots,1,2)^T$, where 'IT', 'CPU', 'RES', respectively, denote elapsed CPU time in seconds, the iteration steps and the relative residual error. From these numerical results confirm that GMMS is convergent for $\Omega_2 = 4I$.

References

[1] R.W. Cottle, J.-S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic, San Diego, 1992.

[2] K.G. Murty, Linear Complementarity, Linear and Nonlinear Programming. Heldermann, Berlin, 1988.

[3] W. Li, A general modulus-based matrix splitting method for linear complementarity problems of H-matrices, Appl. Math. Lett., 26 (2013) 1159-1164.

[4] Z.-Z. Bai, Modulus-based matrix splitting iteration methods for linear complementarity problems, Numer. Linear Algebra Appl., 17 (2010) 917-933.

[5] Z.-Z. Bai, L.-L. Zhang, Modulus-based synchronous two-stage multisplitting iteration methods for linear complementarity problems, Numer. Algor., 62 (2013) 59-77.

[6] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic, New York, 1979.

[7] A. Frommer, G. Mayer, Convergence of relaxed parallel multisplitting methods, Linear Algebra Appl., 119 (1989) 141-152.

[8] J.-G. Hu, Estimates of $\|B^{-1}A\|_\infty$ and their applications, Math. Numer. Sinica., 4 (1982) 272-282.

[9] L.-L. Zhang, Z.-R. Ren, Improved convergence theorems of modulus-based matrix splitting iteration methods for linear complementarity problems, Appl. Math. Lett., 26 (2013) 638-642.