A two-grid multipoint flux mixed finite element method for nonlinear parabolic problems

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Abstract

This paper introduces a two-grid multipoint flux mixed finite element (MFMFE) method for solving nonlinear parabolic problems. The MFMFE method is advantageous due to its ability to decouple saddle point algebraic systems. The two-grid algorithm transforms nonlinear problems into smaller nonlinear systems on coarse grids and linear problems on fine grids, facilitating rapid decoupling of nonlinear equations. We present semi-discrete and fully discrete backward Euler schemes for the model problem. Theoretical results demonstrate the convergence order of velocity and pressure. A numerical example validates the effectiveness of the proposed algorithm, showing that the two-grid MFMFE method significantly reduces CPU running time compared to the standard MFMFE method.

Keywords: nonlinear parabolic problems, multipoint flux mixed finite element, two-grid method, error estimates, numerical example

1 Introduction

We consider the following nonlinear parabolic problem:

$$\frac{\partial p}{\partial t} - \nabla \cdot (K\nabla p) = f(p), \quad (x,y,t) \in \Omega \times (0,T], \tag{1.1}$$

$$p(x,y,0) = p_0(x,y), \quad (x,y) \in \Omega, \tag{1.2}$$

$$K\nabla p \cdot \boldsymbol{n} = 0, \quad (x,y,t) \in \partial\Omega \times (0,T], \tag{1.3}$$

where the polygon domain $\Omega \subset \mathbb{R}^2$ has a boundary $\partial\Omega$, in flow in porous media modeling, p denotes the fluid pressure, n is the outward unit normal on $\partial\Omega$, K is the symmetric positive definite tensor, and $f(p)$ represents the external flow rate.

The assumptions for the solution of (1.1)-(1.3) are given as follows:
(1) For some positive constants k_0, k_1,

$$k_0\xi^T\xi \leq \xi^T K(x)\xi \leq k_1\xi^T\xi, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^2. \tag{1.4}$$

(2) We assume that the solution $p \in L^2(0,T;W^{2,4}(\Omega))$, and $f(p)$ is twice continuously differentiable.

Parabolic equations have been widely used in physical phenomena such as heat conduction processes, electromagnetic field transmission, and wave propagation problems in porous media. In the past few decades, scholars have conducted extensive research, and classic numerical methods include the finite difference method [1, 2], finite element method [3, 4], finite volume element method [5, 6], and so on.

Moreover, to obtain local mass conservation and to accurately approximate the gradient of the principal variable, the mixed finite element (MFE) method is widely used [7, 8]. However, the disadvantage of the MFE method is that it requires solving a saddle point type algebraic system, which is computationally intensive. Mary F. Wheeler and Ivan Yotov proposed a multipoint flux mixed finite element (MFMFE) method [9]. This method can not only keep the advantages of the MFE method but also decouple the saddle point type algebraic system. The development and application of this method can be found in [10–16]. In [10], a posterior error estimation for MFMFE method was studied. In [11, 14], the MFMFE method was presented to solve the Darcy-Forchheimer model. The MFMFE method of decoupling miscible displacement problem was studied in [12, 15].

The model problem is a large nonlinear system. It is necessary to study an efficient algorithm. Inspired by Xu [17, 18], the two-grid algorithm is a suitable candidate. The main idea of the algorithm is to generate a rough approximation of the solution using the coarse grid space, then correct it by solving a linear system on the fine grid space. Many scholars have applied this method to different model problems [19–25]. As far as we know, no one has used the two-grid method to the MFMFE approximation schemes for strongly nonlinear parabolic problems to achieve equation decoupling and accelerate solutions.

In this paper, we will consider a novel two-grid MFMFE method for nonlinear parabolic problems (1.1)-(1.3). Solving a large nonlinear system on the fine grid is reduced to solving a linear problem on the fine grid space and a small nonlinear problem on the coarse grid space. Theoretical deduction and numerical experiment show that the new method can decouple nonlinear equations quickly and have certain theoretical and practical application values. The rest of this paper is arranged as follows. In section 2, the MFMFE spaces and the semi-discrete and fully discrete approximation scheme for nonlinear parabolic problems are presented. A two-grid algorithm of the MFMFE discretization is proposed and error estimates of the schemes are derived in section 3. In section 4, a numerical example is given to illustrate the theoretical analysis and to indicate that the computing time is greatly reduced.

2 Multipoint flux mixed finite element method

2.1 Some notations and weak formulation

For the domain $\Omega \subset \mathbb{R}^2$, let $W^{k,q}(\Omega)$ be a standard Sobolev space, where $1 \leq q \leq \infty$. The norm is defined as

$$\|u\|_{W^{k,q}(\Omega)} = \|u\|_{k,q} = (\int_\Omega \sum_{|\alpha|\leq k} |\partial^\alpha u|^q)^{1/q}, 1 \leqslant q < \infty,$$
$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|\leqslant k} \|\partial^\alpha u\|_{0,\infty} = \max_{|\alpha|\leqslant k} (\operatorname*{ess\,sup}_{x\in\Omega} |\partial^\alpha u|).$$

When $q = 2$, let $W^{k,2}(\Omega) = H^k(\Omega)$ be a Hilbert space equipped with the norm $\|\cdot\|_{k,2}$. We denote by $L^s(0,T;W^{k,q}(\Omega))$ the Banach spaces of all L^s integrable functions from $[0,T]$ into $W^{k,q}(\Omega)$ with norm

$$\|u\|_{L^s(0,T;W^{k,q}(\Omega))} = (\int_0^T \|u\|^s_{W^{k,q}(\Omega)} dt)^{\frac{1}{s}}.$$

Let $0 = t^0 < t^1 < \cdots < t^N = T$ be the partition of time interval $[0,T]$ with $t^n = n\Delta t$. We will also use the space

$$H(\text{div};\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2 : \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\},$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\text{div}} = (\|\boldsymbol{v}\|^2 + \|\nabla\cdot\boldsymbol{v}\|^2)^{1/2}. \tag{2.1}$$

Denoting $\boldsymbol{u} = -K\nabla p$, the weak formulation of (1.1)-(1.3) is the following: find $(\boldsymbol{u}, p) \in V \times W$ such that

$$(K^{-1}\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in V, \tag{2.2}$$
$$(\frac{\partial p}{\partial t}, w) + (\nabla\cdot\boldsymbol{u}, w) = (f(p), w), \quad \forall w \in W, \tag{2.3}$$

where

$$V = \{\boldsymbol{v} \in H(\text{div};\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \quad W = L^2(\Omega).$$

In this paper, we will use C to represent a general positive constant that is independent of the discretization parameter.

2.2 Multipoint flux finite element spaces

Let Γ_h be a shape regular and quasi-uniform finite element partition [26] of Ω consisting of convex quadrilaterals, where $h = \max_{E\in\Gamma_h} \text{diam}(E)$. For any element $E \in \Gamma_h$, there exists a bijection mapping $F_E : \hat{E} \to E$, where $\hat{E}$ is the reference unit square with vertices $\hat{r}_1 = (0,0)^T, \hat{r}_2 = (1,0)^T, \hat{r}_3 = (1,1)^T, \hat{r}_4 = (0,1)^T$. Denote by $r_i = (x_i, y_i)^T (i = 1, ..., 4)$ the four corresponding vertices of element E, the Jacobian matrix by DF_E and $J_E = |\det(DF_E)|$. Denote the inverse mapping by F_E^{-1}, its Jacobian matrix by DF_E^{-1}, and $J_{F_E^{-1}} = |\det(DF_E^{-1})|$. We have that

$$DF_E^{-1}(x) = (DF_E)^{-1}(\hat{x}),\ J_{F_E^{-1}}(x) = \frac{1}{J_E(\hat{x})}.$$

The bilinear mapping given by

$$F_E(\hat{r}) = \boldsymbol{r}_1(1-\hat{x})(1-\hat{y}) + \boldsymbol{r}_2\hat{x}(1-\hat{y}) + \boldsymbol{r}_3\hat{x}\hat{y} + \boldsymbol{r}_4(1-\hat{x})\hat{y}. \tag{2.4}$$

We define the lowest order BDM_1 mixed finite element space [27, 28], which is defined on the reference unit as

$$\hat{V}(\hat{E}) = P_1(\hat{E})^2 + r\ \mathrm{curl}(\hat{x}^2\hat{y}) + s\ \mathrm{curl}(\hat{x}\hat{y}^2) = \begin{pmatrix} \alpha_1\hat{x} + \beta_1\hat{y} + \gamma_1 + r\hat{x}^2 + 2s\hat{x}\hat{y} \\ \alpha_2\hat{x} + \beta_2\hat{y} + \gamma_2 - 2r\hat{x}\hat{y} - s\hat{y}^2 \end{pmatrix}, \tag{2.5}$$

$$\hat{W}(\hat{E}) = P_0(\hat{E}), \tag{2.6}$$

where $r, s, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are real constants and P_k represents the space of polynomials of degree less than or equal to k.

The outward unit normal vectors to the edges of E and $\hat{E}$ are denoted by $\boldsymbol{n}_i$ and $\hat{\boldsymbol{n}}_i$, $i = 1, ..., 4$. The degrees of freedom for $\hat{\boldsymbol{v}} \in \hat{V}(\hat{E})$ can be chosen to be the values of $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}_{\hat{e}}$ at any two points on each edge $\hat{e}$. We can obtain the velocity space on any element E by Piola transformation to $\boldsymbol{v} \leftrightarrow \hat{\boldsymbol{v}} : \boldsymbol{v} = \frac{1}{J_E} DF_E \hat{\boldsymbol{v}} \circ F_E^{-1}$, and the pressure space can be obtained by transformation to $w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1}$.

The BDM_1 spaces are given by

$$V_h = \{v \in V : v\ |_E \leftrightarrow \hat{v}, \hat{v} \in \hat{V}(\hat{E}), \forall E \in \varGamma_h\}, \tag{2.7}$$

$$W_h = \{w \in W : w\ |_E \leftrightarrow \hat{w}, \hat{w} \in \hat{W}(\hat{E}), \forall E \in \varGamma_h\}. \tag{2.8}$$

We define a BDM_1 projection operator $\Pi_h : V \to V_h$, which satisfies

$$(\nabla \cdot (\boldsymbol{u} - \Pi_h\boldsymbol{u}), w) = 0, \quad \forall w \in W_h, \tag{2.9}$$

$$\|\boldsymbol{u} - \Pi_h\boldsymbol{u}\|_{0,q} \le C\|\boldsymbol{u}\|_{r,q}h^r, \quad \frac{1}{q} \le r \le 2, \tag{2.10}$$

$$\|\nabla \cdot (\boldsymbol{u} - \Pi_h\boldsymbol{u})\|_{0,q} \le C\|\nabla \cdot \boldsymbol{u}\|_{r,q}h^r, \quad 0 \le r \le 1. \tag{2.11}$$

In addidition, we define L^2 orthogonal projection $Q_h : W \to W_h$, which satisfies

$$(p - Q_hp, w) = 0, \quad \forall w \in W_h, \tag{2.12}$$

$$\|p - Q_hp\|_{0,q} \le C\|p\|_{r,q}h^r, \quad 0 \le r \le 1. \tag{2.13}$$

For $\boldsymbol{u}, \boldsymbol{v} \in V_h$, we introduce the global quadrature rule

$$(K^{-1}\boldsymbol{u}, \boldsymbol{v})_Q = \sum_{E \in \Gamma_h} (K^{-1}\boldsymbol{u}, \boldsymbol{v})_{Q,E}. \tag{2.14}$$

The integration on any element E is performed by mapping to the reference element $\hat{E}$. The quadrature rule is defined on $\hat{E}$. Using the transformation of the reference unit and the physical unit, we have

$$\begin{aligned}
(K^{-1}\boldsymbol{u}, \boldsymbol{v})_{Q,E} &\equiv \int_E K^{-1}\boldsymbol{u} \cdot \boldsymbol{v} dx \\
&= \int_{\hat{E}} \hat{K}^{-1} \frac{1}{J_E} DF_E \hat{\boldsymbol{u}} \cdot \frac{1}{J_E} DF_E \hat{\boldsymbol{v}} J_E d\hat{x} \\
&= \int_{\hat{E}} \frac{1}{J_E} DF_E^T \hat{K}^{-1} DF_E \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} d\hat{x} \\
&= \int_{\hat{E}} \varkappa^{-1} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} d\hat{x} \\
&\equiv \frac{|\hat{E}|}{4} \sum_{i=1}^{4} \varkappa^{-1}(\hat{\boldsymbol{r}}_{\boldsymbol{i}}) \hat{\boldsymbol{u}}(\hat{\boldsymbol{r}}_{\boldsymbol{i}}) \cdot \hat{\boldsymbol{v}}(\hat{\boldsymbol{r}}_{\boldsymbol{i}}),
\end{aligned} \tag{2.15}$$

where

$$\varkappa = J_E DF_E^{-1} \hat{K} (DF_E^{-1})^T. \tag{2.16}$$

We define the quadrature error on the element to be

$$\sigma_E(K^{-1}\boldsymbol{u}, \boldsymbol{v}) \equiv (K^{-1}\boldsymbol{u}, \boldsymbol{v})_E - (K^{-1}\boldsymbol{u}, \boldsymbol{v})_{Q,E}. \tag{2.17}$$

For the subsequent error analysis, the following lemmas are listed.
Lemma 2.1. *[9] If $\boldsymbol{u} \in V_h(E)$, for all constant vectors $\boldsymbol{v}_0$, then*

$$\sigma_E(\boldsymbol{u}, \boldsymbol{v}_0) = 0. \tag{2.18}$$

Lemma 2.2. *[9] There exists a positive constant C independent of h, such that*

$$(K^{-1}\boldsymbol{v}, \boldsymbol{v})_Q \geq C\|\boldsymbol{v}\|^2, \quad \forall \boldsymbol{v} \in V_h. \tag{2.19}$$

Lemma 2.3. *[29] If $K^{-1} \in W^{1,\infty}$ for all elements E, then there exists a constant C independent of h, such that*

$$|\sigma(K^{-1}\Pi_h \boldsymbol{u}, \boldsymbol{v})| \leq Ch\|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|. \tag{2.20}$$

Lemma 2.4. *[30] Suppose g is the fragment smooth function on partition Γ_h, and $\bar{g}$ is the mean value on the partition unit and $\|\nabla g\|_{0,\infty} \leq M$, then it has the following form*

$$|(g(p)\theta, \phi) - (\bar{g}\theta, \phi)| \leq CMh\|\theta\|\|\phi\|. \tag{2.21}$$

Next, two MFMFE approximation schemes for the model problem are proposed, where the semi-discrete scheme is: Find $(\boldsymbol{u}_h, p_h) \in V_h \times W_h$, such that

$$(K^{-1}\boldsymbol{u}_h, \boldsymbol{v}_h)_Q - (p_h, \nabla \cdot \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in V_h, \tag{2.22}$$

$$(\frac{\partial p_h}{\partial t}, w_h) + (\nabla \cdot \boldsymbol{u}_h, w_h) = (f(p_h), w_h), \quad \forall w_h \in W_h. \tag{2.23}$$

And the fully discrete backward Euler scheme is: Find $(\boldsymbol{u}_h^n, p_h^n) \in V_h \times W_h$, such that

$$(K^{-1}\boldsymbol{u}_h^n, \boldsymbol{v}_h)_Q - (p_h^n, \nabla \cdot \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in V_h, \tag{2.24}$$

$$(\frac{p_h^n - p_h^{n-1}}{\Delta t}, w_h) + (\nabla \cdot \boldsymbol{u}_h^n, w_h) = (f(p_h^n), w_h), \quad \forall w_h \in W_h. \tag{2.25}$$

In order to prove the main theorem of this article, we will use some projection techniques and estimates between the solution of the formulation (2.22)-(2.23) and the elliptic-mixed projection defined below.

We define the following mixed elliptic projection $(R_h\boldsymbol{u}, R_hp) \in V_h \times W_h$ by

$$(K^{-1}R_h\boldsymbol{u}, \boldsymbol{v}_h)_Q - (R_hp, \nabla \cdot \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in V_h, \tag{2.26}$$

$$(\nabla \cdot R_h\boldsymbol{u}, w_h) = (f(p) - \frac{\partial p}{\partial t}, w_h), \quad \forall w_h \in W_h, \tag{2.27}$$

in order to perform error estimation related to the next section of the two-grid algorithm, we need the following theorems.

Theorem 2.1. *$(\boldsymbol{u}, p)$ is the solution of (2.2)-(2.3), $(R_h\boldsymbol{u}, R_hp)$ is the solution of (2.26)-(2.27), then there exists the following convergence*

$$\|\boldsymbol{u} - R_h\boldsymbol{u}\| \leq Ch\|\boldsymbol{u}\|_1, \tag{2.28}$$

$$\|\nabla \cdot (\boldsymbol{u} - R_h\boldsymbol{u})\| \leq Ch\|\nabla \cdot \boldsymbol{u}\|_1. \tag{2.29}$$

Proof. Subtracting the scheme (2.2)-(2.3) from (2.26)-(2.27), we have the error equation

$$\begin{aligned}(K^{-1}(\Pi_h\boldsymbol{u} - R_h\boldsymbol{u}), \boldsymbol{v}_h)_Q - (Q_hp - R_hp, \nabla \cdot \boldsymbol{v}_h) &= -(K^{-1}(\boldsymbol{u} - \Pi_h\boldsymbol{u}), \boldsymbol{v}_h) \\ &\quad - \sigma(K^{-1}\Pi_h\boldsymbol{u}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in V_h, \end{aligned} \tag{2.30}$$

$$(\nabla \cdot (\boldsymbol{u} - R_h\boldsymbol{u}), w_h) = 0, \quad \forall w_h \in W_h. \tag{2.31}$$

From (2.30)-(2.31), taking $\boldsymbol{v}_h = \Pi_h\boldsymbol{u} - R_h\boldsymbol{u}$ and $w_h = Q_hp - R_hp$, we can obtain that

$$(K^{-1}(\Pi_h\boldsymbol{u} - R_h\boldsymbol{u}), \Pi_h\boldsymbol{u} - R_h\boldsymbol{u}) = -(K^{-1}(\boldsymbol{u} - \Pi_h\boldsymbol{u}), \Pi_h\boldsymbol{u} - R_h\boldsymbol{u})$$

$$-\sigma(K^{-1}\Pi_h \boldsymbol{u}, \Pi_h \boldsymbol{u} - R_h \boldsymbol{u}). \tag{2.32}$$

Combining (2.10) and Lemma 2.3, we deduce that

$$\|\Pi_h \boldsymbol{u} - R_h \boldsymbol{u}\|^2 \leq Ch\|\boldsymbol{u}\|_1 \|\Pi_h \boldsymbol{u} - R_h \boldsymbol{u}\|, \tag{2.33}$$

(2.28) can be easily derived.

Then, it is obvious that $(\nabla \cdot (\boldsymbol{u} - R_h \boldsymbol{u}), w_h) = (\nabla \cdot (\Pi_h \boldsymbol{u} - R_h \boldsymbol{u}), w_h) = 0$, so that $\nabla \cdot (\Pi_h \boldsymbol{u} - R_h \boldsymbol{u}) = 0$. Hence, according to (2.11), we have

$$\|\nabla \cdot (\boldsymbol{u} - R_h \boldsymbol{u})\| \leq \|\nabla \cdot (\boldsymbol{u} - \Pi_h \boldsymbol{u})\| \leq Ch\|\nabla \cdot \boldsymbol{u}\|_1, \tag{2.34}$$

so (2.29) is proved, thus we complete the proof of the theorem. □

Theorem 2.2. *$R_h p$ is the mixed elliptic solution of (2.26)-(2.27), then we have the following estimate*

$$\|Q_h p - R_h p\| \leq Ch^2(\|\boldsymbol{u}\|_1 + \|\nabla \cdot \boldsymbol{u}\|_1). \tag{2.35}$$

Proof. We suppose the Dirchlet problem as follows [8]

$$\begin{cases} -\Delta \phi = \psi, & in\ \Omega, \\ \phi = 0, & on\ \partial\Omega. \end{cases} \tag{2.36}$$

For $\psi \in L^q(\Omega)$, the system (2.36) has a unique solution ϕ, and for $\psi \in W^{r,q}(\Omega)$, there has

$$\|\phi\|_{r+2,q} \leq \|\psi\|_{r,q}. \tag{2.37}$$

Let $\psi \in L^2(\Omega)$ and $\phi \in W_0^{1,2}(\Omega)$ satisfy (2.36), then by (2.9) and (2.30)-(2.31), we derive

$$\begin{aligned}
&(Q_h p - R_h p, \psi) \\
=&(Q_h p - R_h p, -\Delta \phi) \\
=&(Q_h p - R_h p, -\nabla \cdot (\Pi_h(\nabla \phi)) \\
=&-(K^{-1}(\boldsymbol{u} - R_h \boldsymbol{u}), \Pi_h(\nabla \phi)) - \sigma(K^{-1}(R_h \boldsymbol{u}), \Pi_h(\nabla \phi)) \\
=&-(K^{-1}(\boldsymbol{u} - R_h \boldsymbol{u}), \nabla \phi - Q_h(\nabla \phi)) + (K^{-1}\nabla \cdot (\boldsymbol{u} - R_h \boldsymbol{u}), \phi - Q_h \phi) \\
&+ (K^{-1}(\boldsymbol{u} - R_h \boldsymbol{u}), \nabla \phi - Q_h(\nabla \phi)) + (K^{-1}(\boldsymbol{u} - R_h \boldsymbol{u}), \nabla \phi - \Pi_h(\nabla \phi)) \\
&- \sigma(K^{-1}(R_h \boldsymbol{u}), \Pi_h(\nabla \phi)).
\end{aligned} \tag{2.38}$$

By using the approximation properties (2.11) and (2.13), we see that the first four terms on the right side of (2.38) are estimated as

$$C(\|\nabla \cdot (\boldsymbol{u} - R_h \boldsymbol{u})\| \|\phi - Q_h \phi\| + \|\boldsymbol{u} - R_h \boldsymbol{u}\| \|\nabla \phi - \Pi_h(\nabla \phi)\|). \tag{2.39}$$

For the last term on the right side of (2.38), by (2.18), we can derive that

$$\sigma(K^{-1}(R_h\boldsymbol{u}),\Pi_h(\nabla\phi))=\sigma(K^{-1}(R_h\boldsymbol{u}),\Pi_h(\nabla\phi)-\overline{\Pi_h(\nabla\phi)}), \tag{2.40}$$

where $\overline{\Pi_h(\nabla\phi)}$ are the mean value of $\Pi_h(\nabla\phi)$ on E. Therefore,

$$\begin{aligned}|\sigma(K^{-1}(R_h\boldsymbol{u}),\Pi_h(\nabla\phi))|&\le Ch\|\boldsymbol{u}\|_1\|\Pi_h(\nabla\phi)-\overline{\Pi_h(\nabla\phi)}\|\\&\le Ch^2\|\boldsymbol{u}\|_1\|\phi\|_2.\end{aligned} \tag{2.41}$$

Combining (2.39) and (2.41), by using the approximation properties (2.10) and (2.13), we see that

$$\begin{aligned}|(Q_hp-R_hp,\psi)|&\le C(h\|\boldsymbol{u}-R_h\boldsymbol{u}\|\|\nabla\phi\|_1+h\|\nabla\cdot(\boldsymbol{u}-R_h\boldsymbol{u})\|\|\phi\|_1+h^2\|\boldsymbol{u}\|_1\|\phi\|_2)\\&\le C(h^2\|\boldsymbol{u}\|_1+h^2\|\nabla\cdot\boldsymbol{u}\|_1)\|\phi\|_2\\&\le Ch^2(\|\boldsymbol{u}\|_1+\|\nabla\cdot\boldsymbol{u}\|_1)\|\psi\|,\end{aligned} \tag{2.42}$$

hence, we derive formula (2.35). □

Theorem 2.3. *Suppose that $(\boldsymbol{u}_h,p_h)\in V_h\times W_h$ is the solution of the semi-discrete scheme (2.22)-(2.23), and $(R_h\boldsymbol{u},R_hp)\in V_h\times W_h$ is the mixed element projection (2.26)-(2.27), then there exists a constant C independent of h, which satisfies*

$$\|R_hp-p_h\|\le Ch^2. \tag{2.43}$$

Proof. Subtracting the scheme (2.22)-(2.23) from the formulation (2.26)-(2.27), then we can get

$$(K^{-1}R_h\boldsymbol{u},\boldsymbol{v}_h)_Q-(K^{-1}\boldsymbol{u}_h,\boldsymbol{v}_h)_Q=(R_hp-p_h,\nabla\cdot\boldsymbol{v}_h),\quad\forall\boldsymbol{v}_h\in V_h, \tag{2.44}$$

$$(\nabla\cdot(R_h\boldsymbol{u}-\boldsymbol{u}_h),w_h)=(f(p)-f(p_h)-\frac{\partial p}{\partial t}+\frac{\partial p_h}{\partial t},w_h),\quad\forall w_h\in W_h. \tag{2.45}$$

Taking the test functions $w_h=R_hp-p_h$ and $\boldsymbol{v}_h=R_h\boldsymbol{u}-\boldsymbol{u}_h$, combining (2.12), adding (2.44) and (2.45), we can derive that

$$\begin{aligned}&(K^{-1}(R_h\boldsymbol{u}-\boldsymbol{u}_h),R_h\boldsymbol{u}-\boldsymbol{u}_h)_Q+(\frac{\partial}{\partial t}(R_hp-p_h),R_hp-p_h)\\&=(f(p)-f(p_h)-\frac{\partial}{\partial t}(Q_hp-R_hp),R_hp-p_h).\end{aligned} \tag{2.46}$$

The left side of (2.46) can be estimated as follows

$$\begin{aligned}&(K^{-1}(R_h\boldsymbol{u}-\boldsymbol{u}_h),R_h\boldsymbol{u}-\boldsymbol{u}_h)_Q+(\frac{\partial}{\partial t}(R_hp-p_h),R_hp-p_h)\\&\ge\|R_h\boldsymbol{u}-\boldsymbol{u}_h\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|R_hp-p_h\|^2.\end{aligned} \tag{2.47}$$

For the right side of (2.46), we see that

$$\begin{aligned}
&(f(p) - f(p_h) - \frac{\partial}{\partial t}(Q_h p - R_h p), R_h p - p_h) \\
&= (f(p) - f(Q_h p) + f(Q_h p) - f(R_h p) + f(R_h p) - f(p_h) \\
&- \frac{\partial}{\partial t}(Q_h p - R_h p), R_h p - p_h),
\end{aligned} \tag{2.48}$$

Due to Taylor expansion $f(p) = f(Q_h p) + f_p(Q_h p)(p - Q_h p) + \bar{f}_{pp}(p)(p - Q_h p)^2$, we have

$$\begin{aligned}
(f(p) - f(Q_h p), R_h p - p_h) \leq\ & (f_p(p)(p - Q_h p), R_h p - p_h) \\
& + (\bar{f}_{pp}(p - Q_h p)^2, R_h p - p_h).
\end{aligned} \tag{2.49}$$

Because of (2.13), (2.21) and ε-inequality,

$$\begin{aligned}
&|(f(p) - f(Q_h p), R_h p - p_h)| \\
&\leq Ch\|p - Q_h p\|\|R_h p - p_h\| + C\|p - Q_h p\|_{0,4}^2\|R_h p - p_h\| \\
&\leq Ch^4 + \varepsilon\|R_h p - p_h\|^2.
\end{aligned} \tag{2.50}$$

Combining Theorem 2.3 and ε-inequality, we deduce that

$$\begin{aligned}
|(f(Q_h p) - f(R_h p), R_h p - p_h)| &\leq C\|Q_h p - R_h p\|\|R_h p - p_h\| \\
&\leq Ch^4 + \varepsilon\|R_h p - p_h\|^2.
\end{aligned} \tag{2.51}$$

$$|(f(R_h p) - f(p_h), R_h p - p_h)| \leq C\|R_h p - p_h\|^2, \tag{2.52}$$

$$\begin{aligned}
|(-\frac{\partial}{\partial t}(Q_h p - R_h p), R_h p - p_h)| &\leq C\|(Q_h p - R_h p)_t\|\|R_h p - p_h\| \\
&\leq Ch^4 + \varepsilon\|R_h p - p_h\|^2.
\end{aligned} \tag{2.53}$$

Substituting (2.47)-(2.53) into (2.46), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|R_h p - p_h\|^2 \leq C(h^4 + \|R_h p - p_h\|^2), \tag{2.54}$$

integrating over $[0, T]$ and combining the Gronwall inequality, we can get the theorem. □

Theorem 2.4. *Suppose that $(\boldsymbol{u}_h^n, p_h^n) \in V_h \times W_h$ is the solution of the fully discrete backward Euler scheme (2.24)-(2.25), and $(R_h\boldsymbol{u}^n, R_h p^n) \in V_h \times W_h$ is their mixed element projection (2.26)-(2.27), then there exists a constant C independent of h, which satisfies*

$$\|R_h p^n - p_h^n\| \leq C(h^2 + \Delta t). \tag{2.55}$$

Proof. At the time $t = t^n$, we rewrite (2.26)-(2.27) as

$$(K^{-1}R_h\boldsymbol{u}^n, \boldsymbol{v}_h)_Q - (R_hp^n, \nabla\cdot\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in V_h, \tag{2.56}$$

$$(\nabla\cdot R_h\boldsymbol{u}^n, w_h) = (f(p^n) - \frac{\partial p^n}{\partial t}, w_h), \quad \forall w_h \in W_h \tag{2.57}$$

Subtracting the scheme (2.24)-(2.25) from the formulation (2.56)-(2.57), then we can get

$$(K^{-1}(R_h\boldsymbol{u}^n - \boldsymbol{u}_h^n), \boldsymbol{v}_h)_Q - (R_hp^n - p_h^n, \nabla\cdot\boldsymbol{v}_h) = 0, \tag{2.58}$$

$$(\nabla\cdot(R_h\boldsymbol{u}^n - \boldsymbol{u}_h^n), w_h) - (\frac{p_h^n - p_h^{n-1}}{\Delta t}, w_h) = (f(p^n) - \frac{\partial p^n}{\partial t} - f(p_h^n), w_h). \tag{2.59}$$

Taking $\boldsymbol{v}_h = \eta^n = R_h\boldsymbol{u}^n - \boldsymbol{u}_h^n$ and $w_h = \zeta^n = R_hp^n - p_h^n$, combining (2.58) and (2.58), then we have

$$\begin{aligned}&(K^{-1}\eta^n, \eta^n)_Q + (\frac{\zeta^n - \zeta^{n-1}}{\Delta t}, \zeta^n)\\ =&((f(p^n) - f(p_h^n)) + (\frac{1}{\Delta t}(R_hp^n - R_hp^{n-1}) - p_t^n), \zeta^n).\end{aligned} \tag{2.60}$$

The left side of (2.60) can be estimated as

$$(K^{-1}\eta^n, \eta^n)_Q + (\frac{\zeta^n - \zeta^{n-1}}{\Delta t}, \zeta^n) \geq \frac{1}{2\Delta t}(\|\zeta^n\|^2 - \|\zeta^{n-1}\|^2) + C\|\eta^n\|^2. \tag{2.61}$$

For the first item on the right side of (2.60), we have

$$\begin{aligned}|(f(p^n) - f(p_h^n), \zeta^n)| \leq&|(f(p^n) - f(Q_hp^n), \zeta^n)| + |(f(Q_hp^n) - f(R_hp^n), \zeta^n)|\\ &+ |(f(R_hp^n) - f(p_h^n), \zeta^n)|\\ \leq&|(f_p(p^n)(p^n - Q_hp^n), \zeta^n)| + |(\bar{f}_{pp}(p^n - Q_hp^n)^2, \zeta^n)|\\ &+ C\|Q_hp^n - R_hp^n\|\|\zeta^n\| + C\|\zeta^n\|^2\\ \leq&Ch\|p^n - Q_hp^n\|\|\zeta^n\| + C\|p^n - Q_hp^n\|_{0,4}^2\|\zeta^n\|\\ &+ C(h^4 + \|\zeta^n\|^2)\\ \leq&C(h^4 + \|\zeta^n\|^2),\end{aligned} \tag{2.62}$$

which follows from Taylor expansion, (2.13), (2.35) and schwarz inequality.

Taking $\alpha^n = p^n - R_hp^n$, we see that

$$\frac{1}{\Delta t}(R_hp^n - R_hp^{n-1}) - p_t^n = \frac{1}{\Delta t}(p^n - p^{n-1}) - p_t^n - \frac{1}{\Delta t}(\alpha^n - \alpha^{n-1}). \tag{2.63}$$

Let $d^n = Q_h p^n - R_h p^n$, from (2.35) and schwarz inequality, we get

$$\begin{aligned}|(\frac{1}{\Delta t}(p^n - p^{n-1}) - p_t^n, \zeta^n)| &\leq \| \int_{t^{n-1}}^{t^n} \frac{\partial^2 p}{\partial t^2} dt \| \|\zeta^n\| \\ &\leq \Delta t \int_{t^{n-1}}^{t^n} \| \frac{\partial^2 p}{\partial t^2} \|^2 dt + \|\zeta^n\|^2 \\ &\leq C(\Delta t + \|\zeta^n\|^2),\end{aligned} \tag{2.64}$$

$$\begin{aligned}|(\frac{1}{\Delta t}(\alpha^n - \alpha^{n-1}), \zeta^n)| &= |(\frac{1}{\Delta t}(d^n - d^{n-1}), \zeta^n)| \\ &\leq \| \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \frac{\partial d}{\partial t} dt \| \|\zeta^n\| \\ &\leq C(\frac{1}{\Delta t} h^4 + \|\zeta^n\|^2),\end{aligned} \tag{2.65}$$

Now, by (2.63)-(2.65), it follows that

$$|(\frac{1}{\Delta t}(R_h p^n - R_h p^{n-1}) - p_t^n, \zeta^n)| \leq C(\frac{1}{\Delta t} h^4 + \Delta t + \|\zeta^n\|^2). \tag{2.66}$$

Combining (2.62) and (2.66), we can get

$$\frac{1}{2\Delta t}(\|\zeta^n\|^2 - \|\zeta^{n-1}\|^2) + \|\eta^n\|^2 \leq C(\frac{1}{\Delta t} h^4 + \Delta t + \|\zeta^n\|^2). \tag{2.67}$$

Multiplying $2\Delta t$ on both sides of (2.67), then summing for n from 1 to N, we can obtain

$$\|\zeta^N\|^2 - \|\zeta^0\|^2 \leq C(h^4 + (\Delta t)^2 + \sum_{n=1}^{N} \Delta t \|\zeta^n\|^2). \tag{2.68}$$

We choose the initial function $\zeta^0 = 0$ and by the discrete Gronwall inequality, we can derive that

$$\|\zeta^N\| \leq C(h^2 + \Delta t). \tag{2.69}$$

Thus, we complete the proof of the theorem. □

3 Two-grid algorithm and error estimates

3.1 The two-grid scheme

In this section, we introduce an efficient two-grid method for the above MFMFE discrete approximation scheme to the problems (1.1)-(1.3). We present two quadrilateral mesh partiton of Ω, denoted as Γ_h and Γ_H with mesh sizes h and H ($h \ll H < 1$). Based on the partitons Γ_h and Γ_H, we define two finite element spaces $V_H \times W_H$ and $V_h \times W_h$, which are called the coarse grid space and the fine grid space, respectively. The main idea of the two-grid algorithm involves a nonlinear solver on the coarse grid space and a linear solver on the fine grid space.

We give the semi-discrete approximation scheme of the MFMFE method based on the two-grid algorithm for the original problem:

Algorithm3.1:

Step 1: find $(\boldsymbol{u}_H, p_H) \in V_H \times W_H$, such that

$$(K^{-1}\boldsymbol{u}_H, \boldsymbol{v}_H)_Q - (p_H, \nabla \cdot \boldsymbol{v}_H) = 0, \quad \forall \boldsymbol{v}_H \in V_H, \tag{3.1}$$

$$(\frac{\partial p_H}{\partial t}, w_H) + (\nabla \cdot \boldsymbol{u}_H, w_H) = (f(p_H), w_H), \quad \forall w_H \in W_H. \tag{3.2}$$

Step 2: find $(\boldsymbol{u}_h, p_h) \in V_h \times W_h$, such that

$$(K^{-1}\boldsymbol{u}_h, \boldsymbol{v}_h)_Q - (p_h, \nabla \cdot \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in V_h, \tag{3.3}$$

$$(\frac{\partial p_h}{\partial t}, w_h) + (\nabla \cdot \boldsymbol{u}_h, w_h) = (f(p_H) + f'(p_H)(p_h - p_H), w_h), \quad \forall w_h \in W_h, \tag{3.4}$$

where the $V_H \times W_H$ and $V_h \times W_h$ are the BDM_1 mixed element space.

A two-grid algorithm of the fully discrete backward Euler approximation scheme by the MFMFE method is given as follows.

Algorithm3.2:

Step 1: find $(\boldsymbol{u}_H^n, p_H^n) \in V_H \times W_H$, such that

$$(K^{-1}\boldsymbol{u}_H^n, \boldsymbol{v}_H)_Q - (p_H^n, \nabla \cdot \boldsymbol{v}_H) = 0, \quad \forall \boldsymbol{v}_H \in V_H, \tag{3.5}$$

$$(\frac{p_H^n - p_H^{n-1}}{\Delta t}, w_H) + (\nabla \cdot \boldsymbol{u}_H^n, w_H) = (f(p_H^n), w_H), \quad \forall w_H \in W_H. \tag{3.6}$$

Step 2: find $(\boldsymbol{u}_h^n, p_h^n) \in V_h \times W_h$, such that

$$(K^{-1}\boldsymbol{u}_h^n, \boldsymbol{v}_h)_Q - (p_h^n, \nabla \cdot \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in V_h, \tag{3.7}$$

$$(\frac{p_h^n - p_h^{n-1}}{\Delta t}, w_h) + (\nabla \cdot \boldsymbol{u}_h^n, w_h) = (f(p_H^n) + f'(p_H^n)(p_h^n - p_H^n), w_h), \quad \forall w_h \in W_h. \tag{3.8}$$

3.2 Error estimate

Theorem 3.1. *Let $(\boldsymbol{u}, p) \in V \times W$ be the solution of problem (2.2)-(2.3), and $(\boldsymbol{u}_H, p_H) \in V_H \times W_H$ be the solution of step 1 of Algorithm 3.1(3.1)-(3.2), then there exists a constant C independent of H, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_H\|_{L^2(0,T;L^2(\Omega))} + \|p - p_H\|_{L^\infty(0,T;L^2(\Omega))} \le CH, \tag{3.9}$$

Proof. Subtracting the numerical scheme (3.1)-(3.2) from the weak formulation (2.2)-(2.3), then the error equation is that

$$(K^{-1}\boldsymbol{u}, \boldsymbol{v}_H) - (K^{-1}\boldsymbol{u}_H, \boldsymbol{v}_H)_Q = (p - p_H, \nabla \cdot \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in V_H, \tag{3.10}$$

$$(\frac{\partial}{\partial t}(p - p_H), w_H) + (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_H), w_H) = (f(p) - f(p_H), w_H), \quad \forall w_H \in W_H. \tag{3.11}$$

For (3.10), we derive that

$$\begin{aligned}
&(K^{-1}(\Pi_H \boldsymbol{u} - \boldsymbol{u}_H), \boldsymbol{v}_H)_Q - (Q_H p - p_H, \nabla \cdot \boldsymbol{v}_H) \\
=& -(K^{-1}(\boldsymbol{u} - \Pi_H \boldsymbol{u}), \boldsymbol{v}_H) - \sigma(K^{-1}\Pi_H \boldsymbol{u}, \boldsymbol{v}_H),
\end{aligned} \tag{3.12}$$

which can be easily get from (2.12) and (2.17). For (3.11), we derive that

$$\begin{aligned}
&(\frac{\partial}{\partial t}(Q_H p - p_H), w_H) + (\nabla \cdot (\Pi_H \boldsymbol{u} - \boldsymbol{u}_H), w_H) \\
=&(\frac{\partial}{\partial t}(Q_H p - p), w_H) + (f(p) - f(p_H), w_H).
\end{aligned} \tag{3.13}$$

Taking $\boldsymbol{v}_H = \Pi_H \boldsymbol{u} - \boldsymbol{u}_H$ and $w_H = Q_H p - p_H$, combining (3.12) and (3.13), then we get

$$\begin{aligned}
&(K^{-1}(\Pi_H \boldsymbol{u} - \boldsymbol{u}_H), \Pi_H \boldsymbol{u} - \boldsymbol{u}_H)_Q + (\frac{\partial}{\partial t}(Q_H p - p_H), Q_H p - p_H) \\
=& -(K^{-1}(\boldsymbol{u} - \Pi_H \boldsymbol{u}), \Pi_H \boldsymbol{u} - \boldsymbol{u}_H) - \sigma(K^{-1}\Pi_H \boldsymbol{u}, \Pi_H \boldsymbol{u} - \boldsymbol{u}_H) \\
&+ (f(p) - f(p_H), Q_H p - p_H) - (\frac{\partial}{\partial t}(p - Q_H p), Q_H p - p_H).
\end{aligned} \tag{3.14}$$

From (2.19), the left side of (3.14) can be estimated as follows

$$\begin{aligned}
&(K^{-1}(\Pi_H \boldsymbol{u} - \boldsymbol{u}_H), \Pi_H \boldsymbol{u} - \boldsymbol{u}_H)_Q + (\frac{\partial}{\partial t}(Q_H p - p_H), Q_H p - p_H) \\
\geq& C\|\Pi_H \boldsymbol{u} - \boldsymbol{u}_H\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|Q_H p - p_H\|^2.
\end{aligned} \tag{3.15}$$

Now, we estimate the right side of (3.14), from (2.10), (2.13), (2.20), and ε-inequality, we have

$$\begin{aligned}
|(K^{-1}(\boldsymbol{u} - \Pi_H \boldsymbol{u}), \Pi_H \boldsymbol{u} - \boldsymbol{u}_H)| &\leq C\|\boldsymbol{u} - \Pi_H \boldsymbol{u}\|\|\Pi_H \boldsymbol{u} - \boldsymbol{u}_H\| \\
&\leq CH^2\|\boldsymbol{u}\|_1^2 + \varepsilon\|\Pi_H \boldsymbol{u} - \boldsymbol{u}_H\|^2,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
|\sigma(K^{-1}\Pi_H \boldsymbol{u}, \Pi_H \boldsymbol{u} - \boldsymbol{u}_H)| &\leq CH\|\boldsymbol{u}\|_1\|\Pi_H \boldsymbol{u} - \boldsymbol{u}_H\| \\
&\leq CH^2\|\boldsymbol{u}\|_1^2 + \varepsilon\|\Pi_H \boldsymbol{u} - \boldsymbol{u}_H\|^2,
\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
|(\frac{\partial}{\partial t}(p-Q_Hp),Q_Hp-p_H)| &\le C\|(p-Q_Hp)_t\|\|Q_Hp-p_H\| \\
&\le CH^2\|p_t\|_1^2+\varepsilon\|Q_Hp-p_H\|^2.
\end{aligned}
\tag{3.18}
$$

By using Taylor expansion, $f(p)=f(p_H)+f'(\bar{p}_H)(p-p_H)$, where $\bar{p}_H$ is between p and p_H, we can derive that

$$
\begin{aligned}
|(f(p)-f(p_H),Q_Hp-p_H)| &\le C\|f(p)-f(p_H)\|\|Q_Hp-p_H\| \\
&\le C\|f'(p_H)(p-p_H)\|\|Q_Hp-p_H\| \\
&\le CH^2\|p\|_1^2+C\|Q_Hp-p_H\|^2.
\end{aligned}
\tag{3.19}
$$

Substituting (3.15)-(3.19) into (3.14), we get

$$
\begin{aligned}
&\|\Pi_H\boldsymbol{u}-\boldsymbol{u}_H\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|Q_Hp-p_H\|^2 \\
\le& CH^2(\|\boldsymbol{u}\|_1^2+\|p\|_1^2+\|p_t\|_1^2)+C\|Q_Hp-p_H\|^2.
\end{aligned}
\tag{3.20}
$$

Integrating over $[0,T]$ and combining the Gronwall inequality, we can deduce that

$$
\begin{aligned}
&\int_0^T\|\Pi_H\boldsymbol{u}-\boldsymbol{u}_H\|^2dt+\|Q_Hp-p_H\|^2 \\
\le& CH^2\int_0^T\|\boldsymbol{u}\|_1^2dt+CH^2\int_0^T\|p\|_1^2dt+CH^2\int_0^T\|p_t\|_1^2.
\end{aligned}
\tag{3.21}
$$

Then, we can obtain

$$
\begin{aligned}
&\int_0^T\|\Pi_H\boldsymbol{u}-\boldsymbol{u}_H\|^2dt+\|Q_Hp-p_H\|^2 \\
\le& CH^2(\|\boldsymbol{u}\|^2_{L^2(0,T;H^1(\Omega))}+\|p\|^2_{L^2(0,T;H^1(\Omega))}+\|p_t\|^2_{L^2(0,T;H^1(\Omega))}).
\end{aligned}
\tag{3.22}
$$

Combining (2.10) and (2.13) , the theorem 3.1 can be proved. □

Proposition 3.2. *For the solution $p_H\in W_H$ in step 1, there is the following L^4 error estimate*

$$
\|p-p_H\|_{0,4}\le CH. \tag{3.23}
$$

Proof. By using (2.13) and (2.43), we can derive that

$$
\begin{aligned}
\|p-p_H\|_{0,4} &= \|p-Q_Hp\|_{0,4}+\|Q_Hp-R_Hp\|_{0,4}+\|R_Hp-p_H\|_{0,4} \\
&\le CH+CH^{\frac{2}{4}-1}(\|Q_Hp-R_Hp\|+\|R_Hp-p_H\|) \\
&\le CH+CH^{\frac{2}{4}-1}H^2 \\
&\le CH.
\end{aligned}
\tag{3.24}
$$

□

Theorem 3.3. *Let* $(\boldsymbol{u}, p) \in V \times W$ *be the solution of problem (2.2)-(2.3), and* $(\boldsymbol{u}_h, p_h) \in V_h \times W_h$ *be the solution of step 2 of Algorithm 3.1(3.3)-(3.4), then there exists a constant* C *independent of* h *and* H*, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))} + \|p - p_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C(h + H^2). \tag{3.25}$$

Proof. Subtracting the numerical scheme (3.3)-(3.4) from the weak formulation (2.2)-(2.3), then the error equation is that

$$\begin{aligned} &(K^{-1}(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{v}_h)_Q - (Q_h p - p_h, \nabla \cdot \boldsymbol{v}_h) \\ =& -(K^{-1}(\boldsymbol{u} - \Pi_h \boldsymbol{u}), \boldsymbol{v}_h) - \sigma(K^{-1}\Pi_h \boldsymbol{u}, \boldsymbol{v}_h), \end{aligned} \tag{3.26}$$

$$\begin{aligned} &(\frac{\partial}{\partial t}(Q_h p - p_h), w_h) + (\nabla \cdot (\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), w_h) \\ =& -(\frac{\partial}{\partial t}(p - Q_h p), w_h) + (f(p) - f(p_H) + f'(p_H)(p_H - p_h), w_h). \end{aligned} \tag{3.27}$$

By using taylor expansion $f(p) = f(p_H) + f'(p_H)(p - p_H) + f''(\tilde{p}_H)(p - p_H)^2$, where $\tilde{p}_H$ is between p and p_H , we derive that

$$\begin{aligned} &(\frac{\partial}{\partial t}(Q_h p - p_h), w_h) + (\nabla \cdot (\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), w_h) \\ =& -(\frac{\partial}{\partial t}(p - Q_h p), w_h) + (f'(p_H)(p - p_h) + f''(\tilde{p}_H)(p - p_H)^2, w_h). \end{aligned} \tag{3.28}$$

Taking $\boldsymbol{v}_h = \Pi_h \boldsymbol{u} - \boldsymbol{u}_h$ and $w_h = Q_h p - p_h$, combining (3.26) and (3.28), then we get

$$\begin{aligned} &(K^{-1}(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), \Pi_h \boldsymbol{u} - \boldsymbol{u}_h)_Q + (\frac{\partial}{\partial t}(Q_h p - p_h), Q_h p - p_h) \\ =& -(K^{-1}(\boldsymbol{u} - \Pi_h \boldsymbol{u}), \Pi_h \boldsymbol{u} - \boldsymbol{u}_h) - \sigma(K^{-1}\Pi_h \boldsymbol{u}, \Pi_h \boldsymbol{u} - \boldsymbol{u}_h) - (\frac{\partial}{\partial t}(p - Q_h p), Q_h p - p_h) \\ &+ (f'(p_H)(p - p_h), Q_h p - p_h) + (f''(\tilde{p}_H)(p - p_H)^2, Q_h p - p_h). \end{aligned} \tag{3.29}$$

From (2.19), (2.10), (2.13), (2.20) and ε-inequality, we derive that

$$\begin{aligned} &\|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Q_h p - p_h\|^2 \leq Ch^2(\|\boldsymbol{u}\|_1^2 + \|p\|_1^2 + \|p_t\|_1^2) \\ &+ C(\|(p - p_H)^2\|^2 + \|Q_h p - p_h\|^2) + \varepsilon\|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|^2. \end{aligned} \tag{3.30}$$

Integrating over $[0, T]$, combining (3.24) and the Gronwall inequality, we can deduce that

$$\int_0^T \|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|^2 dt + \|Q_h p - p_h\|^2 \leq C(h^2 + H^4). \tag{3.31}$$

By (2.10) and (2.13), the theorem 3.3 can be proved. $\square$

Theorem 3.4. *Let $(\boldsymbol{u}, p) \in V \times W$ be the solution of problem (2.2)-(2.3), and $(\boldsymbol{u}_H^n, p_H^n) \in V_H \times W_H$ be the solution of Step 1 of Algorithm 3.2 (3.5)-(3.6), then there exists a constant C independent of H, such that*

$$\max_{1\le n\le N}\|p^n - p_H^n\|^2 + \sum_{n=1}^{N}\Delta t\|\boldsymbol{u}^n - \boldsymbol{u}_H^n\|^2 \le C(H^2 + (\Delta t)^2). \tag{3.32}$$

Proof. Subtracting the numerical scheme (3.5)-(3.6) from the weak formulation (2.2)-(2.3), then the error equation is that

$$(K^{-1}\boldsymbol{u}^n, \boldsymbol{v}_H) - (K^{-1}\boldsymbol{u}_H^n, \boldsymbol{v}_H)_Q = (p^n - p_H^n, \nabla\cdot\boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in V_H, \tag{3.33}$$

$$(\frac{\partial p^n}{\partial t} - \frac{p_H^n - p_H^{n-1}}{\Delta t}, w_H) + (\nabla\cdot(\boldsymbol{u}^n - \boldsymbol{u}_H^n), w_H) = (f(p^n) - f(p_H^n), w_H), \quad \forall w_H \in W_H. \tag{3.34}$$

By using (2.9), (2.12) and (2.17), we get

$$\begin{aligned}&(K^{-1}(\Pi_H\boldsymbol{u}^n - \boldsymbol{u}_H^n), \boldsymbol{v}_H)_Q - (Q_H p^n - p_H^n, \nabla\cdot\boldsymbol{v}_H)\\ &= -(K^{-1}(\boldsymbol{u}^n - \Pi_H\boldsymbol{u}^n), \boldsymbol{v}_H) - \sigma(K^{-1}\Pi_H\boldsymbol{u}^n, \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in V_H,\end{aligned} \tag{3.35}$$

$$\begin{aligned}&(\frac{(Q_H p^n - p_H^n) - (Q_H p^{n-1} - p_H^{n-1})}{\Delta t}, w_H) + (\nabla\cdot(\Pi_H\boldsymbol{u}^n - \boldsymbol{u}_H^n), w_H)\\ &= -(\frac{\partial p^n}{\partial t} - \frac{p^n - p^{n-1}}{\Delta t}, w_H) + (f(p^n) - f(p_H^n), w_H), \quad \forall w_H \in W_H.\end{aligned} \tag{3.36}$$

Denoting $\beta^n = Q_H p^n - p_H^n$, taking $\boldsymbol{v}_H = \Pi_H\boldsymbol{u}^n - \boldsymbol{u}_H^n$ and $w_H = \beta^n$, using Taylor expansion

$$\begin{aligned}f(p^n) &= f(p_H^n) + f'(\bar{p}_H^n)(p^n - p_H^n)\\ &= f(p_H^n) + f'(\bar{p}_H^n)(p^n - Q_H p^n + Q_H p^n - p_H^n)\\ &= f(p_H^n) + f'(\bar{p}_H^n)(p^n - Q_H p^n + \beta^n),\end{aligned} \tag{3.37}$$

then adding the two formulas (3.35) and (3.36) together, we get

$$\begin{aligned}&(K^{-1}(\Pi_H\boldsymbol{u}^n - \boldsymbol{u}_H^n), \Pi_h\boldsymbol{u}^n - \boldsymbol{u}_H^n)_Q + (\frac{\beta^n - \beta^{n-1}}{\Delta t}, \beta^n)\\ &= -(K^{-1}(\boldsymbol{u}^n - \Pi_H\boldsymbol{u}^n), \Pi_H\boldsymbol{u}^n - \boldsymbol{u}_H^n) - \sigma(K^{-1}\Pi_H\boldsymbol{u}^n, \Pi_H\boldsymbol{u}^n - \boldsymbol{u}_H^n)\\ &\quad + (f'(\bar{p}_H^n)(p^n - Q_H p^n), \beta^n) + (f'(\bar{p}_H^n)\beta^n, \beta^n)\\ &\quad - (\frac{\partial p^n}{\partial t} - \frac{p^n - p^{n-1}}{\Delta t}, \beta^n) = \sum_{i=1}^{5} R_i.\end{aligned} \tag{3.38}$$

Multiplying Δt on both sides of (3.38), then summing for n from 1 to N, the left side of (3.38) can be estimated as follows

$$\begin{aligned}&\Delta t\sum_{n=1}^{N}(K^{-1}(\Pi_H\boldsymbol{u}^n-\boldsymbol{u}_H^n),\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_H^n)_Q+\Delta t\sum_{n=1}^{N}(\frac{\beta^n-\beta^{n-1}}{\Delta t},\beta^n)\\ \geq& C\Delta t\sum_{n=1}^{N}\|\Pi_H\boldsymbol{u}^n-\boldsymbol{u}_H^n\|^2+\frac{1}{2}(\beta^N,\beta^N).\end{aligned}\tag{3.39}$$

Now, we estimate the right side of (3.38), we gain the following results

$$\begin{aligned}\Delta t\sum_{n=1}^{N}R_1&=\Delta t\sum_{n=1}^{N}(K^{-1}(\boldsymbol{u}^n-\Pi_H\boldsymbol{u}^n),\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_H^n)\\ &\leq CH^2\|\boldsymbol{u}\|^2_{L^\infty(0,T;H^1)}+\varepsilon\Delta t\sum_{n=1}^{N}\|\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_H^n\|^2,\end{aligned}\tag{3.40}$$

$$\begin{aligned}\Delta t\sum_{n=1}^{N}R_2&=\Delta t\sum_{n=1}^{N}\sigma(K^{-1}\Pi_H\boldsymbol{u}^n,\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_H^n)\\ &\leq C\Delta t\sum_{n=1}^{N}H\|\boldsymbol{u}^n\|_1\|\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_H^n\|\\ &\leq CH^2\|\boldsymbol{u}\|^2_{L^\infty(0,T;H^1)}+\varepsilon\Delta t\sum_{n=1}^{N}\|\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_H^n\|^2,\end{aligned}\tag{3.41}$$

$$\begin{aligned}\Delta t\sum_{n=1}^{N}R_3+\Delta t\sum_{n=1}^{N}R_4&=\Delta t\sum_{n=1}^{N}(f'(\bar{p}_H^n)(p^n-Q_Hp^n),\beta^n)+\Delta t\sum_{n=1}^{N}(f'(\bar{p}_H^n)\beta^n,\beta^n)\\ &\leq C\Delta t\sum_{n=1}^{N}(\|p^n-Q_Hp^n\|\|\beta^n\|)+C\Delta t\sum_{n=1}^{N}\|\beta^n\|^2\\ &\leq C\Delta t\sum_{n=1}^{N}(H\|p^n\|_1\|\beta^n\|)+C\Delta t\sum_{n=1}^{N}\|\beta^n\|^2\\ &\leq CH^2\|p\|^2_{L^\infty(0,T;H^1)}+C\Delta t\sum_{n=1}^{N}\|\beta^n\|^2,\end{aligned}\tag{3.42}$$

$$\Delta t\sum_{n=1}^{N}R_5 = \Delta t\sum_{n=1}^{N}(\frac{\partial p^n}{\partial t}-\frac{p^n-p^{n-1}}{\Delta t},\beta^n)$$
$$\le C(\Delta t\sum_{n=1}^{N}\|\frac{\partial p^n}{\partial t}-\frac{p^n-p^{n-1}}{\Delta t}\|^2+\Delta t\sum_{n=1}^{N}\|\beta^n\|^2) \tag{3.43}$$
$$\le C((\Delta t)^2\int_0^T\|p_{tt}\|^2dt+C\Delta t\sum_{n=1}^{N}\|\beta^n\|^2).$$

Substituting (3.39)-(3.43) into (3.38), combining the Gronwall inequality, then we can get

$$\begin{aligned}&\Delta t\sum_{n=1}^{N}\|\Pi_H\boldsymbol{u}^n-\boldsymbol{u}_H^n\|^2+\|\beta^N\|^2\\ \le& C(H^2\|\boldsymbol{u}\|^2_{L^\infty(0,T;H^1(\Omega))}+H^2\|p\|^2_{L^\infty(0,T;H^1(\Omega))}+(\Delta t)^2\|p_{tt}\|_{L^2(0,T;H^0(\Omega))}).\end{aligned} \tag{3.44}$$

Thus, we complete the proof of theorem 3.4. □

Theorem 3.5. *Let $(\boldsymbol{u}^n,p^n)\in V\times W$ be the solution of problem (2.2)-(2.3) at the time $t=t^n$, and $(\boldsymbol{u}_h^n,p_h^n)\in V_h\times W_h$ be the solution of Step 2 of Algorithm 3.2 (3.7)-(3.8), then there exists a constant C independent of h and Δt, such that*

$$\max_{1\le n\le N}\|p^n-p_h^n\|^2+\sum_{n=1}^{N}\Delta t\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\|^2\le C(h^2+H^4+(\Delta t)^2). \tag{3.45}$$

Proof. Subtracting the numerical scheme (3.7)-(3.8) from the weak formulation (2.2)-(2.3), then the error equation is that

$$\begin{aligned}&(K^{-1}(\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_h^n),\boldsymbol{v}_h)_Q-(Q_Hp^n-p_h^n,\nabla\cdot\boldsymbol{v}_h)\\ =&-(K^{-1}(\boldsymbol{u}^n-\Pi_h\boldsymbol{u}^n),\boldsymbol{v}_h)-\sigma(K^{-1}\Pi_h\boldsymbol{u}^n,\boldsymbol{v}_h),\quad \forall\boldsymbol{v}_h\in V_h,\end{aligned} \tag{3.46}$$

$$\begin{aligned}&(\frac{(Q_hp^n-p_h^n)-(Q_hp^{n-1}-p_h^{n-1})}{\Delta t},w_h)+(\nabla\cdot(\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_h^n),w_h)\\ =&(\frac{\partial p^n}{\partial t}-\frac{p^n-p^{n-1}}{\Delta t},w_h)+(f(p^n)-f(p_H^n)-f'(p_H^n)(p_h^n-p_H^n),w_H),\quad \forall\boldsymbol{v}_h\in V_h.\end{aligned} \tag{3.47}$$

Taking $\boldsymbol{v}_h=\Pi_h\boldsymbol{u}^n-\boldsymbol{u}_h^n$ and $w_h=\xi^n=Q_hp^n-p_h^n$, using Taylor expansion

$$f(p^n)=f(p_H^n)+f'(p_H^n)(p^n-p_H^n)+f''(\bar{p}_H)(p^n-p_H^n)^2, \tag{3.48}$$

then adding (3.46) and (3.47) together, we get

$$\begin{aligned}
&(K^{-1}(\Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n), \Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n)_Q + (\frac{\xi^n - \xi^{n-1}}{\Delta t}, \xi^n) \\
=& - (K^{-1}(\boldsymbol{u}^n - \Pi_h \boldsymbol{u}^n), \Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n) - \sigma(K^{-1}\Pi_h \boldsymbol{u}^n, \Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n) \\
&+ (f'(p_H^n)(p^n - Q_h p^n + \xi^n), \xi^n) + (f''(\bar{p}_H)(p^n - p_H^n)^2, \xi^n) \\
&- (\frac{\partial p^n}{\partial t} - \frac{p^n - p^{n-1}}{\Delta t}, \xi^n) = \sum_{i=1}^{5} \varphi_i.
\end{aligned} \tag{3.49}$$

Multiplying Δt on both sides of (3.49), then summing for n from 1 to N, the left side of (3.49) (denoted as L.S.) can be estimated

$$L.S. \geqslant C\Delta t \sum_{n=1}^{N} \|\Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n\|^2 + \frac{1}{2}\|\xi^N\|^2. \tag{3.50}$$

Now, we estimate the right side of (3.49), because of (2.10), (2.13), (2.20) and $\varepsilon-$ inequality, we gain the following results

$$\begin{aligned}
&\Delta t \sum_{n=1}^{N} \varphi_1 + \Delta t \sum_{n=1}^{N} \varphi_2 + \Delta t \sum_{n=1}^{N} \varphi_5 \\
\leq& C(h^2 \|\boldsymbol{u}\|^2_{L^\infty(0,T;H^1(\Omega))} + (\Delta t)^2 \|p_{tt}\|^2_{L^2(0,T;H^0(\Omega))}) \\
&+ \varepsilon(\Delta t \sum_{n=1}^{N} \|\Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n\|^2 + \Delta t \sum_{n=1}^{N} \|\xi^n\|^2),
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
&\Delta t \sum_{n=1}^{N} \varphi_3 + \Delta t \sum_{n=1}^{N} \varphi_4 \\
=& \Delta t \sum_{n=1}^{N} (f'(p_H^n)(p^n - Q_h p^n + \xi^n), \xi^n) + \Delta t \sum_{n=1}^{N} (f''(\tilde{p}_H^n)(p^n - p_H^n)^2, \xi^n) \\
\leq& C(h^2 \|p\|^2_{L^\infty(0,T;H^1(\Omega))} + \Delta t \sum_{n=1}^{N} \|\xi^n\|^2 + \|p^n - p_H^n\|^4_{0,4}).
\end{aligned} \tag{3.52}$$

Substituting (3.50)-(3.52) into (3.49), combining the Gronwall inequality and (3.23), we can get

$$\Delta t \sum_{n=1}^{N} \|\Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n\|^2 + \|\xi^N\|^2 \leq C(h^2 + H^4 + (\Delta t)^2). \tag{3.53}$$

Thus, we complete the proof of the theorem. □

4 Numerical example

In this section, we provide a numerical example to illustrate the efficiency and accuracy of our proposed two-grid algorithm. For simplicity, we take the domain $\Omega = (0,1) \times (0,1)$, and the exact solution satisfies

$$p = e^t \cos(\pi x)\cos(\pi y)/\pi,$$

then the function f is obtained as

$$f(p) = p^2 + f(x,y). \tag{4.1}$$

Based on two families Γ_H and Γ_h with $h = H^2$, we use BDM_1 mixed finite element space. In order to confirm the efficiency of the two-grid MFMFE method, we compare this method with the MFMFE method. The error and CPU time results are shown in Tables 1-2. The exact solution, the MFMFE solution, and the two-grid MFMFE solution are shown in Fig. 1-6, the comparison of CPU time is presented in Fig. 7.

From these data, we can see that the two-grid MFMFE method can have the same convergence order as the MFMFE method, when the coarse grid size and the fine grid size satisfy $h = O(H^2)$. However, the two-grid MFMFE method is more effective than the MFMFE method judging from the CPU time, for example, when $H = 1/12$, the latter requires almost three times the running time of the former, therefore, the two-grid MFMFE method has significant advantages over the MFMFE method.

Table 1 The error and CPU time of the MFMFE method

h	$\|p - p_h\|$	$Order$	$\|u - u_h\|$	$Order$	$CPUtime/s$
1/36	0.01542		0.04842		2.7
1/64	0.008672	1.00	0.02724	0.99	13.8
1/100	0.005550	1.00	00.01743	1.00	51.2
1/144	0.003958	1.00	0.01211	1.00	150.3

Table 2 The error and CPU time of the two-grid MFMFE method

H	h	$\|p - p_h\|$	$Order$	$\|u - u_h\|$	$Order$	$CPUtime/s$
1/6	1/36	0.01583		0.04842		1.0
1/8	1/64	0.008904	1.00	0.02724	0.99	4.3
1/10	1/100	0.005699	1.00	0.01743	1.00	15.4
1/12	1/144	0.003957	1.00	0.01211	1.00	51.3

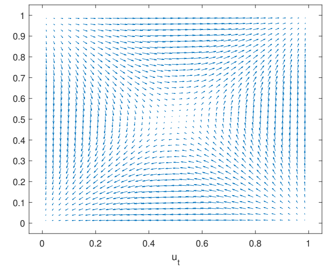

Fig. 1 The exact solution of velocity $\boldsymbol{u}$

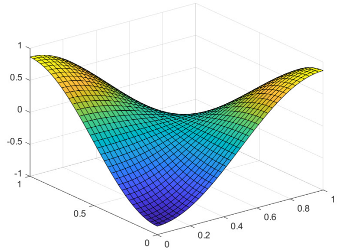

Fig. 2 The exact solution of pressure p

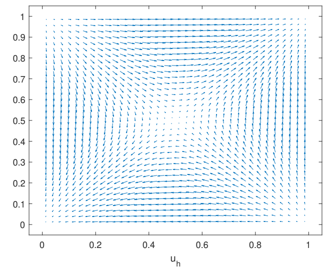

Fig. 3 The MFMFE solution of velocity $\boldsymbol{u}_h$

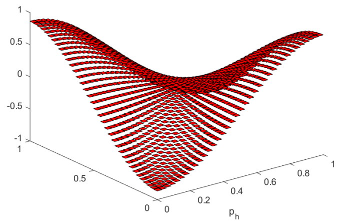

Fig. 4 The MFMFE solution of pressure p_h

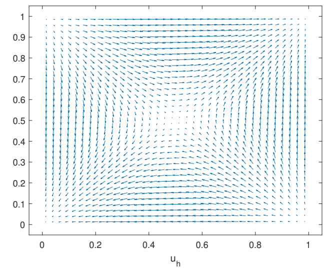

Fig. 5 The two-grid MFMFE solution of velocity $\boldsymbol{u}_h$

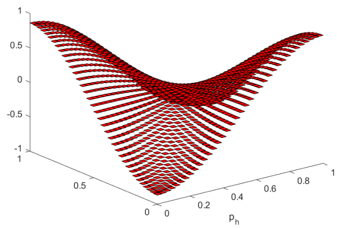

Fig. 6 The two-grid MFMFE solution of pressure p_h

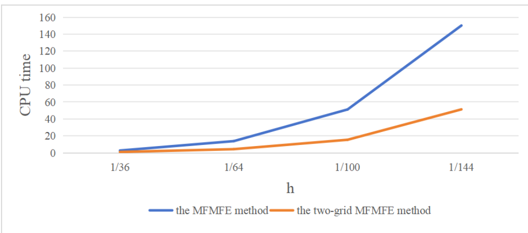

Fig. 7 The comparison of CPU time

References

[1] Dawson, C.N., Wheeler, M.F., Woodward, C.S.: A two-grid finite difference scheme for nonlinear parabolic equations. SIAM Journal on Numerical Analysis, **35(2)**, 435–452 (1998)

[2] Liu, Y.Y., Shu, C.W., Zhang, M.: High order finite difference WENO schemes for nonlinear degenerate parabolic equations. SIAM Journal on Scientific Computing, **33(2)**, 939–965 (2011)

[3] Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Berlin, second edition, 2007.

[4] Bergam, A., Bernardi, C., Mghazli, Z.: A posteriori analysis of the finite element discretization of some parabolic equations. Mathematics of computation, **74(251)**, 1117–1138 (2005)

[5] Eymard, R., Gallouït, T., Herbin, R., *et al.*: Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. Numerische Mathematik, **92(1)**, 41–82 (2002)

[6] Chen, C.J., Liu, W.: Two-grid finite volume element methods for semilinear parabolic problems. Applied Numerical Mathematics, **60(1-2)**, 10–18 (2010)

[7] Johnson, C., Thomée, V.: Error estimates for some mixed finite element methods for parabolic type problems. RAIRO. Analyse numérique, **15**, 41–78 (1981)

[8] Liu, W., Rui, H.X., Long, X.Y.: A two-grid algorithm for mixed finite element solution of nonlinear parabolic equations. Acta Mathematicae Applicatae Sinica, **4**, 635–643 (2007)

[9] Wheeler, M.F., Yotov, I.: A multipoint flux mixed finite element method. SIAM Journal on Numerical Analysis, **44**, 2082–2106 (2006)

[10] Du, S.H., Sun, S.Y., Xie, X.P.: Residual-based a posteriori error estimation for multipoint flux mixed finite element methods. Numerische Mathematik, **134**, 197–222 (2016)

[11] Xu, W.W., Liang, D., Rui, H.X.: A multipoint flux mixed finite element method for the compressible darcy-forchheimer models. Applied Mathematics and Computaion, **315**, 259–277 (2017)

[12] Xu, W.W., Liang, D., Rui, H.X., *et al.*: A multipoint flux mixed finite element method for Darcy–Forchheimer incompressible miscible displacement problem. Journal of Scientific Computing, **82**, 1–20 (2020)

[13] Egger, H., Radu, B.: On a second-order multipoint flux mixed finite element methods on hybrid meshes. SIAM Journal on Numerical Analysis, **58**, 1822–1844

(2020)

[14] He, Z.K., Chung, E.T., Chen, J., *et al.*: Generalized multiscale approximation of a multipoint flux mixed finite element method for darcy-forchheimer model. Journal of Computational and Applied Mathematics, **391**, 112466 (2021)

[15] Li, X.D., Du, M.Y., Xu, W.W.: A multipoint flux mixed finite element method with mass-conservative characteristic finite element method for incompressible miscible displacement problem. Numerical Algorithms, **933**, 1795–1810 (2023)

[16] He, Z.K., Chen, J., Chen, X. Z, *et al.*: An online generalized multiscale approximation of the multipoint flux mixed finite element method. Journal of Computational and Applied Mathematics, **437**, 115498 (2024)

[17] Xu, J.C.: A novel two-grid method for semilinear elliptic equations. SIAM Journal on Scientific Computing, **15**, 231–237 (1994)

[18] Xu, J.C.: Two-grid discretization techniques for linear and nonlinear pdes. SIAM Journal on Numerical Analysis, **33**, 1759–1777 (1996)

[19] Dawson, C.N., Wheeler, M.F., Soodward, W.C.: A two-grid finite difference scheme for nonlinear parabolic equations. SIAM Journal on Numerical Analysis, **35**, 435–452 (1998)

[20] Chen, L.P., Chen, Y.P.: Two-Grid Method for Nonlinear Reaction-Diffusion Equations by Mixed Finite Element Methods. Journal of Scientific Computing, **49**, 383–401 (2011)

[21] Rui, H.X., Liu, W.: A two-grid block-centered finite difference method for Darcy–Forchheimer flow in porous media. SIAM Journal on Numerical Analysis, **53(4)**, 1941–1962 (2015)

[22] Li, X.L., Rui, H.X.: A two-grid block-centered finite difference method for nonlinear non-Fickian flow model. Applied Mathematics and Computation, **281**, 300–313 (2016)

[23] Chen, Y.P., Li, Q.F., Wang, Y.: Two-grid methods of finite element solutions for semi-linear elliptic interface problems. Numerical Algorithms, **84**, 307–330 (2020)

[24] Liu, Y., Chen, Y.P., Huang, Y.Q., *et al.*: Analysis of a two-grid method for semiconductor device problem. Applied Mathematics and Mechanics, **42**, 143–158 (2021)

[25] Tan, Z.J., Zeng, Y.H.: Temporal second-order fully discrete two-grid methods for nonlinear time-fractional variable coefficient diffusion-wave equations. Applied Mathematics and Computation, **466**, 128457 (2024)

[26] Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North–Holland,

New York (1978)

[27] Brezzi, F., Douglas Jr., J., Marini, L.D.: Two families of mixed elements for second order elliptic problems. Numerische Mathematik, **88**, 217–235 (1985)

[28] Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer, New York (1991)

[29] Wheeler, M.F., Yotov, I.: A cell-centered finite difference method for quadrilaterals. In Compatible Spatial Discretizations, IMA Vol. Math. Appl. 142, Springer-Verlag, New York, 189–208 (2006)

[30] Garcia, S.: Improved error estimates for mixed finite-element approximations for nonlinear parabolic equations: the continuous-time case. Numercial Methods for Partial Differential Equations, **10**, 139 (1994)