A PARALLEL TSENG'S SPLITTING METHOD FOR COMMON SOLUTION OF VARIATIONAL INCLUSIONS AND FIXED POINT PROBLEMS ON HADAMARD MANIFOLDS

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ABSTRACT. In this manuscript, we propose an inertial forward-backward-forward splitting method for common solution of variational inclusions and fixed point problems of nonexpansive mappings in the framework of a Hadamard manifold. Using our iterative method together with a self-adaptive method which generates dynamic step-size converging to a positive constant, we establish that the sequence generated by our method converges to a common solution of variational inclusions and fixed point problems. Also, we illustrate a numerical example to show the performance of our method. The result discuss in this article extends and complements many related results in the literature.

1. Introduction

The theory of variational inclusion problems are known to be used as mathematical programming models to study a large number of optimization problem arising in finance, economics, network, transportation and engineering sciences (see [11, 13, 15, 18, 30] and the references therein). In real world application, many nonlinear problems arising in applied areas are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators.

Let $\mathcal{H}$ be a real Hilbert space, $\Psi : \mathcal{H} \to \mathcal{H}$ be an operator and $\varphi : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued operator. The variational inclusion problem (in short, VIP) is to find $u^* \in \mathcal{H}$ such that

$$0 \in (\Psi + \varphi)(u^*). \tag{1.1}$$

If $\Psi = 0$, then the problem (1.1) becomes the inclusion problem introduced by Rockafellar [27]. One of the most important method for solving problem (1.1) goes back to the work of Browder [8]. In the framework of real Hilbert space $\mathcal{H}$, one of the basic ideas in case is reducing the VIP (1.1) to a fixed point problem of the operator J_Ψ defined by $J_\Psi = (I + \Psi)^{-1}$ which is known as the classical resolvent of Ψ. If Ψ has some monotonicity properties, the classical resolvent of Ψ is with full domain and firmly nonexpansive. This is also applicable to nonlinear spaces (Hadamard space and manifolds, to be precise). Methods for approximating zero points of monotone operators in the framework of real Hilbert spaces and based on good properties of the resolvent J_ψ, but these properties are not available in the framework of Banach spaces.

The proximal point algorithm (in short, PPA) introduced by Lions and Mercier [18] has been extensively used to approximate the solution of VIP (1.1). Owing to fixed point formulation, Lions and Mercier [18] introduced the following PPA as follows: let $w_0 \in \mathcal{H}$ be an initial point and

$$w_{k+1} = J_\lambda^\varphi(w_k - \lambda \Psi(w_k)), \ \forall \ k \in \mathbb{N}, \tag{1.2}$$

where $\lambda > 0$. Several other methods have been employed to approximate solution of the problem VIP (1.1) including the Tseng method and the forward-backward splitting method. A well-known modified backward algorithm is the Tseng's splitting algorithm [31]. In 2018, Gibali and Tseng [16] introduced the following forward-backward-forward splitting algorithm as follows:

Algorithm 1.1. *Mann Tseng type algorithm (MTTA)*

Initialization: *Given* $\{\beta^k\}, \{\delta^k\} \subset (0,1), \ \lambda \in (0,1)$ *and* $\gamma^1 > 0$. *Let* $u_1 \in \mathcal{H}$ *and set* $k := 1$

Iterative steps: *Construct* $\{u_k\}$ *by using the following steps:*

step 1: *Compute*

$$r_k = J_{\gamma^k}^\varphi (I - \gamma^k \Psi) u_k$$

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If $u_k = r_k$, then stop and r_k is a solution of VIP (1.1). Otherwise

step 2: *Compute*

$$t_k = r_k - \gamma^k(\Psi r_k - \Psi u_k)$$

and

$$u_{k+1} = (1 - \beta^k - \delta^k)u_k + \delta^k t_k$$

$$\gamma^{k+1} = \begin{cases} \min\Big\{\lambda \frac{\|u_k - r_k\|}{\|\Psi u_k - \Psi r_k\|},\ \gamma^k\Big\} & if\ \Psi u_k \neq \Psi r_k \\ \gamma^k & otherwise. \end{cases}$$

Replace k by $k+1$ and then go to step 1.

Weak convergence of the above algorithm was established under Lipschitz continuity and monotonicity of the operator Ψ.

We observed that in the above method, single variational inclusion problem was considered. In many real-world problems, it is important to find a solution to problems that satisfies multiple constraints as such problems can be applied in machine maintenance, system reliability in military systems and all engineering system designs (see [17] and the references there in). These constraints can be reformulated using a nonlinear functional model, and thus be utilized to solve real-world problems such as signal recovery and image processing problems with various blurred filters, (see [28, 29] and the references confirmed in).

We now define the common variational inclusion problem, which is to find a point $u^* \in \mathcal{H}$ such that

$$0 \in (\Psi_j + \varphi_j)u^*, \tag{1.3}$$

where $\Psi_j : \mathcal{H} \to \mathcal{H}$ are single-valued mappings and $\varphi_j : \mathcal{H} \to 2^{\mathcal{H}}$ and multi-valued mappings for all $j = 1, 2, \cdots, N$.

Many authors have introduced several iterative methods for solving VIP (1.3). For instance, recently Suparatulatorn *et al.* [29] proposed a parallel Tseng's splitting method for solving VIP (1.3) under the Lipschitz continuity and monotonicity of Ψ_j, and maximal monotonicity of φ_j for all $j \in \mathbb{N}$. They established a strong convergence theorem of their proposed method under suitable assumptions and illustrate the applicability of the new method to signal recovering problem arising in compressed sensing.

Very recently, Mouktonglang *et al.* [23] proposed the following method to solve VIP (1.3) and common fixed point problem in a real Hilbert space as follows:

Initialization: Given $\lambda_j \in (0, 1)$ and $\gamma_1^j > 0$ for all $j \in \mathbb{N}$. Select arbitrary element $v_0, v_1 \in \mathcal{H}$ and set $k := 1$.
Iterative step: Construct $\{v_k\}$ by using the following steps :

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step 1: set

$$s_k = v_k + \Theta_k(v_k - v_{k-1})$$

and compute

$$r_k^j = J_{\gamma_k^j}^{\varphi_j}(I - \gamma_k^j \Psi_j)s_k.$$

step 2: Compute, for all $j \in \mathbb{N}$

$$t_k^j = r_k^j - \gamma_k^j(\psi r_k^j - \psi_j s_k)$$

and

$$u_k^j = \alpha_k^j t_k^j + (1 - \alpha_k^j)S_j t_t^j$$

step 3: Compute

$$v_{k+1} = arg\max\{\|u_k^j - s_k\| : j \in \mathbb{N}\} \tag{1.4}$$

and update, for all $j \in \mathbb{N}$,

$$\gamma_{k+1}^j = \begin{cases} \min\Big\{\frac{\lambda_j q_k^j \|s_k - r_k^j\|}{\|\Psi_j s_k - \Psi_j r_k^j\|}, \gamma_k^j + p_k^j\Big\} & if\ \Psi_j s_k \neq \Psi_j r_k^j, \\ \gamma_k^j + p_k^j, & \text{otherwise.} \end{cases} \tag{1.5}$$

Replace k by $k+1$ and then repeat step 1 .

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where $\Psi_j : \mathcal{H} \to \mathcal{H}$ is L_j Lipschitz continuous and monotone mapping, $\varphi_j : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator, $S_j : \mathcal{H} \to \mathcal{H}$ is μ_j-demicontractive mapping such that $I - S_j$ is demiclosed at $0, \{p_k^j\} \subset [0, \infty), \{q_k^j\} \subset [1, \infty)$ such that $\sum_{k=1}^{\infty} p_k < \infty$ and $\lim_{k\to\infty} q_k = 1$ with $\{\theta_k\} \subset [0, \theta), \{\alpha_k^j\} \subset (\mu_j, \bar{\alpha}_j) \subset (0, 1)$, for some $\theta, \bar{\alpha}_j > 0$. They established a weak convergence theorem by using their proposed method.

Let Δ be a nonempty closed geodisic convex subset of a Hadamard manifold $\mathbb{M}$, $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$ of $x \in \mathbb{M}$ and $T\mathbb{M}$ in the tangent bundle of $\mathbb{M}$. The common variational inclusion problem is to find an element $u^* \in \Delta$ such that

$$0 \in (\Psi_j + \varphi_j)u^*, \ \textit{for all} \ j = 1, 2 \cdots, N. \tag{1.6}$$

If $j = 1$, then problem (1.6) reduces to the variational inclusion problem which is to find a point $u^* \in \Delta$ such that

$$0 \in (\Psi + \varphi)u^*, \tag{1.7}$$

where $\Psi : \Delta \to T\mathbb{M}$ is a single-valued vector field, $\varphi : \Delta \to 2^{T\mathbb{M}}$ is a multivalued vector field and 0 denotes the zero sector of $T\mathbb{M}$.

Remark 1.2. If $\Psi \equiv 0$ in (1.7), then (1.7) reduces to monotone inclusion problem (In short, MIP) which is to find:

$$x \in \Delta \text{ such that } 0 \in \varphi u^* \tag{1.8}$$

In 2021, Chaipunya *et al.* [9] introduced the following iterative algorithm for solving (1.7) in the setting of Hadamard manifold as follows: Choose $x_0 \in \Delta$, and define $\{w^k\}$ and $\{u^k\}$ in the following manner:

$$\begin{cases} w^k = J^{\varphi}_{\lambda_k}(\exp_{u^k}(-\lambda^k \Psi(u^k))), \\ u^{k+1} = \exp_{u^k}(1 - \gamma^k) \exp^{-1}_{u^k} w^k, \end{cases} \tag{1.9}$$

for all $k \in \mathbb{N}$, where Δ is a nonempty, closed and geodesic convex subset of a Hadamard manifold $\mathbb{M}$, Ψ is an α-inverse strongly monotone vector field, where $\alpha > 0$ and φ is a maximal monotone vector field with $(\Psi + \varphi)^{-1}(0) \neq \emptyset$. They proved that $\{u^k\}$ converges to a solution of (1.7) under the following condition:

(i) $0 < \gamma^1 \leq \gamma^k \leq \gamma^2 < 1, \ k \in \mathbb{N}$
(ii) $0 < \hat{\lambda} \leq \lambda^k \leq 2\alpha < \infty, \ k \in \mathbb{N}.$

Very recently, Khammahawong *et al.* [20] Introduced two Tseng's methods for finding a singularity point of an inclusion problem defined by means of sum of a single-valued vector field and multi-valued vector field on a Hadamard manifold. One of the method employed is defined as follows:

Algorithm 1.3. ***Initialization:*** *Choose $\tau^0 > 0$ and $\lambda \in (0, 1)$. Let $x_0 \in \mathbb{M}$ be an initial point*

Iterative steps: *Given $x^k \in \Delta$, calculate x^{k+1} by*

step 1: *Compute w^k such that*

$$0 \in \Gamma_{w^k, x^k} + \Psi(x^k) + \varphi(w^k) - \frac{1}{\tau^k} \exp^{-1}_{w^k} x^k. \tag{1.10}$$

If $w^k = x^k$, then stop and x^k is a solution of problem (1.7). Otherwise

step 2: *Compute*

$$x^{k+1} = \exp_{w^k}(\tau^k(\Gamma_{w^k, x^k} \Psi(x^k) - \Psi(w^k))), \tag{1.11}$$

and

$$\tau^{k+1} = \begin{cases} \min\left\{ \frac{\lambda d(x^k, w^k)}{\|\Gamma_{w_k, x_k} \Psi(x^k) - \Psi(w_k)} \|, \tau^k \right\} & \textit{if} \ \|\Gamma_{w^k, x^k} \Psi(x^k) - \Psi(w^k)\| \neq 0, \\ \tau^k & \textit{otherwise}. \end{cases} \tag{1.12}$$

Set $k =: k + 1$ and return to step 1.

They established that the sequence generated by Algorithm 1.3 converges to the solution of (1.7).

Iterative algorithms with inertial extrapolations have been of interest due to fast convergence rate brought about by the addition of inertial extrapolation terms. Note that the inertia is a induced by the term $\theta^k(x^k - x^{k-1})$ and it can be seen as a process of accelerating the rate of convergence (see [6, 25]). Recently, Alvarez and Attouch [3] employed the heavy ball method which was studied in [25] for maximal monotone operator by the proximal point algorithm and it is defined as follows:

$$\begin{cases} y^k = x^k + \theta^k(x^k - x^{k-1}) \\ x^{k+1} = (I + r^k\varphi)^{-1}y^k, \ k \geq 1. \end{cases} \tag{1.13}$$

It was established that if $\{r^k\}$ is non-decreasing and $\{\theta^k\} \subset [0,1)$ with

$$\sum_{k=1}^{\infty} \theta^k \|x^k - x^{k-1}\|^2 < \infty, \tag{1.14}$$

then algorithm (1.13) converges weakly to a zero of φ. It is remarkable that the inertial methodology greatly improves the performance of the algorithm and has a nice convergence properties(see [3, 5, 16] and the references there in).

Motivated by the works of Gibali and Thong [16], Suparatulatorn *et al.* [29], Mouktonglang *et al.* [23], Chaipunya *et al.* [9], Khammahawong *et al.* [20] and some other related results in literature, we propose an inertial forward-backward-forward splitting method for approximating a common solution of variational inclusions and fixed point problems in the setting of Hadamard manifolds. Under suitable condition, we prove that the sequence generated by our method converges to a solution of common variation inclusion problem. Also, we employ a self-adoptive procedure which generates dynamic step-size converging to a positive constant. A numerical example was displayed to show the performance of our iterative method. The result discussed in this article extends and generalizes many related results in literature.

2. Preliminaries

Let $\mathbb{M}$ be an m-dimensional manifold. For $x \in \mathbb{M}$, let $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$ at $x \in \mathbb{M}$. We denote by $T\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x\mathbb{M}$ the tangent bundle of $\mathbb{M}$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian metric on $\mathbb{M}$ if $\langle\cdot,\cdot\rangle_x : T_x\mathbb{M}\times T_x\mathbb{M} \to \mathbb{R}$ is an inner product for all $x \in \mathbb{M}$. The corresponding norm induced by the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x\mathbb{M}$ is denoted by $\|\cdot\|_x$. We will drop the subscript x and adopt $\|\cdot\|$ for the corresponding norm induced by the inner product. A differentiable manifold $\mathbb{M}$ endowed with a Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. In what follows, we denote the Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle$ when no confusion arises. Given a piecewise smooth curve $\gamma : [a,b] \to \mathbb{M}$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$), we define the length $l(\gamma)$ of γ by $l(\gamma) := \int_a^b \|\gamma'(t)\|dt$. The Riemannian distance $d(x,y)$ is the minimal length over the set of all such curves joining x to y. The metric topology induced by d coincides with the original topology on $\mathbb{M}$. We denote by ∇ the Levi-Civita connection associated with the Riemannian metric [26].

Let γ be a smooth curve in $\mathbb{M}$. A vector field X along γ is said to be parallel if $\nabla_{\gamma'}X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' itself is parallel along γ, then we say that γ is a geodesic and $\|\gamma'\|$ is a constant. If $\|\gamma'\| = 1$, then the geodesic γ is said to be normalized. A geodesic joining x to y in $\mathbb{M}$ is called a minimizing geodesic if its length equals $d(x,y)$. A Riemannian manifold $\mathbb{M}$ equipped with a Riemannian distance d is a metric space $(\mathbb{M},d)$. A Riemannian manifold $\mathbb{M}$ is said to be complete if for all $x \in \mathbb{M}$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$. The Hopf-Rinow theorem [26], posits that if $\mathbb{M}$ is complete, then any pair of points in $\mathbb{M}$ can be joined by a minimizing geodesic. Moreover, if $(\mathbb{M},d)$ is a complete metric space, then every bounded and closed subset of $\mathbb{M}$ is compact. If $\mathbb{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ at $x \in \mathbb{M}$ is defined by

$$\exp_x v := \gamma_v(1,x) \ \forall \ v \in T_x\mathbb{M},$$

where $\gamma_v(\cdot,x)$ is the geodesic starting from x with velocity v (that is, $\gamma_v(0,x) = x$ and $\gamma_v'(0,x) = v$). Then, for any t, we have $\exp_x tv = \gamma_v(t,x)$ and $\exp_x \mathbf{0} = \gamma_v(0,x) = x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{M}$ for every $x \in \mathbb{M}$. The exponential map $\exp_x$ has an inverse $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$. For any $x,y \in \mathbb{M}$, we have $d(x,y) = \|\exp_y^{-1} x\| = \|\exp_x^{-1} y\|$ (see [26] for more details). The parallel transport $\Gamma_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{M} \to T_{\gamma(b)}\mathbb{M}$ on the tangent bundle $T\mathbb{M}$ along $\gamma : [a,b] \to \mathbb{R}$ with respect to ∇ is defined by

$$\Gamma_{\gamma,\gamma(b),\gamma(a)}v = F(\gamma(b)), \ \forall \ a,b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}\mathbb{M},$$

where F is the unique vector field such that $\nabla_{\gamma'(t)} v = \mathbf{0}$ for all $t \in [a,b]$ and $F(\gamma(a)) = v$. If γ is a minimizing geodesic joining x to y, then we write $\Gamma_{y,x}$ instead of $\Gamma_{\gamma,y,x}$. Note that for every $a,b,r,s \in \mathbb{R}$, we have

$$\Gamma_{\gamma(s),\gamma(r)} \circ \Gamma_{\gamma(r),\gamma(a)} = \Gamma_{\gamma(s),\gamma(a)} \text{ and } \Gamma^{-1}_{\gamma(b),\gamma(a)} = \Gamma_{\gamma(a),\gamma(b)}.$$

Also, $\Gamma_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{M}$ to $T_{\gamma(b)}\mathbb{M}$, that is, the parallel transport preserves the inner product

$$\langle \Gamma_{\gamma(b),\gamma(a)}(u), \Gamma_{\gamma(b),\gamma(a)}(v)\rangle_{\gamma(b)} = \langle u, v\rangle_{\gamma(a)}, \ \forall\, u, v \in T_{\gamma(a)}\mathbb{M}. \tag{2.1}$$

A subset $\mathcal{K} \subset \mathbb{M}$ is said to be convex if for any two points $x, y \in \mathcal{K}$, the geodesic γ joining x to y is contained in $\mathcal{K}$. That is, if $\gamma : [a,b] \to \mathbb{M}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in \mathcal{K}$ for all $t \in [0,1]$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called an Hadamard manifold. We denote by $\mathbb{M}$ a finite dimensional Hadamard manifold. Henceforth, unless otherwise stated, we represent by $\mathcal{K}$ a nonempty, closed and convex subset of $\mathbb{M}$.

Next, let $\mathcal{H}(\mathcal{K})$ denote the set of all single-valued vector fields $U : \mathcal{K} \to T\mathbb{M}$ such that $U(p) \in T_p\mathbb{M}$, for each $p \in \mathcal{K}$. Let $\mathcal{X}(\mathcal{K})$ denote to the set of all multivalued vector fields $V : \mathcal{K} \to 2^{T\mathbb{M}}$ such that $V(p) \subseteq T_p\mathbb{M}$ for each $p \in \mathcal{K}$, and the denote $Dom(V)$ the domain of V defined by $Dom(V) = \{p \in \mathcal{K} : V(p) \neq \emptyset\}$.
We state some results and definitions which are needed in the next section.

Definition 2.1. [32] A vector field $U \in \mathcal{H}(\mathcal{K})$ is said to be

(i) monotone, if
$$\langle U(p), \exp_p^{-1} q\rangle \leqslant \langle U(q), -\exp_q^{-1} p\rangle, \ \forall\, p, q \in \mathcal{K},$$
(ii) L-Lipschitz continuous if there exists $L > 0$ such that
$$\|\Gamma_{p,q} U(q) - U(p)\| \leqslant L d(p,q), \ \forall\, p, q \in \mathcal{K}.$$

Definition 2.2. [10] A vector field $V \in \mathcal{X}(\mathcal{K})$ is said to be

(i) monotone, if for all $p, q \in Dom(V)$
$$\langle u, \exp_p^{-1} q\rangle \leqslant \langle v, -\exp_q^{-1} p\rangle, \ \forall\, u \in V(p) \text{ and } \forall\, v \in V(q),$$
(ii) maximal monotone if it is monotone and $\forall\, p \in \mathcal{K}$ and $u \in T_p\mathcal{M}$, the condition
$$\langle u, \exp_p^{-1} q\rangle \leqslant \langle v, -\exp_q^{-1} p\rangle, \ \forall\, q \in Dom(V) \text{ and } \forall\, v \in V(q) \text{ implies that } u \in V(p).$$

Definition 2.3. [14] Let $\mathcal{K}$ be a nonempty, closed and subset of $\mathbb{M}$ and $\{x_n\}$ be a sequence in $\mathbb{M}$. Then $\{x_n\}$ is said to be Fejèr convergent with respect to $\mathcal{K}$ if for all $p \in \mathcal{K}$ and $n \in \mathbb{N}$,
$$d(x_{n+1}, p) \leqslant d(x_n, p).$$

Definition 2.4. [21] Let $V \in \mathcal{X}(\mathcal{K})$ be a vector field and $x_0 \in \mathcal{K}$. Then V is said to be upper Kuratowki semicontinuous at x_0 if for any sequences $\{x_n\} \subseteq \mathcal{K}$ and $\{v_n\} \subset T\mathbb{M}$ with each $v_n \in V(x_n)$, the relations $\lim\limits_{n\to\infty} v_n = v_0$ imply that $v_0 \in V(x_0)$. Moreover, V is said to be upper Kuratowski semicontinuous on $\mathcal{K}$ if it is upper Kuratowski semicontinuous for each $x \in \mathcal{K}$.

Proposition 2.5. *[26]. Let $x \in \mathbb{M}$. The exponential mapping $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is a diffeomorphism. For any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic joining x to y, which is given by*
$$\gamma(t) = \exp_x t \exp_x^{-1} y \ \forall\, t \in [0,1].$$

A geodesic triangle $\Delta(p,q,r)$ of a Riemannian manifold $\mathbb{M}$ is a set containing three points p, q, r and three minimizing geodesics joining these points.

Proposition 2.6. *[26]. Let $\Delta(p,q,r)$ be a geodesic triangle in $\mathbb{M}$. Then*
$$d^2(p,q) + d^2(q,r) - 2\langle \exp_q^{-1} p, \exp_q^{-1} r\rangle \leqslant d^2(r,q) \tag{2.2}$$
and
$$d^2(p,q) \leqslant \langle \exp_p^{-1} r, \exp_p^{-1} q\rangle + \langle \exp_q^{-1} r, \exp_q^{-1} p\rangle. \tag{2.3}$$
Moreover, if θ is the angle at p, then we have
$$\langle \exp_p^{-1} q, \exp_p^{-1} r\rangle = d(q,p) d(p,r) \cos\theta. \tag{2.4}$$

Also,

$$\|\exp_p^{-1} q\|^2 = \langle \exp_p^{-1} q, \exp_p^{-1} q\rangle = d^2(p,q). \tag{2.5}$$

Remark 2.7. [21] If $x, y \in \mathbb{M}$ and $v \in T_y\mathbb{M}$, then

$$\langle v, -\exp_y^{-1} x\rangle = \langle v, \Gamma_{y,x}\exp_x^{-1} y\rangle = \langle \Gamma_{x,y} v, \exp_x^{-1} y\rangle. \tag{2.6}$$

Lemma 2.8. *[19] Let $\mathbb{M}$ be an Hadamard manifold and let $u, v, w \in \mathbb{M}$. Then*

$$\|\exp_u^{-1} w - \Gamma_{u,v}\exp_v^{-1} w\| \leqslant d(u,v).$$

Lemma 2.9. *[21] Let $x_0 \in \mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ with $x_n \to x_0$. Then the following assertions hold:*

(i) *For any $y \in \mathbb{M}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} x_n$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$,*
(ii) *If $v_n \in T_{x_n}\mathbb{M}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathbb{M}$,*
(iii) *Given $u_n, v_n \in T_{x_n}\mathbb{M}$ and $u_0, v_0 \in T_{x_0}\mathbb{M}$, if $u_n \to u_0$, then $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$,*
(iv) *For any $u \in T_{x_0}\mathbb{M}$, the function $F : \mathbb{M} \to T\mathbb{M}$, defined by $F(x) = \Gamma_{x,x_0} u$ for each $x \in \mathbb{M}$ is continuous on $\mathbb{M}$.*

The next lemma presents the relationship between triangles in $\mathbb{R}^2$ and geodesic triangles in Riemannian manifolds (see [7]).

Lemma 2.10. *[7]. Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in $\mathbb{M}$. Then there exists a triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ corresponding to $\Delta(x_1, x_2, x_3)$ such that $d(x_i, x_{i+1}) = \|\bar{x}_i - \bar{x}_{i+1}\|$ with the indices taken modulo 3. This triangle is unique up to isometries of $\mathbb{R}^2$.*

The triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Lemma 2.10 is said to be the comparison triangle for $\Delta(x_1, x_2, x_3) \subset \mathbb{M}$. The points $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are called comparison points to the points x_1, x_2 and x_3 in $\mathbb{M}$.

A function $h : \mathbb{M} \to \mathbb{R}$ is said to be geodesic if for any geodesic $\gamma \in \mathbb{M}$, the composition $h \circ \gamma : [u, v] \to \mathbb{R}$ is convex, that is,

$$h \circ \gamma(\lambda u + (1-\lambda)v) \leqslant \lambda h \circ \gamma(u) + (1-\lambda) h \circ \gamma(v), \ \ u, v \in \mathbb{R}, \ \ \lambda \in [0,1].$$

Lemma 2.11. *[21] Let $\Delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$ and $\Delta(p^{'}, q^{'}, r^{'})$ be its comparison triangle.*

(i) *Let α, β, γ (resp. $\alpha^{'}, \beta^{'}, \gamma^{'}$) be the angles of $\Delta(p, q, r)$ (resp. $\Delta(p^{'}, q^{'}, r^{'})$) at the vertices p,q,r (resp. $p^{'}, q^{'}, r^{'}$). Then, the following inequalities hold:*

$$\alpha^{'} \geqslant \alpha, \ \ \beta^{'} \geqslant \beta, \ \ \gamma^{'} \geqslant \gamma,$$

(ii) *Let z be a point in the geodesic joining p to q and $z^{'}$ its comparison point in the interval $[p^{'}, q^{'}]$. Suppose that $d(z, p) = \|z^{'} - p^{'}\|$ and $d(z^{'}, q^{'}) = \|z^{'} - q^{'}\|$. Then the following inequality holds:*

$$d(z, r) \leqslant \|z^{'} - r^{'}\|.$$

Lemma 2.12. *[21] Let $x_0 \in \mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ be such that $x_n \to x_0$. Then, for any $y \in \mathbb{M}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$;*

The following propositions (see [14]) are very useful in our convergence analysis:

Proposition 2.13. *Let $\mathbb{M}$ be an Hadamard manifold and $d : \mathbb{M} \times \mathbb{M} :\to \mathbb{R}$ be the distance function. Then the function d is convex with respect to the product Riemannian metric. In other words, given any pair of geodesics $\gamma_1 : [0,1] \to \mathbb{M}$ and $\gamma_2 : [0,1] \to \mathbb{M}$, then for all $t \in [0,1]$, we have*

$$d(\gamma_1(t), \gamma_2(t)) \leqslant (1-t) d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in \mathbb{M}$, the function $d(\cdot, y) : \mathbb{M} \to \mathbb{R}$ is a convex function.

Proposition 2.14. *Let $\mathbb{M}$ be a Hadamard manifold and $x \in \mathbb{M}$. The map $\Phi_x = d^2(x, y)$ satisfying the following:*

(1) *Φ_x is convex. Indeed, for any geodesic $\gamma : [0,1] \to \mathbb{M}$, the following inequality holds for all $t \in [0,1]$:*

$$d^2(x, \gamma(t)) \leqslant (1-t) d^2(x, \gamma(0)) + t d^2(x, \gamma(1)) - t(1-t) d^2(\gamma(0), \gamma(1)).$$

(2) *Φ_x is smooth. Moreover, $\partial \Phi_x(y) = -2\exp_y^{-1} x$.*

Lemma 2.15. *[33] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality;*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n, \ \forall \ n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\} \subset \mathbb{R}$ *such that*
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$, *or* $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.16. *[22] Let* $\{s_n\}$ *be a real sequence which does not decrease at infinity in the sense that there exists a subsequence* $\{s_{n_k}\}$ *such that*

$$s_{n_k} \leq s_{n_k+1} \ \forall \ k \geq 0$$

Define an integer sequence $\{\tau(n)\}$, *where* $n > n_0$, *by*

$$\tau(n) := \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$ *and for all* $n > n_0$, *we have*

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}.$$

3. Main Result

In this section, we present an iterative method for solving variational inclusion problem in the settings of Hadamard manifolds. We state the following assumptions:

Assumption 3.1. (A1) $\Psi_j \in \mathcal{H}(\mathcal{K})$ *is monotone and L-Lipschitz continuous and* $\varphi_j \in \Phi(\mathcal{K})$ *is maximal monotone for* $j = 1, 2, \cdots, N$.

(A2) $U : \mathcal{K} \to \mathcal{K}$ *is a nonexpansive mapping such that* $F(U) \neq \emptyset$, *and* $\Omega := F(U) \cap (\Psi_j + \varphi_j)^{-1}(0)$ *is nonempty.*
(A3) *Let* $h : \mathcal{K} \to \mathcal{K}$ *be a contraction mapping with constant* $\phi \in (0,1)$.

Assumption 3.2. (B1) $\{\alpha_k\} \in (0,1)$ *and* $0 < \liminf \alpha_k \leq \limsup \alpha_k < 1$,

(B2) $\{\eta_k\}$ *is a nonnegative real numbers sequence such that* $\sum_{k=1}^{\infty} \eta_k < \infty$,
(B3) *Let* $\{\delta_k\} \in (0,1)$ *such that* $\lim_{k\to\infty} \delta_k = 0$ *and* $\sum_{k=1}^{\infty} \delta_k = \infty$,
(B4) $\{\epsilon_k\}$ *is a positive sequence such that* $\epsilon_k = \circ(\delta_k)$, *that is,* $\lim_{k\to\infty} \frac{\epsilon_k}{\delta_k} = 0$.

Algorithm 3.3. *Parallel Tseng's method for common variational inclusion problem*

Initialization: *Choose* $\beta_0 > 0, \mu, \theta \in (0,1)$ *and let* $x_0, x_1 \in \mathbb{M}$ *be arbitrary starting points.*

Iterative step: *Given* x_{k-1}, x_k, *and* β_k, *choose* $\theta_k \in [0, \bar{\theta_k}]$ *where*

$$\theta_k = \begin{cases} \min\left\{\dfrac{\epsilon_k}{d(x_k, x_{k-1})}, \quad \theta\right\}, & \text{if } x_k \neq x_{k-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 1: *Compute*

$$\begin{cases} v_k = \exp_{x_k}(-\theta_k \exp^{-1}_{x_k} x_{k-1}) \\ \mathbf{0} \in \Gamma_{w_k^j, v_k} \Psi_j(v_k) + \varphi_j(w_k^j) - \frac{1}{\beta_k} \exp^{-1}_{w_k^j} v_k, \ j = 1, 2, \cdots, N \end{cases} \tag{3.2}$$

Step 2: *Calculate*

$$y_k^j = \exp_{w_k^j}(\beta_k(\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j))), j = 1, 2, \cdots, N. \tag{3.3}$$

Step 3: *Find the farthest element from* v_k *among* y_k^j. *That is*

$$j_k = arg\max\left\{d(y_k^j, v_k) : j = 1, 2, \cdots, N\right\}, \ y_k = y_k^{j_k}. \tag{3.4}$$

Step 4: *Calculate*

$$u_k = \exp_{y_k}(1-\alpha_k)\exp^{-1}_{y_k} U(y_k). \tag{3.5}$$

Step 5: *Calculate x_{k+1} and β_{k+1} by*

$$x_{k+1} = \exp_{h(x_k)}(1-\delta_k)\exp^{-1}_{h(x_k)} u_k. \tag{3.6}$$

and

$$\beta_{k+1} = \begin{cases} \min\limits_{1\le k\le N}\left\{ \dfrac{\mu d(y_k^j, v_k)}{\|\Gamma_{w_k^j, v_k}\Psi_j(v_k) - \Psi_j(w_k^j)\|}\ , \ \beta_k+\eta_k \right\}; & \text{if } \|\Gamma_{w_k^j}\Psi_j(v_k) - \Psi_j(w_k^j)\| \neq 0,\ j=1,2,\cdots,N, \\ \beta_k+\eta_k, & \text{otherwise.} \end{cases} \tag{3.7}$$

Stopping criterion *If $w_k^j = v_k$ for some $k\ge 1$ then stop. Otherwise set $k := k+1$ and return to* **Iterative step 1**.

Remark 3.4. (1) Our algorithm (3.3) extends the algorithms in [16, 23, 29] from linear spaces to nonlinear spaces.

(2) The result in [20] is a special case of our result when $j = 1$.

Lemma 3.5. *Let $\{x_k\}$ be a sequence generated by Algorithm 3.3 and the sequence $\{\beta_k\}$ is generated by (3.47). Then $\lim\limits_{k\to\infty}\beta_k = \beta$ and $\beta \in \big[\min\limits_{1\le k\le n}\{\frac{\mu}{L_j}, \beta_0\}, \beta_0+\eta\big]$, where $\eta = \sum\limits_{k=0}^{\infty}\eta_k$.*

Proof. The proof of Lemma 3.5 is similar to the ones in [1, 2], so we omit it. □

Lemma 3.6. *Suppose that Assumptions 3.1 holds and let $\{y_k^j\}$ be a sequence generated by Algorithm 3.3, then*

$$d^2(y_k^j, a) \le d^2(v_k, a) - (1-\mu^2\frac{\beta_k^2}{\beta_{k+1}^2})d^2(v_k, w_k^j),\ \forall\ a\in\Omega.$$

Proof. Let $a\in\Omega$, then $-\Psi_j(a)\in\varphi_j(a), j=1,2,\cdots,N$. Using step 1 of Algorithm 3.3, we get $\frac{1}{\beta_k}\exp^{-1}_{w_k^j} v_k - \Gamma_{w_k^j, v_k}\Psi_j(v_k)\in\varphi_j(w_k^j)$.
By employing the monotonicity property of $\varphi_j, j=1,2,\cdots,N$, we have

$$\begin{aligned}\langle\frac{1}{\beta_k}\exp^{-1}_{w_k^j} v_k - \Gamma_{w_k^j, v_k}\Psi_j(v_k), \exp_{w_k^j} a\rangle &\le \langle -\Psi_j(a), -\exp^{-1}_a w_k^j\rangle \\ &= \langle\Psi_j(a), \exp^{-1}_a w_k^j\rangle. \end{aligned}\tag{3.8}$$

Since $\Psi_j, j=1,2,\cdots,N$ is a monotone vector field, we have

$$\langle\Psi_j(a), \exp^{-1}_a w_k^j\rangle \le \langle -\Psi_j(w_k^j), \exp^{-1}_{w_k^j} a\rangle. \tag{3.9}$$

On combining (3.8) and (3.9), we get

$$\langle\frac{1}{\beta_k}\exp^{-1}_{w_k^j} v_k - \Gamma_{w_k^j, v_k}\Psi_j(v_k), \exp_{w_k^j} a\rangle \le \langle -\Psi_j(w_k^j), \exp_{w_k^j} a\rangle,$$

which further yields

$$\langle\exp^{-1}_{w_k^j} v_k, \exp^{-1}_{w_k^j} a\rangle \le \beta_k\langle\Gamma_{w_k^j, v_k}\Psi_j(v_k) - \Psi_j(w_k^j), \exp^{-1}_{w_k^j} a\rangle. \tag{3.10}$$

Now, for $k\in\mathbb{N}$. Let $\Delta(w_k^j, v_k, a)\subseteq\mathbb{M}$ be a geodesic with vertices w_k^j, v_k and a, and let $\Delta(\bar{w}_k^j, \bar{v}_k, \bar{a})\subseteq\mathbb{R}^2$ be the corresponding comparison triangle, thus we have from Lemma 2.11 (ii) that $d(w_k^j, a) = \|\bar{w}_k^j - \bar{a}\|$, $d(v_k, a) = \|\bar{v}_k - \bar{a}\|$ and $d(w_k^j, v_k) = \|\bar{w}_k^j - \bar{v}_k\|$. Also, let $\Delta(y_k^j, w_k^j, a)\subseteq\mathbb{M}$ be a geodesic triangle with vertices y_k^j, v_k and a, then $\Delta(\bar{y}_k^j, \bar{w}_k^j, \bar{a})\subseteq\mathbb{R}^2$ is the corresponding comparison triangle. Hence, we have $d(y_k^j, a) = \|\bar{y}_k^j - \bar{a}\|$, $d(w_k^j, a) =$

$\|\bar{w}_k^j - \bar{a}\|$ and $d(y_k^j, w_k^j) = \|\bar{y}_k^j - \bar{w}_k^j\|$.

$$\begin{aligned}
d^2(y_k^j, a) &\leq \|\bar{y}_k^j - \bar{a}\|^2 \\
&= \|\bar{y}_k^j - \bar{w}_k^j + \bar{w}_k^j - \bar{a}\|^2 \\
&= \|\bar{w}_k^j - \bar{a}\|^2 + \|\bar{y}_k^j - \bar{w}_k^j\|^2 + 2\langle \bar{y}_k^j - \bar{w}_k^j, \bar{w}_k^j - \bar{a}\rangle \\
&= \|(\bar{w}_k^j - \bar{v}_k) + (\bar{v}_k - \bar{a})\|^2 + \|\bar{y}_k^j - \bar{w}_k^j\|^2 + 2\langle \bar{y}_k^j - \bar{w}_k^j, \bar{w}_k^j - \bar{a}\rangle \\
&= \|\bar{w}_k^j - \bar{v}_k\|^2 + \|\bar{v}_k - \bar{a}\|^2 + \|\bar{y}_k^j - \bar{w}_k^j\|^2 + 2\langle \bar{w}_k^j - \bar{v}_k, \bar{v}_k - \bar{a}\rangle \\
&+ 2\langle \bar{w}_k^j - \bar{a}, \bar{w}_k^j - \bar{a}\rangle - 2\|\bar{w}_k^j - \bar{a}\|^2 + 2\langle \bar{y}_k^j - \bar{w}_k^j, \bar{w}_k^j - \bar{a}\rangle \\
&+ 2\langle \bar{w}_k^j - \bar{v}_k, \bar{w}_k^j - \bar{v}_k\rangle - 2\langle \bar{w}_k^j - \bar{v}_k, \bar{w}_k^j - \bar{v}_k\rangle \\
&= d^2(v_k, a) - d^2(w_k^j, v_k) + \|\bar{y}_k^j - \bar{w}_k^j\|^2 + 2\langle \bar{w}_k^j - \bar{v}_k, \bar{w}_k^j - \bar{a}\rangle \\
&+ 2\langle \bar{y}_k^j - \bar{w}_k^j, \bar{w}_k^j - \bar{a}\rangle + 2\langle \bar{w}_k^j - \bar{a}, \bar{w}_k^j - \bar{a}\rangle - 2d^2(w_k^j, a) \\
&= d^2(v_k, a) - d^2(w_k^j, v_k) + \|\bar{y}_k^j - \bar{w}_k^j\|^2 + 2\langle \bar{w}_k^j - \bar{v}_k, \bar{w}_k^j - \bar{a}\rangle \\
&+ 2\langle \bar{y}_k^j - \bar{a}, \bar{w}_k^j - \bar{a}\rangle - 2d^2(w_k^j, a). \qquad (3.11)
\end{aligned}$$

Let θ and $\bar{\theta}$ be the angles of the vertices w_k^j and $\bar{w}_k^j$ respectively. By Lemma 2.11 (i), we get that $\bar{\theta} \geq \theta$. Thus, we obtain from Lemma 2.10 and (2.4) that

$$\begin{aligned}
\langle \bar{w}_k^j - \bar{v}_k, \bar{w}_k^j - \bar{a}\rangle &= \|\bar{w}_k^j - \bar{v}_k\| \ \|\bar{w}_k^j - \bar{a}\| \cos\bar{\theta} \\
&= d(w_k^j, v_k) d(a, w_k^j) \cos\bar{\theta} \\
&\leq d(w_k^j, v_k) d(a, w_k^j) \cos\theta \\
&= \langle \exp^{-1}_{w_k^j} v_k, \exp^{-1}_{w_k^j} a\rangle. \qquad (3.12)
\end{aligned}$$

Using the same approach as in (3.12), we have

$$\langle \bar{y}_k^j - \bar{a}, \bar{w}_k^j - \bar{a}\rangle = \langle \exp^{-1}_a y_k^j, \exp^{-1}_a w_k^j\rangle. \qquad (3.13)$$

Hence, we conclude that from step 2 of Algorithm 3.3 that

$$\|\bar{y}_k^j - \bar{w}_k^j\|^2 \leq \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2. \qquad (3.14)$$

By substituting (3.12), (3.13) and (3.14) into (3.11), we obtain

$$\begin{aligned}
d^2(y_k^j, a) &\leq d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 \\
&+ 2\langle \exp^{-1}_{w_k^j} v_k, \exp^{-1}_{w_k^j} a\rangle - 2d^2(w_k^j, a) + 2\langle \exp^{-1}_a y_k^j, \exp^{-1}_a w_k^j\rangle. \qquad (3.15)
\end{aligned}$$

Using Remark 2.7 and Lemma 2.8 in (3.15), we get

$$\begin{aligned}
d^2(y_k^j, a) &\leq d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 \\
&- 2d^2(w_k^j, a) + 2\langle \exp^{-1}_a y_k^j - \Gamma_{a, w_k^j} \exp^{-1}_{w_k^j} y_k^j + \Gamma_{a, w_k^j} \exp^{-1}_{w_k^j} y_k^j, \exp^{-1}_a w_k^j\rangle \\
&+ 2\langle \exp^{-1}_{w_k^j} v_k, \exp^{-1}_{w_k^j} a\rangle \\
&= d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 \\
&- 2d^2(w_k^j, a) + 2\langle \exp^{-1}_{w_k^j} v_k, \exp^{-1}_{w_k^j} a\rangle + 2\langle \exp^{-1}_a y_k^j - \Gamma_{a, w_k^j} \exp^{-1}_{w_k^j} y_k^j, \exp^{-1}_a w_k^j\rangle \\
&+ 2\langle \Gamma_{a, w_k^j} \exp^{-1}_{w_k^j} y_k^j, \exp^{-1}_a w_k^j\rangle \\
&\leq d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 - 2d^2(w_k^j, a) \\
&+ 2\| \exp^{-1}_a y_k^j - \Gamma_{a, w_k^j} \exp^{-1}_{w_k^j} y_k^j\| \ \| \exp^{-1}_a w_k^j\| + 2\langle \exp^{-1}_{w_k^j} v_k, \exp^{-1}_{w_k^j} a\rangle \\
&- 2\langle \exp^{-1}_{w_k^j} y_k^j, \exp^{-1}_{w_k^j} a\rangle, \qquad (3.16)
\end{aligned}$$

this also implies that

$$\begin{aligned} d^2(y_k^j, a) &\le d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 - 2d^2(w_k^j, a) \\ &+ 2d^2(a, w_k^j) + 2\langle \exp_{w_k^j}^{-1} v_k, \exp_{w_k^j}^{-1} a\rangle - 2\langle \exp_{w_k^j}^{-1} y_k^j, \exp_{w_k^j}^{-1} a\rangle \\ &= d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 + 2\langle \exp_{w_k^j}^{-1} v_k, \exp_{w_k^j}^{-1} a\rangle \\ &- 2\langle \exp_{w_k^j}^{-1} y_k^j, \exp_{w_k^j}^{-1} a\rangle. \end{aligned} \tag{3.17}$$

Using the definition of y_k^j that $\exp_{w_k^j}^{-1} y_k^j = \beta_k(\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j))$. From the last inequality, we obtain

$$\begin{aligned} d^2(y_k^j, a) &\le d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 \\ &+ 2\langle \exp_{w_k^j}^{-1} v_k, \exp_{w_k^j}^{-1} a\rangle - 2\beta_k\langle \Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j), \exp_{w_k^j}^{-1} a\rangle \\ &= d^2(v_k, a) - d^2(w_k^j, v_k) + \beta_k^2 \|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|^2 + 2\langle \exp_{w_k^j}^{-1} v_k, \exp_{w_k^j}^{-1} a\rangle \\ &+ 2\beta_k\langle \Psi_j(w_k^j) - \Gamma_{w_k^j, v_k} \Psi_j(v_k), \exp_{w_k^j}^{-1} a\rangle. \end{aligned} \tag{3.18}$$

On substituting (3.47) and (3.10) into (3.18), we have

$$\begin{aligned} d^2(y_k^j, a) &\le d^2(v_k, a) - d^2(w_k^j, v_k) + \mu^2 \frac{\beta_k^2}{\beta_{k+1}^2} d^2(w_k^j, v_k) + 2\beta_k\langle \Psi_j(w_k^j) - \Gamma_{w_k^j, v_k} \Psi_j(v_k), \exp_{w_k^j}^{-1} a\rangle \\ &- 2\beta_k\langle \Psi_j(w_k^j) - \Gamma_{w_k^j, v_k} \Psi_j(v_k), \exp_{w_k^j}^{-1} a\rangle \\ &= d^2(v_k, a) - (1 - \mu^2 \frac{\beta_k^2}{\beta_{k+1}^2}) d^2(w_k^j, v_k). \end{aligned} \tag{3.19}$$

Thus, the proof completes. □

Lemma 3.7. *Suppose that Assumptions 3.1 holds and let $\{x_k\}$ be a sequence generated by Algorithm 3.3. Then $\{x_k\}$ is bounded.*

Proof. By the property of exp function, we can re-write u_k defined in Algorithm 3.3 as $u_k = \gamma_k(1 - \alpha_k)$, where $\gamma_k^a : [0, 1] \to \mathbb{Z}$ is a geodesic sequence joining y_k to Uy_k. Using Proposition 2.13 and the nonexpansive property of U, we get

$$\begin{aligned} d^2(u_k, a) &= d^2(\gamma_k^a(1 - \alpha_k), a) \\ &\le (1 - \alpha_k) d^2(\gamma_k^a(0), a) + \alpha_k d^2(\gamma_k^a(1), a) - \alpha_k(1 - \alpha_k) d^2(\gamma_k^a(0), \gamma_k^a(1)) \\ &= (1 - \alpha_k) d^2(y_k, a) + \alpha_k d^2(Uy_k, a) - \alpha_k(1 - \alpha_k) d^2(y_k, Uy_k) \\ &\le (1 - \alpha_k) d^2(y_k, a) + \alpha_k d^2(y_k, a) - \alpha_k(1 - \alpha_k) d^2(y_k, Uy_k) \\ &= d^2(y_k, a) - \alpha_k(1 - \alpha_k) d^2(y_k, Uy_k) && (3.20) \\ &\le d^2(y_k, a). && (3.21) \end{aligned}$$

It is obvious from (3.19) and (3.20) that

$$\begin{aligned} d^2(u_k, a) &\le d^2(v_k, a) - \alpha_k(1 - \alpha_k) d^2(y_k, Uy_k) - (1 - \mu^2 \frac{\beta_k^2}{\beta_{k+1}^2}) d^2(w_k^j, v_k) && (3.22) \\ &\le d^2(v_k, a). && (3.23) \end{aligned}$$

By considering the geodesic triangles $\triangle(v_k, x_k, a)$ and $\triangle(x_k, x_{k-1}, a)$ with their respective comparison triangle $\triangle(\bar{v}_k, \bar{x}_k, \bar{a}) \subseteq \mathbb{R}^2$ and $\triangle(\bar{x}_k, \bar{x}_{k-1}, \bar{a})$. Then by Lemma 2.11, we have $d(v_k, x_k) = \|\bar{v}_k - \bar{x}_k\|$, $d(v_k, a) = \|\bar{v}_k - \bar{a}\|$ and $d(x_k, x_{k-1}) = \|\bar{x}_k - \bar{x}_{k-1}\|$. From step 1 of Algorithm 3.3, we get

$$\begin{aligned} d(v_k, a) &= \|\bar{v}_k - \bar{a}\| \\ &= \|\bar{x}_k + \theta_k(\bar{x}_k - \bar{x}_{k-1}) - \bar{a}\| \\ &\le \|\bar{x}_k - \bar{a}\| + \theta_k \|\bar{x}_k - \bar{x}_{k-1}\| \\ &= \|\bar{x}_k - \bar{a}\| + \delta_k \cdot \frac{\theta_k}{\delta_k} \|\bar{x}_k - \bar{x}_{k-1}\|. \end{aligned} \tag{3.24}$$

Since $\frac{\theta_k}{\delta_k}\|\bar{x}_k - \bar{x}_{k-1}\| = \frac{\theta_k}{\delta_k}d(x_k, x_{k-1}) \to 0$ as $k \to \infty$, then there exists a constant $N_2 > 0$ such that $\frac{\theta_k}{\delta_k}d(x_k, x_{k-1}) \le N_2$. Thus, we obtain from (3.24) that

$$d(v_k, a) \le d(x_k, a) + \delta_k N_2. \tag{3.25}$$

By simple computation, it is easy to see that

$$\begin{aligned}
d^2(v_k, a) &= \|\bar{v_k} - \bar{a}\|^2 \\
&\le \left(\|\bar{x_k} - \bar{a}\|^2 + \theta_k\|\bar{x_k} - x_{\bar{k-1}}\|\right)^2 \\
&= \|\bar{x_k} - \bar{a}\|^2 + 2\theta_k\|\bar{x_k} - \bar{a}\| \ \|\bar{x_k} - x_{\bar{k-1}}\| + \theta_k^2\|\bar{x_k} - x_{\bar{k-1}}\| \\
&= \|\bar{x_k} - \bar{a}\|^2 + \theta_k\|\bar{x_k} - x_{\bar{k-1}}\|\left(2\|\bar{x_k} - \bar{a}\| + \theta_k\|\bar{x_k} - x_{\bar{k-1}}\|\right) \\
&\le \|\bar{x_k} - \bar{p}\|^2 + \theta_k\|\bar{x_k} - x_{\bar{k-1}}\|N_3 \\
&\le d^2(x_k, a) + \theta_k d(x_k, x_{k-1})N_3,
\end{aligned} \tag{3.26}$$

where $2d(x_k, a) + \theta_k d(x_k, x_{k-1}) \le N_3$ for some constant $N_3 > 0$. It can be deduce from (3.22) and (3.26) that

$$d^2(u_k, a) \le d^2(x_k, a) + \theta_k d(x_k, x_{k-1})N_3 - \alpha_k(1-\alpha_k)d^2(y_k, Uy_k) - (1 - \mu^2\frac{\beta_k^2}{\beta_{k+1}^2})d^2(w_k^j, v_k). \tag{3.27}$$

Now, since x_{k+1} defined in Algorithm 3.3 can be re-written as $x_{k+1} = \gamma_k^b(1-\delta_k)$, where $\gamma_k^b : [0,1] \to \mathbb{Z}$ is a sequence of geodesic joining $h(x_k)$ to u_k, then, we deduce from (3.23) and (3.25) that

$$\begin{aligned}
d(x_{k+1}, a) &= d(\gamma_k^b(1-\delta_k), a) \\
&= \delta_k d(\gamma_k^b(0), a) + (1-\delta_k)d(\gamma_k^b(1), a) \\
&\le \delta_k d(h(x_k), a) + (1-\delta_k)d(u_k, a) \\
&\le \delta_k(d(h(x_k), h(a)) + d(h(a), a)) + (1-\delta_k)d(u_k, a) \\
&\le \phi\delta_k d(x_k, a) + (1-\delta_k)d(x_k, a) + \delta_k d(h(a), a) + \delta_k N_2 \\
&= (1 - \delta_k(1-\phi))d(x_k, a) + \delta_k(N_2 + d(h(a), a)) \\
&\le \max\left\{d(x_k, a), \frac{N_2 + d(h(a), a)}{(1-\phi)}\right\} \\
&\ \ \vdots \\
&\le \max\left\{d(x_1, a), \frac{N_2 + d(h(a), a)}{(1-\phi)}\right\} < \infty.
\end{aligned}$$

Hence, the sequence $\{x_k\}$ is bounded. Consequently, $\{v_k\}$, $\{y_k^j\}$, $\{w_k^j\}$ and $\{u_k\}$ are also bounded for all $j \in I$. □

Theorem 3.8. *Suppose that Assumptions 3.1 and 3.2 holds, then the sequence $\{x_k\}$ generated by Algorithm 3.3 converges to $a \in \Omega$, where $a = P_\Omega h(a)$.*

Proof. For $k \ge 1$, let $s = h(x_k), t = h(a)$ and $b = u_k$. We then consider the geodesic triangles $\triangle(s,t,b)$, $\triangle(b,t,s)$, $\triangle(b,t,a)$ with their respective comparison $\triangle(\bar{s},\bar{t},\bar{b})$, $\triangle(\bar{b},\bar{t},\bar{s})$ and $\triangle(\bar{b},\bar{t},\bar{a})$. By Lemma 2.11, we have $d(s,t) = \|\bar{s} - \bar{t}\|$, $d(s,b) = \|\bar{s} - \bar{b}\|$, $d(s,a) = \|\bar{s} - \bar{a}\|$, $d(b,a) = \|barb - \bar{a}\|$ and $d(t,a) = \|\bar{t} - \bar{a}\|$. Thus, the comparison point $x_{k+1} \in \mathbb{R}^2$ is $\bar{x}_{k+1} = \delta_k\bar{s} + (1-\delta_k)\bar{b}$. Let τ and $\bar{\tau}$ denote the angle and comparison angle at a and $\bar{a}$ in the triangles $\triangle(t, x_{k+1}, a)$ and $\triangle(\bar{t}, \bar{x}_{k+1}, \bar{a})$ respectively. Hence $\tau \le \bar{\tau}$ and $\cos\bar{\tau} \le \cos\tau$. Then

$$\begin{aligned}
d^2(x_{k+1}, a) &= \|\bar{x}_{k+1} - \bar{a}\|^2 \\
&= \|\delta_k(\bar{s} - \bar{a}) + (1-\delta_k)(\bar{b} - \bar{a})\|^2 \\
&\le \|\delta_k(\bar{s} - \bar{t}) + (1-\delta_k)(\bar{b} - \bar{a})\|^2 + 2\delta_k\langle\bar{x}_{k+1} - \bar{a}, \bar{t} - \bar{a}\rangle \\
&\le (1-\delta_k)\|\bar{b} - \bar{a}\|^2 + \delta_k\|\bar{s} - \bar{t}\|^2 + 2\delta_k\|\bar{x}_{k+1} - \bar{a}\| \ \|\bar{t} - \bar{a}\|\cos\bar{\tau} \\
&\le (1-\delta_k)d^2(b, a) + \delta_k d^2(s, t) + 2\delta_k d(x_{k+1}, a)d(t, a)\cos\tau \\
&= (1-\delta_k)d^2(u_k, a) + \delta_k d^2(h(x), h(a)) + 2\delta_k d(x_{k+1}, a)d(t, a)\cos\tau.
\end{aligned} \tag{3.28}$$

It is easy to see that $d(x_{k+1},a)d(h(a),a)\cos\tau = \langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k+1}\rangle$, thus on substituting (3.27) into (3.28), we get

$$\begin{aligned}
d^2(x_{k+1},a) &\le (1-\delta_k)d^2(u_k,a) + \delta_k d^2(h(x_k),h(a)) + 2\delta_k\langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k+1}\rangle \\
&\le (1-\delta_k)d^2(x_k,a) + (1-\delta_k)\theta_k d(x_k,x_{k-1})N_3 - \alpha_k(1-\alpha_k)(1-\delta_k)d^2(y_k,Uy_k) \\
&\quad - (1-\delta_k)(1-\mu^2\frac{\beta_k^2}{\beta_{k+1}^2})d^2(w_k^j,v_k) + \delta_k d^2(h(x_k),h(a)) + 2\delta_k\langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k+1}\rangle \\
&= (1-\delta_k(1-\phi))d^2(x_k,a) + \delta_k\left[\frac{\theta_k}{\delta_k}d(x_k,x_{k-1})N_3 + d^2(h(x_k),h(a)) + 2\langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k+1}\rangle\right] \\
&\quad - \alpha_k(1-\alpha_k)(1-\delta_k)d^2(y_k,Uy_k) - (1-\delta_k)(1-\mu^2\frac{\beta_k^2}{\beta_{k+1}^2})d^2(w_k^j,v_k) \qquad (3.29)\\
&= (1-\phi_k)d^2(x_k,a) + \phi_k M_k, \qquad (3.30)
\end{aligned}$$

where $M_k = \frac{1}{(1-\phi)}\left[\frac{\theta_k}{\delta_k}d(x_k,x_{k-1})N_3 + d^2(h(x),h(a)) + 2\langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k+1}\rangle\right]$ and $\phi_k = \delta_k(1-\phi)$.

We now prove that $d(x_k,a)\to 0$ by considering the following two possible cases.

Case 1: Suppose that there exists $k_0\in\mathbb{N}$ such that $\{d(x_k,a)\}$ is non-increasing. Since $\{d(x_k,a)\}$ is bounded, it is convergent and therefore

$$d(x_k,a) - d(x_{k+1},a)\to 0, \text{ as } k\to\infty.$$

From (3.29), we have

$$\begin{aligned}
\alpha_k(1-\alpha_k)(1-\delta_k)d^2(y_k,Uy_k) + (1-\delta_k)(1-\mu^2\frac{\beta_k^2}{\beta_{k+1}^2})d^2(w_k^j,v_k) &\le (1-\delta_k(1-\phi))d^2(x_k,a) - d^2(x_{k+1},a) \\
&\quad + \delta_k\Bigg[\frac{\theta_k}{\delta_k}d(x_k,x_{k-1})N_3 + d^2(h(x_k),h(a)) \\
&\quad + 2\langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k+1}\rangle\Bigg]. \qquad (3.31)
\end{aligned}$$

Observe that $\beta_k\to\beta,\ \alpha_k\in(0,1)$ and $\mu\in(0,1)$. Thus, there exists $m\ge 0$ such that for $k\ge m, 0<\frac{\mu\beta_k}{\beta_{k+1}}<1$. Thus

$$\lim_{k\to\infty}(1-\frac{\mu\beta_k}{\beta_{k+1}}) = (1-\mu) > 0.$$

Therefore, we obtain from (3.31) that

$$\lim_{k\to\infty} d(w_k^j,v_k) = \lim_{k\to\infty} d(y_k,Uy_k) = 0. \qquad (3.32)$$

Also, from step 2 of Algorithm 3.3, we have

$$\begin{aligned}
d(y_k^j,w_k^j) &= \|\exp_{w_k^j}^{-1} y_k^j\| = \beta_k\|\Gamma_{w_k^j,v_k}\Psi_j(v_k) - \Psi_j(w_k^k)\| \\
&\le \mu\frac{\beta_k}{\beta_{k+1}}d(w_k^j,v_k)\to 0,\ k\to\infty. \qquad (3.33)
\end{aligned}$$

By replacing a with x_k in (3.24), we obtain that

$$d(v_k,x_k)\le \delta_k\cdot\frac{\theta_k}{\delta_k}d(x_k,x_{k-1})\to 0,\ k\to\infty. \qquad (3.34)$$

By applying the convexity of Riemannian distance and the conditions on α_k and δ_k, we obtain

$$\begin{aligned}
d(u_k,y_k) &= d(\gamma_k^a(1-\alpha_k),y_k) \\
&\le \alpha_k d(\gamma_k^a(0),y_k) + (1-\alpha_k)d(\gamma_k^a(1),y_k) \\
&\le \alpha_k d(y_k,y_k) + (1-\alpha_k)d(Uy_k,y_k)\ \to 0 \text{ as } k\to\infty. \qquad (3.35)
\end{aligned}$$

and

$$\begin{aligned}
d(x_{k+1},u_k) &= d(\gamma_k^b(1-\delta_k),u_k) \\
&\le \delta_k d(\gamma_k^b(0),u_k) + (1-\delta_k)d(\gamma_k^b(1),u_k) \\
&\le \delta_k d(h(x_k),u_k) + (1-\delta_k)d(u_k,u_k)\to 0 \text{ as } k\to\infty. \qquad (3.36)
\end{aligned}$$

We obtain from (3.32)-(3.36) that

$$\begin{cases} \lim_{k\to\infty} d(w_k^j, x_k) = 0, \\ \lim_{k\to\infty} d(y_k^j, x_k) = 0, \\ \lim_{k\to\infty} d(u_k, x_k) = 0, \\ \lim_{k\to\infty} d(x_{k+1}, x_k) = 0. \end{cases} \tag{3.37}$$

Since $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_l}\}$ which converges to $p \in \mathbb{M}$. Also, from (3.34) and (3.36), there exist subsequences $\{v_{k_l}\}$ and $\{w_{k_l}^j\}$ which converges to $p \in \mathbb{M}$ respectively. From Algorithm 3.3, we get

$$\Upsilon_{k_l} := -\Gamma_{w_{k_l}^j, v_{k_l}} \Psi_j(v_{k_l}) - \frac{1}{\beta_{k_l}} \exp^{-1}_{w_{k_k}^j} v_{k_l} \in \varphi_j(w_{k_l}^j). \tag{3.38}$$

Hence, using (3.32), we have

$$\lim_{l\to\infty} \frac{1}{\beta_{k_l}} \| \exp^{-1}_{w_{k_l}^j} v_{k_l} \| = \lim_{l\to\infty} \frac{1}{\beta_{k_l}} d(w_{k_l}^j, v_{k_l}) = 0,$$

so

$$\lim_{l\to\infty} \frac{1}{\beta_{k_l}} \exp^{-1}_{w_{k_l}^j} v_{k_l} = 0. \tag{3.39}$$

Since $\Psi_j, j = 1, 2, \cdots, N$ are Lipschitz continuous vector field and $v_{k_l} \to p$ as $l \to \infty$. By combining (3.38) and (3.39), we obtain that

$$\lim_{l\to\infty} \Upsilon_{k_l} = -\Psi_j(p), j = 1, 2, \cdots, N. \tag{3.40}$$

Since $\Psi_j, j = 1, 2, \cdots, N$ are Lipschitz continuous vector field, so it is upper kuratowski semi-continuous. Hence $-\Psi_j(p) \in \varphi_j(p), j = 1, 2, \cdots, N$, which implies that p solves Ω. Also, since $\{x_{k_l}\}$ is bounded, there exists a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ which converges to $p \in \mathbb{M}$ such that

$$\begin{aligned} \lim_{k\to\infty} \langle \exp_a^{-1} h(x_k), \exp_a^{-1} x_{k_l} \rangle &= \limsup_{l\to\infty} \langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k_l} \rangle \\ &= \langle \exp_a^{-1} h(a), \exp_a^{-1} p \rangle \\ &\leq 0. \end{aligned} \tag{3.41}$$

Thus,

$$\lim_{l\to\infty} \langle \exp_a^{-1} h(a), \exp_a^{-1} x_{k_l} \rangle \leq 0.$$

Applying the last inequality and Lemma 2.15 to (3.30), we obtain that $d(x_k, a) \to 0$ as $k \to \infty$. Hence, $\lim_{k\to\infty} x_k = a$.

Case 2: Suppose $g_k = d^2(x_k, a)$, then there exists a subsequence $\{g_{k_l}\}$ of $\{g_k\}$ such that

$$g_{k_l} \leq g_{k_{l+1}},$$

for all $l \in \mathbb{N}$. Following this, we can defined $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(k) : \max\{k \leq j : g_k < g_{k+1}\}.$$

It follows from Lemma 2.16 that $\tau(k) \to \infty$ and

$$0 < g_{\tau(k)} \leq g_{\tau(k)+1}.$$

Following the same process as in case 1 by replacing k by $\tau(k)$, we can easily show that

$$\lim_{\tau(k)\to\infty} d(w_{\tau(k)}^j, v_{\tau(k)}) = 0 = \lim_{\tau(k)\to\infty} d(y_{\tau(k)}, Uy_{\tau(k)}).$$

Also,

$$\lim_{\tau(k)\to\infty} d(y_{\tau(k)}^j, w_{\tau(k)}^j) = 0 = \lim_{\tau(k)\to\infty} d(v_{\tau(k)}, x_{\tau(k)}).$$

From (3.30), we get

$$\begin{aligned} \phi_{\tau(k)} g_{\tau(k)} &\leq g_{\tau(k)} - g_{\tau(k)+1} + \phi_{\tau(k)} M_{\tau(k)} \\ &\leq \phi_{\tau(k)} M_{\tau(k)}. \end{aligned}$$

Since $\phi_{\tau(k)} = \delta_{\tau(k)}(1-\phi) > 0$, we obtain

$$g_{\tau(k)} \leq M_{\tau(k)},$$

and

$$\limsup_{\tau(k)\to\infty} \leq \limsup_{\tau(k)\to\infty} M_{\tau(k)} \leq 0.$$

It follows from (3.30), that $\lim_{\tau(k)\to\infty} = 0$. For $k \geq k_0$, it is clear that $g_{\tau(k)} \leq g_{\tau(k)+1}$ if $\tau(k) \neq k$ (that is $\tau(k) < k$), since $g_j \geq g_{j+1}$ for $\tau(k)+1 \leq j \leq k$. Consequently, we obtain for all $k \geq k_0$ that

$$0 \leq g_k \leq \max\{g_{\tau(k)}, g_{\tau(k)+1}\} = g_{\tau(k)+1}.$$

Thus,

$$\lim_{k\to\infty} g_k = 0.$$

We therefore conclude that $\{x_k\}$ converges to a. Hence, the proof completes. □

Corollary 3.9. *In the following result, we consider the common solution of variational inclusion problem only. Hence, we have the following algorithm:*

Algorithm 3.10. *Parallel Tseng's method for common variational inclusion problem*

Initialization: *Choose $\beta_0 > 0, \mu, \theta \in (0,1)$ and let $x_0, x_1 \in \mathbb{M}$ be arbitrary starting points.*

Iterative step: *Given x_{k-1}, x_k, and β_k, choose $\theta_k \in [0, \bar{\theta_k}]$ where*

$$\theta_k = \begin{cases} \min\left\{\dfrac{\epsilon_k}{d(x_k, x_{k-1})}, \quad \theta\right\}, & \text{if } x_k \neq x_{k-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.42}$$

Step 1: *Compute*

$$\begin{cases} v_k = \exp_{x_k}(-\theta_k \exp^{-1}_{x_k} x_{k-1}) \\ \mathbf{0} \in \Gamma_{w_k^j, v_k} \Psi_j(v_k) + \varphi_j(w_k^j) - \frac{1}{\beta_k} \exp^{-1}_{w_k^j} v_k, \;\; j = 1, 2, \cdots, N \end{cases} \tag{3.43}$$

Step 2: *Calculate*

$$y_k^j = \exp_{w_k^j}(\beta_k(\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j))), j = 1, 2, \cdots, N. \tag{3.44}$$

Step 3: *Find the farthest element from v_k among y_k^j. That is*

$$j_k = arg\max\left\{d(y_k^j, v_k) : j = 1, 2, \cdots, N\right\}, \;\; y_k = y_k^{j_k}. \tag{3.45}$$

Step 4: *Calculate x_{k+1} and β_{k+1} by*

$$x_{k+1} = \exp_{h(x_k)}(1-\delta_k) \exp^{-1}_{h(x_k)} y_k. \tag{3.46}$$

and

$$\beta_{k+1} = \begin{cases} \min\limits_{1\leq k\leq N}\left\{\dfrac{\mu d(y_k^j, v_k)}{\|\Gamma_{w_k^j, v_k} \Psi_j(v_k) - \Psi_j(w_k^j)\|}, \;\; \beta_k + \eta_k\right\}, & \text{if } \|\Gamma_{w_k^j} \Psi_j(v_k) - \Psi_j(w_k^j)\| \neq 0, \;\; j = 1, 2, \cdots, N. \\ \beta_k + \eta_k, & \text{otherwise.} \end{cases} \tag{3.47}$$

Stopping criterion *If $w_k^j = v_k$ for some $k \geq 1$ then stop. Otherwise set $k := k+1$ and return to* **Iterative step 1**.

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4. Numerical Example

In this section, we give a numerical implementation of our method. The simulation was done on a personal Dell laptop with 8gig RAM and 256gig ROM with MATLAB 2024.

Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $M = (\mathbb{R}_{++}, \langle \cdot, \cdot \rangle)$ be the Riemannian manifold with Riemannian metric defined by $\langle u, v\rangle = \frac{1}{x^2}uv, \in \mathbb{R}_{++}, \; u, v \in T_xM$. The Riemannian distance $d : M \times M \to \mathbb{R}_+$ is given by $d(x, y) = |\ln \frac{y}{x}|$ for all $x, y \in M$. Let $x \in M$, then the exponential map $\exp_x : T_xM \to M$ is defined by $\exp_x tv = xe^{\frac{vt}{x}}$ for all $v \in T_xM$. The inverse of the exponential map, $\exp_x^{-1} : M \to T_xM$ is defined by $\exp_x^{-1} y = x \ln \frac{y}{x}$ for all $x, y \in M$. The parallel transport is the identity on M. Let $C = (0, +\infty)$, $\Psi_j : C \to \mathbb{R}$ and $\varphi_j : C \to TM$ be defined by $\Psi_j x = \frac{x \ln x}{j}$ and $\varphi_j x = x(j + \ln x)$, respectively for each $j = 1, 2, \cdots, N$. Then, the mappings Ψ_j are maximal monotone on C and φ_j are continuous and monotone vector fields on C. It is not difficult to see that w_n in Algorithm 3.3 can be expressed as

$$w_k^j = \left(\frac{v_k}{e^{j\beta_k}}\right)^{\frac{j}{j+\beta_k}}, \; \beta_k > 0, \; j = 1, 2, \cdots, N.$$

Now, let $T : C \to C$ be defined by $Tx = x$. Define the mapping $h : C \to C$ by $h(x) = \frac{x}{4}$ for all $x \in C$. For this example, we let $\delta_n = \frac{1}{n+1}$, $\alpha_k = \frac{1}{2k+3}$, $\eta_k = \frac{1}{k\sqrt{k}}$, $\mu = \frac{1}{2}, \theta = \frac{1}{3}$ and $\beta_1 = 2.5$. We terminate the execution of the process at $E_n = d(x_{n+1}, x_n) = 10^{-3}$ and make a comparison of Algorithm 3.3 with an unaccelerated version of it (i.e $\theta_n = 0$). The result of this experiment is shown in Figure **??**.

Case i: $x_0 = 1.8$ and $x_1 = 0.6$;
Case ii: $x_0 = 1.5$ and $x_1 = 0.5$;
Case iii: $x_0 = 2.5$ and $x_1 = 0.5$;
Case iv: $x_0 = 0.7$ and $x_1 = 0.6$.

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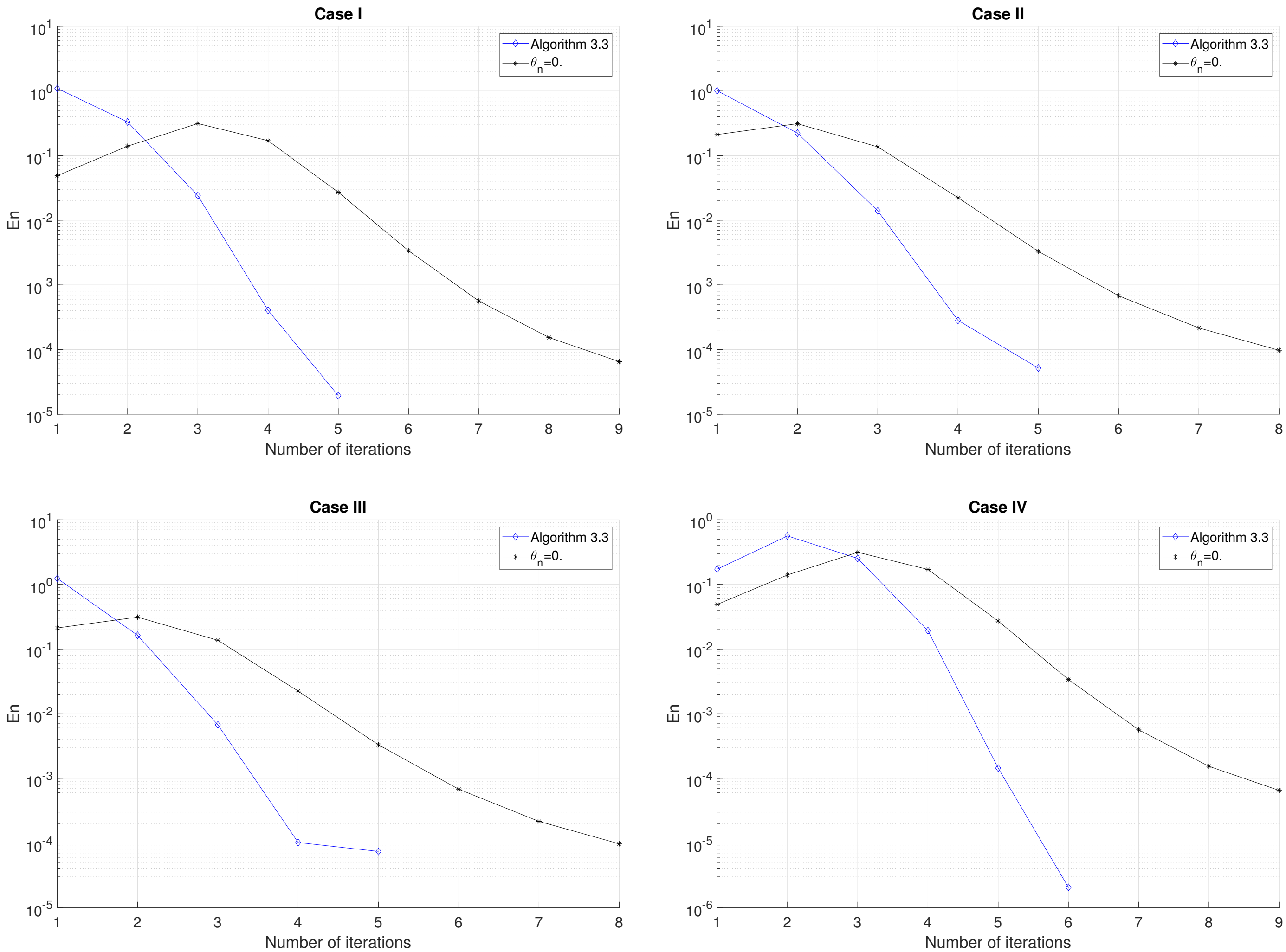

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