HERMITE-HADAMARD INEQUALITIES OF CONFORMABLE FRACTIONAL INTEGRALS FOR STRONGLY h-CONVEX FUNCTIONS

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Abstract. In this paper, the Hermite-Hadamard type inequalities of left and right conformable fractional integrals via strongly h-convex functions are established. Furthermore, by studying some elegant properties of Beta type functions, we obtain some identities related to the two class fractional integrals with m-times differentiable functions, and then gain midpoint type and trapezoid type error estimates connected with the Hermite-Hadamard type inequalities, which generalize some known results.

1. Introduction

Let I be an interval in $\mathbb{R}$ and $h : (0,1) \to [0,\infty)$ be a given function. A function $f : I \to \mathbb{R}$ is called *strongly h-convex with modulus $c > 0$, or f belongs to the class $SX(h,c,I)$*, provided that

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) - ct(1-t)(x-y)^2 \tag{1.1}$$

holds for all $x, y \in I$ and $t \in (0,1)$. The concept of the strongly h-convex function was introduced by Angulo, Gimenez, Moros and Nikodem [2] in 2011 and it generalized the well known classes of h-convex functions, strongly convex functions and etc. Choosing $c \to 0$ in (1.1), the notion reduces to *the h-convex function*, which was introduced by Varošanec [43]. If taking $h(t) = t,\ h(t) = t^s\ (s \in (0,1)),\ h(t) = 1/t$ and $h(t) = 1$ in (1.1), then f is said to be *strongly convex function* [28]*, strongly s-convex function (in the second sense), strongly Godunova-Levin function* and *strongly P-function*, respectively, and furthermore, letting $c \to 0$ in the proceeding definitions, the corresponding functions are sequentially called *convex function, s-convex function* (*in the second sense*) [4]*, Godunova-Levin function* [13] and *P-function* [31]. These functions are important in branches of mathematics and its applications, and were studied extensively, for instance, see [3, 5, 6, 10, 12, 14, 15, 16, 19, 23, 24, 25, 27, 29, 33, 44].

A significant application about the convex function is the famous Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : [a,b] \to \mathbb{R}$ is a convex function. This inequality has been extended under various convex type functions. Dragomir, Pečarić and Persson [11] obtained similar results for Godunova-Levin functions and P-functions in 1995. Dragomir and Fitzpatrick [9] extended it for s-convex function later. Sarikaya, Saglam and Yildirim [34] set up an analogous

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inequality for h-convex functions in 2008. The authors [2] proved the following inequalities for strongly h-convex functions.

Theorem A. *Let $h:(0,1)\to[0,\infty)$ be a Lebesgue integrable function with $h(1/2)>0$. If f is Lebesgue integrable on $[a,b]$ and $f\in SX(h,c,[a,b])$, then*

$$\begin{aligned} \frac{1}{2h(1/2)}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right] &\le \frac{1}{b-a}\int_a^b f(x)dx \\ &\le (f(a)+f(b))\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2. \end{aligned} \tag{1.3}$$

Particulary, Theorem A reduces to Theorem 6 in [34] as $c\to 0$.

On the other hand, one of the interesting and important applications is to give some error estimates with respect to Hermite-Hadamard type inequalities.

Lemma A. *Let f be a differentiable function and f' be Lebesgue integrable on the interval $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt, \tag{1.4}$$

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right) &= (b-a)\int_0^{1/2} tf'(ta+(1-t)b)dt \\ &\quad -\int_{1/2}^1(1-t)f'(ta+(1-t)b)dt. \end{aligned} \tag{1.5}$$

(1.4) and (1.5) were obtained in [7, 30] and [18], respectively. And using these identities, the authors proved some trapezoid and midpoint type estimates when $|f'|$ is convex on $[a,b]$.

Let $\alpha>0$ and f be Lebesgue integrable on $[a,b]$. *The left-sided and the right-sided Riemann-Liouville fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order α* are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\quad x\in(a,b], \tag{1.6}$$

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad x\in[a,b), \tag{1.7}$$

respectively, where $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dx$ is the Gamma function.

Inspired by the conformable fractional derivative [17], Abdeljawad [1] developed the following new fractional integration. Let $n=0,1,2,\cdots,\alpha\in(n,n+1]$ and f be a Lebesgue integrable function on $[a,b]$. *The left and right conformable fractional integrals $I^a_\alpha f$ and ${}^bI_\alpha f$* are defined by

$$I^a_\alpha f(x) = \frac{1}{n!}\int_a^x (x-t)^n(t-a)^{\alpha-n-1}f(t)dt,\quad x\in(a,b], \tag{1.8}$$

$${}^bI_\alpha f(x) = \frac{1}{n!}\int_x^b (t-x)^n(b-t)^{\alpha-n-1}f(t)dt,\quad x\in[a,b), \tag{1.9}$$

respectively. If taking $\alpha=n+1$, one easily to check that the conformable fractional integrals reduce to the Riemann-Liouville fractional integrals for $\alpha\in\mathbb{N}$.

For any $0<p,q<\infty$, the famous Beta function is given by

$$B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}du.$$

And we define two Beta type functions

$$B_t(p,q)=\int_0^t u^{p-1}(1-u)^{q-1}du, \quad \widetilde{B}_t(p,q)=\int_0^t (1-u)^{p-1}u^{q-1}du, \quad 0\le t\le 1.$$

It is easy to see that

$$B_1(p,q)=\widetilde{B}_1(p,q)=B(p,q)=B(q,p), \tag{1.10}$$

$$B_t(p,q)+\widetilde{B}_{1-t}(p,q)=B(p,q),\ 0\le t\le 1, \tag{1.11}$$

and is well known that

$$B\left(p+1,q\right)=\frac{p}{p+q}B\left(p,q\right), \quad B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.12}$$

In 2013, Sarikaya, Set, Yaldiz and Başak [35] first proved the following remarkable inequality of Hermite-Hadamard type via Riemannn-Liouville fractional integrals.

Theorem B. *Let f be a Lebesgue integrable convex function on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\le\frac{f(a)+f(b)}{2}.$$

And the authors [35] gave some identities analogues to (1.4) and provided some trapezoid type estimates for differentiable convex functions. Meanwhile, Tunç [39] extended them to differentiable h-convex functions. Later, similar estimates were obtained for twice differentiable convex functions in [8] and for n-times differentiable convex functions in[42]. On the other hand, some midpoint type inequalities were also studied in [32] for P-convex functions with second derivatives. Recently, [45] gave another extension for trapezoid and midpoint type estimates via strongly h-convex functions with any times derivatives. In 2018, Set, Akdemir and Mumcu [36] established a new inequalities via conformable integrals as follows.

Theorem C. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable convex function. Then*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[I^{a}_{\alpha}f(b)+{}^{b}I_{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2}.$$

It is easy to check that Theorem C reduces to Theorem B with $\alpha=n-1$.

At the same time, the authors [38] obtained similar identities as Lemma A and proved some error estimates involving conformable integrals for differentiable convex functions. Set, Gözpinar [37] extended Theorem C to s-convex functions and also got some error inequalities.

In all of the above statements, the left-sided and right-sided fractional integrals are used together. On the other hand, the study for the Hermite-Hermite type inequalities by using only left-sided or right-sided fractional integrals has also attracted lots of attention. Recently, Kunt, Karapinar, Turhan and İşcan proved the following conclusions.

Theorem D.[21, 22] *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable convex function. Then*

$$\begin{aligned} f\left(\frac{\alpha a+b}{\alpha+1}\right)&\le \tfrac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{a^+}f(b) &\le\frac{\alpha f(a)+f(b)}{\alpha+1},\\ f\left(\frac{a+\alpha b}{\alpha+1}\right)&\le \tfrac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{b^-}f(a) &\le\frac{f(a)+\alpha f(b)}{\alpha+1}.\end{aligned}$$

Furthermore, the authors obtained similar equalities as Lemma A and proved some trapezoid and midpoint type inequalities involving the left-sided or right-sided Riemann-Liouville fractional integral for differentiable convex functions.

Lately, Tuan and his co-authors extended the above inequalities for the left conformable fractional integral in [40] and for the right one in [41].

Theorem E.[40, 41] *Let f be an integrable convex function on $[a,b]$. Then*

$$\begin{aligned} f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I_{\alpha}^{a} f(b) \leq \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}, \\ f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} {}^{b}I_{\alpha} f(a) \leq \frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}. \end{aligned}$$

And they also proved the following equalities.

Lemma B.[40, 41] *Let f be a differentiable function and f' be Lebesgue integrable on $[a,b]$. Then*

$$\begin{aligned} &\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I_{\alpha}^{a} f(b) \tag{1.13} \\ &= \frac{b-a}{B(n+1,\alpha-n)} \int_0^1 \left[B(n+1,\alpha-n+1) - B_t(n+1,\alpha-n)\right] f'(ta+(1-t)b)dt, \end{aligned}$$

$$\begin{aligned} &\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} {}^{b}I_{\alpha} f(a) \tag{1.14} \\ &= \frac{b-a}{B(n+1,\alpha-n)} \int_0^1 \left[B_{1-t}(n+1,\alpha-n) - B(n+1,\alpha-n+1)\right] f'(ta+(1-t)b)dt, \end{aligned}$$

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I_{\alpha}^{a} f(b) - f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \tag{1.15} \\ &= \frac{b-a}{B(n+1,\alpha-n)} \left[\int_0^{\frac{n+1}{\alpha+1}} B_t(n+1,\alpha-n) f'(ta+(1-t)b)dt \right. \\ &\quad \left. - \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] f'(ta+(1-t)b)dt\right], \end{aligned}$$

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} {}^{b}I_{\alpha} f(a) - f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right) \tag{1.16} \\ &= \frac{b-a}{B(n+1,\alpha-n)} \left[-\int_0^{\frac{n+1}{\alpha+1}} B_t(n+1,\alpha-n) f'((1-t)a+tb)dt \right. \\ &\quad \left. + \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] f'((1-t)a+tb)dt\right]. \end{aligned}$$

As a consequence, the authors obtained some error estimates as follows.

Theorem F. [40, 41] *Let f be a Lebesgue integrable function on $[a,b]$. If f is differentiable and $|f'|^q$ $(1 \leq q < \infty)$ is convex on $[a,b]$, then*

$$\left| \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I_{\alpha}^{a} f(b) \right|$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)} K_1^{1-\frac{1}{q}} \left[K_2 \left|f'(a)\right|^q + K_3 \left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

$$\left|\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha \Gamma(\alpha-n)} {}^{b}I_\alpha f(a)\right|$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)} \widetilde{K}_1^{1-\frac{1}{q}} \left[\widetilde{K}_2 \left|f'(a)\right|^q + \widetilde{K}_3 \left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$K_1 = \int_0^1 \left|B(n+1,\alpha-n+1) - B_t(n+1,\alpha-n)\right| dt,$$

$$K_2 = \int_0^1 \left|B(n+1,\alpha-n+1) - B_t(n+1,\alpha-n)\right| t dt, \quad K_3 = K_1 - K_2$$

$$\widetilde{K}_1 = \int_0^1 \left|B(n+1,\alpha-n+1) - B_{1-t}(n+1,\alpha-n)\right| dt,$$

$$\widetilde{K}_2 = \int_0^1 \left|B(n+1,\alpha-n+1) - B_{1-t}(n+1,\alpha-n)\right| t dt, \quad \widetilde{K}_3 = \widetilde{K}_1 - \widetilde{K}_2.$$

Theorem G. [40, 41] *Let f be a Lebesgue integrable function on $[a,b]$. If f is differentiable and $|f'|^q$ $(1 \leq q < \infty)$ is convex on $[a,b]$, then*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha \Gamma(\alpha-n)} I_\alpha^a f(b) - f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\right|$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)} \left\{K_4^{1-\frac{1}{q}} \left[K_5 \left|f'(a)\right|^q + K_6 \left|f'(b)\right|^q\right]^{\frac{1}{q}} + K_7^{1-\frac{1}{q}} \left[K_8 \left|f'(a)\right|^q + K_9 \left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\},$$

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha \Gamma(\alpha-n)} {}^{b}I_\alpha f(a) - f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)\right|$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)} \left\{K_4^{1-\frac{1}{q}} \left[K_5 \left|f'(b)\right|^q + K_6 \left|f'(a)\right|^q\right]^{\frac{1}{q}} + K_7^{1-\frac{1}{q}} \left[K_8 \left|f'(b)\right|^q + K_9 \left|f'(a)\right|^q\right]^{\frac{1}{q}}\right\},$$

where

$$K_4 = \int_0^{\frac{n+1}{\alpha+1}} B_t(n+1,\alpha-n) dt, \quad K_5 = \int_0^{\frac{n+1}{\alpha+1}} B_t(n+1,\alpha-n) t dt,$$

$$K_6 = K_4 - K_5, \quad K_7 = \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] dt,$$

$$K_8 = \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] t dt, \quad K_9 = K_7 - K_8.$$

Remark. *K_4-K_9 have more explicit expressions, the details can be refer to Corollary 3 and the remark below it in section 4.*

In what follows, we assume that the function h in the above definitions is always Lebesgue integrable on $[0,1]$. Denote $L(I)$ be the set of Lebesgue integrable functions on the interval I and let $C^m(I)$ be the space of functions f with m-times derivatives on I, $m = 0, 1, 2, \ldots$. The aim of this paper is to extend the above results to strongly $h-$convex functions and obtain some error estimates related to these inequalities for any order differentiable functions.

2. New Hermite-Hadamard inequality of Conformable fractional integrals

In this section, we establish analogue results as Theorem E for left-sided and right-sided conformable integrals of strongly h-convex functions.

Theorem 1. *Let $f \in L([a,b])$ and $f \in SX(h,c,[a,b])$. Then*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I_{\alpha}^{a} f(b) \leq \frac{\mathcal{A} f(a) + \widetilde{\mathcal{A}} f(b)}{B(n+1,\alpha-n)} - c\frac{(\alpha-n)(n+1)}{(\alpha+1)(\alpha+2)}(b-a)^2, \tag{2.1}$$

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} {}^{b}I_{\alpha} f(a) \leq \frac{\widetilde{\mathcal{A}} f(a) + \mathcal{A} f(b)}{B(n+1,\alpha-n)} - c\frac{(\alpha-n)(n+1)}{(\alpha+1)(\alpha+2)}(b-a)^2, \tag{2.2}$$

where

$$\mathcal{A} = \int_0^1 h(t) t^n (1-t)^{\alpha-n-1} dt, \quad \widetilde{\mathcal{A}} = \int_0^1 h(1-t) t^n (1-t)^{\alpha-n-1} dt.$$

Especially, if f is an integrable strongly convex function with modulus $c > 0$ on $[a,b]$, we have

$$f\left(\frac{(n+1)a + (\alpha-n)b}{\alpha+1}\right) + c\frac{(\alpha-n)(n+1)}{(\alpha+1)^2(\alpha+2)}(b-a)^2$$

$$\leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I_{\alpha}^{a} f(b) \tag{2.3}$$

$$\leq \frac{(n+1)f(a) + (\alpha-n)f(b)}{\alpha+1} - c\frac{(\alpha-n)(n+1)}{(\alpha+1)(\alpha+2)}(b-a)^2, \tag{2.4}$$

and

$$f\left(\frac{(\alpha-n)a + (n+1)b}{\alpha+1}\right) + c\frac{(\alpha-n)(n+1)}{(\alpha+1)^2(\alpha+2)}(b-a)^2$$

$$\leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} {}^{b}I_{\alpha} f(a) \tag{2.5}$$

$$\leq \frac{(\alpha-n)f(a) + (n+1)f(b)}{\alpha+1} - c\frac{(\alpha-n)(n+1)}{(\alpha+1)(\alpha+2)}(b-a)^2. \tag{2.6}$$

Remark. *As $c \to 0$, (2.3)-(2.6) reduce to Theorem E.*

In order to prove the inequalities (2.3) and (2.5), we first introduce the following quadratic support theorem of strongly convex functions.

Lemma C.[3] *Let $f : (a,b) \to \mathbb{R}$ be a strongly convex function with modulus $c > 0$. Then, for every point $x_0 \in (a,b)$, f has support of the form*

$$g(x) = c(x-x_0)^2 + c_0(x-x_0) + f(x_0),$$

such that $g(x) \leq f(x)$ for all $x \in (a,b)$, where c_0 is a constant depending on x_0.

Since the proof of the right conformable fractional integrals is almost same as the left one, for brevity, we just give the proof of inequalities (2.1) and (2.3), (2.4).

Proof. $f \in SX(h,c,[a,b])$ and the definition of the Beta function tell us that

$$\frac{n!}{(b-a)^{\alpha}} I_{\alpha}^{a} f(b) = \int_0^1 t^n (1-t)^{\alpha-n-1} f(ta + (1-t)b) dt$$

$$\begin{aligned} &\leq f(a)\int_0^1 h(t)t^n(1-t)^{\alpha-n-1}dt + f(b)\int_0^1 h(1-t)t^n(1-t)^{\alpha-n-1}dt \\ &\quad -c(b-a)^2B(n+2,\alpha-n+1), \end{aligned}$$

which, combing with the basic property of (1.12), means that

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b) &= \frac{n!}{B(n+1,\alpha-n)(b-a)^\alpha}I_\alpha^a f(b) \\ &\leq \frac{1}{B(n+1,\alpha-n)}\left[\mathcal{A}f(a)+\widetilde{\mathcal{A}}f(b)\right] - \frac{(\alpha-n)(n+1)}{(\alpha+1)(\alpha+2)}c(b-a)^2. \end{aligned}$$

This completes the proof of (2.1).

If $h(t)=t$, it is easy to see that $\mathcal{A}=B(n+2,\alpha-n), \widetilde{\mathcal{A}}=B(n+1,\alpha-n+1)$. Then we infer from (1.12) and (2.1) that

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b) \\ \leq\ & \frac{B(n+2,\alpha-n)}{B(n+1,\alpha-n)}f(a) + \frac{B(n+1,\alpha-n+1)}{B(n+1,\alpha-n)}f(b) - c\frac{(n+1)(\alpha-n)}{(\alpha+2)(\alpha+1)}(b-a)^2 \\ =\ & \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} - c\frac{(\alpha-n)(n+1)}{(\alpha+1)(\alpha+2)}(b-a)^2, \end{aligned}$$

which finishes the proof of inequality (2.4).

Now we turn to prove the inequality of (2.3). Taking $x_0=[(n+1)a+(\alpha-n)b]/(\alpha+1)$ in Lemma C, the strongly convexity of f shows that there is a support

$$c\left(x-\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)^2 + c_0\left(x-\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) + f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \leq f(x)$$

holds for all $x\in(a,b)$. This implies that

$$\begin{aligned} &\frac{n!}{(b-a)^\alpha}I_\alpha^a f(b) = \int_0^1 t^n(1-t)^{\alpha-n-1}f(ta+(1-t)b)dt \\ \geq\ & \int_0^1 t^n(1-t)^{\alpha-n-1}\left[c\left(ta+(1-t)b-\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)^2\right. \\ &\left. +c_0\left(ta+(1-t)b-\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) + f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\right]dt \\ =\ & c\int_0^1 t^n(1-t)^{\alpha-n-1}\left(ta+(1-t)b-\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)^2 dt \\ &+f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)B(n+1,\alpha-n). \end{aligned}$$

On the other hand, a direct calculate yields that

$$\begin{aligned} &\int_0^1 t^n(1-t)^{\alpha-n-1}\left(ta+(1-t)b-\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)^2 dt \\ =\ & a^2B(n+3,\alpha-n)+b^2B(n+1,\alpha-n+2)+2abB(n+2,\alpha-n+1) \\ & -2a\frac{(n+1)a+(\alpha-n)b}{\alpha+1}B(n+2,\alpha-n) - 2b\frac{(n+1)a+(\alpha-n)b}{\alpha+1}B(n+1,\alpha-n+1) \end{aligned}$$

$$
\begin{aligned}
& +\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)^2 B(n+1,\alpha-n) \\
= \; & B(n+1,\alpha-n)\left\{a^2\frac{(n+2)(n+1)}{(\alpha+2)(\alpha+1)}+b^2\frac{(\alpha-n+1)(\alpha-n)}{(\alpha+2)(\alpha+1)}+2ab\frac{(n+1)(\alpha-n)}{(\alpha+2)(\alpha+1)}\right. \\
& -2a\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\cdot\frac{n+1}{\alpha+1}-2b\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\cdot\frac{\alpha-n}{\alpha+1} \\
& \left.+\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)^2\right\} \\
= \; & B(n+1,\alpha-n)\left\{a^2\frac{(n+2)(n+1)}{(\alpha+2)(\alpha+1)}+b^2\frac{(\alpha-n+1)(\alpha-n)}{(\alpha+2)(\alpha+1)}+2ab\frac{(n+1)(\alpha-n)}{(\alpha+2)(\alpha+1)}\right. \\
& \left.-\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)^2\right\} \\
= \; & \frac{B(n+1,\alpha-n)}{(\alpha+2)(\alpha+1)^2}\left\{a^2\left[(n+2)(n+1)(\alpha+1)-(n+1)^2(\alpha+2)\right]\right. \\
& +b^2\left[(\alpha-n+1)(\alpha-n)(\alpha+1)-(\alpha-n)^2(\alpha+2)\right] \\
& \left.+2ab\left[(n+1)(\alpha-n)(\alpha+1)-(n+1)(\alpha-n)(\alpha+2)\right]\right\} \\
= \; & \frac{(\alpha-n)(n+1)}{(\alpha+2)(\alpha+1)^2}B(n+1,\alpha-n)(b-a)^2.
\end{aligned}
$$

Therefore, it follows from the proceeding estimates that

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b) \geq f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+c\frac{(\alpha-n)(n+1)}{(\alpha+2)(\alpha+1)^2}(b-a)^2.$$

Thus we finish the proof of the theorem. □

3. Trapezoid type inequalities for m-times differentiable functions

In this section, we will extend Theorem F to the case of strongly h-convex functions with any order derivative. First, we give some basic properties for Beta type functions.

Lemma 1. *Let $0 < p, q < \infty$, $m = 0, 1, 2, \ldots$. For any $t \in [0, 1]$, define*

$$B_{t,m}(p,q) = \sum_{k=0}^{m}(-1)^k C_m^k t^{m-k} B_t(p+k,q), \tag{3.1}$$

$$\widetilde{B}_{t,m}(p,q) = \sum_{k=0}^{m}(-1)^k C_m^k t^{m-k} \widetilde{B}_t(p,q+k). \tag{3.2}$$

Then $B_{t,m}(p,q), \widetilde{B}_{t,m}(p,q)$ are both strictly increasing functions about t on $[0,1]$ with

$$B_{0,m}(p,q) = \widetilde{B}_{0,m}(p,q) = 0, B_{t,0}(p,q) = B_t(p,q), \widetilde{B}_{t,0}(p,q) = \widetilde{B}_t(p,q), \tag{3.3}$$

$$B_{1,m}(p,q) = B(p,q+m), \quad \widetilde{B}_{1,m}(p,q) = B(p+m,q). \tag{3.4}$$

And the derivatives of them are

$$\frac{d}{dt}B_{t,m}(p,q) = mB_{t,m-1}(p,q), \quad \frac{d}{dt}\widetilde{B}_{t,m}(p,q) = m\widetilde{B}_{t,m-1}(p,q), \ m = 1, 2, \cdots. \tag{3.5}$$

Proof. Obviously, the equations of (3.3) hold. The definition of $B_{t,m}(p,q)$ implies that

$$\begin{aligned}
B_{1,m}(p,q) &= \sum_{k=0}^{m}(-1)^k C_m^k B(p+k,q) = \sum_{k=0}^{m}(-1)^k C_m^k \int_0^1 u^{p+k-1}(1-u)^{q-1}du \\
&= \int_0^1 u^{p-1}(1-u)^{q-1}\left(\sum_{k=0}^{m}(-1)^k C_m^k u^k\right)du \\
&= \int_0^1 u^{p-1}(1-u)^{q-1}(1-u)^m\,du = B(p,q+m).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widetilde{B}_{1,m}(p,q) &= \int_0^1 (1-u)^{p-1}u^{q-1}\left(\sum_{k=0}^{m}(-1)^k C_m^k u^k\right)du \\
&= \int_0^1 (1-u)^{p+m-1}u^{q-1}du = B(q,p+m) = B(p+m,q).
\end{aligned}$$

These finish the proof of (3.4).

Now we turn to prove the equalities (3.5) by induction.

It is clearly that $B_{t,0}(p,q) = B_t(p,q)$, $\widetilde{B}_{t,0}(p,q) = \widetilde{B}_t(p,q)$. If $m = 1$, it is easy to see that

$$\frac{d}{dt}B_{t,1}(p,q) = \frac{d}{dt}\left[tB_t(p,q) - B_t(p+1,q)\right] = B_t(p,q) + t\cdot t^{p-1}(1-t)^{q-1} - t^p(1-t)^{q-1} = B_t(p,q).$$

If (3.5) holds for $m = k$, then, for $m = k+1$, we have

$$\begin{aligned}
\frac{d}{dt}B_{t,k+1}(p,q) &= \frac{d}{dt}\left(\sum_{j=0}^{k+1}(-1)^j C_{k+1}^j t^{k+1-j}B_t(p+j,q)\right) \\
&= \sum_{j=0}^{k}(-1)^j C_{k+1}^j (k+1-j)t^{k-j}B_t(p+j,q) + \sum_{j=0}^{k+1}(-1)^j C_{k+1}^j t^{p+k}(1-t)^{q-1} \\
&= (k+1)\sum_{j=0}^{k}(-1)^j C_k^j t^{k-j}B_t(p+j,q) + t^{p+k}(1-t)^{q-1}\left(\sum_{j=0}^{k+1}(-1)^j C_{k+1}^j\right) \\
&= (k+1)B_{t,k}(p,q). \qquad (3.6)
\end{aligned}$$

By the same discussion, we can obtain the conclusion for $\widetilde{B}_{t,m}(p,q)$ and leave the details to readers. Thus we derive the identity of (3.5)

Noting that $B_t(p,q) \geq 0$, $\widetilde{B}_t(p,q) \geq 0$ on $[0,1]$, by the equation (3.5), it is not difficult to check that $B_{t,m}(p,q)$, $\widetilde{B}_{t,m}(p,q)$ are strictly increasing on $[0,1]$. Therefore we complete the proof Lemma 1. □

Let $f : [a,b] \to \mathbb{R}$ be an $m-$times differentiable function and $\lambda \in [0,1]$. We define

$$\mathfrak{L}^a_{\alpha;m,\lambda}f(b) = \frac{(n+1)f(a) + (\alpha-n)f(b)}{\alpha+1}$$

$$-\frac{1}{B(n+1,\alpha-n)}\sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}\Big[\lambda\mathfrak{B}_{1,k}(n+1,\alpha-n)f^{(k)}(a) \tag{3.7}$$
$$+(-1)^k(1-\lambda)\widetilde{\mathfrak{B}}_{1,k}(n+1,\alpha-n)f^{(k)}(b)\Big]-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b),$$

$$^b\mathfrak{R}_{\alpha;m,\lambda}f(a) = \frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}$$
$$-\frac{1}{B(n+1,\alpha-n)}\sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}\Big[\lambda\widetilde{\mathfrak{B}}_{1,k}(n+1,\alpha-n)f^{(k)}(a) \tag{3.8}$$
$$+(-1)^k(1-\lambda)\mathfrak{B}_{1,k}(n+1,\alpha-n)f^{(k)}(b)\Big]-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}{}^bI_\alpha f(a),$$

here we denote $\sum_{k=1}^{0}(\cdots)\equiv 0$ and

$$\mathfrak{B}_{t,k}(p,q) = t^kB(p,q+1)-B_{t,k}(p,q),\ k=0,1,2,\dots,\ 0<p,q<\infty, \tag{3.9}$$
$$\widetilde{\mathfrak{B}}_{t,k}(p,q) = t^kB(p+1,q)-\widetilde{B}_{t,k}(p,q),\ k=0,1,2,\dots,\ 0<p,q<\infty. \tag{3.10}$$

Lemma 2. *Let $0<p,q<\infty, k=1,2,\cdots$. Then for any $t\in[0,1]$, we have*

$$\tfrac{d}{dt}\mathfrak{B}_{t,k}(p,q)=k\mathfrak{B}_{t,k-1},\quad \tfrac{d}{dt}\widetilde{\mathfrak{B}}_{t,k}(p,q)=k\widetilde{\mathfrak{B}}_{t,k-1}(p,q), \tag{3.11}$$
$$\mathfrak{B}_{0,k}(p,q)=\widetilde{\mathfrak{B}}_{0,k}(p,q)=0, \tag{3.12}$$
$$\mathfrak{B}_{1,k}(p,q)=B(p,q+1)-B(p,q+k),\ \widetilde{\mathfrak{B}}_{1,k}(p,q)=B(p+1,q)-B(p+k,q), \tag{3.13}$$
$$\mathfrak{B}_{t,k}(p,q)\ge 0,\quad \widetilde{\mathfrak{B}}_{t,k}(p,q)\ge 0. \tag{3.14}$$

Proof. According to (3.3)-(3.5) and the definitions of $\mathfrak{B}_{t,k}(p,q)$, $\widetilde{\mathfrak{B}}_{t,k}(p,q)$, we easily verify the equations of (3.11)-(3.13). Next we prove the identities of (3.14) by induction. Since the proof of the second inequality is almost the same as the first one, we just prove $\mathfrak{B}_{t,k}(p,q)\ge 0$ for all $t\in[0,1]$. The definition of Beta type function shows that there is some $t_0\in(0,1)$ satisfying

$$\mathfrak{B}_{t_0,0}(p,q)=B(p,q+1)-B_{t_0,0}(p,q)=B(p,q+1)-B_{t_0}(p,q)=0.$$

and

$$\mathfrak{B}_{t,0}(p,q)>0, t\in[0,t_0),\quad \text{and}\quad \mathfrak{B}_{t,0}(p,q)<0, t\in(t_0,1].$$

Then (3.11) shows that $\mathfrak{B}_{t,1}(p,q)$ is strictly increasing on $[0,t_0]$ and strictly decreasing on $[t_0,1]$. Clearly, (3.13) tells us that $\mathfrak{B}_{1,1}(p,q)=0$, which, together with the fact of $\mathfrak{B}_{0,1}(p,q)=0$, means that

$$\mathfrak{B}_{t,1}(p,q)>0 \tag{3.15}$$

holds for all $t\in(0,1)$. Therefore, we infer from (3.11), (3.12) and (3.15) that $\mathfrak{B}_{t,2}(p,q)>0, t\in(0,1)$. By the same way, we continue the proceeding procedure and complete the proof. □

Noting that

$$\mathfrak{B}_{1,1}(p,q)=\widetilde{\mathfrak{B}}_{1,1}(p,q)=0, \tag{3.16}$$

if $m = 1$ or 2, then $\mathfrak{L}^a_{\alpha;m,\lambda}f(b)$ and ${}^b\mathfrak{R}_{\alpha;m,\lambda}f(a)$ have the same concise form, respectively, as follows.

$$\mathfrak{L}^a_{\alpha;m,\lambda}f(b) = \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)} I^a_\alpha f(b), \tag{3.17}$$

$$ {}^b\mathfrak{R}_{\alpha;m,\lambda}f(a) = \frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)} {}^bI_\alpha f(a), \tag{3.18}$$

which are independent of the parameter λ and the derivatives $f'(a),\ f'(b)$.

Now we introduce the following key lemma.

Lemma 3. *Let $f \in C^m([a,b])$ and $f^{(m)} \in L([a,b]),\ m \in \mathbb{Z}^+$. Then, for any $\lambda \in [0,1]$,*

$$\begin{aligned}\mathfrak{L}^a_{\alpha;m,\lambda}f(b) &= \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\left[\lambda\int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right.\\ &\quad\left.+(-1)^m(1-\lambda)\int_0^1 \widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt\right],\end{aligned}\tag{3.19}$$

$$\begin{aligned}{}^b\mathfrak{R}_{\alpha;m,\lambda}f(a) &= \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\left[\lambda\int_0^1 \widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right.\\ &\quad\left.+(-1)^m(1-\lambda)\int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt\right].\end{aligned}\tag{3.20}$$

Remark. *Especially, we infer from* (1.11) *and* (1.12) *that Lemma 3 reduces to* (1.13) *and* (1.14) *with $m = \lambda = 1$.*

Proof. First, we prove the identity (3.19). Without loss of generality, we may assume that $m \geq 2$. By (3.12)-(3.16) and Lemma B, integration by parts m times shows that

$$\begin{aligned}&\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)} I^a_\alpha f(b)\\
=\ & \frac{b-a}{B(n+1,\alpha-n)}\int_0^1 \mathfrak{B}_{t,0}(n+1,\alpha-n)f'(ta+(1-t)b)dt\\
=\ & \frac{b-a}{B(n+1,\alpha-n)}\int_0^1 f'(ta+(1-t)b)d\mathfrak{B}_{t,1}(n+1,\alpha-n)\\
=\ & \frac{(b-a)^2}{B(n+1,\alpha-n)}\int_0^1 \mathfrak{B}_{t,1}(n+1,\alpha-n)f''(ta+(1-t)b)dt\\
=\ & \frac{1}{B(n+1,\alpha-n)}\left\{\frac{(b-a)^2}{2}\mathfrak{B}_{1,2}(n+1,\alpha-n)f''(a)\right.\\
&\left.+\frac{(b-a)^3}{2}\int_0^1 \mathfrak{B}_{t,2}(n+1,\alpha-n)f^{(3)}(ta+(1-t)b)dt\right\}\\
=\ & \frac{1}{B(n+1,\alpha-n)}\left\{\frac{(b-a)^2}{2}\mathfrak{B}_{1,2}(n+1,\alpha-n)f''(a)+\frac{(b-a)^3}{3!}\mathfrak{B}_{1,3}(n+1,\alpha-n)f^{(3)}(a)\right.\\
&\left.+\frac{(b-a)^4}{3!}\int_0^1 \mathfrak{B}_{t,3}(n+1,\alpha-n)f^{(4)}(ta+(1-t)b)dt\right\}\\
=\ & \cdots\end{aligned}$$

$$= \frac{1}{B(n+1,\alpha-n)}\left\{\sum_{k=2}^{m-1}\frac{(b-a)^k}{k!}\mathfrak{B}_{1,k}(n+1,\alpha-n)f^{(k)}(a)\right.$$
$$\left.+\frac{(b-a)^m}{(m-1)!}\int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right\}.$$

That is

$$\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b)$$
$$-\frac{1}{B(n+1,\alpha-n)}\sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}\mathfrak{B}_{1,k}(n+1,\alpha-n)f^{(k)}(a)$$
$$= \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt. \tag{3.21}$$

On the other hand, taking changing of variable and integration by parts, it follows from (3.4) that

$$\begin{aligned}\frac{n!}{(b-a)^\alpha}I_\alpha^a f(b) &= \int_0^1 (1-t)^n t^{\alpha-n-1} f((1-t)a+tb)dt \\ &= f(b)\widetilde{B}_1(n+1,\alpha-n)-(b-a)\int_0^1 \widetilde{B}_t(n+1,\alpha-n)f^{'}((1-t)a+tb)dt \\ &= f(b)B(n+1,\alpha-n)-(b-a)\int_0^1 \widetilde{B}_t(n+1,\alpha-n)f^{'}((1-t)a+tb)dt.\end{aligned}$$

Then, a similar argument as above shows that

$$\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b)$$
$$\begin{aligned} &= -\frac{n+1}{\alpha+1}[f(b)-f(a)]+\frac{b-a}{B(n+1,\alpha-n)}\int_0^1 \widetilde{B}_t(n+1,\alpha-n)f^{'}((1-t)a+tb)dt \\ &= -\frac{b-a}{B(n+1,\alpha-n)}\int_0^1 \widetilde{\mathfrak{B}}_{t,0}(n+1,\alpha-n)f^{'}((1-t)a+tb)dt \\ &= \frac{(b-a)^2}{B(n+1,\alpha-n)}\int_0^1 \widetilde{\mathfrak{B}}_{t,1}(n+1,\alpha-n)f^{''}((1-t)a+tb)dt \\ &= \frac{1}{B(n+1,\alpha-n)}\left\{\frac{(b-a)^2}{2}\widetilde{\mathfrak{B}}_{1,2}(n+1,\alpha-n)f^{''}(b)\right. \\ &\quad \left.-\frac{(b-a)^3}{2}\int_0^1 \widetilde{\mathfrak{B}}_{t,2}(n+1,\alpha-n)f^{(3)}((1-t)a+tb)dt\right\} \\ &= \cdots \\ &= \frac{1}{B(n+1,\alpha-n)}\left\{\sum_{k=2}^{m-1}\frac{(-1)^k(b-a)^k}{k!}\widetilde{\mathfrak{B}}_{1,k}(n+1,\alpha-n)f^{(k)}(b)\right. \\ &\quad \left.+\frac{(-1)^m(b-a)^m}{(m-1)!}\int_0^1 \widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt\right\},\end{aligned}$$

which means that

$$\begin{aligned}
&\frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}I_{\alpha}^{a}f(b)\\
&-\frac{1}{B(n+1,\alpha-n)}\sum_{k=1}^{m-1}\frac{(-1)^k(b-a)^k}{k!}\widetilde{\mathfrak{B}}_{1,k}(n+1,\alpha-n)f^{(k)}(b)\\
=\;&\frac{(-1)^m(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\int_0^1\widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt. \qquad (3.22)
\end{aligned}$$

Therefore, we complete the proof of (3.19) by (3.21) and (3.22).

Now we turn to prove (3.20). A changing of variable and integration by parts show that

$$\begin{aligned}
&\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}-\frac{n!}{(b-a)^{\alpha}}{}^{b}I_{\alpha}f(a)\\
=\;&\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}-\frac{1}{B(n+1,\alpha-n)}\int_0^1(1-t)^nt^{\alpha-n-1}f(ta+(1-t)b)dt\\
=\;&\frac{n+1}{\alpha+1}[f(b)-f(a)]-\frac{b-a}{B(n+1,\alpha-n)}\int_0^1\widetilde{B}_t(n+1,\alpha-n)f'(ta+(1-t)b)dt\\
=\;&\frac{b-a}{B(n+1,\alpha-n)}\int_0^1\Big[B(n+2,\alpha-n)-\widetilde{B}_t(n+1,\alpha-n)\Big]f'(ta+(1-t)b)dt.
\end{aligned}$$

Then, integration by parts $m-1$ times imply that

$$\begin{aligned}
&\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}{}^{b}I_{\alpha}f(a)\\
=\;&\frac{1}{B(n+1,\alpha-n)}\left\{\sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}\widetilde{\mathfrak{B}}_{1,k}(n+1,\alpha-n)f^{(k)}(a)\right. \qquad (3.23)\\
&\left.+\frac{(b-a)^m}{(m-1)!}\int_0^1\widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right\}.
\end{aligned}$$

On the other hand, similarly, we have

$$\begin{aligned}
&\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}{}^{b}I_{\alpha}f(a)\\
=\;&\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}-\frac{1}{B(n+1,\alpha-n)}\int_0^1t^n(1-t)^{\alpha-n-1}f((1-t)a+tb)dt\\
=\;&-\frac{b-a}{B(n+1,\alpha-n)}\int_0^1[B(n+1,\alpha-n+1)-B_t(n+1,\alpha-n)]f'((1-t)a+tb)dt\\
=\;&\cdots\\
=\;&\frac{1}{B(n+1,\alpha-n)}\left\{\sum_{k=1}^{m-1}\frac{(-1)^k(b-a)^k}{k!}\mathfrak{B}_{1,k}(n+1,\alpha-n)f^{(k)}(b)\right.\\
&\left.+\frac{(-1)^m(b-a)^m}{(m-1)!}\int_0^1\mathfrak{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt\right\}. \qquad (3.24)
\end{aligned}$$

Thus we finish the proof of (3.20) by (3.24) and (3.24). □

According to Lemma 3, we obtain the following error estimates.

Theorem 2. *Let $\lambda \in [0,1]$ and $f \in C^m([a,b])$, $m \in \mathbb{Z}^+$. Suppose that $f^{(m)} \in L([a,b])$ and $\left|f^{(m)}\right|^q \in SX(h,c,[a,b])$ with$1 \leq q < \infty$.*

(i) *If $q=1$, then*

$$\begin{aligned}\left|\mathfrak{L}^a_{\alpha;m,\lambda}f(b)\right| &\leq \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\left\{\left[\lambda\mathcal{C}_2+(1-\lambda)\widetilde{\mathcal{C}}_2\right]\left|f^{(m)}\right|(a)\right.\\&\left.+\left[\lambda\mathcal{C}_3+(1-\lambda)\widetilde{\mathcal{C}}_3\right]\left|f^{(m)}\right|(b)-c\left[\lambda\mathcal{C}_4+(1-\lambda)\widetilde{\mathcal{C}}_4\right](b-a)^2\right\},\\ \left|{}^b\mathfrak{R}_{\alpha;m,\lambda}f(a)\right| &\leq \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\left\{\left[\lambda\widetilde{\mathcal{C}}_2+(1-\lambda)\mathcal{C}_2\right]\left|f^{(m)}\right|(a)\right.\\&\left.+\left[\lambda\widetilde{\mathcal{C}}_3+(1-\lambda)\mathcal{C}_3\right]\left|f^{(m)}\right|(b)-c\left[\lambda\widetilde{\mathcal{C}}_4+(1-\lambda)\mathcal{C}_4\right](b-a)^2\right\}.\end{aligned}$$

(ii) *If $1<q<\infty$, then*

$$\begin{aligned}&\left|\mathfrak{L}^a_{\alpha;m,\lambda}f(b)\right|\\ \leq\;& \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\left\{\lambda\mathcal{C}_1^{1-1/q}\left[\mathcal{C}_2\left|f^{(m)}\right|^q(a)+\mathcal{C}_3\left|f^{(m)}\right|^q(b)-c\mathcal{C}_4(b-a)^2\right]^{1/q}\right.\\&\left.+(1-\lambda)\widetilde{\mathcal{C}}_1^{1-1/q}\left[\widetilde{\mathcal{C}}_2\left|f^{(m)}\right|^q(a)+\widetilde{\mathcal{C}}_3\left|f^{(m)}\right|^q(b)-c\widetilde{\mathcal{C}}_4(b-a)^2\right]^{1/q}\right\},\\ &\left|{}^b\mathfrak{R}_{\alpha;m,\lambda}f(a)\right|\\ \leq\;& \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\left\{\lambda\widetilde{\mathcal{C}}_1^{1-1/q}\left[\widetilde{\mathcal{C}}_2\left|f^{(m)}\right|^q(a)+\widetilde{\mathcal{C}}_3\left|f^{(m)}\right|^q(b)-c\widetilde{\mathcal{C}}_4(b-a)^2\right]^{1/q}\right.\\&\left.+(1-\lambda)\mathcal{C}_1^{1-1/q}\left[\mathcal{C}_2\left|f^{(m)}\right|^q(a)+\mathcal{C}_3\left|f^{(m)}\right|^q(b)-c\mathcal{C}_4(b-a)^2\right]^{1/q}\right\}.\end{aligned}$$

Here

$$\begin{aligned}\mathcal{C}_1&=\int_0^1\left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right|dt, & \widetilde{\mathcal{C}}_1&=\int_0^1\left|\widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)\right|dt,\\ \mathcal{C}_2&=\int_0^1\left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right|h(t)dt, & \widetilde{\mathcal{C}}_2&=\int_0^1\left|\widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)\right|h(t)dt,\\ \mathcal{C}_3&=\int_0^1\left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right|h(1-t)dt, & \widetilde{\mathcal{C}}_3&=\int_0^1\left|\widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)\right|h(1-t)dt,\\ \mathcal{C}_4&=\int_0^1\left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right|t(1-t)dt, & \widetilde{\mathcal{C}}_4&=\int_0^1\left|\widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)\right|t(1-t)dt.\end{aligned}$$

Remark. *If taking $h(t)=t$ and $c\to 0$, i.e. f' is convex on $[a,b]$, by (3.17), it is not difficult to check that Theorem 2 reduces to Theorem F with $m=1, \lambda=1$.*

Proof. Without loss of generality, we just prove the result for $\mathfrak{L}^a_{\alpha;m,\lambda}f(b)$ in (ii), and the proof for ${}^b\mathfrak{R}_{\alpha;m,\lambda}f(a)$ is only notation difference and it does not require new idea. The

Hölder inequality shows that

$$\begin{aligned}
&\left|\int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right| \\
\leq\ & \left(\int_0^1 \left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right| dt\right)^{1-\frac{1}{q}} \\
& \left(\int_0^1 \left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right| \left|f^{(m)}\right|^q (ta+(1-t)b)dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.25}$$

On the other hand, the fact of $|f|^q \in SX(h,c,[a,b])$ tells us that

$$\begin{aligned}
&\int_0^1 \left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right| \left|f^{(m)}\right|^q (ta+(1-t)b)dt \\
\leq\ & \int_0^1 h(t)\left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right| dt \left|f^{(m)}\right|^q (a) \\
& + \int_0^1 h(1-t)\left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right| dt \left|f^{(m)}\right|^q (b) \\
& - c(b-a)^2 \int_0^1 t(1-t)\left|\mathfrak{B}_{t,m-1}(n+1,\alpha-n)\right| dt.
\end{aligned} \tag{3.26}$$

Thus, we infer from (3.25) and (3.26) that

$$\begin{aligned}
&\left|\int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right| \\
\leq\ & \mathcal{C}_1^{1-1/q}\left[\mathcal{C}_2\left|f^{(m)}\right|^q (a) + \mathcal{C}_3\left|f^{(m)}\right|^q (b) - c\mathcal{C}_4(b-a)^2\right]^{1/q}.
\end{aligned} \tag{3.27}$$

Similarly,

$$\begin{aligned}
&\left|\int_0^1 \widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right| \\
\leq\ & \widetilde{\mathcal{C}}_1^{1-1/q}\left[\widetilde{\mathcal{C}}_2\left|f^{(m)}\right|^q (a) + \widetilde{\mathcal{C}}_3\left|f^{(m)}\right|^q (b) - c\widetilde{\mathcal{C}}_4(b-a)^2\right]^{1/q}.
\end{aligned} \tag{3.28}$$

Then we finish the proof of Theorem 2 (ii) by (3.27) and (3.28). □

If taking $h(t)=t$ and $m \geq 2$ in Theorem 2, we have the following explicit expression.

Corollary 2. *Let $1 \leq q < \infty$, $\lambda \in [0,1]$ and $f \in C^m([a,b])$, $m = 2,3,\dots$. Suppose that $f^{(m)} \in L([a,b])$ and $\left|f^{(m)}\right|^q$ be a strongly convex function on $[a,b]$ with modulus $c>0$. Then $\mathfrak{L}^a_{\alpha;m,\lambda}f(b)$, ${}^b\mathfrak{R}_{\alpha;m,\lambda}f(a)$ have the same formal estimates as Theorem 2, and the constants $\mathcal{C}_i, \widetilde{\mathcal{C}}_i, i=1,2,3,4$ have the following more direct identities.*

$$\mathcal{C}_1 = \frac{\mathfrak{B}_{1,m-1}(n+1,\alpha-n)}{m}, \quad \widetilde{\mathcal{C}}_1 = \frac{\widetilde{\mathfrak{B}}_{1,m-1}(n+1,\alpha-n)}{m},$$

$$\mathcal{C}_2 = \frac{\mathfrak{B}_{1,m}(n+1,\alpha-n)}{m} - \frac{\mathfrak{B}_{1,m+1}(n+1,\alpha-n)}{m(m+1)},$$

$$\widetilde{\mathcal{C}}_2 = \frac{\widetilde{\mathfrak{B}}_{1,m}(n+1,\alpha-n)}{m} - \frac{\widetilde{\mathfrak{B}}_{1,m+1}(n+1,\alpha-n)}{m(m+1)},$$

$$\mathcal{C}_3 = \frac{\mathfrak{B}_{1,m+1}(n+1,\alpha-n)}{m(m+1)}, \quad \widetilde{\mathcal{C}}_3 = \frac{\widetilde{\mathfrak{B}}_{1,m+1}(n+1,\alpha-n)}{m(m+1)},$$

$$\mathcal{C}_4 = \frac{\mathfrak{B}_{1,m+1}(n+1,\alpha-n)}{m(m+1)} - \frac{\mathfrak{B}_{1,m+2}(n+1,\alpha-n)}{m(m+1)(m+2)},$$
$$\widetilde{\mathcal{C}}_4 = \frac{\widetilde{\mathfrak{B}}_{1,m+1}(n+1,\alpha-n)}{m(m+1)} - \frac{\widetilde{\mathfrak{B}}_{1,m+2}(n+1,\alpha-n)}{m(m+1)(m+2)}.$$

Proof. It follows (3.14) that $\mathfrak{B}_{t,k}(n+1,\alpha-n) \geq 0$ and $\widetilde{\mathfrak{B}}_{t,k}(n+1,\alpha-n) \geq 0$ for all $t \in [0,1]$ and $m \geq 2$. Then, by Lemma 2, we have

$$\mathcal{C}_1 = \int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)dt = \frac{\mathfrak{B}_{1,m-1}(n+1,\alpha-n)}{m}.$$

$$\begin{aligned}
\mathcal{C}_2 &= \int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)tdt = \frac{1}{m}\int_0^1 td\mathfrak{B}_{t,m}(n+1,\alpha-n) \\
&= \frac{1}{m}\left[\mathfrak{B}_{1,m}(n+1,\alpha-n) - \int_0^1 \mathfrak{B}_{t,m}(n+1,\alpha-n)dt\right] \\
&= \frac{\mathfrak{B}_{1,m}(n+1,\alpha-n)}{m} - \frac{\mathfrak{B}_{1,m+1}(n+1,\alpha-n)}{m(m+1)}.
\end{aligned}$$

$$\mathcal{C}_3 = \int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)(1-t)dt = \mathcal{C}_1 - \mathcal{C}_2 = \frac{\mathfrak{B}_{1,m+1}(n+1,\alpha-n)}{m(m+1)}.$$

$$\begin{aligned}
\mathcal{C}_4 &= \int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)t(1-t)dt = \mathcal{C}_2 - \int_0^1 \mathfrak{B}_{t,m-1}(n+1,\alpha-n)t^2dt \\
&= \mathcal{C}_2 - \frac{1}{m}\int_0^1 t^2 d\mathfrak{B}_{t,m}(n+1,\alpha-n) \\
&= \mathcal{C}_2 - \frac{1}{m}\left[\mathfrak{B}_{1,m}(n+1,\alpha-n) - 2\int_0^1 \mathfrak{B}_{t,m}(n+1,\alpha-n)tdt\right] \\
&= \mathcal{C}_2 - \frac{1}{m}\mathfrak{B}_{1,m}(n+1,\alpha-n) + \frac{2}{m}\int_0^1 \mathfrak{B}_{t,m}(n+1,\alpha-n)tdt \\
&= -\frac{\mathfrak{B}_{1,m+1}(n+1,\alpha-n)}{m(m+1)} \\
&\quad + \frac{2}{m}\left[\frac{1}{m+1}\left(\mathfrak{B}_{1,m+1}(n+1,\alpha-n) - \frac{1}{m+2}\mathfrak{B}_{1,m+2}(n+1,\alpha-n)\right)\right] \\
&= \frac{\mathfrak{B}_{1,m+1}(n+1,\alpha-n)}{m(m+1)} - \frac{\mathfrak{B}_{1,m+2}(n+1,\alpha-n)}{m(m+1)(m+2)}.
\end{aligned}$$

And a similar arguments as above, we derive the conclusion for $\widetilde{\mathcal{C}}_i, i = 1,2,3,4$. □

4. Midpoint type inequalities for m-times differentiable functions

In this section, we will extend Theorem G to strongly h-convex functions with m order derivatives.

Let $f:[a,b]\to\mathbb{R}$ be an $m-$times differentiable function and $\lambda\in[0,1]$. We denote

$$
\begin{aligned}
\widetilde{\mathfrak{L}}^{a}_{\alpha;m,\lambda}f(b) &= \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I^{a}_{\alpha}f(b) \\
&-\sum_{k=0}^{m-1}\frac{(b-a)^k}{k!}\left[\lambda\left(\frac{n+1}{\alpha+1}\right)^k+(1-\lambda)\left(-\frac{\alpha-n}{\alpha+1}\right)^k\right]f^{(k)}\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \\
&+\sum_{k=0}^{m-1}\frac{(b-a)^k}{k!}\left[\lambda\left(1-\frac{B(n+1,\alpha-n+k)}{B(n+1,\alpha-n)}\right)f^{(k)}(a)\right. \\
&\left.+(1-\lambda)(-1)^k\left(1-\frac{B(n+1+k,\alpha-n)}{B(n+1,\alpha-n)}\right)f^{(k)}(b)\right],
\end{aligned} \tag{4.1}
$$

and

$$
\begin{aligned}
{}^{b}\widetilde{\mathfrak{R}}_{\alpha;m,\lambda}f(a) &= \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} {}^{b}I_{\alpha}f(a) \\
&-\sum_{k=0}^{m-1}\frac{(b-a)^k}{k!}\left[\lambda\left(\frac{\alpha-n}{\alpha+1}\right)^k+(1-\lambda)\left(-\frac{n+1}{\alpha+1}\right)^k\right]f^{(k)}\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right) \\
&+\sum_{k=0}^{m-1}\frac{(b-a)^k}{k!}\left[\lambda\left(1-\frac{B(n+1+k,\alpha-n)}{B(n+1,\alpha-n)}\right)f^{(k)}(a)\right. \\
&\left.+(1-\lambda)(-1)^k\left(1-\frac{B(n+1,\alpha-n+k)}{B(n+1,\alpha-n)}\right)f^{(k)}(b)\right].
\end{aligned} \tag{4.2}
$$

Particulary, if $m=1$, then $\widetilde{\mathfrak{L}}^{a}_{\alpha;m,\lambda}f(b)$ and ${}^{b}\widetilde{\mathfrak{R}}_{\alpha;m,\lambda}f(a)$ reduce to

$$
\widetilde{\mathfrak{L}}^{a}_{\alpha;1,\lambda}f(b) = \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} I^{a}_{\alpha}f(b) - f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right), \tag{4.3}
$$

$$
{}^{b}\widetilde{\mathfrak{R}}_{\alpha;1,\lambda}f(a) = \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} {}^{b}I_{\alpha}f(a) - f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right). \tag{4.4}
$$

Lemma 4. *Let $f\in C^m([a,b])$ and $f^{(m)}\in L([a,b])$, $m\in\mathbb{Z}^+$. Then, for any $\lambda\in[0,1]$,*

$$
\begin{aligned}
\widetilde{\mathfrak{L}}^{a}_{\alpha;m,\lambda}f(b) = \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}&\left\{\lambda\left[\int_0^{\frac{n+1}{\alpha+1}} B_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right.\right. \\
&\left.+\int_{\frac{n+1}{\alpha+1}}^{1}\left(B_{t,m-1}(n+1,\alpha-n)-B(n+1,\alpha-n)t^{m-1}\right)f^{(m)}(ta+(1-t)b)dt\right] \\
&+(1-\lambda)(-1)^m\left[\int_0^{\frac{\alpha-n}{\alpha+1}}\widetilde{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt\right. \\
&\left.\left.+\int_{\frac{\alpha-n}{\alpha+1}}^{1}\left(\widetilde{B}_{t,m-1}(n+1,\alpha-n)-B(n+1,\alpha-n)t^{m-1}\right)f^{(m)}((1-t)a+tb)dt\right]\right\},
\end{aligned} \tag{4.5}
$$

and

$$\begin{aligned}
{}^{b}\widetilde{\mathfrak{R}}_{\alpha;m,\lambda}f(a) \;=\; & \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\Bigg\{\lambda\Bigg[\int_0^{\frac{\alpha-n}{\alpha+1}}\widetilde{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt \\
& +\int_{\frac{\alpha-n}{\alpha+1}}^{1}\left(\widetilde{B}_{t,m-1}(n+1,\alpha-n)-B(n+1,\alpha-n)t^{m-1}\right)f^{(m)}(ta+(1-t)b)dt\Bigg] \qquad (4.6)\\
& +(1-\lambda)(-1)^m\Bigg[\int_0^{\frac{n+1}{\alpha+1}}B_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt \\
& +\int_{\frac{n+1}{\alpha+1}}^{1}\left(B_{t,m-1}(n+1,\alpha-n)-B(n+1,\alpha-n)t^{m-1}\right)f^{(m)}((1-t)a+tb)dt\Bigg]\Bigg\}.
\end{aligned}$$

Remark. *Particulary, taking $m=1$ in Lemma 4, it follows easily from* (4.3) *and* (4.4) *that* (4.5) *becomes* (1.15) *with $\lambda=1$ and* (4.6) *reduces to* (1.16) *with $\lambda=0$.*

Proof. First, we prove the identity (4.5). Without loss of generality, we may assume that $m\geq 2$. Changing of variable yields that

$$\begin{aligned}
(4.7)\quad & \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b)-f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\\
= \;& \frac{1}{B(n+1,\alpha-n)}\int_0^1 t^n(1-t)^{\alpha-n-1}f(ta+(1-t)b)dt-f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\\
= \;& f(a)-f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\\
& +\frac{b-a}{B(n+1,\alpha-n)}\int_0^1 B_t(n+1,\alpha-n)f'(ta+(1-t)b)dt\\
= \;& -(b-a)\int_{\frac{n+1}{\alpha+1}}^1 f'(ta+(1-t)b)dt\\
& +\frac{b-a}{B(n+1,\alpha-n)}\int_0^1 B_t(n+1,\alpha-n)f'(ta+(1-t)b)dt.
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
& -(b-a)\int_{\frac{n+1}{\alpha+1}}^1 f'(ta+(1-t)b)dt\\
= \;& (b-a)\left[\frac{n+1}{\alpha+1}f'\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)-f'(a)\right]\\
& -(b-a)^2\int_{\frac{n+1}{\alpha+1}}^1 tf''(ta+(1-t)b)dt\\
= \;& (b-a)\left[\frac{n+1}{\alpha+1}f'\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)-f'(a)\right]\\
& +\frac{(b-a)^2}{2}\left[\left(\frac{n+1}{\alpha+1}\right)^2f''\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)-f''(a)\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(b-a)^3}{2}\int_{\frac{n+1}{\alpha+1}}^1 t^2 f^{(3)}(ta+(1-t)b)dt \\
&= \cdots \\
&= \sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}\left[\left(\frac{n+1}{\alpha+1}\right)^k f^{(k)}\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)-f^{(k)}(a)\right] \\
&-\frac{(b-a)^m}{(m-1)!}\int_{\frac{n+1}{\alpha+1}}^1 t^{m-1}f^{(m)}(ta+(1-t)b)dt. \qquad (4.8)
\end{aligned}
$$

By a similar argument as the proof of Lemma 2, it follows from integration by parts $m-1$ times and Lemma 1 that

$$
\begin{aligned}
&\frac{b-a}{B(n+1,\alpha-n)}\int_0^1 B_t(n+1,\alpha-n)f'(ta+(1-t)b)dt \\
&= \frac{1}{B(n+1,\alpha-n)}\left[\sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}B(n+1,\alpha-n+k)f^{(k)}(a)\right. \\
&\left.+\frac{(b-a)^m}{(m-1)!}\int_0^1 B_{t,m-1}(n+1,\alpha-n)f^{(m)}(ta+(1-t)b)dt\right]. \qquad (4.9)
\end{aligned}
$$

Thus, we infer from (4.7)-(4.9) that

$$
\begin{aligned}
&\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b)+\sum_{k=0}^{m-1}\frac{(b-a)^k}{k!}\left[1-\frac{B(n+1,\alpha-n+k)}{B(n+1,\alpha-n)}\right]f^{(k)}(a) \\
&-\sum_{k=0}^{m-1}\frac{(b-a)^k}{k!}\left(\frac{n+1}{\alpha+1}\right)^k f^{(k)}\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \\
&= \frac{(b-a)^m}{(m-1)!}\left\{\int_0^{\frac{n+1}{\alpha+1}}\frac{B_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}f^{(m)}(ta+(1-t)b)dt\right. \qquad (4.10) \\
&\left.+\int_{\frac{n+1}{\alpha+1}}^1\left[\frac{B_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}-t^{m-1}\right]f^{(m)}(ta+(1-t)b)dt\right\}.
\end{aligned}
$$

On the other hand, taking changing of variable, we also have

$$
\begin{aligned}
&\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b)-f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \qquad (4.11) \\
&= \frac{1}{B(n+1,\alpha-n)}\int_0^1(1-t)^n t^{\alpha-n-1}f((1-t)a+tb)dt-f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \\
&= (b-a)\int_{\frac{\alpha-n}{\alpha+1}}^1 f'((1-t)a+tb)dt \\
&-\frac{b-a}{B(n+1,\alpha-n)}\int_0^1\widetilde{B}_t(n+1,\alpha-n)f'((1-t)a+tb)dt.
\end{aligned}
$$

By a similar discussion as above,

$$
(b-a)\int_{\frac{\alpha-n}{\alpha+1}}^1 f'((1-t)a+tb)dt
$$

$$
\begin{aligned}
(4.12) \quad = & \sum_{k=1}^{m-1} \frac{(-1)^{k-1}(b-a)^k}{k!}\left[f^{(k)}(b)-\left(\frac{\alpha-n}{\alpha+1}\right)^k f^{(k)}\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)\right] \\
& +\frac{(-1)^{m-1}(b-a)^m}{(m-1)!}\int_{\frac{\alpha-n}{\alpha+1}}^1 t^{m-1}f^{(m)}((1-t)a+tb)dt,
\end{aligned}
$$

and

$$
\begin{aligned}
& -\frac{b-a}{B(n+1,\alpha-n)}\int_0^1 \widetilde{B}_t(n+1,\alpha-n)f'((1-t)a+tb)dt \\
(4.13) \quad = & \frac{1}{B(n+1,\alpha-n)}\left[\sum_{k=1}^{m-1}\frac{(-1)^k(b-a)^k}{k!}B(n+1+k,\alpha-n)f^{(k)}(b)\right. \\
& \left. +\frac{(-1)^m(b-a)^m}{(m-1)!}\int_0^1 \widetilde{B}_{t,m-1}(n+1,\alpha-n)f^{(m)}((1-t)a+tb)dt\right].
\end{aligned}
$$

Therefore, (4.11)-(4.13) give that

$$
\begin{aligned}
& \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}I_\alpha^a f(b)+\sum_{k=0}^{m-1}\frac{(-1)^k(b-a)^k}{k!}\left[1-\frac{B(n+1+k,\alpha-n)}{B(n+1,\alpha-n)}\right]f^{(k)}(b) \\
& -\sum_{k=0}^{m-1}\frac{(-1)^k(b-a)^k}{k!}\left(\frac{\alpha-n}{\alpha+1}\right)^k f^{(k)}\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \\
(4.14) = & \frac{(-1)^m(b-a)^m}{(m-1)!}\left\{\int_0^{\frac{\alpha-n}{\alpha+1}}\frac{\widetilde{B}_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}f^{(m)}((1-t)a+tb)dt\right. \\
& \left. +\int_{\frac{\alpha-n}{\alpha+1}}^1\left[\frac{\widetilde{B}_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}-t^{m-1}\right]f^{(m)}((1-t)a+tb)dt\right\}.
\end{aligned}
$$

Thus we finish the proof of (4.5) by (4.10) and (4.14).

Since the identity (4.6) holds for the same reason as (4.5), we only give an outline of the proof of (4.6). Taking changing of variable and integration by parts m times, we have

$$
\begin{aligned}
& \frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}{}^bI_\alpha f(a)-f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right) \\
= & -(b-a)\int_{\frac{\alpha-n}{\alpha+1}}^1 f'(ta+(1-t)b)dt \\
& +\frac{b-a}{B(n+1,\alpha-n)}\int_0^1 \widetilde{B}_t(n+1,\alpha-n)f'(ta+(1-t)b)dt \\
(4.15) \quad = & \sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}\left(\frac{\alpha-n}{\alpha+1}\right)^k f^{(k)}\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right) \\
& -\sum_{k=1}^{m-1}\frac{(b-a)^k}{k!}\left[1-\frac{B(n+1+k,\alpha-n)}{B(n+1,\alpha-n)}\right]f^{(k)}(a) \\
& +\frac{(b-a)^m}{(m-1)!}\left\{\int_0^{\frac{\alpha-n}{\alpha+1}}\frac{\widetilde{B}_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}f^{(m)}(ta+(1-t)b)dt\right.
\end{aligned}
$$

$$+\int_{\frac{\alpha-n}{\alpha+1}}^{1}\left[\frac{\widetilde{B}_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}-t^{m-1}\right]f^{(m)}(ta+(1-t)b)dt\Bigg\}.$$

Similarly,

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}{}^{b}I_{\alpha}f(a)-f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)\\
=\;&(b-a)\int_{\frac{n+1}{\alpha+1}}^{1}f'(ta+(1-t)b)dt\\
&-\frac{b-a}{B(n+1,\alpha-n)}\int_{0}^{1}B_{t}(n+1,\alpha-n)f'((1-t)a+tb)dt\\
=\;&\sum_{k=1}^{m-1}\frac{(-1)^{k}(b-a)^{k}}{k!}\left(\frac{n+1}{\alpha+1}\right)^{k}f^{(k)}\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)\\
&-\sum_{k=1}^{m-1}\frac{(-1)^{k}(b-a)^{k}}{k!}\left[1-\frac{B(n+1,\alpha-n+k)}{B(n+1,\alpha-n)}\right]f^{(k)}(b)\\
&+\frac{(-1)^{m}(b-a)^{m}}{(m-1)!}\left\{\int_{0}^{\frac{n+1}{\alpha+1}}\frac{B_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}f^{(m)}((1-t)a+tb)dt\right.\\
&\left.+\int_{\frac{n+1}{\alpha+1}}^{1}\left[\frac{B_{t,m-1}(n+1,\alpha-n)}{B(n+1,\alpha-n)}-t^{m-1}\right]f^{(m)}((1-t)a+tb)dt\right\}.
\end{aligned}\tag{4.16}$$

Thus we get the conclusion of (4.6) by (4.15) and (4.16). □

As a consequence of Lemma 4, we obtain some midpoint type estimates as follows.

Theorem 3. *Let $\lambda\in[0,1]$ and $f\in C^{m}([a,b])$, $m\in\mathbb{Z}^{+}$. Suppose that $f^{(m)}\in L([a,b])$ and $\left|f^{(m)}\right|^{q}\in SX(h,c,[a,b])$ with$1\le q<\infty$.*

(i) *If $q=1$, then*

$$\begin{aligned}
\left|\widetilde{\mathfrak{L}}^{a}_{\alpha;m,\lambda}f(b)\right|\le\;&\frac{(b-a)^{m}}{(m-1)!B(n+1,\alpha-n)}\left\{\left[\lambda\left(\mathcal{E}_{2}+\mathcal{E}_{6}\right)+(1-\lambda)\left(\widetilde{\mathcal{E}}_{3}+\widetilde{\mathcal{E}}_{7}\right)\right]\left|f^{(m)}(a)\right|\right.\\
&+\left[\lambda\left(\mathcal{E}_{3}+\mathcal{E}_{7}\right)+(1-\lambda)\left(\widetilde{\mathcal{E}}_{2}+\widetilde{\mathcal{E}}_{6}\right)\right]\left|f^{(m)}(b)\right|\\
&\left.-c(b-a)^{2}\left[\lambda\left(\mathcal{E}_{4}+\mathcal{E}_{8}\right)+(1-\lambda)\left(\widetilde{\mathcal{E}}_{4}+\widetilde{\mathcal{E}}_{8}\right)\right]\right\},
\end{aligned}\tag{4.17}$$

$$\begin{aligned}
\left|{}^{b}\widetilde{\mathfrak{R}}_{\alpha;m,\lambda}f(a)\right|\le\;&\frac{(b-a)^{m}}{(m-1)!B(n+1,\alpha-n)}\left\{\left[\lambda\left(\widetilde{\mathcal{E}}_{2}+\widetilde{\mathcal{E}}_{6}\right)+(1-\lambda)\left(\mathcal{E}_{3}+\mathcal{E}_{7}\right)\right]\left|f^{(m)}(a)\right|\right.\\
&+\left[\lambda\left(\widetilde{\mathcal{E}}_{3}+\widetilde{\mathcal{E}}_{7}\right)+(1-\lambda)\left(\mathcal{E}_{2}+\mathcal{E}_{6}\right)\right]\left|f^{(m)}(b)\right|\\
&\left.-c(b-a)^{2}\left[\lambda\left(\widetilde{\mathcal{E}}_{4}+\widetilde{\mathcal{E}}_{8}\right)+(1-\lambda)\left(\mathcal{E}_{4}+\mathcal{E}_{8}\right)\right]\right\}.
\end{aligned}\tag{4.18}$$

(ii) *If $1<q<\infty$, then*

$$\left|\widetilde{\mathfrak{L}}^{a}_{\alpha;m,\lambda}f(b)\right|\le\frac{(b-a)^{m}}{(m-1)!B(n+1,\alpha-n)}\times$$

$$\left\{\lambda\left[\mathcal{E}_1^{1-\frac{1}{q}}\left(\mathcal{E}_2\left|f^{(m)}(a)\right|^q+\mathcal{E}_3\left|f^{(m)}(b)\right|^q-c\mathcal{E}_4(b-a)^2\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\mathcal{E}_5^{1-\frac{1}{q}}\left(\mathcal{E}_6\left|f^{(m)}(a)\right|^q+\mathcal{E}_7\left|f^{(m)}(b)\right|^q-c\mathcal{E}_8(b-a)^2\right)^{\frac{1}{q}}\right]$$
$$+(1-\lambda)\left[\widetilde{\mathcal{E}}_1^{1-\frac{1}{q}}\left(\widetilde{\mathcal{E}}_3\left|f^{(m)}(a)\right|^q+\widetilde{\mathcal{E}}_2\left|f^{(m)}(b)\right|^q-c\widetilde{\mathcal{E}}_5(b-a)^2\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\widetilde{\mathcal{E}}_5^{1-\frac{1}{q}}\left(\widetilde{\mathcal{E}}_7\left|f^{(m)}(a)\right|^q+\widetilde{\mathcal{E}}_6\left|f^{(m)}(b)\right|^q-c\widetilde{\mathcal{E}}_8(b-a)^2\right)^{\frac{1}{q}}\right]\right\}, \tag{4.19}$$

$$\left|{}^{b}\widetilde{\mathfrak{R}}_{\alpha;m,\lambda}f(a)\right| \leq \frac{(b-a)^m}{(m-1)!B(n+1,\alpha-n)}\times$$
$$\left\{\lambda\left[\widetilde{\mathcal{E}}_1^{1-\frac{1}{q}}\left(\widetilde{\mathcal{E}}_2\left|f^{(m)}(a)\right|^q+\widetilde{\mathcal{E}}_3\left|f^{(m)}(b)\right|^q-c\widetilde{\mathcal{E}}_4(b-a)^2\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\widetilde{\mathcal{E}}_5^{1-\frac{1}{q}}\left(\widetilde{\mathcal{E}}_6\left|f^{(m)}(a)\right|^q+\widetilde{\mathcal{E}}_7\left|f^{(m)}(b)\right|^q-c\widetilde{\mathcal{E}}_8(b-a)^2\right)^{\frac{1}{q}}\right]$$
$$+(1-\lambda)\left[\mathcal{E}_1^{1-\frac{1}{q}}\left(\mathcal{E}_3\left|f^{(m)}(a)\right|^q+\mathcal{E}_2\left|f^{(m)}(b)\right|^q-c\mathcal{E}_5(b-a)^2\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\mathcal{E}_5^{1-\frac{1}{q}}\left(\mathcal{E}_7\left|f^{(m)}(a)\right|^q+\mathcal{E}_6\left|f^{(m)}(b)\right|^q-c\mathcal{E}_8(b-a)^2\right)^{\frac{1}{q}}\right]\right\}. \tag{4.20}$$

Here we denote

$$\begin{aligned}
\mathcal{E}_1 &= \int_0^{\frac{n+1}{\alpha+1}} B_{t,m-1}(n+1,\alpha-n)dt,\\
\mathcal{E}_2 &= \int_0^{\frac{n+1}{\alpha+1}} B_{t,m-1}(n+1,\alpha-n)h(t)dt,\\
\mathcal{E}_3 &= \int_0^{\frac{n+1}{\alpha+1}} B_{t,m-1}(n+1,\alpha-n)h(1-t)dt,\\
\mathcal{E}_4 &= \int_0^{\frac{n+1}{\alpha+1}} B_{t,m-1}(n+1,\alpha-n)t(1-t)dt,\\
\mathcal{E}_5 &= \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1}-B_{t,m-1}(n+1,\alpha-n)\right]dt,\\
\mathcal{E}_6 &= \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1}-B_{t,m-1}(n+1,\alpha-n)\right]h(t)dt,\\
\mathcal{E}_7 &= \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1}-B_{t,m-1}(n+1,\alpha-n)\right]h(1-t)dt,\\
\mathcal{E}_8 &= \int_{\frac{n+1}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1}-B_{t,m-1}(n+1,\alpha-n)\right]t(1-t)dt,\\
\widetilde{\mathcal{E}}_1 &= \int_0^{\frac{\alpha-n}{\alpha+1}} \widetilde{B}_{t,m-1}(n+1,\alpha-n)dt,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{E}}_2 &= \int_0^{\frac{\alpha-n}{\alpha+1}} \widetilde{B}_{t,m-1}(n+1,\alpha-n)h(t)dt,\\
\widetilde{\mathcal{E}}_3 &= \int_0^{\frac{\alpha-n}{\alpha+1}} \widetilde{B}_{t,m-1}(n+1,\alpha-n)h(1-t)dt,\\
\widetilde{\mathcal{E}}_4 &= \int_0^{\frac{\alpha-n}{\alpha+1}} \widetilde{B}_{t,m-1}(n+1,\alpha-n)t(1-t)dt,\\
\widetilde{\mathcal{E}}_5 &= \int_{\frac{\alpha-n}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1} - \widetilde{B}_{t,m-1}(n+1,\alpha-n)\right] dt,\\
\widetilde{\mathcal{E}}_6 &= \int_{\frac{\alpha-n}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1} - \widetilde{B}_{t,m-1}(n+1,\alpha-n)\right] h(t)dt,\\
\widetilde{\mathcal{E}}_7 &= \int_{\frac{\alpha-n}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1} - \widetilde{B}_{t,m-1}(n+1,\alpha-n)\right] h(1-t)dt,\\
\widetilde{\mathcal{E}}_8 &= \int_{\frac{\alpha-n}{\alpha+1}}^1 \left[B(n+1,\alpha-n)t^{m-1} - \widetilde{B}_{t,m-1}(n+1,\alpha-n)\right] |t(1-t)dt.
\end{aligned}$$

For any $t \in [0,1]$ and $m = 1, 2, \ldots$, it follows from Lemma 3 that

$$\begin{aligned}
B(n+1,\alpha-n)t^{m-1} - B_{t,m-1}(n+1,\alpha-n) &\geq \mathfrak{B}_{t,m-1}(n+1,\alpha-n) \geq 0,\\
\widetilde{B}(n+1,\alpha-n)t^{m-1} - \widetilde{B}_{t,m-1}(n+1,\alpha-n) &\geq \widetilde{\mathfrak{B}}_{t,m-1}(n+1,\alpha-n) \geq 0.
\end{aligned}$$

Then the remaining proof of Theorem 3 is almost same to that of Theorem 2, we leave it to reads.

Remark. *If taking $h(t) = t$ and letting $c \to 0$, i.e., $|f'|$ is a convex function on $[a,b]$, by* (4.3)*, then* (4.17) *and* (4.19) *reduce to the first inequality in Theorem G with $m = \lambda = 1$, and* (4.18) *and* (4.20) *reduce to the second inequality in Theorem G with $m = 1, \lambda = 0$.*

Before giving some applications of Theorem 3, we first introduce some notations. For any $\gamma \in (0,1]$, $0 < p, q < \infty$ and $m \in \mathbb{Z}^+$, we infer from Lemma 1 that

$$\mathcal{K}_{0;\gamma,m-1}(p,q) := \int_0^\gamma B_{t,m-1}(p,q)dt = \frac{B_{\gamma,m}(p,q)}{m}, \tag{4.21}$$

$$\begin{aligned}
\mathcal{K}_{1;\gamma,m-1}(p,q) &:= \int_0^\gamma B_{t,m-1}(p,q)tdt = \frac{\gamma B_{\gamma,m}(p,q)}{m} - \frac{1}{m}\int_0^\gamma B_{t,m}(p,q)dt\\
&= \frac{\gamma B_{\gamma,m}(p,q)}{m} - \frac{B_{\gamma,m+1}(p,q)}{m(m+1)}, \qquad (4.22)
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_{2;\gamma,m-1}(p,q) &:= \int_0^\gamma B_{t,m-1}(p,q)t^2dt = \frac{\gamma^2 B_{\gamma,m}(p,q)}{m} - \frac{2}{m}\int_0^\gamma B_{t,m}(p,q)tdt\\
&= \frac{\gamma^2 B_{\gamma,m}(p,q)}{m} - \frac{2\gamma B_{\gamma,m+1}(p,q)}{m(m+1)} + \frac{2B_{\gamma,m+2}(p,q)}{m(m+1)(m+2)}. \qquad (4.23)
\end{aligned}$$

Similarly,

$$\widetilde{\mathcal{K}}_{0;\gamma,m-1}(p,q) := \int_0^\gamma \widetilde{B}_{t,m-1}(p,q)dt = \frac{\widetilde{B}_{\gamma,m}(p,q)}{m}, \tag{4.24}$$

$$\widetilde{\mathcal{K}}_{1;\gamma,m-1}(p,q) := \int_0^\gamma \widetilde{B}_{t,m-1}(p,q)tdt = \frac{\gamma\widetilde{B}_{\gamma,m}(p,q)}{m} - \frac{\widetilde{B}_{\gamma,m+1}(p,q)}{m(m+1)}, \tag{4.25}$$

$$\begin{aligned}\widetilde{\mathcal{K}}_{2;\gamma,m-1}(p,q) &:= \int_0^\gamma \widetilde{B}_{t,m-1}(p,q)t^2dt \\ &= \frac{\gamma^2\widetilde{B}_{\gamma,m}(p,q)}{m} - \frac{2\gamma\widetilde{B}_{\gamma,m+1}(p,q)}{m(m+1)} + \frac{2\widetilde{B}_{\gamma,m+2}(p,q)}{m(m+1)(m+2)}.\end{aligned} \tag{4.26}$$

Now choosing $h(t) = t$ in Theorem 3, by (4.21)-(4.26), a direct calculate easily shows that the following more explicit results hold for strongly convex functions and we omit the proof.

Corollary 3. *Let $\lambda \in [0,1], 1 \le q < \infty$ and $f \in C^m([a,b])$, $m \in \mathbb{Z}^+$. Suppose that $f^{(m)} \in L([a,b])$ and $\left|f^{(m)}\right|^q$ be strongly convex on $[a,b]$ with modulus $c > 0$. Then $\widetilde{\mathfrak{L}}^a_{\alpha;m,\lambda} f(b)$, ${}^b\widetilde{\mathfrak{R}}_{\alpha;m,\lambda} f(a)$ have the same formal estimates as Theorem 3, and the constants $\mathcal{E}_i, \widetilde{\mathcal{E}}_i, i = 1,2,\dots,8$, have the following more direct identities.*

$$\begin{aligned}
\mathcal{E}_1 &= \mathcal{K}_{0;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n), \quad \mathcal{E}_2 = \mathcal{K}_{1;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n), \quad \mathcal{E}_3 = \mathcal{E}_1 - \mathcal{E}_2,\\
\mathcal{E}_4 &= \mathcal{K}_{1;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n) - \mathcal{K}_{2;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n),\\
\mathcal{E}_5 &= \left[1-\left(\frac{n+1}{\alpha+1}\right)^m\right]\frac{B(n+1,\alpha-n)}{m} - \left[\mathcal{K}_{0;1,m-1}(n+1,\alpha-n) - \mathcal{K}_{0;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n)\right],\\
\mathcal{E}_6 &= \left[1-\left(\frac{n+1}{\alpha+1}\right)^{m+1}\right]\frac{B(n+1,\alpha-n)}{m+1} - \left[\mathcal{K}_{1;1,m-1}(n+1,\alpha-n) - \mathcal{K}_{1;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n)\right],\\
\mathcal{E}_7 &= \mathcal{E}_5 - \mathcal{E}_6,\\
\mathcal{E}_8 &= \left[1-\left(\frac{n+1}{\alpha+1}\right)^m\right]\frac{B(n+1,\alpha-n)}{m} - \left[1-\left(\frac{n+1}{\alpha+1}\right)^{m+1}\right]\frac{B(n+1,\alpha-n)}{m+1}\\
&\quad - \left[\mathcal{K}_{1;1,m-1}(n+1,\alpha-n) - \mathcal{K}_{1;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n)\right]\\
&\quad + \left[\mathcal{K}_{2;1,m-1}(n+1,\alpha-n) - \mathcal{K}_{2;\frac{n+1}{\alpha+1},m-1}(n+1,\alpha-n)\right],\\
\widetilde{\mathcal{E}}_1 &= \widetilde{\mathcal{K}}_{0;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n), \quad \widetilde{\mathcal{E}}_2 = \widetilde{\mathcal{K}}_{1;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n), \quad \widetilde{\mathcal{E}}_3 = \widetilde{\mathcal{E}}_1 - \widetilde{\mathcal{E}}_2,\\
\widetilde{\mathcal{E}}_4 &= \widetilde{\mathcal{K}}_{1;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n) - \widetilde{\mathcal{K}}_{2;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n),\\
\widetilde{\mathcal{E}}_5 &= \left[1-\left(\frac{\alpha-n}{\alpha+1}\right)^m\right]\frac{B(n+1,\alpha-n)}{m} - \left[\widetilde{\mathcal{K}}_{0;1,m-1}(n+1,\alpha-n) - \widetilde{\mathcal{K}}_{0;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n)\right],\\
\widetilde{\mathcal{E}}_6 &= \left[1-\left(\frac{\alpha-n}{\alpha+1}\right)^{m+1}\right]\frac{B(n+1,\alpha-n)}{m+1} - \left[\widetilde{\mathcal{K}}_{1;1,m-1}(n+1,\alpha-n) - \widetilde{\mathcal{K}}_{1;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n)\right],\\
\widetilde{\mathcal{E}}_7 &= \widetilde{\mathcal{E}}_5 - \widetilde{\mathcal{E}}_6,\\
\widetilde{\mathcal{E}}_8 &= \left[1-\left(\frac{\alpha-n}{\alpha+1}\right)^m\right]\frac{B(n+1,\alpha-n)}{m} - \left[1-\left(\frac{\alpha-n}{\alpha+1}\right)^{m+1}\right]\frac{B(n+1,\alpha-n)}{m+1}\\
&\quad - \left[\widetilde{\mathcal{K}}_{1;1,m-1}(n+1,\alpha-n) - \widetilde{\mathcal{K}}_{1;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n)\right]
\end{aligned}$$

$$+\left[\widetilde{\mathcal{K}}_{2;1,m-1}(n+1,\alpha-n)-\widetilde{\mathcal{K}}_{2;\frac{\alpha-n}{\alpha+1},m-1}(n+1,\alpha-n)\right].$$

Remark. *Obviously, taking $c \to 0$ in Corollary 3, i.e., $|f'|$ is a convex function on $[a,b]$, the proceeding conclusion generalizes Theorem G.*

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