EXISTENCE RESULTS FOR A NONLINEAR ELLIPTIC PROBLEM DRIVEN BY A NON-HOMOGENEOUS OPERATOR

JOHN R. GRAEF, SHAPOUR HEIDARKHANI, LINGJU KONG, AND SHAHIN MORADI

ABSTRACT. In this paper, the authors discuss the existence of at least one weak solution and infinitely many weak solutions to a parametric nonlinear Dirichlet problem involving a nonhomogeneous differential operator of p-Laplacian type. Their approach is based on variational methods. Some recent results are extended and improved, and an example is presented to demonstrate the application of the main results.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with a smooth boundary $\partial\Omega$ and consider the problem

$$\begin{cases} -\mathrm{div}\mathbf{A}(x, \nabla u) = \lambda f(x, u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{P_λ^f}$$

where $\mathbf{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a function admitting a potential and satisfying some natural conditions such that the differential operator $\mathrm{div}\mathbf{A}(x, \nabla u)$ includes the usual p-Laplacian $(p > 1)$. Here, λ is a positive parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a suitable Carathéodory function.

Recently, many authors have applied variational methods to study the existence of multiple solutions of nonlinear Dirichlet problems involving a nonhomogeneous differential operator of p-Laplacian type and containing a parameter; we refer the reader to [5, 7, 8, 9, 10, 11, 18, 19] and references cited therein as examples of such results. Based on a recent abstract critical point theorem proved by Bonanno [3], Kristály *et al.* [18] established the existence of three weak solutions to the problem

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) = \lambda f(x, u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases}$$

where $N \geq 2$ and the nonlinearities $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy certain structural conditions. Bonanno *et al.* [5] used variational methods to obtain some new contributions on the problem (P_λ^f).

In the present paper, motivated by the papers of Bonanno, D'Aguì, and Livrea [5], Colasuonno, Pucci, and Varga [9], and Kristály, Lisei, and Varga [18], we study the problem (P_λ^f) in the case where A admits a potential $\mathscr{A} : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ such that

---

2020 *Mathematics Subject Classification.* 34B15, 35J15.

Key words and phrases. Nonlinear elliptic problem; One weak solution; Infinitely many solutions; Variational methods.

(A) $\mathscr{A} = \mathscr{A}(x, \xi)$ is a continuous function on $\overline{\Omega} \times \mathbb{R}^N$ with a continuous derivative with respect to ξ, $\mathbf{A} = \partial_\xi \mathscr{A}$, and:
  (i) $\mathscr{A}(x, 0) = 0$ and $\mathscr{A}(x, \xi) = \mathscr{A}(x, -\xi)$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.
  (ii) $\mathscr{A}(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \Omega$.
  (iii) There exist constants a_1, a_2, with $0 < a_1 \leq a_2$ such that

$$\mathbf{A}(x, \xi)\xi \geq a_1|\xi|^p \text{ and } |\mathbf{A}(x, \xi)| \leq a_2|\xi|^{p-1} \tag{1.1}$$

for every $(x, \xi) \in \Omega \times \mathbb{R}^N$.

Motivated by the above facts, in the present paper we use a smooth version of Theorem 2.1 of [6] (that is sometimes viewed as a more precise version of Ricceri's Variational Principle [21]) to investigate the existence of at least one weak solution and the existence of infinitely many weak solutions to the problem (P_λ^f). In fact, we shall study the existence of at least one non-trivial weak solution to (P_λ^f) under an assumption on the asymptotic behavior of the nonlinear function f at zero (see Theorem 3.1 below). In addition, under suitable conditions on the oscillatory behavior of f at infinity, we discuss the existence of infinitely many weak solutions to (P_λ^f). We prove the existence of an interval on λ in which the problem (P_λ^f) admits a sequence of solutions that are unbounded in the space $W^{1,p}(\Omega)$ (see Theorem 4.1). Additionally, some consequences of Theorem 4.1 are presented. By replacing the conditions at infinity on the nonlinear term, by a similar one at zero, we obtain a sequence of pairwise distinct solutions strongly converging at zero (see Theorem 4.6).

We have organized the remainder of the paper as follows. In Section 2, we recall some basic definitions and the tools to be used in our proofs. In Sections 3 and 4, we state and prove the main results of the paper.

2. Preliminaries

The key argument used in proving our results is the following version of Ricceri's variational principle [21, Theorem 2.1] as given by Bonanno and Molica Bisci in [6].

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) - \Psi(u)}{r - \Phi(u)}$$

$$\theta := \liminf_{r \to +\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in \left(0, \dfrac{1}{\varphi(r)}\right)$, the restriction of the functional $\mathscr{I}_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of $\mathscr{I}_\lambda$ in X.*

(b) *If $\theta < +\infty$, then for each $\lambda \in \left(0, \dfrac{1}{\theta}\right)$, either*

  (b_1) *$\mathscr{I}_\lambda$ possesses a global minimum,* *or*

(b$_2$) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* $\mathscr{I}_\lambda$ *such that*

$$\lim_{n\to+\infty}\Phi(u_n)=+\infty.$$

(c) *If* $\delta<+\infty$*, then for each* $\lambda\in\left(0,\dfrac{1}{\delta}\right)$*, either*

(c$_1$) *there is a global minimum of* Φ *which is a local minimum of* $\mathscr{I}_\lambda$*,*

or

(c$_2$) *there is a sequence of pairwise distinct critical points (local minima) of* $\mathscr{I}_\lambda$ *that weakly converges to a global minimum of* Φ.

We refer the interested reader to the papers [1, 2, 12, 14, 16] in which Theorem 2.1 has been successfully employed to prove the existence of at least one non-trivial solution to boundary value problems, and to the papers [4, 13, 15, 17] in which Theorem 2.1 was used to show the existence of infinitely many solutions.

Next, we give some pertinent definitions and notations. Throughout the paper, Ω is a bounded domain in $\mathbb{R}^N$ with $1<p<N$, $W_0^{1,p}(\Omega)$ is the usual Sobolev space endowed with the norm

$$\|u\|=\|\nabla u\|_p,$$

and $W^{-1,p'}(\Omega)$ is its dual space, where $\frac{1}{p}+\frac{1}{p'}=1$. It is well known that if $1<p<N$ and $p^*=\dfrac{Np}{N-p}$, then there is a constant $T=T(N,p)$ such that

$$\|u\|_{p^*}\le T\|u\| \tag{2.1}$$

for every $u\in W_0^{1,p}(\Omega)$. Such a constant has been sharply determined by Talenti in [22] as given by the formula

$$T=\pi^{\frac{-1}{2}}N^{\frac{-1}{p}}\left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}}\left\{\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}\right\}^{\frac{1}{N}}, \tag{2.2}$$

where Γ is the gamma function. Clearly, inequality (2.1), in conjunction with Hölder's inequality, implies that for every $s\in[1,p^*]$,

$$\|u\|_s\le T|\Omega|^{\frac{(p^*-s)}{(p^*s)}}\|u\| \tag{2.3}$$

for all $u\in W_0^{1,p}(\Omega)$, where $|\Omega|$ is the Lebesgue measure of Ω. Notice that the embedding $W_0^{1,p}(\Omega)\hookrightarrow L^s(\Omega)$ is compact provided $s\in[1,p^*)$.

Following [9], we will assume that $\mathbf{A}:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a function admitting a smooth potential $\mathscr{A}:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}$ as given in condition (A) above.

In [9], it has been explicitly observed that $(A)(i)$ and $(A)(iii)$ imply that

$$a_1|\xi|^p\le p\mathscr{A}(x,\xi)\le a_2|\xi|^p \tag{2.4}$$

for every $(x,\xi)\in\Omega\times\mathbb{R}^N$. Moreover, it is possible to obtain the following lemma.

Lemma 2.2. ([9, Lemma 2.5]) *Let $\mathscr{A}$ satisfy condition (A). Then the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\Phi(u) = \int_\Omega \mathscr{A}(x, \nabla u(x))dx \tag{2.5}$$

is convex, weakly lower semicontinuous, and belongs to the class C^1 in $W_0^{1,p}(\Omega)$ with

$$\Phi'(u)(v) = \int_\Omega \mathbf{A}(x, \nabla u(x))\nabla v(x)dx$$

for every $u, v \in W_0^{1,p}(\Omega)$. Moreover, $\Phi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ satisfies the condition:

($\mathscr{S}_+$) *For every sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} \int_\Omega \mathbf{A}(x, \nabla u_n) \cdot (\nabla u_n - \nabla u)dx \leq 0,$$

then $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

Given a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and a positive function $a \in L^\alpha(\Omega)$ with $\alpha > N/p$ and $1 < q \leq p$, we say that f is of type $(\mathscr{G}_{f,a,q})$ if it satisfies the growth condition:

$(\mathscr{G}_{f,a,q})$ there exist positive constants M_1 and M_2 such that

$$|f(x,t)| \leq a(x) \left(M_1 + M_2|t|^{q-1}\right) \tag{2.6}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

We will also need the following lemma to prove our main results.

Lemma 2.3. ([9, Lemma 3.2], [5, Lemma 2.2]) *Assume that f is of type $(\mathscr{G}_{f,a,q})$ and set $F(x,t) = \int_0^t f(x,s)ds$. Then, the functional $\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\Psi(u) = \int_\Omega F(x, u(x))dx \tag{2.7}$$

is in the class C^1 with

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)dx.$$

Moreover, the operator $\Psi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is compact and sequentially weakly continuous in $W_0^{1,p}(\Omega)$.

Remark 2.4. As was pointed out in [5, Remark 2.1], Colasuonno, Pucci, and Varga [9, Lemma 3.2], proved the compactness of Ψ' for $1 < q < p$. But the same argument can be adopted in order to show that it is also true if $q = p$.

Next, we define what is meant by a weak solution of (P_λ^f).

Definition 2.5. *A function $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of the BVP (P_λ^f) if*

$$\int_\Omega \mathbf{A}(x, \nabla u(x))\nabla v(x)dx - \lambda \int_\Omega f(x, u(x))v(x)dx = 0$$

for every $v \in W_0^{1,p}(\Omega)$.

Hence, in view of Lemmas 2.2 and 2.3, we consider the functional $\mathscr{I}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by $\mathscr{I}_\lambda = \Phi(u) - \lambda\Psi(u)$ for $\lambda > 0$ and $u \in W_0^{1,p}(\Omega)$.

Lemma 2.6. *A function $u \in W_0^{1,p}(\Omega)$ is a critical point of $\mathscr{I}_\lambda$ in $W_0^{1,p}(\Omega)$ if and only if u is a solution of (P_λ^f).*

3. Existence of One Solution

Here is our main result on the existence of one solution to the problem (P_λ^f). In what follows α' is the conjugate of α, that is, $\dfrac{1}{\alpha} + \dfrac{1}{\alpha'} = 1$.

Theorem 3.1. *Assume that f satisfies $(\mathscr{G}_{f,a,q})$ and there are sets $D \subseteq \Omega$ and $\mathcal{G} \subset D$ such that*

$$\limsup_{\xi\to 0^+} \frac{ess\inf_{x\in\mathcal{G}} F(x,\xi)}{|\xi|^p} = +\infty \tag{3.1}$$

and

$$\liminf_{\xi\to 0^+} \frac{ess\inf_{x\in D} F(x,\xi)}{|\xi|^p} > -\infty. \tag{3.2}$$

Then, for each

$$\lambda \in \Lambda = \left(0, \sup_{\theta>0} \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{\frac{(p^*-\alpha' p)}{(p^*\alpha')}} \left[M_1\theta^{1-p} + \dfrac{M_2}{q}\theta^{q-p}\right]}\right),$$

the problem (P_λ^f) admits at least one non-trivial weak solution $u \in W_0^{1,p}(\Omega)$.

Proof. Our aim is to apply Theorem 2.1(a) to the problem (P_λ^f). We introduce the functionals Φ and Ψ as given in (2.5) and (2.7), respectively. Lemmas 2.2 and 2.3 establish that Φ and Ψ are of class C^1, while condition (2.4) assures that

$$\frac{a_1}{p}\|u\|^p \le \Phi(u) \le \frac{a_2}{p}\|u\|^p \tag{3.3}$$

for every $u \in W_0^{1,p}(\Omega)$. By using the first inequality in (3.3), it follows that

$$\lim_{\|u\|\to+\infty} \Phi(u) = +\infty,$$

so Φ is coercive.

Choose λ such that

$$0 < \lambda < \sup_{\theta>0} \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{\frac{(p^*-\alpha' p)}{(p^*\alpha')}} \left[M_1\theta^{1-p} + \dfrac{M_2}{q}\theta^{q-p}\right]}.$$

Then, there exists $\bar{\theta} > 0$ such that

$$\lambda < \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{\frac{(p^*-\alpha' p)}{(p^*\alpha')}} \left[M_1\bar{\theta}^{1-p} + \dfrac{M_2}{q}\bar{\theta}^{q-p}\right]}.$$

Set

$$r = \frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\bar{\theta}^p.$$

Now since $\alpha > N/p$ implies that $1 < \alpha' < \alpha' q \leq \alpha' p < p^*$, condition $(\mathscr{G}_{f,a,q})$, Hölder's inequality, and (2.3) imply

$$\begin{aligned}\Psi(u) &\leq M_1 \int_\Omega a(x)|u(x)|dx + \frac{M_2}{q}\int_\Omega a(x)|u(x)|^q dx \\ &\leq M_1\|a\|_\alpha \|u\|_{\alpha'} + \frac{M_2}{q}\|a\|_\alpha \|u\|^q_{\alpha' q} \\ &\leq M_1\|a\|_\alpha T|\Omega|^{\frac{(p^*-\alpha')}{(p^*\alpha')}}\|u\| + \frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{\frac{(p^*-\alpha' q)}{(p^*\alpha')}}\|u\|^q \qquad (3.4)\end{aligned}$$

for every $u \in W_0^{1,p}(\Omega)$. Hence, in view of (3.3),

$$\Phi^{-1}(-\infty, r) = \left\{u \in W_0^{1,p} \ \mid\ \Phi(u) < r\right\} \subseteq \left\{u \in W_0^{1,p} \ \mid\ \|u\| \leq \left(\frac{pr}{a_1}\right)^{\frac{1}{p}}\right\},$$

so (3.4) implies that

$$\sup_{\Phi(u)\leq r}\Psi(u) \leq M_1\|a\|_\alpha T|\Omega|^{\frac{(p^*-\alpha')}{(p^*\alpha')}}\left(\frac{pr}{a_1}\right)^{\frac{1}{p}} + \frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{\frac{(p^*-\alpha' q)}{(p^*\alpha')}}\left(\frac{pr}{a_1}\right)^{\frac{q}{p}}.$$

Since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, we have

$$\begin{aligned}\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r} \\ &\leq M_1\|a\|_\alpha T|\Omega|^{\frac{(p^*-\alpha')}{(p^*\alpha')}}\left(\frac{p}{a_1}\right)^{\frac{1}{p}} r^{\frac{1-p}{p}} + \frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{\frac{(p^*-\alpha' q)}{(p^*\alpha')}}\left(\frac{p}{a_1}\right)^{\frac{q}{p}} r^{\frac{q-p}{p}} \\ &= \|a\|_\alpha \frac{p}{a_1}T^p|\Omega|^{\frac{(p^*-\alpha' p)}{(p^*\alpha')}}\left[M_1\left(\frac{T^p pr}{a_1|\Omega|^{\frac{p}{p^*}}}\right)^{\frac{1-p}{p}} + \frac{M_2}{q}\left(\frac{T^p pr}{a_1|\Omega|^{\frac{p}{p^*}}}\right)^{\frac{q-p}{p}}\right] \\ &= \|a\|_\alpha \frac{p}{a_1}T^p|\Omega|^{\frac{(p^*-\alpha' p)}{(p^*\alpha')}}\left[M_1\bar{\theta}^{1-p} + \frac{M_2}{q}\bar{\theta}^{q-p}\right] < \frac{1}{\lambda}.\end{aligned}$$

Hence, by Theorem 2.1, for every $\lambda \in \Lambda \subset \left(0, \frac{1}{\varphi(r)}\right)$ the functional $\mathscr{I}_\lambda$ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$.

To complete the proof we need to show that u_λ is nontrivial. First, we show that

$$\limsup_{\|u\|\to 0^+}\frac{\Psi(u)}{\Phi(u)} = +\infty. \qquad (3.5)$$

By our assumptions at zero, namely, (3.1) and (3.2), we can choose a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants ζ and κ with $\zeta > 0$ such that

$$\lim_{n\to+\infty} \frac{ess\inf_{x\in\mathcal{G}} F(x,\xi_n)}{|\xi_n|^p} = +\infty$$

and

$$ess\inf_{x\in D} F(x,\xi) \geq \kappa|\xi|^p,$$

for $\xi \in [0,\zeta]$. Now for the given $\mathcal{G}$ and D, take the function $v \in W_0^{1,p}(\Omega)$ such that:

(j) $v(x) \in [0,1]$ for every $x \in \Omega$,
(jj) $v(x) = 1$ for every $x \in \mathcal{G}$,
(jjj) $v(x) = 0$ for every $x \in \Omega \setminus D$.

Hence, fix $Y > 0$ and consider a positive real number η with

$$Y < \frac{\eta\,\mathrm{meas}\,(\mathcal{G}) + \kappa\int_{D\setminus\mathcal{G}} |v(x)|^p dx}{\frac{a_2}{p}\|v\|^p}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \zeta$ and

$$ess\inf_{x\in\mathcal{G}} F(x,\xi_n) \geq \eta|\xi_n|^p$$

for all $n > n_0$. Now, for every $n > n_0$, using the properties of the function v (that is $0 \leq \xi_n v(x) < \zeta$ for large n), by (3.3),

$$\begin{aligned}\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\int_{\mathcal{G}} F(x,\xi_n)dx + \int_{D\setminus\mathcal{G}} F(x,\xi_n v(x)dx}{\Phi(\xi_n v)} \\ &> \frac{\eta\ \mathrm{meas}\,(\mathcal{G}) + \kappa\int_{D\setminus\mathcal{G}} |v(x)|^p dx}{\frac{a_2}{p}\|v\|^p} > Y.\end{aligned}$$

Since Y can be arbitrarily large,

$$\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (3.5) clearly follows.

Hence, there exists a sequence $\{w_n\} \subset W_0^{1,p}(\Omega)$ strongly converging to zero with $w_n \in \Phi^{-1}(-\infty,r)$ and

$$\mathscr{I}_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of $\mathscr{I}_\lambda$ to $\Phi^{-1}(-\infty,r)$, we conclude that

$$\mathscr{I}_\lambda(u_\lambda) < 0, \tag{3.6}$$

so u_λ is not trivial. This completes the proof of the theorem. □

Remark 3.2. If in addition to the hypotheses of Theorem 3.1, we ask that

$$\sup_{\theta>0} \frac{a_1}{\|a\|_\alpha p T^p |\Omega|^{\frac{(p^*-\alpha' p)}{(p^*\alpha')}} \left[M_1\theta^{1-p} + \frac{M_2}{q}\theta^{q-p} \right]} > 1, \tag{3.7}$$

then the conclusion of the theorem holds with $\lambda = 1$, that is, for problem (P_λ^f) without a parameter.

Next we present an example to illustrate Theorem 3.1.

Example 3.3. Consider the problem

$$\begin{cases} -\mathrm{div}\mathbf{A}(x, \nabla u) = \lambda f(u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{3.8}$$

where $\Omega \subset \mathbb{R}^4$ with meas $(\Omega) = 1$ and $f(\xi) = 2\xi + 1$ for every $\xi \in \mathbb{R}$. Condition (2.6) is satisfied with $a(t) \equiv 1$, $M_1 = 1$, $M_2 = 2$, and $q = 2$. Note that $p = 2$ and $F(\xi) = \xi^2 + \xi$ for $\xi \in \mathbb{R}$. All the conditions of Theorem 3.1 are satisfied, so there exists $\rho > 0$ such that for any $\lambda \in (0, \rho)$, problem (3.8) has at least one non-trivial weak solution $u \in W_0^{1,2}(\Omega)$.

We now give some remarks on our results.

Remark 3.4. In Theorem 3.1 we searched for the critical points of the functional $\mathscr{I}_\lambda$ naturally associated with the problem (P_λ^f). We note that, in general, $\mathscr{I}_\lambda$ can be unbounded from below in $W_0^{1,p}(\Omega)$. For example, in the case where $f(\xi) = 1 + |\xi|^{\gamma-p}\xi$ for $\xi \in \mathbb{R}$ with $\gamma > p$, for any fixed $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\iota \in \mathbb{R}$, we obtain

$$\mathscr{I}_\lambda(\iota u) = \Phi(\iota u) - \lambda \int_\Omega F(\iota u(x))dx \le \iota^p \frac{a_2}{p}\|u\|^p - \lambda\iota\|u\|^p - \lambda\frac{\iota^\gamma}{\gamma}\|u\|^\gamma \to -\infty$$

as $\iota \to +\infty$. Therefore, condition (I_2) in [20, Theorem 2.2] is not satisfied. Hence, we can not use direct minimization to find the critical points of the functional $\mathscr{I}_\lambda$.

Remark 3.5. We wish to point out that the energy functional $\mathscr{I}_\lambda$ associated with the problem (P_λ^f) may not be coercive. For example, if $F(\xi) = |\xi|^s$ with $s \in (p, +\infty)$ for $\xi \in \mathbb{R}$, for any fixed $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\iota \in \mathbb{R}$ we have

$$\mathscr{I}_\lambda(\iota u) = \Phi(\iota u) - \lambda \int_\Omega F(\iota u(x))dx \le \iota^p \frac{a_2}{p}\|u\|^p - \lambda\iota^s\|u\|^s \to -\infty$$

as $\iota \to -\infty$.

Remark 3.6. From (3.6) we can easily see that the map

$$\lambda \mapsto \mathscr{I}_\lambda(u_\lambda) \text{ for } \lambda \in (0, \lambda^*) \tag{3.9}$$

is negative. Furthermore, we have

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

To see this, by considering that Φ is coercive and that for $\lambda \in (0, \lambda^*)$, the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, we have that there exists a positive constant $\mathcal{L}$ such that $\|u_\lambda\| \le \mathcal{L}$ for every

$\lambda \in (0, \lambda^*)$. It is easy to see that, since f is bounded, an application of Hölder's inequality implies there exists a positive constant $\mathcal{M}$ such that

$$\left| \int_\Omega f(x, u_\lambda(x)) u_\lambda(x) dx \right| \leq \mathcal{M} \tag{3.10}$$

for $\lambda \in (0, \lambda^*)$. Since u_λ is a critical point of $\mathscr{I}_\lambda$, we have $I'_\lambda(u_\lambda)(v) = 0$, for $v \in W_0^{1,p}(\Omega)$ and $\lambda \in (0, \lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_\Omega f(x, u_\lambda(x)) u_\lambda(x) dx \tag{3.11}$$

for $\lambda \in (0, \lambda^*)$. Then, since

$$0 \leq a_1 \|u_\lambda\|^p \leq \Phi'(u_\lambda)(u_\lambda),$$

from (3.11),

$$0 \leq a_1 \|u_\lambda\|^p \leq \Phi'(u_\lambda)(u_\lambda) \leq \lambda \int_\Omega f(x, u_\lambda(x)) u_\lambda(x) dx$$

for $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^+$, by (3.10), we have $\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$.

Finally, we show that the map

$$\lambda \mapsto \mathscr{I}_\lambda(u_\lambda)$$

is strictly decreasing in $(0, \lambda^*)$. To see this, observe that for any $u \in W_0^{1,p}(\Omega)$,

$$\mathscr{I}_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right). \tag{3.12}$$

Now, fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_1} and u_{λ_2} be global minimums of the functional $\mathscr{I}_{\lambda_i}$ restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} = \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right),$$

for $i = 1, 2$.

Since $\lambda > 0$, it is clear that (3.9) together with (3.12) imply

$$m_{\lambda_i} < 0, \text{ for } i = 1, 2. \tag{3.13}$$

Moreover,

$$m_{\lambda_2} \leq m_{\lambda_1} \tag{3.14}$$

since $0 < \lambda_1 < \lambda_2$. Then, (3.12)–(3.14) and the fact that $0 < \lambda_1 < \lambda_2$, implies

$$I_{\lambda_2}(\bar{u}_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(\bar{u}_{\lambda_1}),$$

so that the map $\lambda \mapsto \mathscr{I}_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$.

Remark 3.7. Observe that Theorem 3.1 above is a bifurcation result in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\left\{ (u_\lambda, \lambda) \in W_0^{1,p}(\Omega) \times (0, +\infty) : u_\lambda \text{ is a non-trivial weak solution of } (P_\lambda^f) \right\}$$

in $W_0^{1,p}(\Omega) \times \mathbb{R}$. Indeed, by Remark 3.6, we have that

$$\|u_\lambda\| \to 0 \quad \text{as} \quad \lambda \to 0.$$

Hence, there exist two sequences $\{u_i\}$ in $W_0^{1,p}(\Omega)$ and $\{\lambda_i\}$ in $\mathbb{R}^+$ (here $u_i = u_{\lambda_i}$) such that

$$\lambda_i \to 0^+ \quad \text{and} \quad \|u_i\| \to 0$$

as $i \to +\infty$. Moreover, we want to emphasis that due to the fact that the map

$$\lambda \mapsto \mathscr{I}_\lambda(u_\lambda),\ \lambda \in (0, \lambda^*)$$

is strictly decreasing, for every $\lambda_1,\ \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, the solutions $\bar{u}_{\lambda_1}$ and $\bar{u}_{\lambda_2}$ guaranteed by Remark 3.6 are different.

Remark 3.8. If f is non-negative, then the weak solution obtained by Theorem 3.1 is non-negative. To see this, let u_0 be a non-trivial weak solution of the problem (P_λ^f). For the sake of a contradiction, assume that the set $\mathscr{D} = \{x \in \Omega : u_0(x) < 0\}$ is non-empty and of positive measure. Set $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in \Omega$. Clearly, $\bar{v} \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega \mathbf{A}(x, \nabla u_0(x))\nabla \bar{v}(x)dx - \lambda \int_\Omega f(x, u_0(x))\bar{v}(x)dx = 0.$$

Thus, from our sign assumptions on f, we have

$$0 \leq a_1\|u_0\|_{\mathscr{D}}^p \leq \int_{\mathscr{D}} \mathbf{A}(x, \nabla u_0(x))\nabla u_0(x)dx = \lambda \int_{\mathscr{D}} f(x, u_0(x))u_0(x)dx \leq 0.$$

Hence, $u_0 = 0$ in $\mathscr{D}$, which is a contradiction.

4. Existence of Infinitely Many Solutions

In this section, we formulate our main result on the existence of infinitely many weak solutions to the problem (P_λ^f). First we introduce some additional notation. Let $\mathscr{R} : \Omega \to [0, +\infty)$ be the distance function defined by $\mathscr{R}(x) = d(x, \partial\Omega)$ for each $x \in \Omega$. Thus, for every fixed $x_0 \in \Omega$, $B(x_0, \mathscr{R}(x_0)) = \{x \in \Omega : |x - x_0| < \mathscr{R}(x_0)\} \subseteq \Omega$, and for $a \in L^\alpha(\Omega)$ with $\alpha > N/p$, we let

$$\mathcal{B}^\infty = \limsup_{\xi \to \infty} \frac{\left(\dfrac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \operatorname{essinf}_{B(x_0, \mathscr{R}(x_0)/2)} F(x, \xi)}{\dfrac{a_2}{p}\left(\dfrac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N - 1)|B(0,1)|\xi^p}.$$

Theorem 4.1. *Assume that f satisfies $(\mathscr{G}_{f,a,q})$ and there exist $x_0 \in \Omega$ and two real sequences $\{d_n\}$ and $\{b_n\}$ with*

$$\lim_{n \to \infty} b_n = \infty$$

such that:

(A_1) $f(x, t) \geq 0$ *for* $(x, t) \in B(x_0, \mathscr{R}(x_0)) \times \mathbb{R}$*;*

(A_2) $\dfrac{a_2}{p}\left(\dfrac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N - 1)|B(0,1)|d_n^p < \dfrac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p} b_n^p$ *for* $n \in \mathbb{N}$*;*

(A_3) $\mathscr{A}^\infty = \lim_{n\to\infty} \dfrac{\mathscr{K}_{b_n} - \left(\dfrac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x,d_n)}{\dfrac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p} b_n^p - \left(\dfrac{a_2}{p}\left(\dfrac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|d_n^p\right)} < \mathcal{B}^\infty$, *where*

$$\mathscr{K}_{b_n} = M_1\|a\|_\alpha T|\Omega|^{\frac{(p^*-\alpha')}{(p^*\alpha')}} \left(\frac{|\Omega|^{\frac{p}{p^*}}}{T^p} b_n^p\right)^{\frac{1}{p}} + \frac{M_2}{q}\|a\|_\alpha T^q |\Omega|^{\frac{(p^*-\alpha' q)}{(p^*\alpha')}} \left(\frac{|\Omega|^{\frac{p}{p^*}}}{T^p} b_n^p\right)^{\frac{q}{p}}.$$

Then, for each $\lambda \in \left(\dfrac{1}{\mathcal{B}^\infty}, \dfrac{1}{\mathscr{A}^\infty}\right)$, *the problem* (P_λ^f) *admits an unbounded sequence of solutions.*

Proof. Our wish here is to again apply Theorem 2.1. Consider the functionals Φ and Ψ as given in (2.5) and (2.7). The regularity assumptions on Φ and Ψ required in Theorem 2.1 are satisfied. Take

$$r_n = \frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p} b_n^p$$

for all $n \in \mathbb{N}$. We see that $r_n > 0$ for all $n \in \mathbb{N}$ and $r_n \to \infty$ as $n \to \infty$. From the definition of Φ and in view of (3.3), for every $r_n > 0$,

$$\Phi^{-1}(-\infty, r_n) = \left\{u \in W_0^{1,p} : \ \Phi(u) < r_n\right\} \subseteq \left\{u \in W_0^{1,p} : \ \|u\| \leq \left(\frac{pr_n}{a_1}\right)^{\frac{1}{p}}\right\},$$

and inequality (3.4) assures that

$$\sup_{\Phi(u)\leq r_n} \Psi(u) \leq M_1\|a\|_\alpha T|\Omega|^{\frac{(p^*-\alpha')}{(p^*\alpha')}} \left(\frac{pr_n}{a_1}\right)^{\frac{1}{p}} + \frac{M_2}{q}\|a\|_\alpha T^q|\Omega|^{\frac{(p^*-\alpha' q)}{(p^*\alpha')}} \left(\frac{pr_n}{a_1}\right)^{\frac{q}{p}}.$$

For each $n \in \mathbb{N}$, define

$$w_n(x) = \begin{cases} 0, & x \in \Omega \backslash \bar{B}(x_0, \mathscr{R}(x_0)), \\ \dfrac{2d_n}{\mathscr{R}(x_0)}(\mathscr{R}(x_0) - |x-x_0|), & x \in B(x_0,\mathscr{R}(x_0)) \backslash \bar{B}(x_0, \mathscr{R}(x_0)/2), \\ d_n, & x \in B(x_0, \mathscr{R}(x_0)/2). \end{cases} \tag{4.1}$$

Clearly, $w_n \in W_0^{1,p}(\Omega)$. A direct computation based on (3.3) shows that

$$\begin{aligned} \Phi(w_n) &\geq \frac{a_1}{p}\frac{2^p}{[\mathscr{R}(x_0)]^p}|B(x_0,\mathscr{R}(x_0)) \setminus \bar{B}(x_0,\mathscr{R}(x_0)/2)|d_n^p \\ &= \frac{a_1}{p}\frac{2^p}{[\mathscr{R}(x_0)]^p}|B(0,1)|\left([\mathscr{R}(x_0)]^N - [\mathscr{R}(x_0)/2]^N\right) d_n^p \\ &= \frac{a_1}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|d_n^p. \end{aligned}$$

Similarly,

$$\Phi(w_n) \leq \frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|d_n^p. \tag{4.2}$$

In view of condition (A_1),

$$\begin{aligned}\Psi(w_n) &= \int_\Omega F(x, w_n(x))dx \\ &\geq \int_{B(x_0,\mathscr{R}(x_0)/2)} F(x, d_n)dx \\ &\geq |B(x_0,\mathscr{R}(x_0)/2)|\mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d_n) \\ &= \left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)|\mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d_n).\end{aligned}$$

From (A_2), we have $\Phi(w_n) < r_n$. Also, $\Phi(0) = \Psi(0) = 0$. Therefore, for large n, from (A_3) and (4.2),

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r_n)}\Psi(u)) - \Psi(u)}{r_n - \Phi(u)} \\ &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_n)}\Psi(u) - \left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)|\mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d_n)}{r_n - \Phi(w_n)} \\ &\leq \frac{\mathscr{K}_{b_n} - \left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)|\mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)}F(x,d_n)}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}b_n^p - \left(\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|d_n^p\right)}.\end{aligned}$$

Hence, from (A_2), $\lambda \leq \lim_{n\to\infty}\varphi(r_n) \leq \mathscr{A}^\infty < \infty$ follows.

Next, we show that $\mathscr{I}_\lambda$ is unbounded from below. Let $\{c_n\} \to \infty$ be a sequence of positive numbers to be determined. Let $\{y_n\}$ in $W_0^{1,p}(\Omega)$ be given by

$$y_n(x) = \begin{cases} 0, & x \in \Omega\backslash\bar{B}(x_0,\mathscr{R}(x_0)), \\ \frac{2c_n}{\mathscr{R}(x_0)}(\mathscr{R}(x_0) - |x - x_0|), & x \in B(x_0,\mathscr{R}(x_0))\backslash\bar{B}(x_0,\mathscr{R}(x_0)/2), \\ c_n, & x \in B(x_0,\mathscr{R}(x_0)/2). \end{cases} \tag{4.3}$$

Then

$$\Phi(y_n) \leq \frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|c_n^p. \tag{4.4}$$

Hence,

$$\mathscr{I}_\lambda(y_n) \leq \frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|c_n^p$$

$$- \lambda \left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \text{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x, c_n).$$

Now, we consider the two cases.

If $\mathscr{B}^\infty < \infty$, then since $\lambda > \dfrac{1}{\mathscr{B}^\infty}$, we can fix $\epsilon > 0$ such that $\epsilon < \mathscr{B}^\infty - \dfrac{1}{\lambda}$. Hence, there exists $\nu_\epsilon \in \mathbb{N}$ and a positive sequence $\{h_n\} \to \infty$ such that

$$\begin{aligned} &\left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \text{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x, h_n) \\ &> (\mathscr{B}^\infty - \epsilon) \left(\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N - 1)|B(0,1)| h_n^p\right) \end{aligned}$$

for all $n > \nu_\epsilon$. Hence,

$$\begin{aligned} \mathscr{I}_\lambda(y_n) &\leq \frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N - 1)|B(0,1)| h_n^p \\ &\quad - \lambda \left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \text{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x, h_n) \\ &< \frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N - 1)|B(0,1)| h_n^p (1 - \lambda(\mathscr{B}^\infty - \epsilon)). \end{aligned}$$

Since $1 - \lambda(\mathscr{B}^\infty - \epsilon) < 0$, letting $\{h_n\}$ be the sequence $\{c_n\}$ in (4.3) and taking (4.4) into account, we have

$$\lim_{n\to\infty} \mathscr{I}_\lambda(y_n) = -\infty.$$

Now consider the situation where $\mathscr{B}^\infty = \infty$ and fix $D > \dfrac{1}{\lambda}$. There exists ν_D and a positive sequence $\{k_n\} \to \infty$ such that

$$\begin{aligned} &\left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \text{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x, k_n) \\ &> D \left(\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N - 1)|B(0,1)| k_n^p\right) \end{aligned}$$

for all $n > \nu_D$, and moreover

$$\mathscr{I}_\lambda(y_n) < \frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N - 1)|B(0,1)| k_n^p (1 - \lambda D).$$

Since $1 - \lambda D < 0$, letting $\{k_n\}$ be the sequence $\{c_n\}$ in (4.3), and arguing as before, we see that

$$\lim_{n\to\infty} \mathscr{I}_\lambda(y_n) = -\infty.$$

Hence, in both cases $\mathscr{I}_\lambda$ is unbounded from below. The conclusion of the theorem then follows from Theorem 2.1(b). $\square$

Remark 4.2. If $\{d_n\}$ and $\{b_n\}$ are two real sequences with $\lim_{n\to\infty} b_n = \infty$, such that the condition (A$_2$) in Theorem 4.1 is satisfied, then under the conditions $\mathscr{A}_\infty = 0$ and $\mathcal{B}^\infty = \infty$, Theorem 4.1 assures that for every $\lambda > 0$ the problem (P^f_λ) admits infinitely many weak solutions.

Theorem 4.3. *Assume that f satisfies $(\mathscr{G}_{f,a,q})$, condition (A$_1$) holds, and*

$$(\mathrm{A}_4)\quad \liminf_{\xi\to\infty} \frac{\mathscr{K}_\xi}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\xi^p} < \limsup_{\xi\to\infty} \frac{\left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)|\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x,\xi)}{\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|\xi^p}.$$

Then, for each

$$\lambda \in \left(\frac{\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p}(2^N-1)|B(0,1)|\xi^p}{\left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)|\operatorname{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x,\xi)}, \frac{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\xi^p}{\mathscr{K}_\xi} \right),$$

the problem (P^f_λ) has an unbounded sequence of weak solutions.

Proof. Let $\{d_n\} \equiv 0$ and choose a sequence $\{b_n\}$ of positive numbers tending to ∞ such that

$$\lim_{n\to\infty} \frac{\mathscr{K}_{b_n}}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}b_n^p} = \liminf_{\xi\to\infty} \frac{\mathscr{K}_\xi}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\xi^p}.$$

From Theorem 4.1 the conclusion follows. □

Next, we point out two simple consequences of our main results. First, from Theorem 4.1 we have the following corollary.

Corollary 4.4. *Assume that f satisfies $(\mathscr{G}_{f,a,q})$, there exist two real sequences $\{d_n\}$ and $\{b_n\}$, with $\lim_{n\to\infty} b_n = \infty$, such that (A$_1$) and (A$_2$) hold, $\mathscr{A}_\infty < 1$, and $\mathcal{B}_\infty > 1$. Then, the problem*

$$\begin{cases} -\operatorname{div}\mathbf{A}(x,\nabla u) = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P^f}$$

has an unbounded sequence of weak solutions.

The following corollary is a consequence of Theorem 4.3.

Corollary 4.5. *Assume that $\mathcal{B}_\infty > 1$ and*

$$\liminf_{\xi\to\infty} \frac{\mathscr{K}_\xi}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\xi^p} < 1.$$

Then, the problem (P^f) has an unbounded sequence of weak solutions.

Now put

$$\mathcal{B}^0 = \limsup_{\xi\to 0} \frac{\left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x,\xi)}{\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N-1)|B(0,1)|\xi^p}.$$

Arguing as in the proof of Theorem 4.1, but using conclusion (c) of Theorem 2.1 instead of (b), we can obtain the following result.

Theorem 4.6. *Assume that f satisfies $(\mathscr{G}_{f,a,q})$, (A_1) holds, and there exist two real sequences $\{s_n\}$ and $\{e_n\}$ with $\lim_{n\to\infty} e_n = 0$ such that*

(A_5) $\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N-1)|B(0,1)|s_n^p < \frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p} e_n^p$ *for* $n\in\mathbb{N}$,

(A_6) $$\mathscr{A}^0 = \lim_{n\to\infty} \frac{\mathscr{K}_{e_n} - \left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x,s_n)}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p} e_n^p - \left(\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N-1)|B(0,1)|s_n^p\right)} < \mathcal{B}^0.$$

Then, for each $\lambda \in (\lambda_3, \lambda_4)$ with $\lambda_3 := \frac{1}{\mathcal{B}^0}$ and $\lambda_4 := \frac{1}{\mathscr{A}_0}$, the problem (P_λ^f) has a sequence of pairwise distinct weak solutions that strongly converges to 0 in $W_0^{1,p}(\Omega)$.

Our final existence result is contained in the following theorem.

Theorem 4.7. *Assume that f satisfies $(\mathscr{G}_{f,a,q})$ and assume that*

(A_7) $$\liminf_{\xi\to 0^+} \frac{\mathscr{K}_\xi}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\xi^p} < \limsup_{\xi\to 0} \frac{\left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x,\xi)}{\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N-1)|B(0,1)|\xi^p}.$$

Then, for each

$$\lambda \in \left(\frac{1}{\limsup_{\xi\to 0} \frac{\left(\frac{\mathscr{R}(x_0)}{2}\right)^N |B(0,1)| \mathrm{essinf}_{B(x_0,\mathscr{R}(x_0)/2)} F(x,\xi)}{\frac{a_2}{p}\left(\frac{\mathscr{R}(x_0)}{2}\right)^{N-p} (2^N-1)|B(0,1)|\xi^p}}, \frac{1}{\liminf_{\xi\to 0^+} \frac{\mathscr{K}_\xi}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\xi^p}} \right),$$

the problem (P_λ^f) has a sequence of pairwise distinct weak solutions that strongly converges to 0 in $W_0^{1,p}(\Omega)$.

Proof. Let $\{s_n\} \equiv 0$ and choose a sequence $\{e_n\}$ of positive numbers tending to 0 such that

$$\lim_{n\to\infty} \frac{\mathscr{K}_{e_n}}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p} e_n^p} = \liminf_{\xi\to\infty} \frac{\mathscr{K}_\xi}{\frac{a_1|\Omega|^{\frac{p}{p^*}}}{pT^p}\xi^p}.$$

The conclusion follows from Theorem 4.6. □

Remark 4.8. By applying Theorem 4.6 instead of Theorem 4.1, results similar to Remark 4.2 and Corollaries 4.4 and 4.5 can be obtained.

Declarations: The authors have no competing interests to disclose.

References

[1] Afrouzi G A, Bohner M, Caristi G, Heidarkhani S, Moradi S. An existence result for impulsive multipoint boundary value systems using a local minimization principle. J. Optim. Theo. Appl. 2018; 177: 1–20.
[2] Barilla D, Bohner M, Heidarkhani S, Moradi S. Existence results for dynamic Sturm-Liouville boundary value problems via variational methods. Appl. Math. Comput. 2021; 409:125614, 10pp.
[3] Bonanno G. Some remarks on a three critical points theorem. Nonlinear Anal. 2021; 54:651–665.
[4] Bonanno G, Candito P. Infinitely many solutions for a class of discrete non-linear boundary value problems. Appl. Anal. 2009; 88: 605–616.
[5] Bonanno G, D'Aguì G, Livrea R. Triple solutions for nonlinear elliptic problems driven by a nonhomogeneous operator. Nonlinear Anal. 2020; 197; 111862, 17 pp.
[6] Bonanno G, Molica Bisci G. Infinitely many solutions for a boundary value problem with discontinuous nonlinearities. Bound. Value Probl. 2009; 2009: 670675, 20 pp.
[7] Chen C, Wang H. Ground state solutions for singular p-Laplacian equation in $\mathbb{R}^N$. J. Math. Anal. Appl. 2009; 351: 773–780.
[8] Colasuonno F, Pucci P. Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations. Nonlinear Anal. 2011; 74: 5962–5974.
[9] Colasuonno F, Pucci P. Varga R. Multiple solutions for an eigenvalue problem involving p-Laplacian type operators. Nonlinear Anal. 2012; 75: 4496–4512.
[10] Demengel F. Hebey E. On some nonlinear equations involving the p-Laplacian with critical Sobolev growth. Adv. Differ. Equ. 1998; 3: 533–574.
[11] Deng Y, Pi H. Multiple solutions for p-harmonic type equations. Nonlinear Anal. 2009; 71: 4952–4959.
[12] Galewski M, Molica Bisci G. Existence results for one-dimensional fractional equations. Math. Meth. Appl. Sci. (2016); 39: 1480–1492.
[13] Graef J R, Heidarkhani S, Kong L. Infinitely many solutions for systems of Sturm-Liouville boundary value problems. Results Math. 2014; 66: 327–341.
[14] Graef J R, Heidarkhani S, Kong L, Moradi S. On an anisotropic discrete boundary value problem of Kirchhoff type. J. Differ. Equ. Appl. 2021; 27: 1103–1119.
[15] Heidarkhani S. Infinitely many solutions for systems of n two-point boundary value Kirchhoff-type problems. Ann. Polon. Math. 2013; 107: 133–152.
[16] Heidarkhani S, Afrouzi G A, Moradi S, Caristi G, Ge B. Existence of one weak solution for $p(x)$-biharmonic equations with Navier boundary conditions. Z. Angew. Math. Phys. 2016; 67: 1–13.
[17] Heidarkhani S, Zhao Y, Caristi G, Afrouzi G A, Moradi S. Infinitely many solutions for perturbed impulsive fractional differential systems. Appl. Anal. 2017; 96: 1401–1424.
[18] Kristály A, Lisei H, Varga C. Multiple solutions for p-Laplacian type equations. Nonlinear Anal. 2008; 68: 1375–1381.
[19] Nápoli P D, Mariani M C. Mountain pass solutions to equations of p-Laplacian type. Nonlinear Anal. 2003; 54: 1205–1219.
[20] Rabinowitz P H. Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Reg. Conf. Ser. Math. 1986; 65, Amer. Math. Soc., Providence, RI.
[21] Ricceri B. A general variational principle and some of its applications. J. Comput. Appl. Math. 2000; 113: 401–410.
[22] Talenti G. Best constants in Sobolev inequality. Ann. Mat. Pura Appl. 1976; 110: 353–372.

Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA.
E-mail address: John-Graef@utc.edu

Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran
E-mail address: s.heidarkhani@razi.ac.ir

Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA.
E-mail address: Lingju-Kong@utc.edu

Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran
E-mail address: shahin.moradi86@yahoo.com