Study on the Generalized Three-Parameter Lindley Distribution

Xiaoling Xu [1], Xinyi Xu[2], Beiqing Gu[1,*], Ronghua Wang[3]

(1. School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai 201620, China

2. Facility of Art and Science, University of Toronto, M5S2E8 Toronto, Canada

3. Mathematics and Science College, Shanghai Normal University, Shanghai 200234, China)

Abstract: This paper introduces a new life distribution, the generalized three-parameter Lindley distribution, derived as a product of the inverse power law model and the generalized two-parameter Lindley distribution in a progressive stress accelerated life testing scenario. The study presents the graphical characteristics of the density function, failure rate function, mean failure rate function, and mean residual life function. Point estimates for the three parameters are provided through logarithmic transformation. The paper concludes with two practical examples demonstrating the application of this method.

Keywords: generalized three-parameter Lindley distribution, density function, failure rate function, mean failure rate function, mean residual life, inverse power law model, progressive stress accelerated life testing

MSC(2010) 62N05.

1. Introduction

The Lindley distribution was first introduced by Lindley in 1958, as documented in reference [24]. Since then, numerous scholars have conducted extensive research on both the Lindley distribution and its generalized forms, achieving significant results. These results are not exhaustively enumerated here, but are available in references [1]-[10], [13]-[24], [26]-[32], [35]-[46]. The distribution plays an essential role in the reliability studies of the stress-strength model.

Consider a non-negative continuous random variable that follows a Lindley distribution with parameter θ, denoted as $\text{Lindley}(\theta)$. The density function $f(x)$ and distribution function $F(x)$ of this distribution are defined as follows:

$$f(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\quad F(x)=1-\left(1+\frac{\theta}{\theta+1}x\right)e^{-\theta x}, x>0, \theta>0\,.$$

This article initially extends the single-parameter Lindley distribution to the generalized three-parameter Lindley distribution. Through theoretical derivation, it is demonstrated that under the inverse power law model, the life distribution in a progressive stress accelerated life testing scenario for the generalized two-parameter Lindley distribution exactly corresponds to the generalized three-parameter Lindley distribution. Furthermore, the study investigates the graphical characteristics of the density function and failure rate function of the generalized three-parameter Lindley distribution. Lastly, the paper presents a methodology for point estimation of parameters in a full-sample context and illustrates its application with practical examples.

* The corresponding author. Email: gubeiqing@suibe.edu.cn (Beiqing Gu)

2. Generalized two-parameter and three-parameter Lindley distributions

Consider a non-negative continuous random variable that follows a Lindley distribution with parameter θ, denoted as $\text{Lindley}(\theta)$. Its density function $f(x)$ can be expressed as:

$$f(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}=\left(1-\frac{1}{\theta+1}\right)\theta e^{-\theta x}+\frac{1}{\theta+1}\theta^2 xe^{-\theta x}.$$

Let $f_1(x)=\theta e^{-\theta x}$, $f_2(x)=\theta^2 xe^{-\theta x}$. It is evident that $f_1(x)$ is the density function of the Exponential distribution $\text{Exp}(\theta)$, and $f_2(x)$ is the density function of the Gamma distribution $\Gamma(2,\theta)$. Then we have $f(x)=(1-\beta)f_1(x)+\beta f_2(x)$ where $\beta=\frac{1}{\theta+1}$.

From the above, it can be seen that the single-parameter Lindley distribution can be regarded as a mixture of the Exponential distribution $\text{Exp}(\theta)$ and the Gamma distribution $\Gamma(2,\theta)$, where $\beta=\frac{1}{\theta+1}$. If the parameter β is retained $0\le\beta\le 1$ while assuming that β is independent of θ, the generalized two-parameter Lindley distribution is obtained.

Definition 2.1. A non-negative continuous random variable X is said to follow the generalized two-parameter Lindley distribution $\text{GL}(\theta,\beta)$, with its distribution function $F(x)$ and density function $f(x)$ respectively defined as:

$$F(x)=1-\left(1+\frac{\beta}{\theta}x\right)e^{-x/\theta},\ f(x)=\frac{1}{\theta}\left(1-\beta+\frac{\beta}{\theta}x\right)e^{-x/\theta},\ x>0,$$

where $0\le\beta\le 1$ is referred to as the shape parameter, and $\theta>0$ as the scale parameter.

Specifically, when $\beta=0$, the distribution function is $F(x)=1-e^{-x/\theta}$, that is, the generalized two-parameter Lindley distribution $\text{GL}(\theta,\beta)$ degenerates into the single-parameter Exponential distribution $\text{Exp}(1/\theta)$. When $\beta=1$, the distribution function is $F(x)=1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}$, which is the one-parameter Эрланга distribution. When $\theta'=\theta^{-1}$ and $\beta=\frac{1}{\theta'+1}=\frac{\theta}{\theta+1}$, the distribution function is $F(x)=1-\left(1+\frac{\theta'}{\theta'+1}x\right)e^{-\theta' x}$, which is the single-parameter Lindley distribution.

Building upon Definition 2.1, the introduction of an additional shape parameter m leads to the following generalized three-parameter Lindley distribution.

Definition 2.2. A non-negative continuous random variable X is said to follow the generalized three-parameter Lindley distribution $\mathrm{GL}(\theta,\beta,m)$, with its distribution function $F(x)$ and density function $f(x)$ respectively defined as:

$$F(x)=1-\left[1+\beta\left(\frac{x}{\theta}\right)^m\right]\exp\left[-\left(\frac{x}{\theta}\right)^m\right], x\geq 0,\theta,m>0, 0\leq\beta\leq 1,$$

$$f(x)=-\frac{\beta m x^{m-1}}{\theta^m}\exp\left[-\left(\frac{x}{\theta}\right)^m\right]+\left[1+\beta\left(\frac{x}{\theta}\right)^m\right]\frac{mx^{m-1}}{\theta^m}\exp\left[-\left(\frac{x}{\theta}\right)^m\right]$$

$$=\frac{mx^{m-1}}{\theta^m}\left[\beta\left(\frac{x}{\theta}\right)^m+1-\beta\right]\exp\left[-\left(\frac{x}{\theta}\right)^m\right].$$

In particular, when $\beta=0$, the distribution function is $F(x)=1-\exp\left[-\left(\frac{x}{\theta}\right)^m\right]$, which is the two-parameter Weibull distribution.

When $m=1$, the distribution function is $F(x)=1-\left(1+\frac{\beta}{\theta}x\right)e^{-x/\theta}$, which is the generalized two-parameter Lindley distribution.

When $m=1,\beta=0$, the distribution function is $F(x)=1-e^{-x/\theta}$, which is the Exponential distribution $\mathrm{Exp}(1/\theta)$.

When $m=1,\beta=1$, the distribution function is $F(x)=1-\left(1+\frac{x}{\theta}\right)e^{-x/\theta}$, which is the Эрланга distribution.

When $\beta=1$, the distribution function is $F(x)=1-\left[1+\left(\frac{x}{\theta}\right)^m\right]e^{-\left(\frac{x}{\theta}\right)^m}$, which is the generalized Exponential sum distribution.

When $m=1$, $\theta'=\theta^{-1}$ and $\beta=\frac{1}{\theta'+1}=\frac{\theta}{\theta+1}$, the distribution function is $F(x)=1-\left(1+\frac{\theta'}{\theta'+1}x\right)e^{-\theta' x}$, which is the single-parameter Lindley distribution.

Theorem 2.1. (1) The generalized three-parameter Lindley distribution $\mathrm{GL}(\theta,\beta,m)$ can be viewed as a mixture of the two-parameter Weibull distribution with the density function $\frac{mx^{m-1}}{\theta^m}\exp\left[-\left(\frac{x}{\theta}\right)^m\right]$ and the distribution with the density function $\frac{mx^{m-1}}{\theta^m}\left(\frac{x}{\theta}\right)^m\exp\left[-\left(\frac{x}{\theta}\right)^m\right]$.

(2) For $k>0$, the k-order moment of is $E(X^k)=\theta^k\left[\beta\Gamma\left(2+\frac{k}{m}\right)+(1-\beta)\Gamma\left(1+\frac{k}{m}\right)\right]$.

Proof. (1) By denoting

$$f_1(x)=\frac{mx^{m-1}}{\theta^m}\exp\left[-\left(\frac{x}{\theta}\right)^m\right],\ f_2(x)=\frac{mx^{m-1}}{\theta^m}\left(\frac{x}{\theta}\right)^m\exp\left[-\left(\frac{x}{\theta}\right)^m\right],$$

we have

$$f(x)=(1-\beta)\frac{mx^{m-1}}{\theta^m}\exp\left[-\left(\frac{x}{\theta}\right)^m\right]+\beta\frac{mx^{m-1}}{\theta^m}\left(\frac{x}{\theta}\right)^m\exp\left[-\left(\frac{x}{\theta}\right)^m\right]=(1-\beta)f_1(x)+\beta f_2(x).$$

It is easy to see that the generalized three-parameter Lindley distribution $\mathrm{GL}(\theta,\beta,m)$ can be regarded as a special mixed distribution.

(2) $$E(X^k)=\int_0^{+\infty}x^k\frac{mx^{m-1}}{\theta^m}\left[\beta\left(\frac{x}{\theta}\right)^m+1-\beta\right]\exp\left[-\left(\frac{x}{\theta}\right)^m\right]\mathrm{d}x$$

$$=\theta^k\int_0^{+\infty}(\beta t+1-\beta)t^{k/m}e^{-t}\mathrm{d}t=\theta^k\left[\beta\int_0^{+\infty}t^{k/m+1}e^{-t}\mathrm{d}t+(1-\beta)\int_0^{+\infty}t^{k/m}e^{-t}\mathrm{d}t\right]$$

$$=\theta^k\left[\beta\Gamma\left(2+\frac{k}{m}\right)+(1-\beta)\Gamma\left(1+\frac{k}{m}\right)\right].$$ □

In particular, its mathematical expectation and the second moment are

$$E(X)=\theta\left[\beta\Gamma\left(2+\frac{1}{m}\right)+(1-\beta)\Gamma\left(1+\frac{1}{m}\right)\right],$$

$$E(X^2)=\theta^2\left[\beta\Gamma\left(2+\frac{2}{m}\right)+(1-\beta)\Gamma\left(1+\frac{2}{m}\right)\right].$$

3. Failure mode of progressive stress accelerated test for the generalized two-parameter Lindley distribution under the inverse power law model

3.1. Basic assumptions of step-stress testing and the inverse power law model

Assumption 1. It is assumed that the product life X follows the generalized two-parameter Lindley distribution $\mathrm{GL}(\theta,\beta)$ with the shape parameter β and the scale parameter θ at any stress level V.

Assumption 2. Under various stress levels, the failure mechanism of the product remains the same. That is, the shape parameter β of the product's lifetime distribution is the same for each stress level, while the scale parameter depends on the stress level.

Assumption 3. The scale parameter θ and accelerated stress level V satisfy the inverse power law model.

The inverse power law model refers to the relationship between the scale parameter θ (in hours) and voltage (in volts) when voltage is used as the accelerated stress. This is based on physical principles and empirical summaries from experiments, which have found that for some products (such as insulating materials, capacitors, micro motors, and certain electronic devices), there is the following inverse power law relationship between the scale parameter and voltage: $\theta = \frac{1}{dV^c}$, where $d > 0$ and $c > 0$ are constants. For electronic components, physical experiments have shown that c is only related to the type of component and is independent of its specifications.

After taking the logarithm of both sides of the above equation, the parameter θ satisfies a logarithmic linear relationship: $\ln\theta = a + b\phi(V)$, where $a = -\ln d,\ b = -c$ and $\phi(V) = \ln V$ is a function of stress V.

The statistical analysis of step-stress or progressive-stress accelerated life tests is primarily based on the well-known Nelson assumptions, commonly referred to as the Cumulative Exposure (CE) model.

Assumption 4. The residual life of a product depends solely on the extent of failure that has already accumulated and the current stress level, rather than on the manner in which the failure has accumulated.

The Nelson Assumption essentially represents a form of "time scaling." That is, if a product is continuously subjected to a constant stress, the non-failed products will fail according to the distribution function under that stress, but this failure process starts from the previously accumulated failures.

Assume that under a constant stress $V_i, i = 1, 2$, the lifetime X_i of a product follows a generalized two-parameter Lindley distribution $\mathrm{GL}(\theta_i, \beta)$, with its distribution function given as:

$$F_{V_i}(x) = 1 - \left(1 + \frac{\beta}{\theta_i} x\right) e^{-x/\theta_i}, x > 0, \beta > 0, \theta_i > 0, i = 1, 2.$$

Based on the Nelson assumption $F_{V_1}(x_1) = F_{V_2}(x_2)$, that is,

$$1 - \left(1 + \frac{\beta}{\theta_1} x_1\right) e^{-x_1/\theta_1} = 1 - \left(1 + \frac{\beta}{\theta_2} x_2\right) e^{-x_2/\theta_2}.$$

From which we derive $\frac{x_1}{\theta_1} = \frac{x_2}{\theta_2}$. This is equivalent to $x_1 = \frac{\theta_1}{\theta_2} x_2 = \left(\frac{V_2}{V_1}\right)^c x_2$.

The above equation can be interpreted as: The duration x_2 for which a product operates under stress V_2 is equivalent to the time $x_1 = \left(\frac{V_2}{V_1}\right)^c x_2$ it operates under stress V_1.

3.2. Failure modes under progressive stress ($V(x) = Kx$) in accelerated life testing with the inverse power law model

The statistical analysis of generalized two-parameter Lindley distribution progressive stress accelerated life tests (abbreviated as progressive stress tests) under the inverse power law model is also based on the aforementioned four fundamental assumptions.

First, consider the general progressive stress $V(x) = Kx + V_1, V_1 > 0$ accelerated life testing. It is assumed that under a given stress V_1, the life distribution of a product follows a generalized two-parameter Lindley fatigue life distribution $\mathrm{GL}(\theta_1, \beta)$, and the scale parameter θ_1 conforms to an inverse power law model $\theta_1 = \frac{1}{dV_1^c}$.

According to reference [34], the duration x for which a product operates under a given stress level $V(x) = Kx + V_1, V_1 > 0$ is equivalent to the operational time under a constant stress level V_1, which can be expressed as:

$$\int_0^x \frac{(Kt+V_1)^c}{V_1^c} \mathrm{d}t = \frac{K^c}{V_1^c} \int_0^x \left(t + \frac{V_1}{K}\right)^c \mathrm{d}t = \frac{K^c}{V_1^c} \frac{1}{c+1} \left[\left(x + \frac{V_1}{K}\right)^{c+1} - \left(\frac{V_1}{K}\right)^{c+1}\right]$$

$$= \frac{1}{K(c+1)} \frac{(Kx+V_1)^{c+1} - V_1^{c+1}}{V_1^c}.$$

Therefore, the life distribution of the product under progressive stress $V(x) = Kx + V_1, V_1 > 0$ is

$$F_{V(x)}(x) = 1 - \left\{1 + \beta \frac{d}{K(c+1)} \left[(Kx+V_1)^{c+1} - V_1^{c+1}\right]\right\} \exp\left\{-\frac{d}{K(c+1)} \left[(Kx+V_1)^{c+1} - V_1^{c+1}\right]\right\}.$$

Specifically, when $V_1 = 0$, the life distribution of the product under progressive stress $V(x) = Kx$ is as follows:

$$F_{V(x)}(x)=1-\left(1+\beta\frac{dK^{c}x^{c+1}}{c+1}\right)\exp\left(-\frac{dK^{c}x^{c+1}}{c+1}\right).$$

By denoting $\theta=\left(\frac{c+1}{dK^{c}}\right)^{1/(c+1)}$, $m=c+1$, it is observed that:

$$F(x)=1-\left[1+\beta\left(\frac{x}{\theta}\right)^{m}\right]\exp\left[-\left(\frac{x}{\theta}\right)^{m}\right],$$

which is the generalized three-parameter Lindley distribution.

Note: While the requirement here is $m>1$, in fact, it can be extended to $m>0$.

4. The graphical characteristics of the generalized three-parameter Lindley distribution

Theorem 4.1. Assume that a non-negative random variable X follows the three-parameter generalized Lindley distribution $\mathrm{GL}(\theta,\beta,m)$. Then, its density function $f(x)$ exhibits the following graphical characteristics:

(1)When $0<m\le\frac{1}{3}$, $f(x)$ is strictly monotonically decreasing. (2)When $\frac{1}{3}<m<\frac{1}{2}$, $f(x)$ is strictly monotonically decreasing.(3)When $\frac{1}{2}\le m\le 1$ and $\beta_0=\frac{m(m+1)+2m\sqrt{m(1-m)}}{5m^2-2m+1}$,

①If $\beta\le\frac{m}{3m-1}$, $f(x)$ is strictly monotonically decreasing. ②If $\frac{m}{3m-1}<\beta\le\beta_0$, $f(x)$ is strictly monotonically decreasing. ③If $\beta_0<\beta<1$, $f(x)$ first strictly monotonically decreases, then increases, and eventually decreases again.(4)When $m>1$, $f(x)$ exhibits an "inverted bathtub" shape.

Proof. $f(0)=+\infty,\ f(+\infty)=0$ for $0<m<1$,

$$f(0)=\frac{1-\beta}{\theta},\ f(+\infty)=0 \quad \text{for} \quad m=1,$$

$$f(0)=0,\ f(+\infty)=0 \quad \text{for} \quad m>1,$$

$$f'(x)=\frac{m}{\theta^{m}}\frac{x^{2m-2}}{\theta^{m}}\exp\left[-\left(\frac{x}{\theta}\right)^{m}\right]\left\{(m-1)\frac{\theta^{m}}{x^{m}}\left[\beta\left(\frac{x}{\theta}\right)^{m}+1-\beta\right]+\beta m-m\left[\beta\left(\frac{x}{\theta}\right)^{m}+1-\beta\right]\right\}.$$

By denoting $t=\left(\frac{x}{\theta}\right)^{m}$, we have

$$f'(x)=\frac{m}{\theta^m}\frac{x^{2m-2}}{\theta^m}\exp\left[-\left(\frac{x}{\theta}\right)^m\right]\left[(m-1)t^{-1}(\beta t+1-\beta)+\beta m-m(\beta t+1-\beta)\right]$$

$$=\frac{m}{\theta^m}\frac{x^{2m-2}}{\theta^m}t^{-1}\exp\left[-\left(\frac{x}{\theta}\right)^m\right]\left[-m\beta t^2+(3\beta m-\beta-m)t+(m-1)(1-\beta)\right].$$

Define the function as $g(t)=-m\beta t^2+(3\beta m-\beta-m)t+(m-1)(1-\beta), t\ge 0$.

$$g(0)=(m-1)(1-\beta), g(+\infty)=-\infty,$$

$$\Delta=(3\beta m-\beta-m)^2+4m\beta(m-1)(1-\beta)=5\beta^2m^2-2\beta^2m-2\beta m^2+\beta^2+m^2-2\beta m.$$

(1)When $m\le\frac{1}{3}$, $3\beta m-\beta-m\le 0$, $g(t)<0, f'(x)<0$, implying $f(x)$ is strictly monotonically decreasing.

(2) When $\frac{1}{3}<m<\frac{1}{2}$, at this point $\frac{m}{3m-1}>1, 3m-1<m, (3m-1)\beta<m, 3\beta m-\beta-m<0$, $g(t)<0$, $f'(x)<0$, indicating $f(x)$ is strictly monotonically decreasing.

(3) When $\frac{1}{2}\le m\le 1$, if $\beta\le\frac{m}{3m-1}$, at this point $\frac{m}{3m-1}\le 1$, then $3\beta m-\beta-m\le 0$, $g(t)<0$, $f'(x)<0$, which means $f(x)$ is strictly monotonically decreasing.

When $\frac{1}{2}\le m\le 1$, and $\beta>\frac{m}{3m-1}$, it is evident that $\frac{1}{2}\le\frac{m}{3m-1}\le 1$, at this point, $3\beta m-\beta-m>0$.

Define the function as $h(m,\beta)=5\beta^2m^2-2m\beta^2-2m^2\beta-2m\beta+\beta^2+m^2$.

$$\frac{\partial h(m,\beta)}{\partial\beta}=10m^2\beta-4m\beta-2m^2-2m+2\beta=2(5m^2\beta-2m\beta-m^2-m+\beta).$$

Define the function as $h_1(m,\beta)=5m^2\beta-2m\beta-m^2-m+\beta$.

$$\frac{\partial h_1(m,\beta)}{\partial\beta}=5m^2-2m+1>0,$$

$$\beta>\frac{m}{3m-1}, h_1\left(m,\frac{m}{3m-1}\right)=5m^2\frac{m}{3m-1}-2m\frac{m}{3m-1}-m^2-m+\frac{m}{3m-1}=\frac{2m(m-1)^2}{3m-1}>0.$$

Then $h_1(m,\beta) > h_1\left(m,\dfrac{m}{3m-1}\right) > 0$, indicating that $h(m,\beta)$ is strictly monotonically increasing for β .

$$h\left(m,\frac{m}{3m-1}\right)=\frac{4m^2(2m-1)(m-1)}{(3m-1)^2}<0,\ \ h(m,1)=(2m-1)^2\geq 0.$$

Moreover $h(m,\beta)=(5m^2-2m+1)\beta^2-2m(m+1)\beta+m^2$.

Solving for the root β_0 of β from the equation $h(m,\beta)=0$.

$$\Delta = 4m^2(m+1)^2-4m^2(5m^2-2m+1)=16m^3(1-m)\geq 0.$$

If we choose $\beta_0=\dfrac{m(m+1)-2m\sqrt{m(1-m)}}{5m^2-2m+1}$, since it is required that $\beta_0>\dfrac{m}{3m-1}$, then

$$\frac{m(m+1)-2m\sqrt{m(1-m)}}{5m^2-2m+1}>\frac{m}{3m-1},$$

which means

$$\frac{(m+1)-2\sqrt{m(1-m)}}{5m^2-2m+1}>\frac{1}{3m-1},(m+1)(3m-1)-2(3m-1)\sqrt{m(1-m)}>5m^2-2m+1,$$

$$3m^2+2m-1-2(3m-1)\sqrt{m(1-m)}>5m^2-2m+1,-2(3m-1)\sqrt{m(1-m)}>2(m-1)^2,$$

which is a contradiction.

Therefore, β_0 should be taken as $\beta_0=\dfrac{m(m+1)+2m\sqrt{m(1-m)}}{5m^2-2m+1}$, and $\dfrac{m}{3m-1}<\beta_0<1$.

Hence, when $\dfrac{1}{2}\leq m\leq 1$, and $\dfrac{m}{3m-1}<\beta<\beta_0$, $h(m,\beta)<0$, then $g(t)<0$, $f'(x)<0$, indicating that $f(x)$ is strictly monotonically decreasing.

When $\dfrac{1}{2}\leq m\leq 1$, and $\beta_0<\beta<1$, $h(m,\beta)>0$, then due to the symmetry axis of the equation $g(t)=0$ is $\dfrac{3\beta m-\beta-m}{2m\beta}>0$, it follows that equation $g(t)=0$ has two positive real roots $t_1,t_2,t_1<t_2$, i.e., when $t<t_1$, $g(t)<0$, $f'(x)<0$, $f(x)$ is strictly monotonically decreasing, when $t_1<t<t_2$, $g(t)>0$, $f'(x)>0$, $f(x)$ is strictly monotonically increasing, when $t > t_2$, $g(t) < 0$, $f'(x) < 0$, $f(x)$ is strictly monotonically decreasing.

(4) When $m > 1, g(0) > 0$, there exists t_0, when $t < t_0$, $g(t) > 0$, $f'(x) > 0$, $f(x)$ is strictly monotonically increasing, when $t > t_0$, $g(t) < 0$, $f'(x) < 0$, $f(x)$ is strictly monotonically decreasing. Thus, $f(x)$ exhibits an "inverted bathtub" shape. □

Note: The graph of β_0 as a function of m is shown in Figure 1 below, where $\beta_0 = \frac{m(m+1) + 2m\sqrt{m(1-m)}}{5m^2 - 2m + 1}, \frac{1}{2} \le m \le 1$.

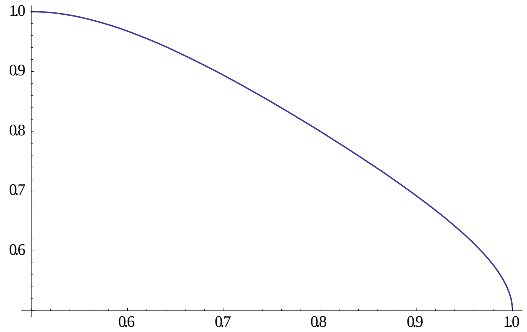

Figure 1. Graph of β_0 variation with respect to m $(0.5 \le m \le 1)$

Set the scale parameter as $\theta = 1$, and for different combinations of the parameters m, β, the graphical representations of the density function $f(x)$ are illustrated in Figures 2 to 7.

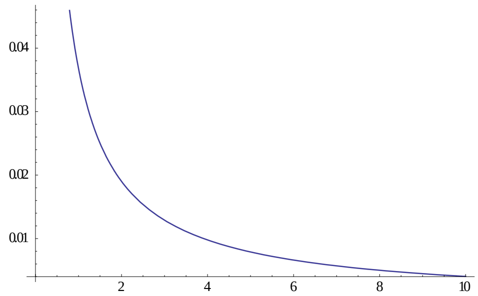

Figure 2. Graph of the density function with $m = 0.1, \beta = 0.5, \theta = 1$

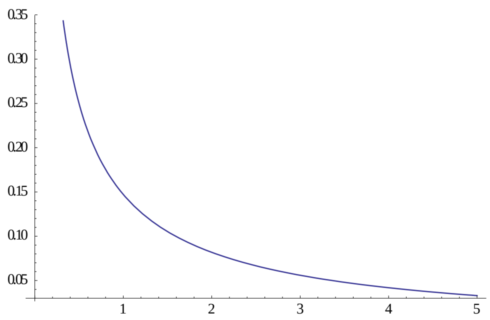

Figure 3. Graph of the density function with $m = 0.4, \beta = 0.5, \theta = 1$

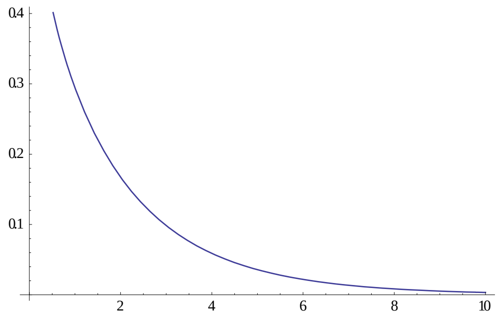

Figure 4. Graph of the density function with $m = 0.8, \beta = 0.5, \theta = 1$ ($\beta_0 = 0.8, \dfrac{m}{3m-1} = \dfrac{4}{7}$)

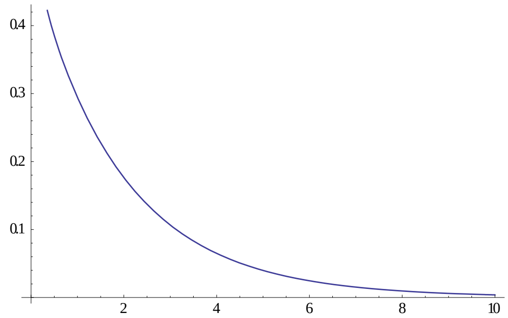

Figure 5. Graph of the density function with $m = 0.8, \beta = 0.6, \theta = 1$ ($\beta_0 = 0.8, \dfrac{m}{3m-1} = \dfrac{4}{7}$)

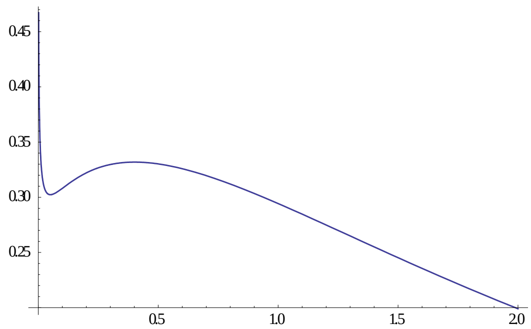

Figure 6. Graph of the density function with $m = 0.8, \beta = 0.85, \theta = 1$ ($\beta_0 = 0.8, \dfrac{m}{3m-1} = \dfrac{4}{7}$)

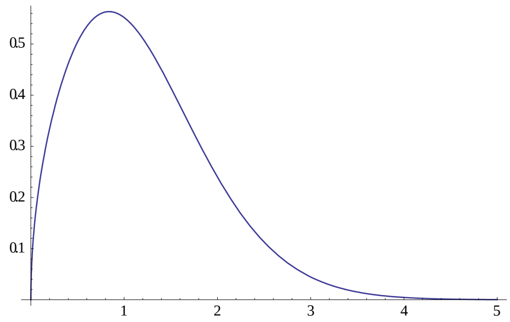

Figure 7. Graph of the density function with $m=1.5, \beta=0.5, \theta=1$

Theorem 4.2. Assume a non-negative random variable X follows a three-parameter generalized Lindley distribution $\mathrm{GL}(\theta, \beta, m)$. The failure rate function $\lambda(x)$ then exhibits the following graphical characteristics:

(1) When $m \le \frac{1}{2}$, $\lambda(x)$ is strictly monotonically decreasing.

(2) When $\frac{1}{2} < m < 1$, ①If $2m-2+\beta \le 0$, $\lambda(x)$ is strictly monotonically decreasing. ②If $2m-2+\beta > 0, \beta = 1$, $\lambda(x)$ exhibits an "inverted bathtub" shape. ③ If $2m-2+\beta > 0, 2(1-m) < \beta < 1, \beta \le 4m(1-m)$, $\lambda(x)$ is strictly monotonically decreasing. ④If $2m-2+\beta > 0, 4m(1-m) < \beta < 1$, $\lambda(x)$ initially decreases strictly monotonically, then increases, and finally decreases again.

(3) When $m \ge 1$, $\lambda(x)$ is strictly monotonically increasing.

Proof. The failure rate function is defined as $\lambda(x) = \frac{m}{\theta^m} \frac{x^{m-1}\left[\beta\left(\frac{x}{\theta}\right)^m + 1 - \beta\right]}{1+\beta\left(\frac{x}{\theta}\right)^m}$.

When $0 < m < 1$, $\lambda(0) = +\infty, \lambda(+\infty) = 0$.

When $m = 1$, $\lambda(0) = \frac{1-\beta}{\theta}, \lambda(+\infty) = \frac{1}{\theta}$.

When $m > 1$, $\lambda(0) = 0, \lambda(+\infty) = +\infty$.

$$\lambda'(x) = \frac{m}{\theta^m} \frac{x^{2m-2}}{\theta^m}\left[1+\beta\left(\frac{x}{\theta}\right)^m\right]^{-2} \left\{(m-1)\frac{\theta^m}{x^m}\left[\beta\left(\frac{x}{\theta}\right)^m + 1 - \beta\right]\left[1+\beta\left(\frac{x}{\theta}\right)^m\right]\right.$$

$$+\beta m\left[1+\beta\left(\frac{x}{\theta}\right)^m\right]-\beta m\left[\beta\left(\frac{x}{\theta}\right)^m+1-\beta\right]\Bigg\}.$$

By denoting $t=\left(\frac{x}{\theta}\right)^m$, we have

$$\lambda'(x)=\frac{mx^{2m-2}}{\theta^{2m}}\frac{1}{t(1+\beta t)^2}\left[\beta^2(m-1)t^2+\beta(2m-2+\beta)t+(1-\beta)(m-1)\right].$$

Define the function as $g(t)=\beta^2(m-1)t^2+\beta(2m-2+\beta)t+(1-\beta)(m-1), t>0$.

$$g(0)=(1-\beta)(m-1),\quad g(+\infty)=\begin{cases}+\infty, & m>1\\ +\infty, & m=1,\\ -\infty, & m<1\end{cases}$$

$$\Delta=\beta^2[2(m-1)+\beta]^2-4\beta^2(1-\beta)(m-1)^2=\beta^3\left[4(m-1)^2+4(m-1)+\beta\right].$$

(1) When $m\le\frac{1}{2}$, $g(t)<0$, $\lambda(x)$ is strictly monotonically decreasing.

(2) When $\frac{1}{2}<m<1$, if $2m-2+\beta\le 0$, that is $\beta\le 2(1-m)$, $g(t)<0$, $\lambda(x)$ is strictly monotonically decreasing.

When $\frac{1}{2}<m<1$, if $2m-2+\beta>0$, that is $2(1-m)<\beta\le 1$.

①If $\beta=1$, at this point

$$g(t)=(m-1)t^2+(2m-1)t=t[(m-1)t+(2m-1)]=t(1-m)\left(-t+\frac{2m-1}{1-m}\right).$$

Let $t_0=\frac{2m-1}{1-m}$. When $t<t_0$, $g(t)>0$, $\lambda'(x)>0$; when $t>t_0$, $g(t)<0$, $\lambda'(x)<0$, indicating $\lambda(x)$ first strictly monotonically increases then decreases, forming an "inverted bathtub" shape.

②If $2(1-m)<\beta<1$, $\Delta=\beta^3[4m(m-1)+\beta]=\beta^3[\beta-4m(1-m)]$.

If $\beta\le 4m(1-m)$, $\Delta\le 0$, $g(t)<0$, $\lambda'(x)<0$, $\lambda(x)$ is strictly monotonically decreasing.

If $4m(1-m)<\beta<1$, $\Delta>0$, there exists t_1,t_2, $0<t_1<t_2$, such that $g(t_1)=g(t_2)=0$.

When $t < t_1$, $g(t) < 0$, $\lambda'(x) < 0$, when $t_1 < t < t_2$, $g(t) > 0$, $\lambda'(x) > 0$, when $t > t_2$, $g(t) < 0$, $\lambda'(x) < 0$, indicating $\lambda(x)$ first strictly monotonically decreases, then increases, and finally decreases again.

(3) When $m \geq 1$, $g(t) > 0$, $\lambda(x)$ is strictly monotonically increasing. □

Set the scale parameter as $\theta = 1$, and for different combinations of the parameters m, β, the graphical representations of the failure rate function $\lambda(x)$ are illustrated in Figures 8 to 13.

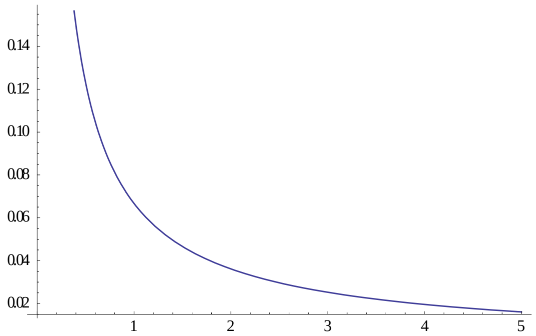

Figure 8. Graph of the failure rate function $\lambda(x)$ with $m = 0.1, \beta = 0.5, \theta = 1$

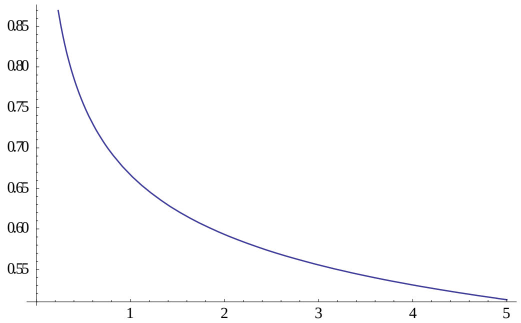

Figure 9. Graph of the failure rate function $\lambda(x)$ with $m = 0.8, \beta = 0.2, \theta = 1$

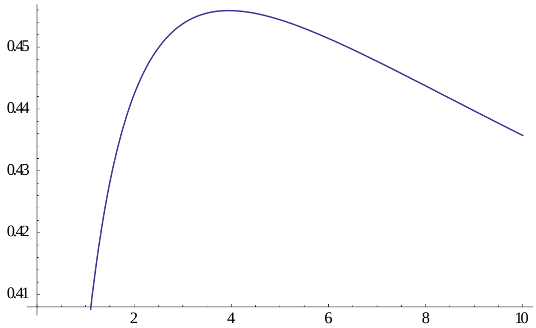

Figure 10. Graph of the failure rate function $\lambda(x)$ with $m = 0.8, \beta = 1, \theta = 1$

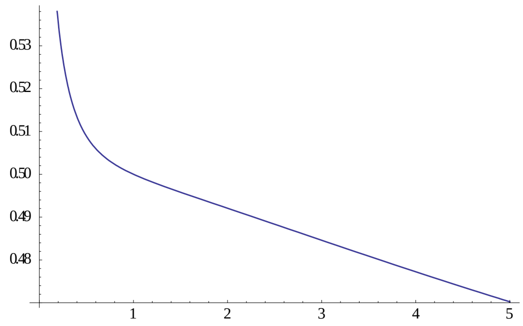

Figure 11. Graph of the failure rate function $\lambda(x)$ **with** $m=0.8, \beta=0.6, \theta=1$

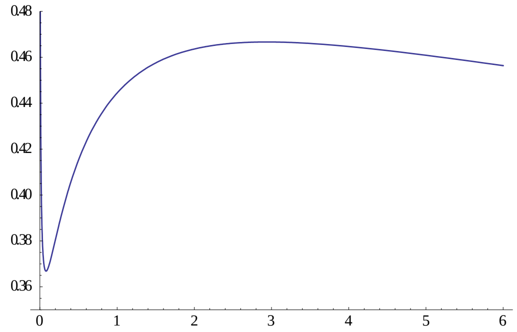

Figure 12. Graph of the failure rate function $\lambda(x)$ **with** $m=0.8, \beta=0.8, \theta=1$

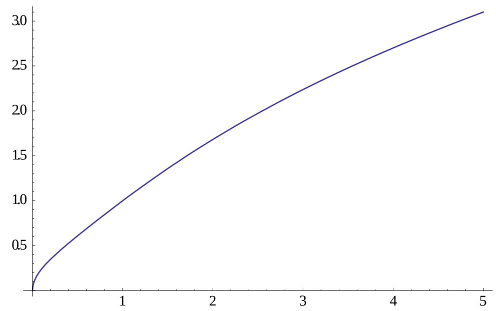

Figure 13. Graph of the failure rate function $\lambda(x)$ **with** $m=1.5, \beta=0.5, \theta=1$

Theorem 4.3. Assume that a non-negative random variable X follows a three-parameter generalized Lindley distribution $\mathrm{GL}(\theta,\beta,m)$. Then, the mean failure rate function $\bar{\lambda}(x)$ exhibits the following graphical characteristics:

(1)When $m \le \frac{1}{2}$, $\bar{\lambda}(x)$ is strictly monotonically decreasing. (2)When $\frac{1}{2} < m < 1$, ①If $2(m-1)+\beta \le 0$, $\bar{\lambda}(x)$ is strictly monotonically decreasing. ②If $2(m-1)+\beta > 0$, denote β_0 as the root of the equation $1-m-\beta m+\beta \ln \frac{\beta}{2(1-m)} = 0$,（i）If $2(1-m) < \beta \le \beta_0$, $\bar{\lambda}(x)$ is strictly monotonically decreasing.（ii）If $\beta_0 < \beta < 1$, the graph of $\bar{\lambda}(x)$ can exhibit two possible behaviors: <1>strictly monotonically decreasing, <2>initially strictly monotonically decreasing, then increasing, and finally decreasing again. ③If $\beta = 1$, $\bar{\lambda}(x)$ initially increases strictly monotonically and then decreases. (3)When $m \geq 1$, $\bar{\lambda}(x)$ is strictly monotonically increasing.

Proof. The mean failure rate is defined as $\bar{\lambda}(x) = -\frac{1}{x}\left\{\ln\left[1+\beta\left(\frac{x}{\theta}\right)^m\right] - \left(\frac{x}{\theta}\right)^m\right\}$.

By denoting $t = \left(\frac{x}{\theta}\right)^m$, we have $\bar{\lambda}(x) = -\frac{\ln(1+\beta t) - t}{\theta t^{1/m}}$, and $\bar{\lambda}(0) = \frac{m}{\theta}\lim_{t\to 0}\frac{\beta t + 1 - \beta}{t^{(1-m)/m} + \beta t^{1/m}}$.

When $m < 1$, $$\bar{\lambda}(0) = \frac{m}{\theta}\lim_{t\to 0}\frac{\beta t + 1 - \beta}{t^{(1-m)/m} + \beta t^{1/m}} = +\infty .$$

When $m = 1$, $$\lambda(0) = \frac{1}{\theta}\lim_{t\to 0}\frac{t - \ln(1+\beta t)}{t} = \frac{1}{\theta}\lim_{t\to 0}\left[1 - \frac{\ln(1+\beta t)}{t}\right] = \frac{1-\beta}{\theta}.$$

When $m > 1$, $$\bar{\lambda}(0) = 0 .$$

$$\bar{\lambda}(+\infty) = \frac{m}{\theta}\lim_{t\to +\infty}\frac{\beta t + 1 - \beta}{t^{(1-m)/m} + \beta t^{1/m}} .$$

When $m < 1$, $$\bar{\lambda}(+\infty) = 0 .$$

When $m = 1$, $$\bar{\lambda}(+\infty) = \frac{1}{\theta}\lim_{t\to 0}\frac{t - \ln(1+\beta t)}{t} = \frac{1}{\theta}\lim_{t\to 0}\left[1 - \frac{\ln(1+\beta t)}{t}\right] = \frac{1}{\theta}.$$

When $m > 1$, $$\bar{\lambda}(+\infty) = +\infty .$$

$$\bar{\lambda}'(x) = \frac{x^{m-1}}{\theta^{m+1}} t^{-1/m-1}\frac{1}{1+\beta t}\left[-m\beta t + (m-1)(1+\beta t)t + (1+\beta t)\ln(1+\beta t)\right].$$

Define the function as $g(t) = -m\beta t + (m-1)(1+\beta t)t + (1+\beta t)\ln(1+\beta t), t > 0$,

then we have $$g(t) = -m\beta t + mt + m\beta^2 t - t - \beta t^2 + (1+\beta t)\ln(1+\beta t) .$$

$$g(0) = 0, g(+\infty) = \begin{cases} +\infty, & m > 1 \\ +\infty, & m = 1. \\ -\infty, & m < 1 \end{cases}$$

In fact, when $m > 1$, $g(t) = t^2\left[\beta(m-1) + (m - m\beta - 1)\frac{1}{t} + \frac{(1+\beta t)\ln(1+\beta t)}{t^2}\right]$.

$$\lim_{t\to+\infty}\frac{(1+\beta t)\ln(1+\beta t)}{t^2}=\beta^2\lim_{y\to+\infty}\frac{(1+y)\ln(1+y)}{y^2}=\beta^2\lim_{y\to+\infty}\frac{\ln(1+y)+1}{2y}=0.$$

Thus, it follows that: $g(+\infty)=+\infty$.

When $m=1$, $g(t)=-\beta t+(1+\beta t)\ln(1+\beta t)=\beta t\left[\frac{(1+\beta t)\ln(1+\beta t)}{\beta t}-1\right]$.

$$\lim_{t\to+\infty}\frac{(1+\beta t)\ln(1+\beta t)}{\beta t}=\lim_{y\to+\infty}\frac{(1+y)\ln(1+y)}{y}=\lim_{y\to+\infty}[\ln(1+y)-1]=+\infty.$$

We have $g(+\infty)=+\infty$.

When $m<1$, $g(+\infty)=-\infty$.

$$g'(t)=-\beta m+(m-1)(1+\beta t)+(m-1)\beta t+\beta+\beta\ln(1+\beta t)$$

$$=2(m-1)\beta t+(m-1)(1-\beta)+\beta\ln(1+\beta t).$$

Define the function as $g_1(t)=2(m-1)\beta t+(m-1)(1-\beta)+\beta\ln(1+\beta t),t>0$.

$$g_1(0)=(m-1)(1-\beta)=\begin{cases}>0, & m>1\\ =0, & m=1\\ <0, & m<1\end{cases},\quad g_1(+\infty)=\begin{cases}+\infty, & m\ge 1\\ -\infty, & m<1\end{cases},$$

$$g_1'(t)=2(m-1)\beta+\frac{\beta^2}{1+\beta t}.$$

Define the function as $g_2(t)=2(m-1)\beta+\frac{\beta^2}{1+\beta t},t>0$.

$$g_2(0)=\beta\left[2(m-1)+\beta\right],g_2(+\infty)=2(m-1)\beta,g_2'(t)=-\frac{\beta^3}{(1+\beta t)^2}<0.$$

(1)When $m\le\frac{1}{2}$, $2(1-m)>1,2(m-1)+\beta<0$, $g_2(t)<0,g_1'(t)<0$, $g_1(t)<0,g'(t)<0$, $g(t)<0$, $\overline{\lambda}(x)$ is strictly monotonically decreasing.

(2)When $\frac{1}{2}<m<1$,

①If $2(m-1)+\beta\le 0$, that is $\beta\le 2(1-m)<1$, $g_2(0)\le 0,g_2(t)<0$, $g_1'(t)<0,g_1(t)<0$,

$g'(t) < 0, g(t) < 0$, $\overline{\lambda}(x)$ is strictly monotonically decreasing.

②If $2(m-1)+\beta > 0$, that is $2(1-m) < \beta < 1$, there exists t_2, $g_2(t_2) = 0$, if $t < t_2$, $g_2(t) > 0, g_1'(t) > 0$; if $t > t_2$, $g_2(t) < 0, g_1'(t) < 0$.

$$2(m-1)(1+\beta t)+\beta = 0, 2(m-1)+2(m-1)\beta t+\beta = 0, t_2 = \frac{2(m-1)+\beta}{2(1-m)\beta},$$

Then when $t = t_2$, $g_1(t_2)$ is the maximum value of $g_1(t)$:

$$g_1(t_2) = 2(m-1)\beta\frac{2(m-1)+\beta}{2(1-m)\beta}+(m-1)(1-\beta)+\beta\ln\left[1+\frac{2(m-1)+\beta}{2(1-m)\beta}\right]$$

$$= 1-m-\beta m+\beta\ln\frac{\beta}{2(1-m)}.$$

Define the function as $h(\beta) = 1-m-\beta m+\beta\ln\frac{\beta}{2(1-m)}, 2(1-m) \le \beta \le 1$.

$$h'(\beta) = -m+\ln\frac{\beta}{2(1-m)}+1 > 0,$$

$$h(2(1-m)) = 1-m-2(1-m)m = 2m^2-3m+1 = (2m-1)(m-1) < 0,$$

$$h(1) = 1-m-m-\ln[2(1-m)] = 1-\ln 2-2m-\ln(1-m).$$

Define the function as $h_1(m) = 1-\ln 2-2m-\ln(1-m), \frac{1}{2} < m < 1$.

$$h_1'(m) = -2+\frac{1}{1-m} = \frac{2m-1}{1-m} > 0, h_1(0.5) = 0, h_1(m) > 0, h(1) > 0.$$

Then, there exists β_0 such that $h(\beta_0) = 0$, when $2(1-m) < \beta < \beta_0$, $h(\beta) < 0$, when $\beta_0 < \beta \le 1$, $h(\beta) > 0$.

(i) When $2(1-m) < \beta \le \beta_0$, $h(\beta) \le 0, g_1(t_2) \le 0, g_1(t) \le 0$, $g'(t) < 0, g(t) < 0$, $\overline{\lambda}(x)$ is strictly monotonically decreasing. (ii) When $\beta_0 < \beta < 1$, $h(\beta) > 0, g_1(t_2) > 0$, there exists $t_{11}, t_{12}, t_{11} < t_{12}$ with $g_1(t_{11}) = g_1(t_{12}) = 0$.

When $t < t_{11}$, $g_1(t) < 0, g'(t) < 0$, when $t_{11} < t < t_{12}$, $g_1(t) > 0, g'(t) > 0$, when $t > t_{12}$,

$g_1(t) < 0, g'(t) < 0$.

It is clear that, <1> if $g(t) \le 0$, then $\bar{\lambda}(x)$ is strictly monotonically decreasing, <2> if there exists t_{01}, t_{02}, $g(t_{01}) = g(t_{02}) = 0$, when $t < t_{01}$, $g(t) < 0$; when $t_{01} < t < t_{02}$, $g(t) > 0$; when $t > t_{02}$, $g(t) < 0$, then $\bar{\lambda}(x)$ is "first strictly monotonically decreasing then increasing and again decreasing".

③Especially when $\beta = 1$, and $\frac{1}{2} < m < 1$, at this time

$$g(t) = -mt + mt + mt^2 - t - t^2 + (1+t)\ln(1+t) = mt^2 - t - t^2 + (1+t)\ln(1+t),$$

$$g(0) = 0, g(+\infty) = -\infty, g'(t) = 2mt - 1 - 2t + 1 + \ln(1+t) = 2mt - 2t + \ln(1+t).$$

Define the function as $g_1(t) = 2mt - 2t + \ln(1+t), t > 0$.

$$g_1(0) = 0, g_1(+\infty) = -\infty, g_1'(t) = 2m - 2 + \frac{1}{1+t} = 2(m-1) + \frac{1}{1+t}.$$

Denote $t_2 = \frac{2m-1}{2(1-m)}$, then $g_1'(t_2) = 0$, when $t < t_2$, $g_1'(t) > 0$, when $t > t_2$, $g_1'(t) < 0$.

There exists $t_1, g_1(t_1) = 0$,when $t < t_1, g_1(t) > 0, g'(t) > 0$, when $t > t_1, g_1(t) < 0, g'(t) < 0$.

There exists t_0, $g(t_0) = 0$, when $t < t_0$, $g(t) > 0$, when $t > t_0$, $g(t) < 0$, then $\bar{\lambda}(x)$ first strictly monotonically increases then decreases.

(3)When $m \ge 1$, $g_1'(t) > 0, g_1(t) > 0, g'(t) > 0, g(t) > 0$, $\bar{\lambda}(x)$ is strictly monotonically increasing. □

Set the scale parameter as $\theta = 1$, and for different combinations of the parameters m, β, the graphical representations of the mean failure rate function $\bar{\lambda}(x)$ are illustrated in Figures 14 to 23.

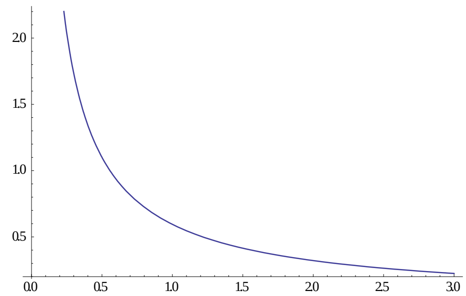

Figure 14. Graph of the Mean Failure Rate Function $\bar{\lambda}(x)$ **with** $m = 0.1, \beta = 0.5, \theta = 1$

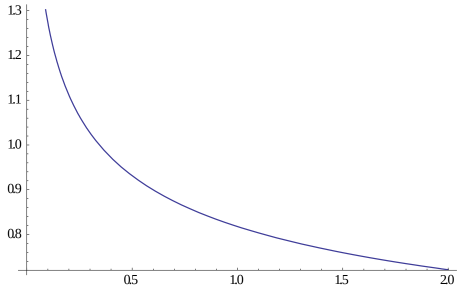

Figure 15. Graph of the Mean Failure Rate Function $\bar{\lambda}(x)$ **with** $m = 0.8, \beta = 0.2, \theta = 1$

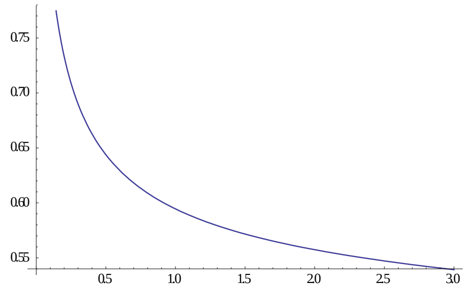

Figure 16. Graph of the Mean Failure Rate Function $\bar{\lambda}(x)$ **with** $m = 0.8, \beta = 0.5, \theta = 1$

($\beta_0 = 0.656402$)

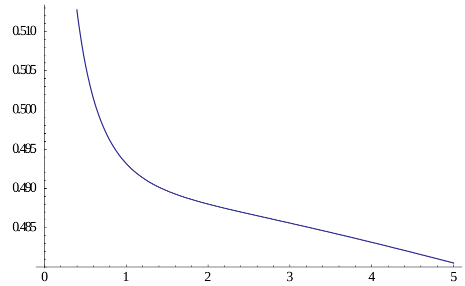

Figure 17. Graph of the Mean Failure Rate Function $\bar{\lambda}(x)$ **with** $m = 0.8, \beta = 0.66, \theta = 1$

($\beta_0 = 0.656402$)

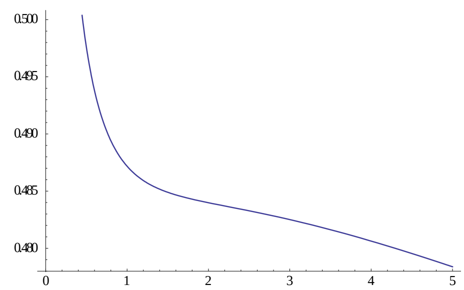

Figure 18. Graph of the Mean Failure Rate Function $\bar{\lambda}(x)$ **with** $m = 0.8, \beta = 0.67, \theta = 1$

($\beta_0 = 0.656402$)

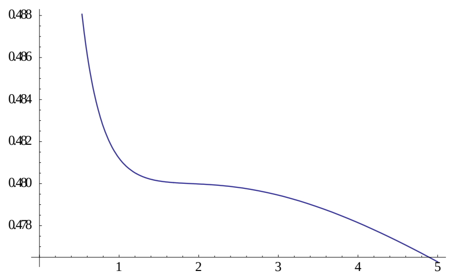

Figure 19. Graph of the Mean Failure Rate Function $\overline{\lambda}(x)$ **with** $m = 0.8, \beta = 0.68, \theta = 1$

($\beta_0 = 0.656402$)

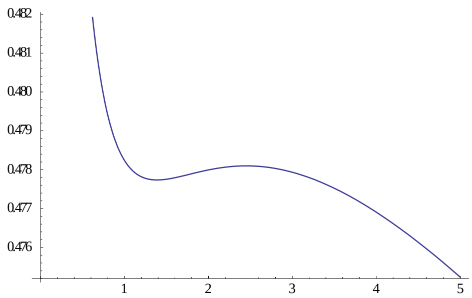

Figure 20. Graph of the Mean Failure Rate Function $\overline{\lambda}(x)$ **with** $m = 0.8, \beta = 0.685, \theta = 1$

($\beta_0 = 0.656402$)

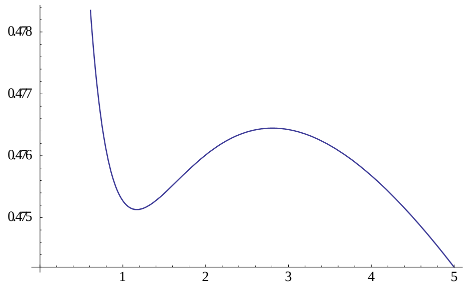

Figure 21. Graph of the Mean Failure Rate Function $\overline{\lambda}(x)$ **with** $m = 0.8, \beta = 0.69, \theta = 1$

($\beta_0 = 0.656402$)

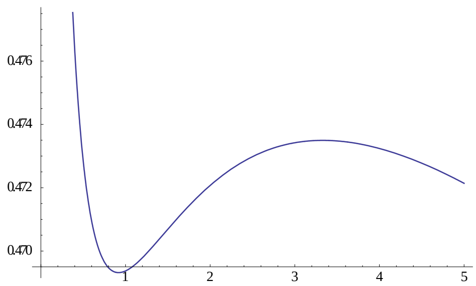

Figure 22. Graph of the Mean Failure Rate Function $\overline{\lambda}(x)$ **with** $m = 0.8, \beta = 0.7, \theta = 1$

($\beta_0 = 0.656402$)

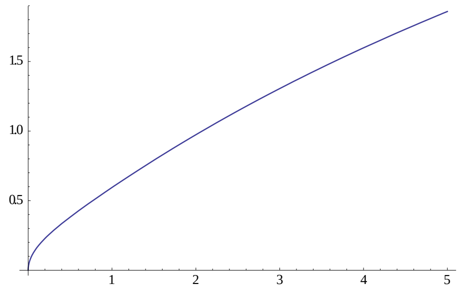

Figure 23. Graph of the Mean Failure Rate Function $\overline{\lambda}(x)$ **with** $m=1.5, \beta=0.5, \theta=1$

Theorem 4.4. Assume a non-negative random variable X follows a three-parameter generalized Lindley distribution $\mathrm{GL}(\theta,\beta,m)$. Then, the mean residual life $M(x)$ exhibits the following graphical characteristics:

(1)When $m<1$, ①If $2(m-1)+\beta \le 0$, <1> If $m \le \frac{1}{2}$, $M(x)$ is strictly monotonically increasing. <2> If $\frac{1}{2}<m<1$, $M(x)$ is strictly monotonically increasing. ② If $2(m-1)+\beta>0$, <1> If $m<\frac{1}{2}$, this condition does not exist. <2> If $\frac{1}{2} \le m<1$, (i)If $\beta \le 4(1-m)m$, $M(x)$ is strictly monotonically increasing. (ii)If $4(1-m)m<\beta<1$, $M(x)$ may strictly monotonically increase, or it may initially increase, then decrease, and increase again. (iii)If $\beta=1$, $M(x)$ initially decreases strictly monotonically, then increases.

(2) When $m \ge 1$, $M(x)$ is strictly monotonically decreasing.

Proof. The mean residual life is defined as:

$$M(x)=\frac{\int_x^{+\infty}[1-F(y)]\mathrm{d}y}{1-F(x)}=\frac{\int_{\left(\frac{x}{\theta}\right)^m}^{+\infty}\frac{(1+\beta z)\theta}{m}e^{-z}z^{1/m-1}\mathrm{d}z}{\left[1+\beta\left(\frac{x}{\theta}\right)^m\right]\exp\left[-\left(\frac{x}{\theta}\right)^m\right]}.$$

By denoting $t=\left(\frac{x}{\theta}\right)^\beta$, we have $M(x)=\frac{\theta}{m}\frac{\int_t^{+\infty}(1+\beta z)e^{-z}z^{1/m-1}\mathrm{d}z}{(1+\beta t)e^{-t}}$, $M(0)=E(X)$,

$$M(+\infty)=\lim_{x\to+\infty}\frac{1-F(x)}{f(x)}=\frac{\theta^m}{m}\lim_{x\to+\infty}\frac{1+\beta\left(\frac{x}{\theta}\right)^m}{x^{m-1}\left[\beta\left(\frac{x}{\theta}\right)^m+1-\beta\right]}=\frac{\theta^m}{m}\lim_{x\to+\infty}\frac{1+\beta t}{\theta^{m-1}t^{1-1/m}(\beta t+1-\beta)}$$

$$=\frac{\theta}{m}\lim_{x\to+\infty}\frac{1+\beta t}{t^{1-1/m}(\beta t+1-\beta)}=\begin{cases}0, & m>1\\ \theta, & m=1\\ +\infty, & m<1\end{cases}.$$

$$M'(x)=\left(\frac{x}{\theta}\right)^{m-1}\frac{\beta t+1-\beta}{(1+\beta t)^2 e^{-t}}\left\{\int_t^{+\infty}(1+\beta z)z^{1/m-1}e^{-z}\mathrm{d}z-\frac{(1+\beta t)^2 t^{1/m-1}e^{-t}}{\beta t+1-\beta}\right\}.$$

Define the function as $g(t)=\int_t^{+\infty}(1+\beta z)z^{1/m-1}e^{-z}\mathrm{d}z-\frac{(1+\beta t)^2 t^{1/m-1}e^{-t}}{\beta t+1-\beta}, t>0$.

$$g'(t)=\frac{1+\beta t}{m}t^{1/m-2}e^{-t}\left[(m-1)(1+\beta t)^2+\beta(1+\beta t)-\beta m\right].$$

$$g(+\infty)=0,\quad g(0)=\int_0^{+\infty}(1+\beta z)z^{1/m-1}e^{-z}\mathrm{d}z-\lim_{t\to 0}\frac{(1+\beta t)^2 t^{1/m-1}e^{-t}}{\beta t+1-\beta}$$

$$=\begin{cases}-\infty, & m>1\\ \int_0^{+\infty}(1+\beta z)e^{-z}\mathrm{d}z-\frac{1}{1-\beta}=1+\beta-\frac{1}{1-\beta}=-\frac{\beta^2}{1-\beta}, & m=1\\ \int_0^{+\infty}(1+\beta z)z^{1/m-1}e^{-z}\mathrm{d}z, & m>1\end{cases}.$$

Define the function as $g_1(t)=(m-1)(1+\beta t)^2+\beta(1+\beta t)-\beta m, t>0$.

$$g_1(0)=m-1+\beta-\beta m=(m-1)(1-\beta)=\begin{cases}>0, & m>1\\ =0, & m=1\\ <0, & m<1\end{cases},\quad g_1(+\infty)=\begin{cases}+\infty, & m>1\\ +\infty, & m=1\\ -\infty, & m<1\end{cases},$$

$$g_1'(t)=2\beta(m-1)(1+\beta t)+\beta^2.$$

(1) When $m<1$, let $g_1'(t)=0$, that is

$$2(m-1)+2\beta(m-1)t+\beta=0, 2\beta(1-m)t=2(m-1)+\beta,\quad g_1'(0)=\beta\left[2(m-1)+\beta\right].$$

① If $2(m-1)+\beta\le 0$, that is $\beta\le 2(1-m)$.

<1> For $m\le\frac{1}{2}$, that is $2(1-m)\ge 1$, then $2(m-1)+\beta\le 0$, furthermore $g_1'(t)<0, g_1(t)<0$,

$g'(t)<0, g(t)>0$, $M(x)$ is strictly monotonically increasing.

<2> If $\frac{1}{2}<m<1$, $\beta\le 2(1-m)$, $g_1'(t)<0, g_1(t)<0, g'(t)<0, g(t)>0$, then $M(x)$ is

strictly monotonically increasing.

② If $2(m-1)+\beta>0$, that is $\beta>2(1-m)$.

<1> If $m<\frac{1}{2}$, $\beta>2(1-m)>1$, which is in contradiction with $0\le\beta\le1$.

<2> If $\frac{1}{2}\le m<1$, $\beta\ge 2(1-m)$, i.e. $2(1-m)\le\beta\le1$, denote $t_2=\frac{2(m-1)+\beta}{2\beta(1-m)}$ at this point. When $t<t_2$, $g_1'(t)>0$; when $t>t_2$, $g_1'(t)<0$, i.e. $g_1(t_2)$ is the maximum value of $g_1(t)$.

$$g_1(t_2)=(m-1)\left[1+\frac{2(m-1)+\beta}{2(1-m)}\right]^2+\beta\left[1+\frac{2(m-1)+\beta}{2(1-m)}\right]-\beta m=\frac{\beta}{4(1-m)}\left[\beta-4(1-m)m\right].$$

(i) If $\beta\le4(1-m)m$, $g_1(t)\le0, g'(t)<0, g(t)>0$, $M(x)$ is strictly monotonically increasing.

(ii) If $4(1-m)m<\beta<1$, there exists $t_{11},t_{12},t_{11}<t_{12}$, $g_1(t_{11})=g_1(t_{12})=0$, when $t<t_{11}$, $g_1(t)<0$, $g'(t)<0$; when $t_{11}<t<t_{12}$, $g_1(t)>0$, $g'(t)>0$; when $t>t_{12}$, $g_1(t)<0$, $g'(t)<0$, then $M(x)$ may strictly monotonically increase or first increase, then decrease, and increase again.

(iii)If $\beta=1$, since $g_1(0)=0$, there exists t_1, $g_1(t_1)=0$, when $t<t_1$, $g_1(t)>0, g'(t)>0$; when $t>t_1$, $g_1(t)<0, g'(t)<0$, then $M(x)$ first strictly monotonically decreases, then increases.

(2) When $m\ge1$, $g_1'(t)>0, g_1(t)>0, g'(t)>0, g(t)<0$, $M(x)$ is strictly monotonically decreasing. □

Set the scale parameter as $\theta=1$, and for different combinations of the parameters m,β, the graphical representations of the mean residual life $M(x)$ are illustrated in Figures 24 to 30.

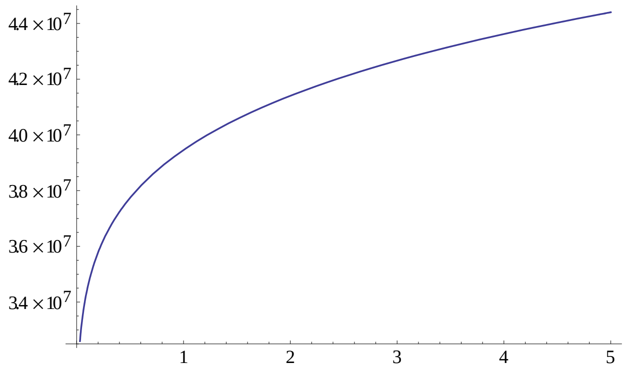

Figure 24. Graph of the mean residual function $M(x)$ **with** $m=0.1, \beta=0.5, \theta=1$

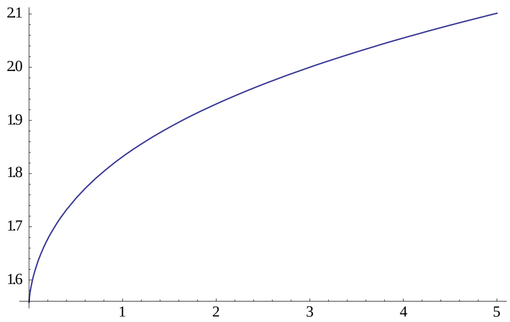

Figure 25. Graph of the mean residual function $M(x)$ **with** $m=0.8, \beta=0.3, \theta=1$

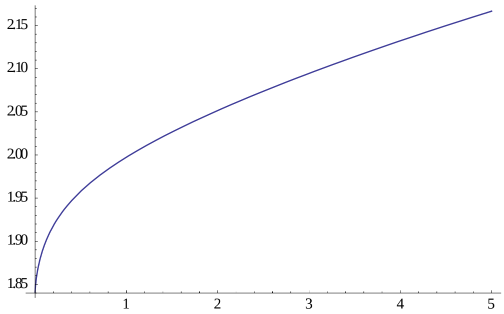

Figure 26. Graph of the mean residual function $M(x)$ **with** $m=0.8, \beta=0.5, \theta=1$

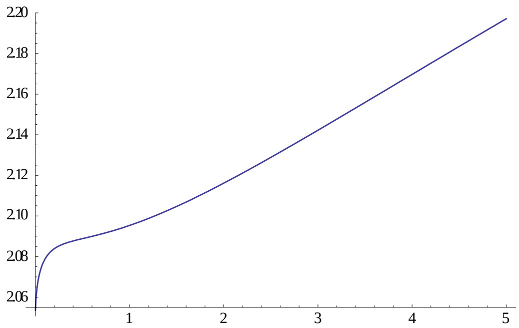

Figure 27. Graph of the mean residual function $M(x)$ **with** $m=0.8, \beta=0.65, \theta=1$

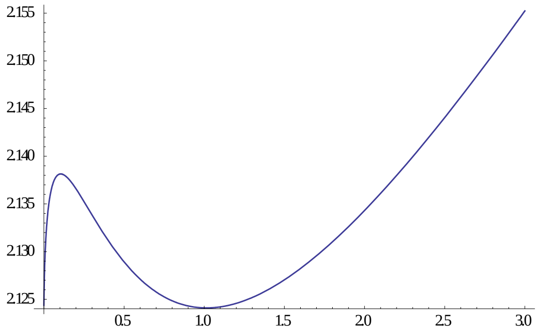

Figure 28. Graph of the mean residual function $M(x)$ **with** $m = 0.8, \beta = 0.7, \theta = 1$

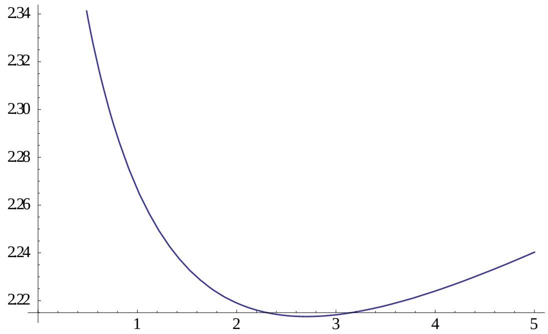

Figure 29. Graph of the mean residual function $M(x)$ **with** $m = 0.8, \beta = 1, \theta = 1$

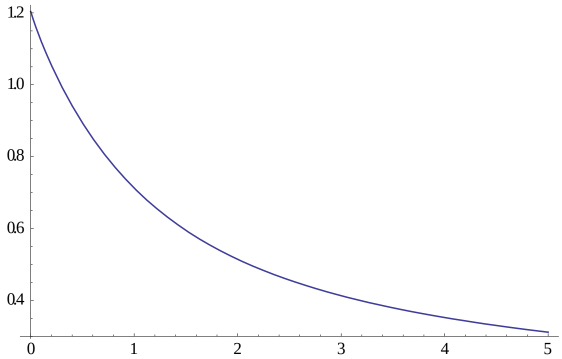

Figure 30. Graph of the mean residual function $M(x)$ **with** $m = 1.5, \beta = 0.5, \theta = 1$

5. Estimation of parameters for the generalized three-parameter Lindley distribution in a full sample context

Let $X_1, X_2, \cdots, X_n$ be a simple random sample of size n from a population following a three-parameter generalized Lindley distribution $X \sim \mathrm{GL}(\theta, \beta, m)$, with sample observations denoted as $x_1, x_2, \cdots, x_n$ and ordered observations as $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$.

Due to the complexity of the distribution function, density function, and higher-order moments of the generalized three-parameter Lindley distribution $\mathrm{GL}(\theta, \beta, m)$, the conventional methods of maximum likelihood estimation and moment estimation involve solving very complex transcendental equations. It is theoretically difficult to study their existence and uniqueness, hence the necessity of finding new methods for parameter estimation.

Let $Y = \ln X$, $Y_i = \ln X_i, y_i = \ln x_i, i = 1, 2, \cdots, 3$, and denote $\mu = \ln\theta, \sigma = \frac{1}{m}$.

For $-\infty < y < +\infty$, the distribution function of Y is

$$F_Y(y) = P(\ln X \le y) = P\left(X \le e^y\right)$$

$$= 1 - \left[1 + \beta\left(\frac{e^y}{\theta}\right)^m\right]\exp\left[-\left(\frac{e^y}{\theta}\right)^m\right] = 1 - \left[1 + \beta\exp\left(\frac{y-\mu}{\sigma}\right)\right]\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right].$$

Let $Z = \frac{\ln X - \mu}{\sigma}$ for $-\infty < z < +\infty$, and then the distribution function of Z is

$$F_Z(z) = 1 - \left(1 + \beta e^z\right)e^{-e^z},\ f_Z(z) = \left(1 + \beta e^z - \beta\right)e^z e^{-e^z}.$$

Lemma 5.1. Suppose the distribution function $F_Z(z)$ and density function $f_Z(z)$ of a random variable Z are respectively:

$$F_Z(z) = 1 - \left(1 + \beta e^z\right)e^{-e^z},\quad f_Z(z) = \left[(1-\beta) + \beta e^z\right]e^z e^{-e^z}.$$

Then $E(Z) = a_1 + \beta$, $E(Z^2) = a_2 + c_2\beta$, $E(Z^3) = a_3 + c_3\beta$, $E(Z^4) = a_4 + c_4\beta$,

$$D(Z) = -\beta^2 + b_2,\ E\left[Z - E(Z)\right]^3 = 2\beta^3 + b_3,\ E\left[Z - E(Z)\right]^4 = -3\beta^4 - 6b_2\beta^2 + b_4,$$

where $a_1 = -0.577216$, $a_2 = 1.97811$, $a_3 = -5.44487$, $a_4 = 23.5615$,

$c_1 = 1$, $c_2 = -1.15443$, $c_3 = 5.93434$, $c_4 = -21.7795$,

$b_2 = a_2 - a_1^2 = 1.64493$, $b_3 = a_3 - 3a_1a_2 + 2a_1^3 = -2.40411$,

$b_4 = a_4 - 4a_3a_1 + 6a_2a_1^2 - 3a_1^4 = 14.6114$.

Proof. It is evident that the k-th moment of Z is

$$E(Z^k) = \int_{-\infty}^{+\infty} z^k\left[(1-\beta) + \beta e^z\right]e^z e^{-e^z}\mathrm{d}z = \int_0^{+\infty}[(1-\beta) + \beta t](\ln t)^k e^{-t}\mathrm{d}t$$

$$= (1-\beta)\int_0^{+\infty}(\ln t)^k e^{-t}\mathrm{d}t + \beta\int_0^{+\infty} t(\ln t)^k e^{-t}\mathrm{d}t$$

$$= \int_0^{+\infty}(\ln t)^k e^{-t}\mathrm{d}t + \beta\left[\int_0^{+\infty} t(\ln t)^k e^{-t}\mathrm{d}t - \int_0^{+\infty}(\ln t)^k e^{-t}\mathrm{d}t\right].$$

Let $a_k = \int_0^{+\infty}(\ln t)^k e^{-t}\mathrm{d}t$, $c_k = \int_0^{+\infty} t(\ln t)^k e^{-t}\mathrm{d}t - \int_0^{+\infty}(\ln t)^k e^{-t}\mathrm{d}t$.

Considering: $\lim_{t\to 0}(\ln t)^k e^{-t} = \lim_{t\to 0}(t+1)(\ln t)^k e^{-t} = 0, \lim_{t\to+\infty}(\ln t)^k e^{-t} = \lim_{t\to+\infty}(t+1)(\ln t)^k e^{-t} = 0$,

$$a_k = \int_0^{+\infty} (\ln t)^k e^{-t} \mathrm{d}t = -(\ln t)^k e^{-t}\Big|_0^{+\infty} + k\int_0^{+\infty} \frac{1}{t}(\ln t)^{k-1} e^{-t}\mathrm{d}t = k\int_0^{+\infty} \frac{1}{t}(\ln t)^{k-1} e^{-t}\mathrm{d}t,$$

$$c_k = \int_0^{+\infty} t(\ln t)^k e^{-t}\mathrm{d}t - a_k = -(t+1)(\ln t)^k e^{-t}\Big|_0^{+\infty} + k\int_0^{+\infty} \frac{t+1}{t}(\ln t)^{k-1} e^{-t}\mathrm{d}t - a_k$$

$$= k\int_0^{+\infty} (\ln t)^{k-1} e^{-t}\mathrm{d}t + k\int_0^{+\infty} \frac{1}{t}(\ln t)^{k-1} e^{-t}\mathrm{d}t - k\int_0^{+\infty} \frac{1}{t}(\ln t)^{k-1} e^{-t}\mathrm{d}t$$

$$= k\int_0^{+\infty} (\ln t)^{k-1} e^{-t}\mathrm{d}t = ka_{k-1}.$$

By calculation, we have $a_1 = -0.577216$, $a_2 = 1.97811$, $a_3 = -5.44487$, $a_4 = 23.5615$,

$$c_1 = 1,\ c_2 = -1.15443,\ c_3 = 5.93434,\ c_4 = -21.7795.$$

Let $b_2 = a_2 - a_1^2 = 1.64493$, $b_3 = a_3 - 3a_1a_2 + 2a_1^3 = -2.40411$,

$$b_4 = a_4 - 4a_3a_1 + 6a_2a_1^2 - 3a_1^4 = 14.6114.$$

Then we have $E(Z) = a_1 + \beta$, $E(Z^2) = a_2 + c_2\beta$, $E(Z^3) = a_3 + c_3\beta$, $E(Z^4) = a_4 + c_4\beta$,

$$D(Z) = E[Z - E(Z)]^2 = a_2 + c_2\beta - (a_1+\beta)^2 = -\beta^2 + (c_2 - 2a_1)\beta + a_2 - a_1^2$$

$$= -\beta^2 + b_2 = -\beta^2 + 1.64493,$$

$$E\left[Z - E(Z)\right]^3 = E(Z^3) - 3E(Z^2)E(Z) + 2[E(Z)]^3$$

$$= a_3 + c_3\beta - 3(a_2 + c_2\beta)(a_1+\beta) + 2(a_1+\beta)^3 = 2\beta^3 + b_3 = 2\beta^3 - 2.40411,$$

$$E\left[Z - E(Z)\right]^4 = E(Z^4) - 4E(Z^3)E(Z) + 6E(Z^3)[E(Z)]^2 - 3[E(Z)]^4$$

$$= a_4 + c_4\beta - 4(a_3 + c_3\beta)(a_1+\beta) + 6(a_2 + c_2\beta)(a_1^2 + 2a_1\beta + \beta^2) - 3(a_1^2 + 2a_1\beta + \beta^2)^2$$

$$= -3\beta^4 - 6(a_2 - a_1^2)\beta^2 + b_4 = -3\beta^4 - 6b_2\beta^2 + b_4 = -3\beta^4 - 9.86958\beta^2 + 14.6114. \quad \square$$

Furthermore, given that $Y = \mu + \sigma Z$, it can be deduced from the aforementioned lemma that:

$$E(Y) = \mu + \sigma E(Z) = \mu + \sigma(a_1 + \beta),\ D(Y) = \sigma^2 D(Z).$$

Let $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i, S_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$, and thus a system of moment equations can be established as follows:

$$\begin{cases} \overline{Y} = \mu + \sigma(a_1 + \beta) \\ S_Y^2 = \sigma^2(b_2 - \beta^2) \end{cases}.$$

If the shape parameter β is known, then the point estimates of the parameters σ, m are:

$$\hat{\sigma} = \sqrt{\frac{S_Y^2}{b_2 - \beta^2}},\ \hat{m} = \sqrt{\frac{b_2 - \beta^2}{S_Y^2}}.$$

And the point estimates for the parameters μ, θ are:

$$\hat{\mu} = \overline{Y} - \hat{\sigma}(a_1 + \beta),\ \hat{\theta} = \exp\left\{\overline{Y} - \frac{a_1 + \beta}{\hat{m}}\right\}.$$

Since $\hat{m}, \hat{\theta}$ depend on the shape parameter β, it can be denoted as $\hat{m}(\beta), \hat{\theta}(\beta)$.

Let $F(x_{(j)}) = 1 - \left[1 + \beta\left(\frac{x_{(j)}}{\hat{\theta}(\beta)}\right)^{\hat{m}(\beta)}\right]\exp\left[-\left(\frac{x_{(j)}}{\hat{\theta}(\beta)}\right)^{\hat{m}(\beta)}\right], j = 1, 2, \cdots, n$ be a function of the shape parameter β.

Define $Q(\beta) = \sum_{j=1}^{n}\left|F(x_{(j)}) - \frac{j}{n}\right|$, and considering the range $[0,1]$ of the shape parameter β, set a step size 0.00001, denote $\beta_j = 0.00001j$, $j = 0, 1, 2, \cdots, 10^5$ and calculate the values of $Q(\beta_j), j = 0, 1, 2, \cdots, 10^5$, taking its minimum value. The corresponding β_j can then be considered as the point estimate of the parameter β, denoted as $\hat{\beta}$.

Note: The step size mentioned above can be determined according to the required computational precision and is not necessarily 0.00001.

Consequently, the point estimates of the parameters m, θ can be obtained as:

$$\hat{m} = \sqrt{\frac{b_2 - \hat{\beta}^2}{S_Y^2}},\ \hat{\theta} = \exp\left\{\overline{Y} - \frac{a_1 + \hat{\beta}}{\hat{m}}\right\}.$$

6. Case Study Analysis

Case 6.1. Reference [18] provides data on the waiting time (in minutes) of 100 customers waiting for service at a bank:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6,
4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3,
5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6,

7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Reference [11], through the K-S test, considers the data to originate from a single-parameter Lindley distribution. In the full sample context, the point estimate of parameter θ is calculated as $\hat{\theta} = 0.1866$, and its distribution function is shown in Figure 31.

Utilizing the method proposed in this paper, the data is fitted with the generalized three-parameter Lindley distribution, resulting in point estimates for the three parameters as $\hat{\beta} = 0.99999$, $\hat{m} = 1.02945$, and $\hat{\theta} = 5.005$. The distribution function is illustrated in Figure 31.

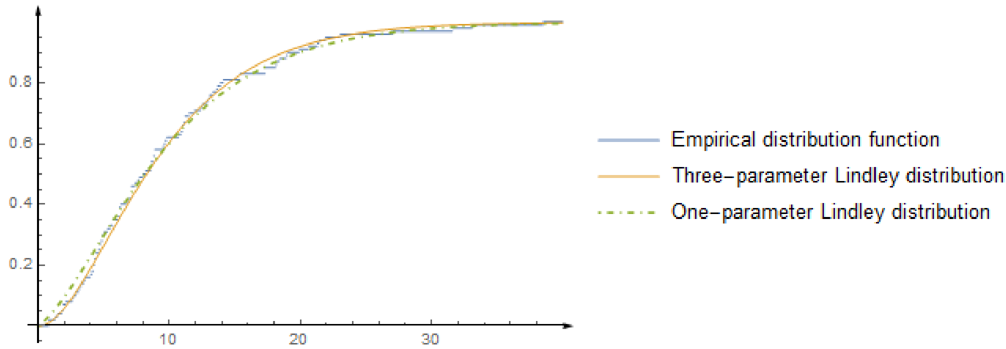

Figure 31. Empirical distribution function and corresponding theoretical distribution function for case study 6.1

Observing Figure 31, it is evident that both the single-parameter Lindley distribution and the generalized three-parameter Lindley distribution fit the batch of data well.

Case 6.2. In reference [25], through 47 observations during the maintenance process of a certain model of tank, the on-site observed values for the primary level preventive maintenance and secondary level upkeep time were obtained as follows (unit: hours):

0.80,	1.00,	1.00,	1.41,	1.50,	1.50,	1.50,	2.00,	2.00,	2.00
2.00,	2.50,	2.50,	2.75,	3.20,	3.30,	3.70,	3.80,	3.80,	4.00
4.00,	4.00,	4.00,	4.00,	4.00,	4.10,	5.00,	5.00,	5.50,	5.50
5.50,	6.00,	6.50,	7.00,	7.16,	7.75,	8.00,	8.00,	9.50,	9.73
10.00,	11.40,	12.00,	12.00,	14.00,	15.21,	15.50			

Reference [12], through fitting tests, arrived at the following two conclusions: (1) It is considered that the primary level preventive maintenance and secondary level upkeep time for this model of tank follows an Erlang distribution with parameter $\hat{\theta} = 2.7299$. (2) It is believed that the primary level preventive maintenance and secondary level upkeep time for this model of tank follows a Lindley distribution with parameter $\hat{\theta} = 0.3217$. The corresponding distribution functions are shown in Figure 32.

Using the method presented in this paper, the data is fitted with the generalized three-parameter Lindley distribution, resulting in point estimates for the three parameters as

$\hat{\beta} = 0.99999$, $\hat{m} = 1.06297$, $\hat{\theta} = 2.823$, The distribution function is illustrated in Figure 32.

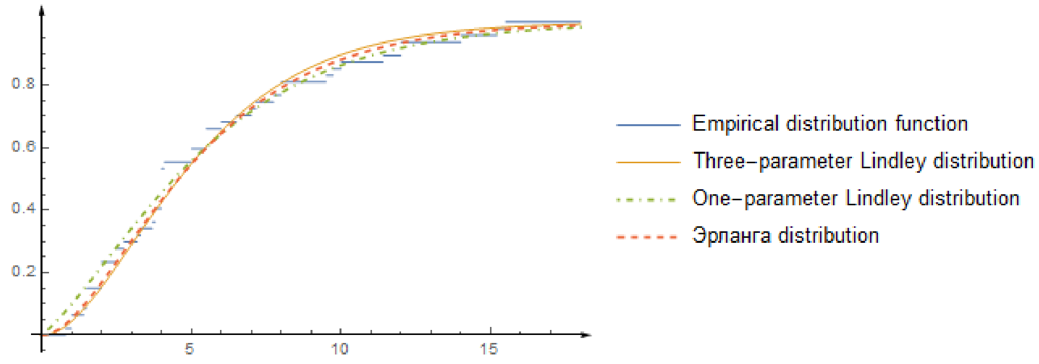

Figure 32. Empirical distribution function and corresponding theoretical distribution function for case study 6.2

Observing Figure 32, it is evident that both the single-parameter Lindley distribution and the generalized three-parameter Lindley distribution fit the batch of data well.

References

[1] M. M. E. Abd El-Monsef, *A new Lindley distribution with location parameter*, Communications in Statistics-Theory and Methods, 2016, 45(17), 5204-5219.

[2] M. M. E. Abd El-Monsef, W. A. Hassanein, N. M. Kilany, *Erlang–Lindley distribution*, Communications in Statistics-Theory and Methods, 2017, 46(19), 9494-9506.

[3] M. Ahsanullah, M. E. Ghitany, D. K. Al-Mutairi, *Characterization of Lindley distribution by truncated moments*, Communications in Statistics-Theory and Methods, 2017, 46(12), 6222-6227.

[4] A. Algarni, *On a new generalized Lindley distribution: Properties estimation and applications*, PLoS ONE, 2021,16(2), 1-19.

[5] T. Arslan, S. Acitas, B. Senoglu, *Generalized Lindley and Power Lindley distributions for modeling the wind speed data*, Energy Conversion and Management, 2017, 152, 300-311.

[6] A. Asgharzadeh, S. Nadarajah, F. Sharafi, *Weibull Lindley Distribution*, Revstat-Statistical Journal, 2018, 16(1), 87-113.

[7] S. K. Ashour, M. A. Eltehiwy, *Exponentiated power Lindley distribution*, Journal of Advanced Research, 2015, 6(6), 895-905.

[8] K. V. P. Barco, J. Mazucheli, V. Janeiro, *The inverse power Lindley distribution*, Communications in Statistics-Simulation and Computation, 2017, 46(8), 6308-6323.

[9] M. Borah, J. Hazarika, *A new quasi Poison-Lindley distribution: Properties and applications*, Journal of Statistical Theory and Applications, 2017, 16(4), 576-588.

[10] C. Chesneau, L. Tomy, M. Jose, *Wrapped modified Lindley distribution*, Journal of Statistics and Management Systems, 2021, 24(5), 1025-1040.

[11] Y. Dai, *Statistical analysis of Lindley distribution*, Master Dissertation of Shanghai Normal University, 2018.

[12] Y. Dai, R. Wang, X. Xu, *Statistical analysis of Lindley distribution under type-I censoring sample*, Statistics & Decision, 2018(1), 84-87.

[13] M. El-Morshedy, M. S. Eliwa, H. Nagy, *A new two-parameter exponentiated discrete Lindley distribution: properties, estimation and applications*, Journal of Applied Statistics, 2020, 47(2), 354-375.

[14] M. Ghica, N. D. Poesina, I. Prasacu, *Exponentiated power quasi Lindley distribution, submodels and some Properties*, Review of the Air Force Academy, 2017, 15(2), 75-84.

[15] M. E. Ghitany, D. K. Al-Mutairi, N. Balakrishnan, L. J. Al-Enezi, *Power Lindley distribution and associated inference*, Computational Statistics and Data Analysis, 2013, 64, 20-33.

[16] M. E. Ghitany, D. K. Al-Mutairi, S. Nadarajah, *Zero-truncated Poisson-Lindley distribution and its application*, Mathematics and Computers in Simulation, 2008, 79(3), 279-287.

[17] M. E. Ghitany, F. Alqallaf, D. K. Al-Mutairi, H. A. Husain, *A two-parameter weighted Lindley distribution and its applications to survival data*, Mathematics and Computers in Simulation, 2011, 81(6), 1190-1201.

[18] M. E. Ghitany, B. Atieh, S. Nadarajah, *Lindley distribution and its application*, Mathematics and Computers in Simulation, 2008, 78(4), 493-506.

[19] T. Hussain, M. Aslam, M. Ahmad, *A two parameter discrete Lindley distribution*, Revista Colombiana de Estadística, 2016, 39(1), 45-61.

[20] S. Joshi, J. Kanichukattu, *Wrapped Lindley distribution*, Communications in Statistics-Theory and Methods, 2018, 47(5), 1013-1021.

[21] S. A. Kemaloglu, M. Yilmaz. *Transmuted two-parameter Lindley distribution*, Communications in Statistics-Theory and Methods, 2017, 46(23), 11866-11879.

[22] A. Khodadadi, M. Shirazi, S. Geedipally, D. Lord, *Evaluating alternative variations of Negative Binormial-Lindley distribution for modelling crash data*, Transportmetrica A: Transport Science, 2022, 1-22.

[23] C. S. Kumar, R. Jose, *On double Lindley distribution and some of its properties*, American Journal of Mathematical and Management Sciences, 2019, 38(1), 23-43.

[24] D. V. Lindley, *Fiducial distributions and Bayes' theorem*, Journal of the Royal Statistical Society, Series B: Methodological, 1958, 20(1), 102–107.

[25] H. Lv, L. Gao, C. Chen, *Эрланга distribution and its application in support data analysis*, Journal of Academy of Armored Force Engineering, 2000, 16(3), 48-52.

[26] E. Mahmoudi, H. Zakerzadeh, *Generalized Poisson-Lindley distribution*, Communications in Statistics-Theory and Methods, 2010, 39(10), 1785-1798.

[27] I. Makhdoom, P. Nasiri, A. Pak, *Bayesian approach for the reliability parameter of power Lindley distribution*, International Journal of System Assurance Engineering and Management, 2016, 7(3), 341-355.

[28] R. Maya, M. R. Irshad, *New extended generalized Lindley distribution: Properties and Applications*, Statistica, 2017, (1), 33-52.

[29] J. Mazucheli, J. A. Achcar, *The Lindley distribution applied to competing risks lifetime data*, Computer Methods and Programs in Biomedicine, 2011, 104(2), 188-192.

[30] M. Mesfioui, A. M. Abouammoh, *On a multivariate Lindley distribution*, Communications in Statistics-Theory and Methods, 2017, 46(16), 8027-8045.

[31] M. Naderi, A. Arabpour, A. Jamalizadeh, *Multivariate normal mean-variance mixture distribution based on Lindley distribution*, Communications in Statistics-Simulation and Computation, 2018, 47(4), 1179-1192.

[32] S. Nedjar, H. Zeghdoudi, *On gamma Lindley distribution: Properties and simulations*,

Journal of Computational and Applied Mathematics, 2016, 298, 167-174.
[33] W. Nelson, *Accelerated life testing-step-stress models and data analyses*, IEEE Transactions on Reliability, 1980, R-29(2), 103-108.
[34] W. Nelson, *Accelerated testing: statistical models, test plans, and data analysis*, John Wiley & Sons, Inc, 2004, 507-509.
[35] J. Nie, W. Gui, *Parameter estimation of Lindley distribution based on progressive type-II censored competing risks data with binomial removals*, Mathmatics, 2019, 7(7), 646-659.
[36] B. Oluyede, G. Warahena-Liyanage, M. Pararai, *A new class of generalized Power Lindley distribution with applications to lifetime data*, Theoretical Mathematics and Applications, 2015, 5(1), 53-96.
[37] H. Papageorgiou, M. Vardaki, *Bivariate discrete Poisson-Lindley distributions*, Journal of Statistical Theory and Practice, 2022, 16(2), 1-23.
[38] R. Roozegar, S. Nadarajah, *On a Generalized Lindley Distribution*, Statistica, 2017, 77(2), 149-157.
[39] M. Sankaran, *The discrete Poisson–Lindley distribution*, Biometrics, 1970, 26(1), 145–149.
[40] R. Shanker, K. K. Shukla, R. Shanker, T. A. Leonida, *A three-Parameter Lindley distribution*, American Journal of Mathematics and Statistics, 2017, 7(1), 15-26.
[41] V. K. Sharma, S. K. Singh, U. Singh, V. Agiwal, *The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data*, Journal of Industrial and Production Engineering, 2015, 32(3), 162-173.
[42] D. S. Shibu, M. R. Irshad, *Extended New generalized Lindley distribution*, Statistica, 2016, 76(1), 41-56.
[43] M. Shrahili, N. Alotaibi, D. Kumar, A. R. Shafay, *Inference on Exponentiated Power Lindley distribution based on order statistics with application*, Complexity, 2020, 1-14.
[44] A. Thongteeraparp, A. Volodin, *Parameter estimation of the negative Binomial-New Weighted Lindley distribution by the method of maximum likelihood*, Journal of Mathematics, 2020, 41(3), 430-434.
[45] M. Yilmaz, M. Hameldarbandi, S. A. Kemaloglu, *Exponential-modified discrete Lindley distribution*, SpringerPlus, 2016, 5, 1660-1680.
[46] H. Zeghdoudi, S. Nedjar, *A pseudo Lindley distribution and its application*, The Statistics and Probability African Society, 2016, 11(1), 923-932.