Dynamics for delay plate equations with multiplicative noise on unbounded domains

Xiaobin Yao[1,†], Yang Bai[1]

Abstract: This paper is concerned with the asymptotic behavior of solutions for plate equations with delay blurred by multiplicative noise in $\mathbb{R}^n$. First of all, we obtain the uniform compactness of pullback random attractors of the problem, then derive the upper semi-continuity of the attractors.

Keywords: random attractor, delay equation, asymptotical compactness, upper semicontinuity.

AMS Subject Classification: 60H15, 34C35, 58F11, 58F36.

0

1 Introduction

This paper is concerned with the following plate equations with time delay blurred by multiplicative noise in $\mathbb{R}^n$:

$$\begin{cases} \partial_{tt}u + \alpha\partial_t u + \partial_{xxxx}(\partial_t u) + \partial_{xxxx}u + \lambda u + F(x, u(t,x)) = f(x, u(t-\rho, x)) \\ \qquad\qquad\qquad +g(t,x) + \epsilon u \circ \frac{dw}{dt}, \quad x \in \mathbb{R}^n, \quad t > \tau, \\ u_\tau(s,x) := u(\tau + s, x) = \phi(s,x), \quad x \in \mathbb{R}^n, \quad s \in [-\rho, 0], \\ \partial_t u_\tau(s,x) = \partial_t \phi(s,x), \quad x \in \mathbb{R}^n, \quad s \in [-\rho, 0], \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}, x \in \mathbb{R}^n, s \in [-\rho, 0], \epsilon \in (0,1]$, α, λ are positive constants, the time delay $\rho > 0$ is a constant, the conditions that F, f satisfying see Section 3, $g(t,\cdot) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ and $\phi \in C([\tau - \rho, \tau], H^2(\mathbb{R}^n))$, W is a two-side real-value Wiener process on a complete probability space which will be specified later. The problem (1.1) is understood in the sense of Stratonovich integration.

Delays in differential systems are used for mathematical modeling in many applications to describe of the dynamics influenced by events from the past. It is known that differential equations with delay appears in physics, biology and other disciplines, and the time delay

[0†] The corresponding author. Email address: yaoxiaobin2008@163.com (Xiaobin Yao)

[1] School of Mathematics and Statistics, Qinghai Minzu University, Xining 810007, China

[1] The authors are supported by the Natural Science Foundation of Qinghai Province(No.2024- ZJ-931) and the National Natural Science Foundation of China (No. 12161071).

are considered in the model of the systems ([5, 8]). Especially, these equations are applied to mathematical modeling in many applications to describe of the dynamics influenced by events from the past, see [10, 11, 13]. Most noteworthy is that the attractors of deterministic differential equations with time delay have been studied in [8], while the stochastic case has been studied in [4, 18].

For the case $\rho \equiv 0$ in (1.1), we derived the existence of results in, see [9, 19]. However, as far as we know, there is little literature dealing with stochastic time-delay plate equations. For all we know, in [17], Wang and Ma sudied the existence of pullback attractors for the non-autonomous suspension bridge equation with time delay.

Motivated by literature above, we study the dynamics of the delay plate equations. The main features of the work are summarized as follows: (i) We prove that (1.1) generate random dynamical systems; (ii) We show the existence random attractors for (1.1); (iii) We obtain the convergence of random attractors for (1.1) as $\rho \to 0$ or $\epsilon \to 0$. A major difficulty in the proof process is to prove the existence random attractors for (1.1), the reason is Sobolev embeddings are no longer compact. To overcome it, we use the uniform estimates and the splitting technique ([15]).

This paper is organised as follows. In the next section, we recall some basic concepts on the theory of random dynamical systems. We then prove an abstract result for upper semicontinuity of random attractors for stochastic delay equations. In Section 3, we establish the continuous random dynamical system for (1.1). Some necessary estimates are given in section 4. We then prove the existence of pullback attractors for (1.1) in section 5. In Section 6-7, we further prove the upper semicontinuity of attractors when $\epsilon \to 0$ and $\rho \to 0$.

2 Notations

Now, we recall some notations and proposition on the theory of random dynamical systems, the reader is referred to [2, 6, 7, 12, 14].

Denote $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space and for $t \in \mathbb{R}, \omega \in \Omega$,

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space, and let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be an ergodic metric dynamical system

Proposition 2.1.([14]) *Let $\mathcal{D}$ be an inclusion closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Then Φ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}$ in $\mathcal{D}$ if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$.*

We denote by C_X the space $C([-\rho, 0], X)$ with the sup-norm

$$\|u\|_{C_X} = \sup_{s\in[-\rho,0]} \|u\|_X, \; u \in C([-\rho, 0], X)$$

.

Denote by $(Y, \|\cdot\|)$ a Banach space satisfies that the injection $X \subset Y$ is continuous, we also denote by $C_{X,Y}$ the Banach space $C_X \cap C^1([-\rho, 0], Y)$ with the norm $\|\cdot\|_{C_{X,Y}}$

$$\|y\|^2_{C_{X,Y}} = \|v\|^2_{C_Y} + \|u\|^2_{C_X}, \quad y = (u, v)^\top, \; u \in C_X, \quad v \in C_Y. \tag{2.1}$$

Denote $-\Delta$ the Laplace operator, $A = \Delta^2$ and the Hilbert spaces $V_\nu = D(A^{\frac{\nu}{4}})$ endowed with inner product and norm

$$(u, v)_\nu = (A^{\frac{\nu}{4}} u, A^{\frac{\nu}{4}} v), \qquad \|\cdot\|_\nu = \|A^{\frac{\nu}{4}} \cdot\|.$$

In particular, $V_0 = L^2(\mathbb{R}^n), V_2 = H^2(\mathbb{R}^n)$.

3 Cocycles for stochastic plate equation

In this section, we discuss the assumptions on F, f and g and define a continuous cocycle in $C_{V_2,V_0}(\mathbb{R}^n)$ for (1.1). Assume $F(x, \cdot) \in C^2(\mathbb{R}), f, g$ satisfy the following conditions:

$$|F(x, u)| \le c_1|u|^\gamma + \phi_1(x), \quad \phi_1 \in L^2(\mathbb{R}^n), \tag{3.1}$$

$$F(x, u)u - c_2\widetilde{F}(x, u) \ge \phi_2(x), \quad \phi_2 \in L^1(\mathbb{R}^n), \tag{3.2}$$

$$\widetilde{F}(x, u) \ge c_3|u|^{\gamma+1} - \phi_3(x), \quad \phi_3 \in L^1(\mathbb{R}^n), \tag{3.3}$$

$$|\frac{\partial F}{\partial u}(x, u)| \le \varpi, \tag{3.4}$$

$$f(x, 0) = 0, \quad \text{and} \quad |f(x, u_1) - f(x, u_2) \le l_f|u_1 - u_2|, \tag{3.5}$$

where $\varpi, l_f > 0, \; 1 \le \gamma \le \frac{n+4}{n-4}, c_i > 0$. It follows from (3.1)-(3.2) that

$$\widetilde{F}(x, u) \le c(|u|^2 + |u|^{\gamma+1} + \phi_1^2 + \phi_2). \tag{3.6}$$

Assume g satisfies

$$\int_{-\infty}^0 e^{\sigma s}\|g(\cdot, s + \tau)\|^2 ds < \infty, \; \forall \; \tau \in \mathbb{R}, \tag{3.7}$$

which implies that

$$\lim_{r\to\infty} \int_{-\infty}^\tau \int_{|x|\ge r} e^{\sigma s}|g(\cdot, s)|^2 dx ds = 0, \; \forall \; \tau \in \mathbb{R}, \tag{3.8}$$

where σ is a positive constant.

For $Y = (u, v)^\top \in C_{V_2,V_0}(\mathbb{R}^n)$, set

$$\|Y\|_{C_{V_2,V_0}(\mathbb{R}^n)} = (\|u\|^2_{C_{V_2}} + \|v\|^2_{C_{V_0}})^{\frac{1}{2}} = (\|v\|^2 + (\delta^2 + \lambda - \delta\alpha)\|u\|^2 + (1 - \delta)\|\Delta u\|^2)^{\frac{1}{2}}. \tag{3.9}$$

In addition, we see that $\|\cdot\|_{C_{V_2,V_0}}(\mathbb{R}^n)$ is equivalent to $\|\cdot\|_{C_{V_2,V_0}(\mathbb{R}^n)}$ in (2.1).

Let $\xi = \partial_t u + \delta u$, where δ is determined by (3.14), then problem (1.1) is equivalent to

$$\begin{cases} \frac{du}{dt} + \delta u = \xi, \\ \frac{d\xi}{dt} + (\alpha - \delta)\xi + \Delta^2 z + (\delta^2 + \lambda - \delta\alpha)u + (1 - \delta)\Delta^2 u + f(x, u) + F(x, u(t, x)) \\ \qquad\qquad = f(x, u(t - \rho, x)) + g(x, t) + \epsilon u \circ \frac{dw}{dt}, \\ u_\tau(s, x) := u(\tau + s, x) = \phi(s, x), \quad x \in \mathbb{R}^n, \quad s \in [-\rho, 0], \\ \xi_\tau(s, x) = \partial_t \phi(s, x) + \delta\phi(s, x), \quad x \in \mathbb{R}^n, \quad s \in [-\rho, 0]. \end{cases} \tag{3.10}$$

Consider the Ornstein-Uhlenbeck equation $dy(\theta_t\omega)+\epsilon y(\theta_t\omega)dt = dw(t)$, and Ornstein-Uhlenbeck process

$$y(\omega) = -\epsilon\int_{-\infty}^{0} e^{\epsilon\tau}\omega(\tau)d\tau.$$

From [3], it is known that the random variable $|y(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega}\subset\Omega$ of full $\mathcal{P}$ measure such that $y(\theta_t\omega)$ is continuous in t for every $\omega\in\widetilde{\Omega}$.

Let $v(t,\tau,\omega) = \xi(t,\tau,\omega) - \epsilon y(\theta_t\omega)u(t,\tau,\omega)$, we obtain the equivalent system of (3.10),

$$\begin{cases} \frac{du}{dt} + \delta u - v = \epsilon y(\theta_t\omega)u, \\ \frac{dv}{dt} + (\alpha-\delta)v + \Delta^2 v + (\delta^2+\lambda-\delta\alpha)u + (1-\delta)\Delta^2 u + \epsilon y(\theta_t\omega)\Delta^2 u + F(x,u(t,x)) \\ = f(x,u(t-\rho,x)) + g(x,t) - \epsilon y(\theta_t\omega)v - \epsilon(\epsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)u, \\ u_\tau(s,x) := u(\tau+s,x) = \phi(s,x), \quad x\in\mathbb{R}^n, \quad s\in[-\rho,0], \\ v_\tau(s,x) = \partial_t\phi(s,x) + \delta\phi(s,x) - \epsilon y(\theta_\tau\omega)\phi(s,x) := \psi(s,x), \quad x\in\mathbb{R}^n, \quad s\in[-\rho,0]. \end{cases} \tag{3.11}$$

For given $\tau\in\mathbb{R},\omega\in\Omega$ and $\phi\in C_{V_2}(\mathbb{R}^n),\psi\in C_{V_0}(\mathbb{R}^n)$, a solution of (3.11) will be written as $(u(\cdot,\tau,\omega,\phi), v(\cdot,\tau,\omega,\psi))$. As usual, the segment of $u(\cdot,\tau,\omega,\phi)$ and $v(\cdot,\tau,\omega,\psi)$ on $[t-\rho,t]$ are written as $u^t(\cdot,\tau,\omega,\phi)$ and $v^t(\cdot,\tau,\omega,\psi)$, respectively; that is,

$$u_t(s,\tau,\omega,\phi) = u(t+s,\tau,\omega,\phi), \quad \text{for all} \quad s\in[-\rho,0];$$
$$v_t(s,\tau,\omega,\psi) = v(t+s,\tau,\omega,\psi), \quad \text{for all} \quad s\in[-\rho,0].$$

Under conditions (3.1)-(3.5), one can verify that for every $\tau\in\mathbb{R},\omega\in\Omega$ and $\phi\in C_{V_2}(\mathbb{R}^n),\psi\in C_{V_0}(\mathbb{R}^n)$, problem (3.11) has a unique continuous solution $(u(\cdot,\tau,\omega,\phi),v(\cdot,\tau,\omega,\psi)) : [\tau-\rho,\infty]\to C_{V_2,V_0}(\mathbb{R}^n)$, and the segment $u_t(\cdot,\tau,\omega,\phi)$ of u is $(\mathcal{F},\ \mathcal{B}(C_{V_2}(\mathbb{R}^n)))$-measurable in $\omega\in\Omega$ and continuous with respect to ϕ in $C_{V_2}(\mathbb{R}^n)$; the segment $v_t(\cdot,\tau,\omega,\psi)$ of v is $(\mathcal{F},\ \mathcal{B}(C_{V_0}(\mathbb{R}^n)))$-measurable in $\omega\in\Omega$ and continuous with respect to ψ in $C_{V_0}(\mathbb{R}^n)$.

Define $\Phi : \mathbb{R}^+\times\mathbb{R}\times\Omega\times C_{V_2,V_0}(\mathbb{R}^n)\to C_{V_2,V_0}(\mathbb{R}^n)$ by

$$\Phi(t,\tau,\omega,(\phi,\psi))(\cdot) = (u_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,\phi), v_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,\psi)), \tag{3.12}$$

where $(t,\tau,\omega,(\phi,\psi))\in\mathbb{R}^+\times\mathbb{R}\times\Omega\times C_{V_2,V_0}(\mathbb{R}^n)$, $u_{t+\tau}(s,\tau,\theta_{-\tau}\omega,\phi) = u(t+\tau+s,\tau,\theta_{-\tau}\omega,\phi)$ for $s\in[-\rho,0]$; $v_{t+\tau}(s,\tau,\theta_{-\tau}\omega,\psi) = v(t+\tau+s,\tau,\theta_{-\tau}\omega,\psi)$ for $s\in[-\rho,0]$. Then Φ is a continuous cocycle on $C_{V_2,V_0}(\mathbb{R}^n)$ over $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$.

Let $D = \{D(\tau,\omega)\subseteq C_{V_2,V_0}(\mathbb{R}^n) : \tau\in\mathbb{R},\omega\in\Omega\}$ be a family of bounded nonempty subsets of $C_{V_2,V_0}(\mathbb{R}^n)$ satisfying

$$\lim_{t\to+\infty} e^{-\gamma t}\|D(\tau-t,\theta_{-t}\omega)\|_{C_{V_2,V_0}(\mathbb{R}^n)} = 0, \quad \forall\ \gamma>0, \tag{3.13}$$

where $\|D(\tau-t,\theta_{-t}\omega)\|_{C_{V_2,V_0}(\mathbb{R}^n)} = \sup\limits_{(u,v)\in D(\tau-t,\theta_{-t}\omega)}\|(u,v)\|_{C_{V_2,V_0}(\mathbb{R}^n)}$. Let $\mathcal{D}$ be the set of all families $D = \{D(\tau,\omega)\subseteq C_{V_2,V_0(\mathbb{R}^n)} : \tau\in\mathbb{R},\omega\in\Omega\}$ which satisfies (3.13).

For later purpose, we assume $\delta\in(0,1)$ be small enough such that

$$\alpha-\delta>0, \quad \delta^2+\lambda-\delta\alpha>0, \quad 1-\delta>0. \tag{3.14}$$

In addition,

$$\lambda > \frac{16l_f^2+\delta(\alpha-\delta)^2}{\delta(\alpha-\delta)}. \tag{3.15}$$

Under (3.15), we can define σ appearing in (3.7) by

$$\sigma = \min\{\frac{\alpha-\delta}{8}, \frac{\delta}{4}, \frac{c_2\delta}{8}\}. \tag{3.16}$$

4 Uniform estimates of solutions

We will obtain some necessary estimates of solutions for (3.11) in this section.

Lemma 4.1 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. Then for every $\varsigma, \tau \in \mathbb{R}, \omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \varsigma) > 0$ such that for all $t \geq T$,*

$$\begin{aligned}
&\|Y(\iota + s, \tau - t, \theta_{-\tau}\omega, Y_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)} \\
&+ \int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma - \epsilon c - \epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho} \|Y(\iota + s, \tau - t, \theta_{-\tau}\omega, Y_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)} \\
&+ \int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma - \epsilon c - \epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho} \|\Delta v(r, \tau - t, \theta_{-\tau}\omega, \psi)\|^2 dr \\
\leq& M\big(1 + R_1(\tau, \omega)\big).
\end{aligned} \tag{4.1}$$

where $Y_0 = (\phi, \psi)^\top \in D(\tau - t, \theta_{-t}\omega)$, M is a positive constant depending on λ, σ, α and δ, but independent of τ, ω, D and ϵ, and $R_1(\tau, \omega)$ is given by (4.16). Moreover, c is uniform with respect to ρ in $(0, \rho_0)$ for every positive ρ_0.

Proof. Taking the inner product of $(3.11)_2$ with v in $L^2(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2 + (\alpha - \delta)(v, v) + (\lambda + \delta^2 - \delta\alpha)(u, v) + (1 - \delta)(\Delta^2 u, v) \\
&\quad + (\Delta^2 v, v) + \epsilon y(\theta_t\omega)(\Delta^2 u, v) + (F(x, u(t, x)), v) \\
&= (f(x, u(t - \rho, x)), v) + (g(x, t), v) - \epsilon y(\theta_t\omega)(v, v) - \epsilon(\epsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)(u, v).
\end{aligned} \tag{4.2}$$

By simple calculation, we can get the following estimates for the right-hand side of (4.2):

$$(u, v) = (u, \frac{du}{dt} + \delta u - \epsilon y(\theta_t\omega)u) = \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - \epsilon y(\theta_t\omega)\|u\|^2, \tag{4.3}$$

$$(\Delta^2 u, v) = (\Delta^2 u, \frac{du}{dt} + \delta u - \epsilon y(\theta_t\omega)u) = \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \delta\|\Delta u\|^2 - \epsilon y(\theta_t\omega)\|\Delta u\|^2, \tag{4.4}$$

$$\begin{aligned}
&(F(x, u(t, x)), v) = (F(x, u(t, x)), \frac{du}{dt} + \delta u - \epsilon y(\theta_t\omega)u) \\
=& \frac{d}{dt}\int_{\mathbb{R}^n} \widetilde{F}(x, u)dx + \delta(F(x, u(t, x)), u) - \epsilon y(\theta_t\omega)(F(x, u(t, x)), u).
\end{aligned} \tag{4.5}$$

By (4.2)-(4.5), we get

$$\begin{aligned}
&\frac{d}{dt}(\|v\|^2 + (\delta^2 + \lambda - \delta\alpha)\|u\|^2 + (1 - \delta)\|\Delta u\|^2 + 2\int_{\mathbb{R}^n} \widetilde{F}(x, u)dx) \\
&+ 2(\alpha - \delta)\|v\|^2 + 2\delta(\delta^2 + \lambda - \delta\alpha)\|u\|^2 + 2\delta(1 - \delta)\|\Delta u\|^2 + 2\delta(F(x, u(t, x)), u) + 2\|\Delta v\|^2 \\
=& 2(f(x, u(t - \rho, x)), v) + 2(g, v) - 2\epsilon(\epsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)(u, v) - 2\epsilon y(\theta_t\omega)(\Delta^2 u, v) \\
&+ 2\epsilon(\delta^2 + \lambda - \delta\alpha)y(\theta_t\omega)\|u\|^2 + 2\epsilon(1 - \delta)y(\theta_t\omega)\|\Delta u\|^2 \\
&- 2\epsilon y(\theta_t\omega)\|v\|^2 + 2\epsilon y(\theta_t\omega)(F(x, u(t, x)), u).
\end{aligned} \tag{4.6}$$

By (3.1) and (3.3), we have

$$\begin{aligned}
&2\epsilon y(\theta_t\omega)(F(x, u(t, x)), u) \leq 2\epsilon c_1|y(\theta_t\omega)|\int_{\mathbb{R}^n} |u|^{\gamma+1}dx + \epsilon|y(\theta_t\omega)|\|\phi_1\|^2 + \epsilon|y(\theta_t\omega)|\|u\|^2 \\
&\leq 2\epsilon c_1 c_3^{-1}|y(\theta_t\omega)|\int_{\mathbb{R}^n} (\widetilde{F}(x, u) + \phi_3(x))dx + \epsilon|y(\theta_t\omega)|\|\phi_1\|^2 + \epsilon|y(\theta_t\omega)|\|u\|^2
\end{aligned}$$

$$\leq \epsilon c|y(\theta_t\omega)|\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx+\epsilon c|y(\theta_t\omega)|+\epsilon|y(\theta_t\omega)|\|u\|^2, \tag{4.7}$$

where c depends on $c_1,\ c_3^{-1},\ \|\phi_3\|_{L^1(\mathbb{R}^n)}$ and $\|\phi_1\|_{L^2(\mathbb{R}^n)}$.

Using Young's inequality, we obtain

$$\begin{aligned}
&2(g,v)-2\epsilon y(\theta_t\omega)\|v\|^2-2\epsilon(\epsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(u,v)-2\epsilon y(\theta_t\omega)(\Delta^2 u,v)\\
&\qquad\qquad+2\epsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|u\|^2+2\epsilon(1-\delta)y(\theta_t\omega)\|\Delta u\|^2\\
&\leq\frac{15(\alpha-\delta)}{16}\|v\|^2+c\|g\|^2+\epsilon c|y(\theta_t\omega)|(1+|y(\theta_t\omega)|)(\|u\|^2+\|v\|^2+\|\Delta u\|^2)+\|\Delta v\|^2.
\end{aligned}\tag{4.8}$$

By (3.5), we get

$$(f(x,u(t-\rho,x)),v)\leq l_f\|u(t-\rho)\|\|v\|\leq\frac{4l_f^2}{\alpha-\delta}\|u(t-\rho)\|^2+\frac{\alpha-\delta}{16}\|v\|^2. \tag{4.9}$$

By (4.6)-(4.9) and (3.2), we have

$$\begin{aligned}
&\frac{d}{dt}(\|v\|^2+(\delta^2+\lambda-\delta\alpha)\|u\|^2+(1-\delta)\|\Delta u\|^2+2\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx)\\
&+(\alpha-\delta)\|v\|^2+2\delta(\delta^2+\lambda-\delta\alpha)\|u\|^2+2\delta(1-\delta)\|\Delta u\|^2+2\delta c_2\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx+\|\Delta v\|^2\\
\leq& c+c\|g\|^2+\epsilon c|y(\theta_t\omega)|\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx+\frac{8l_f^2}{\alpha-\delta}\|u(t-\rho)\|^2\\
&+\epsilon c(1+|y(\theta_t\omega)|^2)(1+\|u\|^2+\|v\|^2+\|\Delta u\|^2).
\end{aligned}\tag{4.10}$$

By (3.3) and (3.16), we get

$$\delta c_2\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx\geq 4\sigma\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx+(4\sigma-\delta c_2)\int_{\mathbb{R}^n}\phi_3(x)dx. \tag{4.11}$$

By (3.16), (4.10)-(4.11) and the norm $\|\cdot\|_{C_{V_2,V_0}(\mathbb{R}^n)}$ in (3.9), we have

$$\begin{aligned}
&\frac{d}{dt}(\|Y\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx)+4\sigma(\|Y\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx)\\
&+\|\Delta v\|^2+\frac{1}{2}\delta(\delta^2+\lambda-\delta\alpha)\|u\|^2\\
\leq&\frac{8l_f^2}{\alpha-\delta}\|u(t-\rho)\|^2+\epsilon c(1+|y(\theta_t\omega)|^2)(\|Y\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}\widetilde{F}(x,u)dx)\\
&+c(1+\|g\|^2+\epsilon|y(\theta_t\omega)|^2).
\end{aligned}\tag{4.12}$$

Considering time $\tau-t$ instead of τ, firstly, multiplying (4.12) by $e^{\int_0^t(2\sigma-\epsilon c-\epsilon c|y(\theta_r\omega)|^2)dr}$, then integrating over $[\tau-t,\iota+s]$ for any fixed $s\in[-\rho,0]$ with $\iota\geq\tau-t+\rho$, last replacing ω by $\theta_{-\tau}\omega$ (i.e., u now denotes $u(\cdot,\tau-t,\theta_{-\tau}\omega,\phi)$ and v now denotes $v(\cdot,\tau-t,\theta_{-\tau}\omega,\psi)$, we have

$$\begin{aligned}
&\|Y(\iota+s,\tau-t,\theta_{-\tau}\omega,Y_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}\widetilde{F}(x,u(\iota+s,\tau-t,\theta_{-\tau}\omega,\phi))dx\\
&+2\sigma\int_{\tau-t}^{\iota+s}e^{\int_\tau^s(2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(\|Y(\iota+s,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\\
&+2\int_{\mathbb{R}^n}\widetilde{F}(x,u(\iota+s,\tau-t,\theta_{-\tau}\omega,\phi))dx)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|\Delta v(r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2 dr\\
&+\frac{1}{2}\delta(\delta^2+\lambda-\delta\alpha)\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^2 dr\\
\le& e^{\int_{\iota+s}^{\tau-t} (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(\|Y_0\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}\widetilde{F}(x,\phi)dx)\\
&+\frac{8l_f^2}{\alpha-\delta}\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(r-\rho,\tau-t,\theta_{-\tau}\omega,\phi)\|^2 dr\\
&+c\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(1+\|g(r,\cdot)\|^2+\epsilon|y(\theta_{r-\tau}\omega)|^2)dr. \qquad (4.13)
\end{aligned}
$$

Combining with the proof of Lemma 4.1 in [9], we have the followings:

$$
e^{\int_{\iota+s}^{\tau-t} (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(\|Y_0\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}\widetilde{F}(x,\phi)dx)\to 0, \quad \text{as} \quad t\to\infty; \qquad (4.14)
$$

$$
\begin{aligned}
&\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(r-\rho,\tau-t,\theta_{-\tau}\omega,\phi)\|^2 dr\\
=&\int_{\tau-t-\rho}^{\iota+s-\rho} e^{\int_{\tau+\rho}^{s+\rho} (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^2 dr\\
=&\int_{\tau-t-\rho}^{\tau-t} e^{\int_{\tau+\rho}^{s+\rho} (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^2 dr\\
&+\int_{\tau-t}^{\iota+s-\rho} e^{\int_{\tau+\rho}^{s+\rho} (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^2 dr\\
\le&\rho e^{\sigma(s-\tau)}\|\phi\|^2_{C_{V_2}(\mathbb{R}^n)}+\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(r,\tau-t,\theta_{-\tau}\omega,\phi)\|^2 dr \qquad (4.15)
\end{aligned}
$$

and the following integral

$$
R_1(\tau,\omega)=c\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(1+\|g(r,\cdot)\|^2+\epsilon|y(\theta_{r-\tau}\omega)|^2)dr \qquad (4.16)
$$

is convergent.

By (4.13)-(4.16) and (3.15) we get

$$
\begin{aligned}
&\|Y(\iota+s,\tau-t,\theta_{-\tau}\omega,Y_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+2\int_{\mathbb{R}^n}\widetilde{F}(x,u(\iota+s,\tau-t,\theta_{-\tau}\omega,\phi))dx\\
&+2\sigma\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(\|Y(\iota+s,\tau-t,\theta_{-\tau}\omega,Y_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\\
&+2\int_{\mathbb{R}^n}\widetilde{F}(x,u(\iota+s,\tau-t,\theta_{-\tau}\omega,\phi))dx)\\
&+\int_{\tau-t}^{\iota+s} e^{\int_\tau^s (2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|\Delta v(r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2 dr\\
\le&1+R_1(\tau,\omega). \qquad (4.17)
\end{aligned}
$$

which along with (3.3) yields (4.1). □

Let $\rho:\mathbb{R}^n\to\mathbb{R}$ be a smooth function such that $0\le\rho(x)\le 1$ for all $x\in\mathbb{R}^n$, and

$$
\rho(x)=0 \quad \text{for} \quad |x|\le\frac{1}{2}; \quad \text{and} \quad \rho(x)=1 \quad \text{for} \quad |x|\ge 1.
$$

For every $k\in\mathbb{N}$, let
$$\rho_k(x)=\rho(x/k),\quad x\in\mathbb{R}^n.$$
We also assume that for all $x\in\mathbb{R}^n$ and $k\in\mathbb{N}$, $|\nabla\rho_k|\le\frac{1}{k}c_4$, $|\Delta\rho_k|\le\frac{1}{k}c_5$, $|\Delta\nabla\rho_k|\le\frac{1}{k}c_6$, $|\Delta^2\rho_k|\le\frac{1}{k}c_7$, where c_4,c_5,c_6 and c_7 are positive constants independent of k.

Given $k\ge1$, denote $\mathbb{H}_k=\{x\in\mathbb{R}^n:|x|<k\}$ and $\mathbb{R}^n\setminus\mathbb{H}_k$ the complement of $\mathbb{H}_k$.

Lemma 4.2 *Assume that (3.1)-(3.5), (3.7)-(3.8), (3.14)-(3.16) hold. Then for every* $\tau\in\mathbb{R},s\in[-\rho,0],\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,D,\eta)>0$ *and* $K=K(\tau,\omega,\eta)\ge1$, *such that for all* $t\ge T,k\ge K$,
$$\|Y(\tau+s,\tau-t,\theta_{-\tau}\omega,Y_0)\|^2_{H^2(\mathbb{R}^n\setminus\mathbb{H}_k)\times L^2(\mathbb{R}^n\setminus\mathbb{H}_k)}\le\eta. \tag{4.18}$$

Proof. Multiplying $(3.11)_2$ with $\rho_k(x)v$, we obtain
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)|v|^2dx+(\alpha-\delta)\int_{\mathbb{R}^n}\rho_k(x)|v|^2dx\\
&+(\lambda+\delta^2-\delta\alpha)\int_{\mathbb{R}^n}\rho_k(x)\cdot u\cdot vdx+(1-\delta)\int_{\mathbb{R}^n}\rho_k(x)\cdot v\cdot\Delta^2udx\\
&+\int_{\mathbb{R}^n}\rho_k(x)\cdot v\cdot\Delta^2vdx+\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)\cdot v\cdot\Delta^2udx+\int_{\mathbb{R}^n}\rho_k(x)\cdot F(x,u)vdx\\
=&\int_{\mathbb{R}^n}\rho_k(x)\cdot g\cdot vdx-\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)|v|^2dx+\int_{\mathbb{R}^n}\rho_k(x)\cdot f(x,u(t-\rho,x))\cdot vdx\\
&-\epsilon(\epsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)uvdx.
\end{aligned}\tag{4.19}$$

Using Young's inequality and interpolation inequality
$$\|\nabla v\|\le\varsigma\|v\|+C_\varsigma\|\Delta v\|,\quad\forall\,\varsigma>0,$$
we have
$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho_k(x)\cdot u\cdot vdx=\int_{\mathbb{R}^n}\rho_k(x)u(\frac{du}{dt}+\delta u-\varepsilon y(\theta_t\omega)u)dx\\
=&\int_{\mathbb{R}^n}\rho_k(x)(\frac{1}{2}\frac{d}{dt}u^2+\delta u^2-\epsilon y(\theta_t\omega)u^2)dx\\
=&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)|u|^2dx+\delta\int_{\mathbb{R}^n}\rho_k(x)|u|^2dx-\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)|u|^2dx,
\end{aligned}\tag{4.20}$$
$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho_k(x)\cdot v\cdot\Delta^2udx=\int_{\mathbb{R}^n}\Delta^2u\cdot\rho_k(x)\cdot(\frac{du}{dt}+\delta u-\epsilon y(\theta_t\omega)u)dx\\
=&\int_{\mathbb{R}^n}\Delta u\cdot\Big(\Delta\rho_k(x)\cdot v+2\nabla\rho_k(x)\cdot\nabla v+\rho_k(x)\cdot\Delta(\frac{du}{dt}+\delta u-\epsilon y(\theta_t\omega)u)\Big)dx\\
\ge&-\frac{c_5}{k}\int_{\mathbb{R}^n}|\Delta u\cdot v|dx-\frac{2c_4}{k}\int_{\mathbb{R}^n}|\Delta u\cdot\nabla v|dx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx+\delta\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx\\
&-\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx\\
\ge&-\frac{c_5}{2k}(\|\Delta u\|^2+\|v\|^2)-\frac{2c_4}{k}\|\Delta u\|(\varsigma\|v\|+C_\varsigma\|\Delta v\|)+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx\\
&+\delta\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx-\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx
\end{aligned}$$

$$\geq -\frac{c_5}{k}(\|\Delta u\|^2+\|v\|^2)-\frac{c_4}{k}(\|\Delta u\|^2+2\varsigma^2\|v\|^2+2C_\varsigma^2\|\Delta v\|^2)$$
$$+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx+\delta\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx-\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx, \tag{4.21}$$

$$\int_{\mathbb{R}^n}\rho_k(x)\cdot v\cdot\Delta^2 vdx=\int_{\mathbb{R}^n}\Delta^2 v\cdot\rho_k(x)\cdot vdx=\int_{\mathbb{R}^n}\Delta v\cdot\Delta(\rho_k(x)\cdot v))dx$$
$$=\int_{\mathbb{R}^n}\Delta v\cdot(\Delta\rho_k(x)\cdot v+2\nabla\rho_k(x)\cdot\nabla v+\rho_k(x)\cdot\Delta v)dx$$
$$\geq -\frac{c_5}{2k}(\|\Delta v\|^2+\|v\|^2)-\frac{2c_4}{k}\|\Delta v\|\|\nabla v\|+\int_{\mathbb{R}^n}\rho_k(x)|\Delta v|^2dx$$
$$\geq -\frac{c_5}{2k}(\|\Delta v\|^2+\|v\|^2)-\frac{2c_4}{k}\|\Delta v\|(\varsigma\|v\|+C_\varsigma\|\Delta v\|)+\int_{\mathbb{R}^n}\rho_k(x)|\Delta v|^2dx$$
$$\geq -\frac{c_5}{2k}(\|\Delta v\|^2+\|v\|^2)-\frac{c_4}{k}(\|\Delta v\|^2+2\varsigma^2\|v\|^2+2C_\varsigma^2\|\Delta v\|^2)+\int_{\mathbb{R}^n}\rho_k(x)|\Delta v|^2dx, \tag{4.22}$$

$$\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)\cdot v\cdot\Delta^2 udx=\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\Delta u\cdot\Delta\big(\rho_k(x)\cdot v\big)dx$$
$$=\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\Delta u\cdot(\Delta\rho_k(x)\cdot v+2\nabla\rho_k(x)\cdot\nabla v+\rho_k(x)\cdot\Delta v)dx$$
$$\geq -\frac{c_5}{2k}\epsilon|y(\theta_t\omega)|(\|\Delta u\|^2+\|v\|^2)-\frac{2c_4}{k}\epsilon|y(\theta_t\omega)|\|\Delta u\|\|\nabla v\|$$
$$-\frac{1}{4}\epsilon|y(\theta_t\omega)|^2\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx-\int_{\mathbb{R}^n}\rho_k(x)|\Delta v|^2dx$$
$$\geq -\frac{c_5}{2k}\epsilon|y(\theta_t\omega)|(\|\Delta u\|^2+\|v\|^2)-\frac{c_4}{k}\epsilon|y(\theta_t\omega)||y(\theta_t\omega)|(2\|\Delta u\|^2+\varsigma^2\|v\|^2$$
$$+C_\varsigma^2\|\Delta v\|^2)-\frac{1}{4}\epsilon|y(\theta_t\omega)|^2\int_{\mathbb{R}^n}\rho_k(x)|\Delta u|^2dx-\int_{\mathbb{R}^n}\rho_k(x)|\Delta v|^2dx, \tag{4.23}$$

$$\int_{\mathbb{R}^n}\rho_k(x)F(x,u)vdx=\int_{\mathbb{R}^n}\rho_k(x)F(x,u)(\frac{du}{dt}+\delta u-\epsilon y(\theta_t\omega)u)dx$$
$$=\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)\widetilde{F}(x,u)dx+\delta\int_{\mathbb{R}^n}\rho_k(x)F(x,u)udx-\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)F(x,u)udx. \tag{4.24}$$

By (3.1)-(3.3), we have

$$\delta\int_{\mathbb{R}^n}\rho_k(x)F(x,u)udx\geq c_2\delta\int_{\mathbb{R}^n}\rho_k(x)\widetilde{F}(x,u)dx+\delta\int_{\mathbb{R}^n}\rho_k(x)\phi_2(x)dx, \tag{4.25}$$

$$\epsilon y(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)F(x,u)udx\leq\epsilon cy(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)\widetilde{F}(x,u)dx+\epsilon cy(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)|u|^2dx$$
$$+\epsilon cy(\theta_t\omega)\int_{\mathbb{R}^n}\rho_k(x)(|\phi_1|^2+|\phi_3|)dx. \tag{4.26}$$

By (3.5) we get

$$\int_{\mathbb{R}^n}\rho_k(x)\cdot f(x,u(t-\rho,x))\cdot vdx\leq l_f\int_{\mathbb{R}^n}\rho_k(x)|u(t-\rho)|\cdot|v|dx$$
$$\leq\frac{4l_f^2}{\alpha-\delta}\int_{\mathbb{R}^n}\rho_k(x)|u(t-\rho)|^2dx+\frac{\alpha-\delta}{16}\int_{\mathbb{R}^n}\rho_k(x)|v|^2dx. \tag{4.27}$$

By (4.19)-(4.27), we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)\big(|v|^2+(\delta^2+\lambda-\delta\alpha)|u|^2+(1-\delta)|\Delta u|^2+2\widetilde{F}(x,u)\big)dx+\big(2\sigma-c|\epsilon|$$

$$
\begin{aligned}
&- c\epsilon|y(\theta_t\omega)|^2\Big)\int_{\mathbb{R}^n}\rho_k(x)\big(|v|^2+(\delta^2+\lambda-\delta\alpha)|u|^2+(1-\delta)|\Delta u|^2+2\widetilde{F}(x,u)\big)dx\\
\leq & c\int_{\mathbb{R}^n}\rho_k(x)(|\phi_2|+|\phi_3|+|g|^2)dx+c\epsilon\big(1+|y(\theta_t\omega)|^2\big)\int_{\mathbb{R}^n}\rho_k(x)(|\phi_1|^2+|\phi_3|)dx\\
&+\frac{8l_f^2}{\alpha-\delta}\int_{\mathbb{R}^n}\rho_k(x)|u(t-\rho)|^2dx-\frac{\delta(\delta^2+\lambda-\delta\alpha)}{2}\int_{\mathbb{R}^n}\rho_k(x)|u|^2dx\\
&+\frac{c}{k}\bigg(\big(1-\delta+\epsilon|y(\theta_t\omega)|\big)\|\Delta u\|^2+\big(2-\delta+\epsilon|y(\theta_t\omega)|\big)\|v\|^2+\|\Delta v\|^2\bigg)\\
&+\frac{c}{k}\bigg(\big(1-\delta+\epsilon|y(\theta_t\omega)|\big)\|\Delta u\|^2+2\varsigma^2\big(2-\delta+\epsilon|y(\theta_t\omega)|\big)\|v\|^2\\
&+\big(2C_\varsigma^2(2-\delta+\epsilon|y(\theta_t\omega)|)+1\big)\|\Delta v\|^2\bigg). \qquad (4.28)
\end{aligned}
$$

As $\phi_1\in L^2(\mathbb{R}^n)$ and $\phi_2,\ \phi_3\in L^1(\mathbb{R}^n)$, we know that for given $\eta>0$, there exists $K_1=K_1(\eta)\geq 1$ such that for all $k\geq K_1$,

$$
\begin{aligned}
&c\int_{\mathbb{R}^n}\rho_k(x)(|\phi_1|^2+|\phi_2|+|\phi_3|)dx=c\int_{|x|\geq k}\rho_k(x)(|\phi_1|^2+|\phi_2|+|\phi_3|)dx\\
\leq & c\int_{|x|\geq k}(|\phi_1|^2+|\phi_2|+|\phi_3|)dx\leq\eta. \qquad (4.29)
\end{aligned}
$$

By (4.28) and (4.29), there exists $K_2=K_2(\eta)\geq K_1$ such that for all $k\geq K_2$,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_k(x)\big(|v|^2+(\delta^2+\lambda-\delta\alpha)|u|^2+(1-\delta)|\Delta u|^2+2\widetilde{F}(x,u)\big)dx+\big(2\sigma-c\epsilon\\
&- c\epsilon|y(\theta_t\omega)|^2\big)\int_{\mathbb{R}^n}\rho_k(x)\big(|v|^2+(\delta^2+\lambda-\delta\alpha)|u|^2+(1-\delta)|\Delta u|^2+2\widetilde{F}(x,u)\big)dx\\
\leq & \epsilon\eta(1+|y(\theta_t\omega)|^2)+c\int_{|x|\geq k}|g(t,x)|^2dx+\eta\ (1+\|\Delta u\|^2+\|\Delta v\|^2+\|v\|^2)\\
&+\frac{8l_f^2}{\alpha-\delta}\int_{\mathbb{R}^n}\rho_k(x)|u(t-\rho)|^2dx-\frac{\delta(\delta^2+\lambda-\delta\alpha)}{2}\int_{\mathbb{R}^n}\rho_k(x)|u|^2dx. \qquad (4.30)
\end{aligned}
$$

Integrating (4.30) over $(\tau-t,\tau+s)$ for any fixed $s\in[-\rho,0]$ with $t>\rho$, then we replace ω by $\theta_{-\tau}\omega$ in the resulting inequality and use a similar calculation with (4.15), we obtain for all $k\geq K_2$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho_k(x)(Y(\tau+s,\tau-t,\theta_{-\tau}\omega,Y_0)+2\widetilde{F}(x,u(\tau+s,\tau-t,\theta_{-\tau}\omega,\phi)))dx\\
\leq & e^{\int_{\tau+s}^{\tau-t}(2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\int_{\mathbb{R}^n}\rho_k(x)(Y_0+2\widetilde{F}(x,\phi))dx\\
&+\epsilon\eta\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(1+|y(\theta_{r-\tau}\omega)|^2)dr\\
&+c\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\int_{|x|\geq k}|g(r,x)|^2dxdr\\
&+\eta\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(1+\|\Delta u(\tau+r,\tau-t,\theta_{-\tau}\omega,\phi)\|^2\\
&\quad+\|\Delta v(\tau+r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2+\|v(\tau+r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2)dr\\
\leq & ce^{\int_0^{-t}(2\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}(\|Y_0\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+\|\phi\|^2_{C([\tau-\rho,\tau],H^2(\mathbb{R}^n))}+\|\phi\|^{\gamma+1}_{C([\tau-\rho,\tau],H^2(\mathbb{R}^n))})
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\eta\int_{-t}^{0} e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_\varrho\omega)|^2)d\varrho}(1+|y(\theta_r\omega)|^2)dr\\
&+c\int_{-t}^{0} e^{\int_0^s(2\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}\int_{|x|\geq k}|g(r+\tau,x)|^2dxdr\\
&+\eta\int_{-t}^{0} e^{\int_0^s(2\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}dr\\
&+\eta\int_{\tau-t}^{\tau+s} e^{\int_\tau^s(2\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}(\|\Delta u(\tau+r,\tau-t,\theta_{-\tau}\omega,\phi)\|^2\\
&\quad+\|\Delta v(\tau+r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2+\|v(\tau+r,\tau-t,\theta_{-\tau}\omega,\psi)\|^2)dr. \qquad (4.31)
\end{aligned}
$$

Due to $(\phi,\psi)^\top\in D(\tau-t,\theta_{-t}\omega)\in\mathcal{D}$, we know that there exists a positive $T=T(\tau,\omega,D,\eta)$ such that for all $t\geq T$,

$$
ce^{\int_0^{-t}(2\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}(\|Y_0\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+\|\phi\|^2_{C([\tau-\rho,\tau],H^2(\mathbb{R}^n))}+\|\phi\|^{\gamma+1}_{C([\tau-\rho,\tau],H^2(\mathbb{R}^n))})\leq\eta. \qquad (4.32)
$$

By (3.8), we know that there is $K_3=K_3(\tau,\eta)>K_2$, such that for all $k>K_3$,

$$
c\int_{-\infty}^{0} e^{\int_0^s(2\sigma-\varepsilon c-\varepsilon c|y(\theta_\varrho\omega)|^2)d\varrho}\int_{|x|\geq k}|g(r+\tau,x)|^2dxdr\leq\eta. \qquad (4.33)
$$

By (4.16), we know that the following integral

$$
R_2(\tau,\omega)=\int_{-\infty}^{0} e^{\int_0^s(2\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}(1+|y(\theta_r\omega)|^2)dr \qquad (4.34)
$$

is convergent.

By (4.31)-(4.34) and Lemma 4.1, we get for all $t\geq T,k\geq K_3$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho_k(x)(Y(\tau+s,\tau-t,\theta_{-\tau}\omega,Y_0)+2\widetilde{F}(x,u(\tau+s,\tau-t,\theta_{-\tau}\omega,\phi)))dx\\
\leq&3\eta(1+R(\tau,\omega)+R_2(\tau,\omega)), \qquad (4.35)
\end{aligned}
$$

where $R(\tau,\omega)$ and $R_2(\tau,\omega)$ are given by (4.16) and (4.34), respectively. It follows from (3.3) and (4.35) that there exists $K_4=K_4(\tau,\eta)\geq K_3$ such that for all $k\geq K_4,t\geq T$, (4.18) hold. □

For every $x\in\mathbb{R}^n$ and $k\geq1$, we denote

$$
\begin{cases}
\widehat{u}(t,\tau,\omega,\widehat{\phi})=\widehat{\rho_k}(x)u(t,\tau,\omega,\phi),\\
\widehat{v}(t,\tau,\omega,\widehat{\psi})=\widehat{\rho_k}(x)v(t,\tau,\omega,\psi),
\end{cases} \qquad (4.36)
$$

where $\widehat{\rho_k}=1-\rho_k$. Then, for $k\geq1,x\in\mathbb{R}^n\backslash\mathbb{H}_k$, we have $\widehat{u}(t,\tau,\omega,\widehat{\phi})=\widehat{v}(t,\tau,\omega,\widehat{\psi})=0$. In addition, there is some constant $c>0$ independent of $k\geq1$, such that $\|\widehat{u}\|_{H^2(\mathbb{R}^n)}\leq c\|u\|_{H^2(\mathbb{R}^n)}$, $\|\widehat{v}\|_{L^2(\mathbb{R}^n)}\leq c\|v\|_{L^2(\mathbb{R}^n)}$. Accordingly, together with (3.11) and (4.36), we get

$$
\begin{cases}
\frac{d\widehat{u}}{dt}+\delta\widehat{u}-\widehat{v}=\epsilon y(\theta_t\omega)\widehat{u},\\
\frac{d\widehat{v}}{dt}+(\alpha-\delta)\widehat{v}+\Delta^2\widehat{v}+(\lambda+\delta^2-\delta\alpha)\widehat{u}+(1-\delta)\Delta^2\widehat{u}+\epsilon y(\theta_t\omega)\Delta^2\widehat{u}+\widehat{\rho_k}(x)F(x,u)\\
=\widehat{\rho_k}(x)g(x,t)+\widehat{\rho_k}(x)f(x,u(t-\rho,x))-\epsilon y(\theta_t\omega)\widehat{v}-\epsilon(\epsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)\widehat{u}\\
+4\Delta\nabla\widehat{\rho_k}(x)\nabla v+6\Delta\widehat{\rho_k}(x)\Delta v+4\nabla\widehat{\rho_k}(x)\Delta\nabla v+v\Delta^2\widehat{\rho_k}(x)+\epsilon y(\theta_t\omega)u\Delta^2\widehat{\rho_k}(x)\\
+4(1-\delta)\Delta\nabla\widehat{\rho_k}(x)\nabla u+6(1-\delta)\Delta\widehat{\rho_k}(x)\Delta u+4(1-\delta)\nabla\widehat{\rho_k}(x)\Delta\nabla u+(1-\delta)u\Delta^2\widehat{\rho_k}(x)\\
+4\epsilon y(\theta_t\omega)\Delta\nabla\widehat{\rho_k}(x)\nabla u+6\epsilon y(\theta_t\omega)\Delta\widehat{\rho_k}(x)\Delta u+4\epsilon y(\theta_t\omega)\nabla\widehat{\rho_k}(x)\Delta\nabla u,\\
\widetilde{u}_\tau(s,x)=\widehat{\rho_k}(x)\phi(s,x),x\in\mathbb{R}^n,s\in[-\rho,0];\ \widetilde{u}_\tau(s,x)=0,x\in\mathbb{R}^n\setminus\mathbb{H}_k,s\in[-\rho,0],\\
\widetilde{v}_\tau(s,x)=\widehat{\rho_k}(x)\psi(s,x),x\in\mathbb{R}^n,s\in[-\rho,0];\ \widetilde{v}_\tau(s,x)=0,x\in\mathbb{R}^n\setminus\mathbb{H}_k,s\in[-\rho,0].
\end{cases} \qquad (4.37)
$$

Considering the eigenvalue problem

$$\lambda\widehat{u} = A\widehat{u} \;\text{ in }\; \mathbb{H}_k, \;\text{ with }\; \frac{\partial\widehat{u}}{\partial n} = \widehat{u} = 0 \;\text{ on }\; \partial\mathbb{H}_k. \tag{4.40}$$

It is easy to see that eigenfunctions $\{e_i\}_{i\in\mathbb{N}}$ and eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ of (4.40) satisfy:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots, \; \lambda_i \to +\infty \; (i \to +\infty).$$

For given n, assume $X_n = \operatorname{span}\{e_1, \cdots, e_n\}$, $P_n : L^2(\mathbb{H}_k) \to X_n$.

Lemma 4.3 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. Then for every $\tau \in \mathbb{R}, s \in [-\rho, 0], \omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\eta) > 0$ and $K = K(\tau,\omega,\eta) \ge 1$ and $N = N(\tau,\omega,\eta) \ge 1$, such that for all $t \ge T, k \ge K$ and $n \ge N$,*

$$\|(I-P_n)\widehat{Y}(\tau+s, \tau-t, \theta_{-\tau}\omega, \widehat{Y}_0)\|^2_{H^2(\mathbb{H}_k)\times L^2(\mathbb{H}_k)} \le \eta. \tag{4.39}$$

Proof. Denote $\widehat{u}_{n,1} = P_n\widehat{u}$, $\widehat{u}_{n,2} = (I-P_n)\widehat{u}$, $\widehat{v}_{n,1} = P_n\widehat{v}$, $\widehat{v}_{n,2} = (I-P_n)\widehat{v}$. Multiplying $(4.37)_1$ with $I - P_n$, we get

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \epsilon y(\theta_t\omega)\widehat{u}_{n,2}. \tag{4.40}$$

Multiplying $(4.37)_2$ with $I - P_n$ then taking inner product with $\widehat{v}_{n,2}$ in $L^2(\mathbb{H}_k)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2 + (\alpha-\delta)\|\widehat{v}_{n,2}\|^2 + \|\Delta\widehat{v}_{n,2}\|^2 + (\delta^2+\lambda-\delta\alpha)(\widehat{u}_{n,2},\widehat{v}_{n,2})\\
&+ (1-\delta)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2}) + \epsilon y(\theta_t\omega)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2}) + (\widehat{\rho_k}(x)F(x,u),\widehat{v}_{n,2})\\
=&(\widehat{\rho_k}(x)g + \widehat{\rho_k}(x)f(x,u(t-\rho,x)) + 4\Delta\nabla\widehat{\rho_k}(x)\nabla v + 6\Delta\widehat{\rho_k}(x)\Delta v + 4\nabla\widehat{\rho_k}(x)\Delta\nabla v + v\Delta^2\widehat{\rho_k}(x)\\
&+ 4(1-\delta)\Delta\nabla\widehat{\rho_k}(x)\nabla u + 6(1-\delta)\Delta\widehat{\rho_k}(x)\Delta u + 4(1-\delta)\nabla\widehat{\rho_k}(x)\Delta\nabla u + (1-\delta)u\Delta^2\widehat{\rho_k}(x)\\
&+ 4\epsilon y(\theta_t\omega)\Delta\nabla\widehat{\rho_k}(x)\nabla u + 6\epsilon y(\theta_t\omega)\Delta\widehat{\rho_k}(x)\Delta u + 4\epsilon y(\theta_t\omega)\nabla\widehat{\rho_k}(x)\Delta\nabla u + \epsilon y(\theta_t\omega)u\Delta^2\widehat{\rho_k}(x), \widehat{v}_{n,2})\\
&- \epsilon y(\theta_t\omega)\|\widehat{v}_{n,2}\|^2 - \epsilon(\epsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)(\widehat{u}_{n,2},\widehat{v}_{n,2}),
\end{aligned} \tag{4.41}$$

together with (4.40), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\widehat{v}_{n,2}\|^2 + (\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2 + (1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\
&+ 2(\alpha-\delta)\|\widehat{v}_{n,2}\|^2 + 2\delta(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2 + 2\delta(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2\\
=&2(4\Delta\nabla\widehat{\rho_k}(x)\nabla v + 6\Delta\widehat{\rho_k}(x)\Delta v + 4\nabla\widehat{\rho_k}(x)\Delta\nabla v + v\Delta^2\widehat{\rho_k}(x) + 4(1-\delta)\Delta\nabla\widehat{\rho_k}(x)\nabla u\\
&+ 6(1-\delta)\Delta\widehat{\rho_k}(x)\Delta u + 4(1-\delta)\nabla\widehat{\rho_k}(x)\Delta\nabla u + (1-\delta)u\Delta^2\widehat{\rho_k}(x) + 4\epsilon y(\theta_t\omega)\Delta\nabla\widehat{\rho_k}(x)\nabla u\\
&+ 6\epsilon y(\theta_t\omega)\Delta\widehat{\rho_k}(x)\Delta u + 4\epsilon y(\theta_t\omega)\nabla\widehat{\rho_k}(x)\Delta\nabla u + \epsilon y(\theta_t\omega)u\Delta^2\widehat{\rho_k}(x), \widehat{v}_{n,2})\\
&+ 2(\widehat{\rho_k}(x)g + \widehat{\rho_k}(x)f(x,u(t-\rho,x)), \widehat{v}_{n,2}) - 2\epsilon y(\theta_t\omega)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2}) - 2\epsilon y(\theta_t\omega)\|\widehat{v}_{n,2}\|^2\\
&- 2\epsilon(\epsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)(\widehat{u}_{n,2},\widehat{v}_{n,2}) + 2\epsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|\widehat{u}_{n,2}\|^2\\
&+ 2\epsilon(1-\delta)y(\theta_t\omega)\|\Delta\widehat{u}_{n,2}\|^2 - 2\|\Delta\widehat{v}_{n,2}\|^2 - 2(\widehat{\rho_k}(x)F(x,u),\widehat{v}_{n,2}).
\end{aligned} \tag{4.42}$$

Now, we estimate all the terms on the right-hand side of (4.42) as follows.

$$\begin{aligned}
&(4\Delta\nabla\widehat{\rho_k}(x)\cdot\nabla v + 6\Delta\widehat{\rho_k}(x)\cdot\Delta v + 4\nabla\widehat{\rho_k}(x)\cdot\Delta\nabla v + v\Delta^2\widehat{\rho_k}(x), \widehat{v}_{n,2})\\
\le&\frac{4c_6}{k}\lambda_{n+1}^{-\frac{1}{4}}\|\Delta v\|\cdot\|\widehat{v}_{n,2}\| + \frac{6c_5}{k}\|\Delta v\|\cdot\|\widehat{v}_{n,2}\| + \frac{4c_4}{k}\lambda_{n+1}^{-\frac{1}{4}}\|\Delta v\|\cdot\|\Delta\widehat{v}_{n,2}\| + \frac{c_7}{k}\|v\|\cdot\|\widehat{v}_{n,2}\|\\
\le&c\lambda_{n+1}^{-\frac{1}{2}}\|\Delta v\|^2 + \frac{1}{6}\|\Delta\widehat{v}_{n,2}\|^2 + \frac{c}{k}(\|v\|^2 + \|\Delta v\|^2) + \frac{\alpha-\delta}{8}\|\widehat{v}_{n,2}\|^2,
\end{aligned} \tag{4.43}$$

$$\begin{aligned}&(1-\delta)(4\Delta\nabla\widehat{\rho_k}(x)\cdot\nabla u+6\Delta\widehat{\rho_k}(x)\cdot\Delta u+4\nabla\widehat{\rho_k}(x)\cdot\Delta\nabla u+u\Delta^2\widehat{\rho_k}(x),\widehat{v}_{n,2})\\ \le& c\lambda_{n+1}^{-\frac{1}{2}}\|\Delta u\|^2+\frac{1}{6}\|\Delta\widehat{v}_{n,2}\|^2+\frac{c}{k}(\|u\|^2+\|\Delta u\|^2)+\frac{\alpha-\delta}{8}\|\widehat{v}_{n,2}\|^2,\end{aligned}\tag{4.44}$$

$$\begin{aligned}&\epsilon y(\theta_t\omega)(4\Delta\nabla\widehat{\rho_k}(x)\cdot\nabla u+6\Delta\widehat{\rho_k}(x)\cdot\Delta u+4\nabla\widehat{\rho_k}(x)\cdot\Delta\nabla u+u\Delta^2\widehat{\rho_k}(x),\widehat{v}_{n,2})\\ \le& c\lambda_{n+1}^{-\frac{1}{2}}\|\Delta u\|^2+\frac{1}{6}\|\Delta\widehat{v}_{n,2}\|^2+\frac{c}{k}(\|u\|^2+\|\Delta u\|^2)+\frac{\alpha-\delta}{8}\|\widehat{v}_{n,2}\|^2,\end{aligned}\tag{4.45}$$

$$\begin{aligned}&-2\epsilon y(\theta_t\omega)(\Delta^2\widehat{u}_{n,2},\widehat{v}_{n,2})-2\epsilon y(\theta_t\omega)\|\widehat{v}_{n,2}\|^2-2\epsilon(\epsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(\widehat{u}_{n,2},\widehat{v}_{n,2})\\ &\quad+2\epsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|\widehat{u}_{n,2}\|^2+2\epsilon(1-\delta)y(\theta_t\omega)\|\Delta\widehat{u}_{n,2}\|^2\\ \le&\epsilon c(1+|y(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)+\frac{1}{2}\|\Delta\widehat{v}_{n,2}\|^2,\end{aligned}\tag{4.46}$$

$$2(\widehat{\rho_k}(x)g,\widehat{v}_{n,2})=2((I-P_n)\widehat{\rho_k}(x)g,\widehat{v}_{n,2})\le\frac{1}{4}(\alpha-\delta)\|\widehat{v}_{n,2}\|^2+c\|(I-P_n)(\widehat{\rho_k}(x)g)\|^2,\tag{4.47}$$

$$2(\widehat{\rho_k}(x)f(x,u(t-\rho,x)),\widehat{v}_{n,2})\le\frac{8l_f^2}{\alpha-\delta}\|\widehat{u}_{n,2}(t-\rho,x)\|^2+\frac{\alpha-\delta}{8}\|\widehat{v}_{n,2}\|^2.\tag{4.48}$$

By (3.1) and Gagliardo-Nirenberg interpolation inequality, we set $\theta=\frac{n(\gamma-1)}{4(\gamma+1)}$, then

$$\begin{aligned}&|-2(\widehat{\rho_k}(x)F(x,u),\widehat{v}_{n,2})|\le c_1\int_{\mathbb{R}^n}\widehat{\rho_k}(x)|u|^\gamma|\widehat{v}_{n,2}|dx+\int_{\mathbb{R}^n}\widehat{\rho_k}(x)|\phi_1(x)||\widehat{v}_{n,2}|dx\\ \le& c_1\|u\|_{\gamma+1}^\gamma\|\widehat{v}_{n,2}\|_{\gamma+1}+\|\phi_1\|\|\widehat{v}_{n,2}\|\\ \le& c_1\|u\|_{\gamma+1}^\gamma\|\Delta\widehat{v}_{n,2}\|^\theta\|\widehat{v}_{n,2}\|^{1-\theta}+\lambda_{n+1}^{-\frac{1}{2}}\|\phi_1\|\|\Delta\widehat{v}_{n,2}\|\\ \le& c_1\lambda_{n+1}^{\frac{\theta-1}{2}}\|u\|_{H^2}^\gamma\|\Delta\widehat{v}_{n,2}\|+\lambda_{n+1}^{-\frac{1}{2}}\|\phi_1\|\|\Delta\widehat{v}_{n,2}\|\\ \le&\lambda_{n+1}^{-\frac{1}{2}}\|\Delta\widehat{v}_{n,2}\|(c_1\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^\gamma+\|\phi_1\|)\\ \le&\frac{1}{2}\|\Delta\widehat{v}_{n,2}\|^2+\frac{1}{2}\lambda_{n+1}^{-1}(c_1\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^\gamma+\|\phi_1\|)^2.\end{aligned}\tag{4.49}$$

By (4.42)-(4.49), we get

$$\begin{aligned}&\frac{d}{dt}(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\ &+7\sigma(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\ \le&\epsilon c(1+|y(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\ &+c\lambda_{n+1}^{-\frac{1}{2}}(\|\Delta v\|^2+\|\Delta u\|^2)+\frac{c}{k}(\|u\|^2+\|\Delta u\|^2+\|v\|^2+\|\Delta v\|^2)\\ &+c_6\|(I-P_n)(\widehat{\rho_k}(x)g)\|^2+\frac{1}{2}\lambda_{n+1}^{-1}(c_1\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^\gamma+\|\phi_1\|)^2\\ &+\frac{8l_f^2}{\alpha-\delta}\|\widehat{u}_{n,2}(t-\rho,x)\|^2-\frac{\delta}{2}(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2.\end{aligned}\tag{4.50}$$

Therefore, for given $\eta>0$, there exist $N_1=N_1(\eta)\ge 1$ and $K_1=K_1(\eta)\ge 1$ such for all $n\ge N_1$ and $k\ge K_1$,

$$\begin{aligned}&\frac{d}{dt}(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\ &+(7\sigma-\epsilon c-\epsilon c|y(\theta_t\omega)|^2)(\|\widehat{v}_{n,2}\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2+(1-\delta)\|\Delta\widehat{u}_{n,2}\|^2)\\ \le&\eta(1+\|u\|_{H^2(\mathbb{R}^n)}^{2\gamma})+c_6\|(I-P_n)(\widehat{\rho_k}(x)g)\|^2\end{aligned}$$

$$+\frac{8l_f^2}{\alpha-\delta}\|\widehat{u}_{n,2}(t-\rho,x)\|^2-\frac{\delta}{2}(\delta^2+\lambda-\delta\alpha)\|\widehat{u}_{n,2}\|^2. \tag{4.51}$$

Integrating (4.51) over $(\tau-t,\tau+s)$ for any fixed $s\in[-\rho,0]$ with $t>\rho$, then we replace ω by $\theta_{-\tau}\omega$ in the resulting inequality and use a similar calculation with (4.15), we get for all $n\geq N_1$ and $k\geq K_1$,

$$\begin{aligned}
&\|\widehat{Y}_{n,2}(\tau+s,\tau-t,\theta_{-\tau}\omega,\widehat{Y}_{n,2,0})\|^2_{H^2(\mathbb{H}_k)\times L^2(\mathbb{H}_k)}\\
\leq & ce^{\int_{\tau+s}^{\tau-t}(7\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|\widehat{Y}_{n,2,0}\|^2_{H^2(\mathbb{H}_k)\times L^2(\mathbb{H}_k)}\\
&+\eta\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(7\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}dr\\
&+\eta\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(7\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,\phi)\|^{2\gamma}_{H^2(\mathbb{R}^n)}dr\\
&+c\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(7\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|(I-P_n)(\widehat{\rho_k}(x)g)(r)\|^2dr\\
\leq & ce^{\int_0^{-t}(7\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}\|\widehat{Y}_{n,2,0}\|^2_{H^2(\mathbb{H}_k)\times L^2(\mathbb{H}_k)}\\
&+\eta\int_{-t}^{0}e^{\int_0^s(7\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho}\omega)|^2)d\varrho}dr\\
&+\eta\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(7\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,\phi)\|^{2\gamma}_{H^2(\mathbb{R}^n)}dr\\
&+c\int_{-t}^{0}e^{\int_0^s(7\sigma-\varepsilon c-\varepsilon c|y(\theta_{\varrho}\omega)|^2)d\varrho}\|(I-P_n)(\widehat{\rho_k}(x)g)(r+\tau)\|^2dr.
\end{aligned}\tag{4.52}$$

For the first term in (4.52), there exists $T_1=T_1(\tau,\omega,D,\eta)>0$ such that for all $t\geq T_1$,

$$ce^{\int_0^{-t}(7\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}\|\widehat{Y}_{n,2,0}\|^2_{H^2(\mathbb{H}_k)\times L^2(\mathbb{H}_k)}\leq\eta. \tag{4.53}$$

By Lemma 4.1 we have

$$\eta\int_{\tau-t}^{\tau+s}e^{\int_\tau^s(7\sigma-\epsilon c-\epsilon c|y(\theta_{\varrho-\tau}\omega)|^2)d\varrho}\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,\phi)\|^{2\gamma}_{H^2(\mathbb{R}^n)}dr\leq\eta R^\gamma(\tau,\omega), \tag{4.54}$$

where $R(\tau,\omega)$ is given by Lemma 4.1.

By (3.7) we see that there exists $N_2=N_2(\tau,\omega,\eta)\geq N_1$ such for all $n\geq N_2$,

$$\int_{-\infty}^{0}e^{\int_0^s(7\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}\|(I-P_n)(\widehat{\rho}g)(r+\tau)\|^2dr\leq\eta, \tag{4.55}$$

which along with (4.52)-(4.55) implies (4.39). □

5 Existence of pullback random attractors.

In this section, we establish the existence and uniqueness of random attractors for problem (3.11). We can easily obtain the existence of random absorbing sets of Φ from Lemma 4.1.

Lemma 5.1 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. Then for every $\epsilon\in(0,1],\tau\in\mathbb{R}$ and $\omega\in\Omega$, the continuous cocycle Φ has a closed measurable $\mathcal{D}$-pullback absorbing set $K_\epsilon=\{K_\epsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ as defined by*

$$K_\epsilon(\tau,\omega)=\{Y\in C_{V_2,V_0(\mathbb{R}^n)}:\|Y\|^2_{C_{V_2,V_0(\mathbb{R}^n)}}\leq R_\epsilon(\tau,\omega)\},$$

with $Y=(u,v)^\top$ and $R_\epsilon(\tau,\omega)$ being a positive number given by

$$R_\epsilon(\tau,\omega)=M+M\int_{-\infty}^{0}e^{\int_\tau^s(2\sigma-\epsilon c-\epsilon c|y(\theta_\varrho\omega)|^2)d\varrho}(1+\|g(r+\tau,\cdot)\|^2+\epsilon|y(\theta_r\omega)|^2)dr. \tag{5.1}$$

Next, we prove the asymptotic compactness of the cocycle Φ in $C_{V_2,V_0}(\mathbb{R}^n)$.

Lemma 5.2 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. Then the cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in $C_{V_2,V_0}(\mathbb{R}^n)$.*

Proof. For every $\tau\in\mathbb{R},\omega\in\Omega$ and $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, we need to prove that the sequence $\{Y_\tau(\cdot,\tau-t_n,\theta_{-\tau}\omega,Y_{0,n})\}$ has a convergent subsequence in $C_{V_2,V_0}(\mathbb{R}^n)$ whenever $t_n\to\infty$ and $Y_{0,n}\in D(\tau-t_n,\theta_{-\tau}\omega)$.

By Lemma 4.1, $\{Y_\tau(\cdot,\tau-t_n,\theta_{-\tau}\omega,Y_{0,n})\}$ is bounded in $C_{V_2,V_0}(\mathbb{R}^n)$; that is, for every $\tau\in\mathbb{R},\omega\in\Omega$, there exists $\widehat{N}_1=\widehat{N}_1(\tau,\omega,D)>0$ such for all $n>\widehat{N}_1$,

$$\|Y_\tau(\cdot,\tau-t_n,\theta_{-\tau}\omega,Y_{0,n})\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\le R_\epsilon(\tau,\omega). \tag{5.2}$$

In addition, it follows from Lemma 4.2 that there exist $k_1=k_1(\tau,\eta,\omega)>0$ and $\widehat{N}_2=\widehat{N}_2(\tau,D,\eta,\omega)>0$, such that for every $n\ge\widehat{N}_2$ and every fixed $s\in[-\rho,0]$,

$$\|Y(\tau+s,\tau-t_n,\theta_{-\tau}\omega,Y_{0,n})\|^2_{H^2(\mathbb{R}^n\setminus\mathbb{H}_{k_1})\times L^2(\mathbb{R}^n\setminus\mathbb{H}_{k_1})}\le\eta. \tag{5.3}$$

Next, by using Lemma 4.3, there are $N=N(\tau,\eta,\omega)>0$, $k_2=k_2(\tau,\eta,\omega)\ge k_1$ and $\widehat{N}_3=\widehat{N}_3(\tau,D,\eta,\omega)>0$, such that for every $n\ge\widehat{N}_3$ and every fixed $s\in[-\rho,0]$,

$$\|(I-P_N)\widehat{Y}(\tau+s,\tau-t_n,\theta_{-\tau}\omega,\widehat{Y}_{0,n})\|^2_{H^2(\mathbb{H}_{2k_2})\times L^2(\mathbb{H}_{2k_2})}\le\eta. \tag{5.4}$$

Using (4.36) and (5.2), we find that $\{P_N\widehat{Y}(\tau+s,\tau-t_n,\theta_{-\tau}\omega,\widehat{Y}_{0,n})\}$ is bounded in the finite-dimensional space $P_NH^2(\mathbb{H}_{2k_2})\times L^2(\mathbb{H}_{2k_2})$, which together with (5.4) implies that $\{\widehat{Y}(\tau+s,\tau-t_n,\theta_{-\tau}\omega,\widehat{Y}_{0,n})\}$ is precompact in $H^2(\mathbb{H}_{2k_2})\times L^2(\mathbb{H}_{2k_2})$.

Note that $\widehat{\rho_{k_2}}(x)=1$ for $|x|\le\frac{k_2}{2}$. Recalling (4.36), we find that $\{Y(\tau+s,\tau-t_n,\theta_{-\tau}\omega,Y_{0,n})\}$ is precompact in $H^2(\mathbb{H}_{k_2})\times L^2(\mathbb{H}_{k_2})$, which along with (5.3) shows that the precompactness of this sequence in $C_{V_2,V_0}(\mathbb{R}^n)$ for every fixed $s\in[-\rho,0]$. This completes the proof. □

As an immediate consequence of Proposition 2.1, Lemma 5.1 and Lemma 5.2, we have

Theorem 5.1 *Assume that (3.1)-(3.5), (3.7)-(3.8), (3.14)-(3.16) hold. Then for every $\epsilon\in(0,1]$, the continuous cocycle Φ associated with (3.11) has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_\epsilon=\{\mathcal{A}_\epsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ in $C_{V_2,V_0}(\mathbb{R}^n)$. If, in addition, there exists $T>0$ such that $g(t)$ is T-periodic in $t\in\mathbb{R}$ in $L^2(\mathbb{R}^n)$, then the attractor $\mathcal{A}_\epsilon$ is also T-periodic.*

6 Upper semi-continuity of attractors as intensity of noise approaches zero.

In this section, we establish the upper semi-continuity of random attractors of the plate equation (3.11) with delay driven by additive noise when $\epsilon\to0$. We write the solution and the corresponding cocycle of (3.11) as u^ϵ, v^ϵ and Φ_ϵ, respectively.

In section 5, we have get that Φ_ϵ has a $\mathcal{D}$-pullback attractor $\mathcal{A}_\epsilon\in\mathcal{D}$ in $C_{V_2,V_0}(\mathbb{R}^n)$ and a closed measurable $\mathcal{D}$-pullback absorbing set $K_\epsilon=\{K_\epsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ with $K_\epsilon(\tau,\omega)\subseteq K(\tau,\omega)$ for all $\epsilon\in(0,1]$, where for every $\tau\in\mathbb{R},\omega\in\Omega$,

$$K(\tau,\omega)=\{Y\in C_{V_2,V_0}(\mathbb{R}^n):\|Y\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\le R(\tau,\omega)\},$$

and

$$R(\tau,\omega)=M+M\int_{-\infty}^{0}e^{\int_\tau^s(2\sigma-c-c|y(\theta_\varrho\omega)|^2)d\varrho}(1+\|g(r+\tau,\cdot)\|^2+|y(\theta_r\omega)|^2)dr.$$

From Lemma 5.1 we know for $\tau\in\mathbb{R},\omega\in\Omega$,

$$\bigcup_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\tau,\ \omega)\subseteq\bigcup_{0<\epsilon\le 1}K_\epsilon(\omega)\subseteq K(\tau,\ \omega). \tag{6.1}$$

As $\epsilon=0$, the random problem (3.11) reduces to a deterministic one:

$$\begin{cases}\frac{du}{dt}+\delta u-v=0,\\ \frac{dv}{dt}+(\alpha-\delta)v+\Delta^2v+(\delta^2+\lambda-\delta\alpha)u+(1-\delta)\Delta^2u+F(x,u(t,x))\\ \qquad\qquad =f(x,u(t-\rho,x))+g(x,t),\quad t>\tau,\\ u_\tau(s,x)=\phi(s,x),\quad x\in\mathbb{R}^n,\quad s\in[-\rho,0],\\ v_\tau(s,x)=\partial_t\phi(s,x)+\delta\phi(s,x):=\widehat{\psi}(s,x),\quad x\in\mathbb{R}^n,\quad s\in[-\rho,0].\end{cases} \tag{6.2}$$

Accordingly, by Theorem 5.1 the cocycle Φ_0 generated by (6.2) is readily verified to admit a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0=\{\mathcal{A}_0(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}_0$ in $C_{V_2,V_0}(\mathbb{R}^n)$ and a $\mathcal{D}_0$-pullback absorbing set $K_0=\{K_0(\tau):\tau\in\mathbb{R}\}$, where

$$\mathcal{D}_0=\left\{D=\{D(\tau)\subseteq C_{V_2,V_0}(\mathbb{R}^n):\tau\in\mathbb{R},\quad \lim_{t\to+\infty}e^{-\gamma t}\|D(\tau-t)\|_{C_{V_2,V_0}(\mathbb{R}^n)}=0,\quad\forall\ \gamma>0\}\right\}$$

and

$$K_0(\tau)=\{Y\in C_{V_2,V_0}(\mathbb{R}^n):\|Y\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\le R_0(\tau)\}, \tag{6.3}$$

with $R_0(\tau)$ being a positive number defined by

$$R_0(\tau)=M+M\int_{-\infty}^{0}e^{2\sigma s}(1+\|g(r+\tau,x)\|^2)dr. \tag{6.4}$$

Note that $R_0(\tau)$ corresponds to the number $R_\epsilon(\tau,\omega)$ given by (5.1) with $\epsilon=0$. By Lemma 5.1, (6.3) and (6.4) we get that for all $\tau\in\mathbb{R},\omega\in\Omega$,

$$\limsup_{\epsilon\to 0}R_\epsilon(\tau,\ \omega)=R_0(\tau),\quad\text{and}\quad\limsup_{\epsilon\to 0}\|K_\epsilon(\tau,\ \omega)\|=\|K_0(\tau)\|. \tag{6.5}$$

Now, we will establish the convergence of solutions of (3.11) as $\epsilon\to 0$ to obtain the upper semi-continuity of the $\mathcal{D}$-pullback attractor $\mathcal{A}_\epsilon$.

Lemma 6.1. *Let $Y^\epsilon=(u^\epsilon,v^\epsilon)$ and $Y=(u,v)$ be the solutions of (3.11) and (6.2) with initial data $Y_0^\epsilon=(\phi^\epsilon,\psi^\epsilon)$ and $Y_0=(\phi,\widehat{\psi})$, respectively. Assume that (3.1)-(3.5) and (3.14) hold. If $\lim_{\epsilon\to 0}(\phi^\epsilon,\psi^\epsilon)=(\phi,\widehat{\psi})\in C_{V_2,V_0}(\mathbb{R}^n)$, then for every $\tau\in\mathbb{R},\omega\in\Omega$, $T>0$ and $t\in[\tau,\tau+T]$,*

$$\|Y_t^\epsilon(\cdot,\tau,\omega,Y_0^\epsilon)-Y_t(\cdot,\tau,Y_0)\|_{C_{V_2,V_0}(\mathbb{R}^n)}\to 0\quad\textit{as}\quad\epsilon\to 0. \tag{6.6}$$

Proof. Let $\big(u^\epsilon(t,\tau,\omega,\phi^\epsilon), v^\epsilon(t,\tau,\omega,\psi^\epsilon)\big)$ be the solution of (3.11) and $\tilde u = u^\epsilon - u, \tilde v = v^\epsilon - v$. Then by (3.11) and (6.2) we get that

$$\begin{cases} \frac{d\tilde u}{dt} + \delta\tilde u - \tilde v = \epsilon y(\theta_t\omega)\tilde u + \epsilon y(\theta_t\omega)u, \\ \frac{d\tilde v}{dt} + (\alpha-\delta)\tilde v + \Delta^2\tilde v + (\delta^2+\lambda-\delta\alpha)\tilde u + (1-\delta)\Delta^2\tilde u + \big(F(x,u^\epsilon) - F(x,u)\big) \\ = \big(f(x,u^\epsilon(t-\rho,x)) - f(x,u(t-\rho,x))\big) - \epsilon y(\theta_t\omega)\Delta^2\tilde u - \epsilon y(\theta_t\omega)\Delta^2 u - \epsilon y(\theta_t\omega)\tilde v \\ -\epsilon y(\theta_t\omega)v - \epsilon(\epsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)\tilde u - \epsilon(\epsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)u, \\ \tilde u_\tau(s,x) = \phi^\epsilon(s,x) - \phi(s,x), \quad x\in\mathbb{R}^n, \quad s\in[-\rho,0], \\ \tilde v_\tau(s,x) = \psi^\epsilon(s,x) - \tilde\psi(s,x), \quad x\in\mathbb{R}^n, \quad s\in[-\rho,0]. \end{cases} \tag{6.7}$$

Taking the inner product of the second equation of $(6.7)_2$ with $\tilde v$ in $L^2(\mathbb{R}^n)$, and then using the first equation of (6.7) to simplify the resulting equality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\tilde v\|^2 + (\delta^2+\lambda-\delta\alpha)\|\tilde u\|^2 + (1-\delta)\|\Delta\tilde u\|^2) \\ &+ (\alpha-\delta)\|\tilde v\|^2 + \delta(\delta^2+\lambda-\delta\alpha)\|\tilde u\|^2 + \delta(1-\delta)\|\Delta\tilde u\|^2 + \|\Delta\tilde v\|^2 \\ =&\big(F(x,u) - F(x,u^\epsilon),\tilde v\big) + \big(f(x,u^\epsilon(t-\rho,x)) - f(x,u(t-\rho,x)),\tilde v\big) \\ &+ \varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)\|\tilde u\|^2 + \varepsilon(\delta^2+\lambda-\delta\alpha)y(\theta_t\omega)(\tilde u,u) + \varepsilon(1-\delta)y(\theta_t\omega)\|\Delta\tilde u\|^2 \\ &+ \varepsilon(1-\delta)y(\theta_t\omega)(\Delta\tilde u,\Delta u) - (\varepsilon y(\theta_t\omega)\Delta^2\tilde u + \varepsilon y(\theta_t\omega)\Delta^2 u,\tilde v) \\ &- \varepsilon y(\theta_t\omega)\|\tilde v\|^2 - \varepsilon y(\theta_t\omega)(\tilde v,v) - \varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(\tilde v,u) \\ &- \varepsilon(\varepsilon y(\theta_t\omega)-2\delta)y(\theta_t\omega)(\tilde v,\tilde u). \end{aligned} \tag{6.8}$$

By (3.4), we get

$$|\big(F(x,u^\epsilon) - F(x,u),\tilde v\big)| \le c\|\tilde u\|^2 + c\|\tilde v\|^2. \tag{6.9}$$

By (3.5), we get

$$\begin{aligned} &|\big(f(x,u^\epsilon(t-\rho,x)) - f(x,u(t-\rho,x)),\tilde v\big)| \le l_f\|\tilde u(t-\rho,x)\|\cdot\|\tilde v\| \\ \le& c\|\tilde u(t-\rho,x)\|^2 + c\|\tilde v\|^2. \end{aligned} \tag{6.10}$$

Thanks to Young's inequality, we find the remaining terms on the right hand side of (6.8) are controlled by $\epsilon c(1+|y(\theta_t\omega)|^2)(\|\tilde u\|^2_{H^2(\mathbb{R}^n)} + \|\tilde v\|^2 + \|u\|^2_{H^2(\mathbb{R}^n)} + \|v\|^2)$ for all $\epsilon\le 1$.

Applying Lemma 4.1, there exists a constant $c_0 = c_0(\tau,\omega,R_1,T) > 0$ such that for all $t\ge T$,

$$\|u\|^2_{H^2(\mathbb{R}^n)} + \|v\|^2 \le c_0. \tag{6.11}$$

It follows from (6.8)-(6.11) that

$$\begin{aligned} &\frac{d}{dt}(\|\tilde v\|^2 + (\delta^2+\lambda-\delta\alpha)\|\tilde u\|^2 + (1-\delta)\|\Delta\tilde u\|^2) \\ \le& c(\|\tilde v\|^2 + (\delta^2+\lambda-\delta\alpha)\|\tilde u\|^2 + (1-\delta)\|\Delta\tilde u\|^2) + c\|\tilde u(t-\rho,x)\|^2 + \epsilon c(1+|y(\theta_t\omega)|^2). \end{aligned} \tag{6.12}$$

Integrating (6.12) over (τ,t) with $t\in[\tau,\tau+T]$, we have

$$\begin{aligned} &\|\tilde v(t)\|^2 + (\delta^2+\lambda-\delta\alpha)\|\tilde u(t)\|^2 + (1-\delta)\|\Delta\tilde u(t)\|^2 \\ \le& \|\tilde v(\tau)\|^2 + (\delta^2+\lambda-\delta\alpha)\|\tilde u(\tau)\|^2 + (1-\delta)\|\Delta\tilde u(\tau)\|^2 \\ &+ c\int_\tau^t (\|\tilde v(r)\|^2 + (\delta^2+\lambda-\delta\alpha)\|\tilde u(r)\|^2 + (1-\delta)\|\Delta\tilde u(r)\|^2)dr + c\int_\tau^t \|\tilde u(r-\rho,x)\|^2 dr \end{aligned}$$

$$+\epsilon c\int_\tau^t(1+|y(\theta_r\omega)|^2)dr. \tag{6.13}$$

Note that

$$\int_\tau^t\|\tilde u(r-\rho,x)\|^2dr=\int_{\tau-\rho}^{t-\rho}\|\tilde u(r,x)\|^2dr=\int_{\tau-\rho}^{\tau}\|\tilde u(r,x)\|^2dr+\int_\tau^{t-\rho}\|\tilde u(r,x)\|^2dr$$
$$\le\rho\|\phi^\epsilon-\phi\|_{C_{V_2}(\mathbb{R}^n)}+\int_\tau^t\|\tilde u(r,x)\|^2dr. \tag{6.14}$$

Therefore, for every $t\in[\tau,\tau+T]$, it follows from (6.13)-(6.14) that

$$\|\tilde Y(t)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\le c\|Y_0^\epsilon-Y_0\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+c\int_\tau^t\|\tilde Y(r)\|^2dr+\epsilon c\int_\tau^t(1+|y(\theta_r\omega)|^2)dr,$$

which along with Gronwall's lemma implies that for all $t\in[\tau,\tau+T]$,

$$\|\tilde Y(t,\tau,\omega,\tilde Y_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\le c\|Y_0^\epsilon-Y_0\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}+\epsilon c\int_\tau^t(1+|y(\theta_r\omega)|^2)dr. \tag{6.15}$$

So if $\lim\limits_{\epsilon\to0}(\phi^\epsilon,\psi^\epsilon)=(\phi,\widehat{\psi})\in C_{V_2,V_0}(\mathbb{R}^n)$, then

$$\|Y_0^\epsilon-Y_0\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\to0\quad\text{as}\quad\epsilon\to0, \tag{6.16}$$

and hence by (6.15), for all $t\in[\tau,\tau+T]$,

$$\sup_{-\rho\le s\le0}\|\tilde Y(t+s,\tau,\omega,\tilde Y_0)\|^2_{C_{V_2,V_0}(\mathbb{R}^n)}\to0\quad\text{as}\quad\epsilon\to0. \tag{6.17}$$

Then (6.6) follows from (6.17) immediately. □

As follows, we establish the uniform compactness of $\mathcal{A}_\epsilon$ in $C_{V_2,V_0}(\mathbb{R}^n)$.

Lemma 6.2 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. Then for every $\tau\in\mathbb{R},\omega\in\Omega$, the union $\bigcup\limits_{0<\epsilon\le1}\mathcal{A}_\epsilon(\tau,\ \omega)$ is precompact in $C_{V_2,V_0}(\mathbb{R}^n)$.*

Proof. Given $\epsilon\in(0,1]$. Firstly, From (6.4), Lemma 4.2 and the invariance of $\mathcal{A}_\epsilon(\tau,\ \omega)$, we know that for $\eta>0$ and $\tau\in\mathbb{R},\ \omega\in\Omega$, there exists $r_0=r_0(\omega,\eta)\ge1$ such that

$$\int_{|x|\ge k_0}(\|u(x)\|^2+\|\Delta u(x)\|^2+\|v(x)\|^2)dx\le\eta,\ \text{for all}\ (u,v)\in\bigcup_{0<\epsilon\le1}\mathcal{A}_\epsilon(\tau,\ \omega). \tag{6.18}$$

Secondly, From (6.1), Lemma 5.2, Lemma 4.3 and the invariance of $\mathcal{A}_\epsilon(\tau,\ \omega)$, we know that there exists $k_1=k_1(\omega,\eta)\ge k_0$ such that for all $k\ge k_1$, the set $\bigcup\limits_{0<\epsilon\le1}\mathcal{A}_\epsilon(\tau,\ \omega)$ is precompact in $C_{V_2,V_0}(\mathbb{H}_k)$, which together with (6.18) implies that $\bigcup\limits_{0<\epsilon\le1}\mathcal{A}_\epsilon(\tau,\ \omega)$ is precompact in $C_{V_2,V_0}(\mathbb{R}^n)$.
□

Now, we are ready to prove the upper semi-continuity of the $\mathcal{A}_\epsilon$ as $\epsilon\to0$. In fact, it's an immediate consequence of Theorem 3.2 in [16] based on (6.5), Lemma 6.1-6.2.

Theorem 6.1 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. Then for every $\tau\in\mathbb{R},\omega\in\Omega$,*

$$\lim_{\epsilon\to0}d_{C_{V_2,V_0}(\mathbb{R}^n)}\big(\mathcal{A}_\epsilon(\tau,\ \omega),\mathcal{A}(\tau)\big)=0,$$

where $d_{C_{V_2,V_0}(\mathbb{R}^n)}$ is the Hausdorff semidistance in $C_{V_2,V_0}(\mathbb{R}^n)$.

7 Upper semi-continuity of attractors as delay approaches zero.

In this section, we establish the upper semi-continuity of random attractors of the plate equation (3.11) when the delay ρ approaches zero for a fixed $\epsilon \in (0,1]$. We write the solution and the corresponding cocycle of (3.11) as u^ρ, v^ρ and Φ^ρ, respectively.

For given $\tau \in \mathbb{R}, \omega \in \Omega$, denote

$$K^\rho(\tau,\omega) = \{Y \in C_{V_2,V_0}(\mathbb{R}^n) : \|Y\|^2_{C_{V_2,V_0}(\mathbb{R}^n)} \leq R^\rho(\tau,\omega)\}, \tag{7.1}$$

where $R^\rho(\tau,\omega)$ is given by the right-hand side of (5.1). From (7.1) and Lemma 5.1 we know that $K^\rho = \{K^\rho(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a $\mathcal{D}$-pullback absorbing set of Φ^ρ in $C_{V_2,V_0}(\mathbb{R}^n)$ for all $\rho \in (0,1]$. In addition, Φ^ρ has a $\mathcal{D}$-pullback attractor $\mathcal{A}^\rho \in \mathcal{D}$ in $C_{V_2,V_0}(\mathbb{R}^n)$ for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\mathcal{A}^\rho(\tau,\omega) \subseteq K^\rho(\tau,\omega). \tag{7.2}$$

As $\rho = 0$, the stochastic delay system (3.11) becomes a stochastic system without delay given by

$$\begin{cases} \frac{du}{dt} + \delta u - v = \epsilon y(\theta_t\omega)u, \\ \frac{dv}{dt} + (\alpha-\delta)v + \Delta^2 v + (\delta^2+\lambda-\delta\alpha)u + (1-\delta)\Delta^2 u + \epsilon y(\theta_t\omega)\Delta^2 u + F(x,u(t,x)) \\ = f(x,u(t,x)) + g(x,t) - \epsilon y(\theta_t\omega)v - \epsilon(\epsilon y(\theta_t\omega) - 2\delta)y(\theta_t\omega)u, \\ u(\tau,x) = \phi(x), \quad x \in \mathbb{R}^n, \\ v(\tau,x) = \partial_t\phi(x) + \delta\phi(x) - \epsilon y(\theta_\tau\omega)\phi(x) := \psi(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{7.3}$$

Accordingly, by Theorem 5.1 the cocycle Φ^0 generated by (7.3) is readily verified to admit a unique $\mathcal{D}^0$-pullback attractor $\mathcal{A}^0 = \{\mathcal{A}^0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}^0$ in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and a $\mathcal{D}^0$-pullback absorbing set $K^0 = \{K^0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where

$$\mathcal{D}^0 = \Big\{ D = \{D(\tau,\omega) \subseteq H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega, \\ \lim_{t\to+\infty} e^{-\gamma t}\|D(\tau-t,\theta_{-t}\omega)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = 0, \quad \forall\, \gamma > 0\} \Big\}$$

and

$$K^0(\tau,\omega) = \{Y \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \|Y\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq R^0(\tau,\omega)\}, \tag{7.4}$$

with $R^0(\tau,\omega)$ is given by the right-hand side of (5.1).

It follows from (7.1) and (7.4) that for all $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\limsup_{\rho\to 0} \|K^\rho(\tau,\omega)\|_{C_{V_2,V_0}(\mathbb{R}^n)} = \|K^0(\tau,\omega)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}. \tag{7.5}$$

In order to prove the upper semi-continuity of the $\mathcal{D}$-pullback attractor $\mathcal{A}^\rho$, we will establish the convergence of solutions of (3.11) as $\rho \to 0$.

Lemma 7.1. *Let $Y^\rho = (u^\rho, v^\rho)^\top$ and $Y = (u,v)^\top$ be the solutions of (3.11) and (7.3) with initial data $Y_0^\rho = (\phi^\rho, \psi^\rho)^\top$ and $Y_0 = (\phi,\psi)^\top$, respectively. Assume that (3.1)-(3.5) and (3.14) hold. If $\lim_{\rho\to 0}\sup_{-\rho\leq s\leq 0}\|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = 0$, then for every $\tau \in \mathbb{R}, \omega \in \Omega$, $T > 0$ and $t \in [\tau, \tau+T]$,*

$$\sup_{-\rho\leq s\leq 0} \|Y^\rho(t+s,\tau,\omega,Y_0^\rho) - Y(t,\tau,\omega,Y_0)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \to 0 \quad as \quad \rho \to 0. \tag{7.6}$$

Proof. For any $s \in [-\rho, 0]$ and $t \geq \tau$, denote by $\widehat{v} = v^\rho(t+s) - v(t)$ and $\widehat{u} = u^\rho(t+s) - u(t)$, where $v^\rho(t,\tau,\omega,\psi^\rho) = \partial_t u^\rho(t,\tau,\omega,\phi^\rho) + \delta u^\rho(t,\tau,\omega,\phi^\rho) - \epsilon y(\theta_t\omega) u^\rho(t,\tau,\omega,\phi^\rho)$ and $v(t,\tau,\omega,\psi) = \partial_t u(t,\tau,\omega,\phi) + \delta u(t,\tau,\omega,\phi) - \epsilon y(\theta_t\omega) u(t,\tau,\omega,\phi)$ with $\psi^\rho(s) = \partial_t \phi^\rho(s) + \delta\phi^\rho(s) - \epsilon y(\theta_t\omega)\phi^\rho(s)$ for $s \in [-\rho, 0]$ and $\psi(s) = \partial_t\phi(s) + \delta\phi(s) - \epsilon y(\theta_t\omega)\phi(s)$.

It follows from (3.11) and (7.3) that for $t > \tau - s$ and $s \in [-\rho, 0]$,

$$\begin{aligned}
&\frac{d\widehat{v}}{dt} + (\alpha-\delta)\widehat{v} + \Delta^2\widehat{v} + (\delta^2+\lambda-\delta\alpha)\widehat{u} + (1-\delta)\Delta^2\widehat{u} + \epsilon y(\theta_{t+s}\omega)\Delta^2 u^\rho(t+s,x)\\
&\quad - \epsilon y(\theta_t\omega)\Delta^2 u(t,x) + \big(F(x,u^\rho(t+s,x)) - F(x,u(t,x))\big)\\
=&\big(f(x,u^\rho(t+s-\rho,x)) - f(x,u(t,x))\big) + g(x,t+s) - g(x,t) - \epsilon y(\theta_{t+s}\omega) v^\rho(t+s,x)\\
&\quad + \epsilon y(\theta_t\omega) v(t,x) - \epsilon(\epsilon y(\theta_{t+s}\omega) - 2\delta) y(\theta_{t+s}\omega) u^\rho(t+s,x) + \epsilon(\epsilon y(\theta_t\omega) - 2\delta) y(\theta_t\omega) u(t,x),
\end{aligned} \tag{7.7}$$

Taking the inner product of (7.7) with $\widehat{v}$ in $L^2(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\widehat{v}\|^2 + (\delta^2+\lambda-\delta\alpha)\|\widehat{u}\|^2 + (1-\delta)\|\Delta\widehat{u}\|^2)\\
&\quad + (\alpha-\delta)\|\widehat{v}\|^2 + \delta(\delta^2+\lambda-\delta\alpha)\|\widehat{u}\|^2 + \delta(1-\delta)\|\Delta\widehat{u}\|^2 + \|\Delta\widehat{v}\|^2\\
=&-\big(F(x,u^\rho(t+s,x)) - F(x,u(t,x),\widehat{v}\big) + \big(f(x,u^\rho(t+s-\rho,x)) - f(x,u(t,x)),\widehat{v}\big)\\
&\quad + \epsilon(\delta^2+\lambda-\delta\alpha) y(\theta_{t+s}\omega)\|\widehat{u}\|^2 + \epsilon(\delta^2+\lambda-\delta\alpha)\big(y(\theta_{t+s}\omega) - y(\theta_t\omega)\big)(\widehat{u},u)\\
&\quad + \epsilon(1-\delta) y(\theta_{t+s}\omega)\|\Delta\widehat{u}\|^2 + \epsilon(1-\delta)\big(y(\theta_{t+s}\omega) - y(\theta_t\omega)\big)(\Delta\widehat{u},\Delta u) - \epsilon y(\theta_{t+s}\omega)(\Delta^2\widehat{u},\widehat{v})\\
&\quad - \epsilon\big(y(\theta_{t+s}\omega) - y(\theta_t\omega)\big)(\Delta^2 u,\widehat{v}) + \big(g(x,t+s) - g(x,t),\widehat{v}\big)\\
&\quad - \epsilon y(\theta_{t+s}\omega)\|\widehat{v}\|^2 - \epsilon\big(y(\theta_{t+s}\omega) - y(\theta_t\omega)\big)(\widehat{v},v) - \epsilon(\epsilon y(\theta_{t+s}\omega) - 2\delta) y(\theta_{t+s}\omega)(\widehat{v},\widehat{u})\\
&\quad - \epsilon\big(y(\theta_{t+s}\omega) - y(\theta_t\omega)\big)\big(\epsilon\big(y(\theta_{t+s}\omega) + y(\theta_t\omega)\big) - 2\delta\big)(\widehat{v},u).
\end{aligned} \tag{7.8}$$

By (3.4), we get

$$|\big(F(x,u^\rho(t+s,x)) - F(x,u(t,x)),\widehat{v}\big)| \leq c\|\widehat{u}\|^2 + c\|\widehat{v}\|^2. \tag{7.9}$$

By (3.5), we get

$$\begin{aligned}
&|\big(f(x,u^\rho(t+s-\rho,x)) - f(x,u(t,x)),\widehat{v}\big)| \leq l_f\|u^\rho(t+s-\rho,x) - u(t,x)\|\cdot\|\widehat{v}\|\\
\leq& c\|u^\rho(t+s-\rho,x) - u(t,x)\|^2 + c\|\widehat{v}\|^2.
\end{aligned} \tag{7.10}$$

In addition, we have

$$\big(g(x,t+s) - g(x,t),\widehat{v}\big) \leq \frac{1}{2}\|g(x,t+s) - g(x,t)\|^2 + \frac{1}{2}\|\widehat{v}\|^2. \tag{7.11}$$

Thanks to Young's inequality, we find the remaining terms on the right hand side of (6.10) are controlled by $\epsilon c|y(\theta_{t+s}\omega) - y(\theta_t\omega)|^2(\|v\|^2 + \|u\|^2 + \|\Delta u\|^2) + c(\|\widehat{v}\|^2 + \|\widehat{u}\|^2 + \|\Delta\widehat{u}\|^2) + \|\Delta\widehat{v}\|^2$, which along with (7.8)-(7.11) implies that for $t > \tau - s$ and $s \in [-\rho, 0]$,

$$\begin{aligned}
&\frac{d}{dt}(\|\widehat{v}\|^2 + (\delta^2+\lambda-\delta\alpha)\|\widehat{u}\|^2 + (1-\delta)\|\Delta\widehat{u}\|^2)\\
\leq& c(\|\widehat{v}\|^2 + (\delta^2+\lambda-\delta\alpha)\|\widehat{u}\|^2 + (1-\delta)\|\Delta\widehat{u}\|^2) + c\|u^\rho(t+s-\rho,x) - u(t,x)\|^2\\
&\quad + \|g(x,t+s) - g(x,t)\|^2 + \epsilon c|y(\theta_{t+s}\omega) - y(\theta_t\omega)|^2(\|v\|^2 + \|u\|^2 + \|\Delta u\|^2).
\end{aligned} \tag{7.12}$$

Integrating (7.12) over $(\tau - s, t)$ with $t \in [\tau, \tau + T]$, we get that

$$\|\widehat{v}(t)\|^2 + (\delta^2+\lambda-\delta\alpha)\|\widehat{u}(t)\|^2 + (1-\delta)\|\Delta\widehat{u}(t)\|^2$$

$$
\begin{aligned}
\leq&\|\widehat{v}(\tau-s)\|^2+(\delta^2+\lambda-\delta\alpha)\|\widehat{u}(\tau-s)\|^2+(1-\delta)\|\Delta\widehat{u}(\tau-s)\|^2\\
&+c\int_{\tau-s}^{t}\|u^\rho(r+s-\rho,x)-u(r,x)\|^2dr+\int_{\tau-s}^{t}\|g(x,r+s)-g(x,r)\|^2dr\\
&+\epsilon c\int_{\tau-s}^{t}|y(\theta_{r+s}\omega)-y(\theta_r\omega)|^2(\|v(r)\|^2+\|u(r)\|^2+\|\Delta u(r)\|^2)dr. \qquad (7.13)
\end{aligned}
$$

Note that for any $s\in[-\rho,0]$ and $t\in[\tau,\tau+T]$ with $t>\tau-s$,

$$
\begin{aligned}
&\int_{\tau-s}^{t}\|u^\rho(r+s-\rho,x)-u(r,x)\|^2dr\\
=&\int_{\tau-s}^{\tau+\rho-s}\|u^\rho(r+s-\rho,x)-u(r,x)\|^2dr+\int_{\tau+\rho-s}^{t}\|u^\rho(r+s-\rho,x)-u(r,x)\|^2dr\\
\leq&2\int_{\tau-\rho}^{\tau}\|u^\rho(r,x)-\phi\|^2dr+2\int_{\tau-s}^{\tau+\rho-s}\|u(r,x)-\phi\|^2dr\\
+&\int_{\tau-s}^{t}\|u^\rho(r+s,x)-u(r+\rho,x)\|^2dr\\
\leq&2\int_{\tau-\rho}^{\tau}\|u^\rho(r,x)-\phi\|^2dr+2\int_{\tau-s}^{\tau+\rho-s}\|u(r,x)-\phi\|^2dr\\
+&2\int_{\tau-s}^{t}\|u^\rho(r+s,x)-u(r,x)\|^2dr+2\int_{\tau-s}^{t}\|u(r+\rho,x)-u(r,x)\|^2dr\\
\leq&2\rho\sup_{-\rho\leq s\leq 0}\|\phi^\rho(s)-\phi\|^2+2\int_{\tau-s}^{\tau+\rho-s}\|u(r,x)-\phi\|^2dr\\
&+2\int_{\tau-s}^{t}\|\widehat{u}(r)\|^2dr+2\int_{\tau-s}^{t}\|u(r+\rho,x)-u(r,x)\|^2dr. \qquad (7.14)
\end{aligned}
$$

Therefore, for every $t\in[\tau,\tau+T]$ with $t>\tau-s$, it follows from (7.13)-(7.14) and (3.9) that

$$
\begin{aligned}
&\|\widehat{Y}(t)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}\\
\leq&\|\widehat{Y}(\tau-s)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}+c\int_{\tau-s}^{t}\|\widehat{Y}(r)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}dr\\
&+\int_{\tau}^{\tau+T}\|g(x,r+s)-g(x,r)\|^2dr+c\rho\sup_{-\rho\leq s\leq 0}\|\phi^\rho(s)-\phi\|^2\\
&+c\int_{\tau}^{\tau+2\rho}\|u(r,x)-\phi\|^2dr+c\int_{\tau}^{\tau+T}\|u(r+\rho,x)-u(r,x)\|^2dr\\
&+\epsilon c\int_{\tau-s}^{t}|y(\theta_{r+s}\omega)-y(\theta_r\omega)|^2(\|v(r)\|^2+\|u(r)\|^2+\|\Delta u(r)\|^2)dr. \qquad (7.15)
\end{aligned}
$$

Note that $y(\theta_r\omega)$ is uniformly continuous in r on $[\tau-T,\tau+T]$, Therefore, given $\eta>0$, there exists $\rho_1\in(0,1]$ such that for all $\rho\leq\rho_1, s\in[-\rho,0]$ and $r\in[\tau,\tau+T]$,

$$|y(\theta_{r+s}\omega)-y(\theta_r\omega)|^2\leq\eta. \qquad (7.16)$$

By $\lim_{\rho\to 0}\int_\tau^{\tau+2\rho}\|u(r,x)-\phi\|^2dr=0$ we know that there exists $\rho_2\leq\rho_1$ such that for all $\rho\leq\rho_2$,

$$\int_\tau^{\tau+2\rho}\|u(r,x)-\phi\|^2dr\leq\eta. \qquad (7.17)$$

Note that u is uniformly continuous from $[\tau, \tau+1+T]$ to $H^2(\mathbb{R}^n)$ we get that there is $\rho_3 \leq \rho_2$ such that for all $\rho \leq \rho_3$ and $r \in [\tau, \tau+T]$,

$$\|u(r+\rho, x) - u(r, x)\| \leq \eta. \tag{7.18}$$

Since $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ we get

$$\lim_{s\to 0} \int_\tau^{\tau+T} \|g(x, r+s) - g(x, r)\|^2 dr = 0, \tag{7.19}$$

which means that there exists $\rho_4 \leq \rho_3$ such that for all $\rho \leq \rho_4$ and $s \in [-\rho, 0]$,

$$\int_\tau^{\tau+T} \|g(x, r+s) - g(x, r)\|^2 dr \leq \eta, \tag{7.20}$$

It follows from (7.15)-(7.20) that

$$\begin{aligned} \|\widehat{Y}(t)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq & c\int_{\tau-s}^t \|\widehat{Y}(r)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} dr + \|\widehat{Y}(\tau-s)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \\ & + c\rho \sup_{-\rho\leq s\leq 0} \|\phi^\rho(s) - \phi\|^2 + c\eta. \end{aligned} \tag{7.21}$$

Accordingly, we have

$$\|\widehat{Y}(t)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq c\|\widehat{Y}(\tau-s)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + c\rho \sup_{-\rho\leq s\leq 0} \|\phi^\rho(s) - \phi\|^2 + c\eta, \tag{7.22}$$

In addition,

$$\begin{aligned} &\|\widehat{Y}(\tau-s)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = \|Y^\rho(\tau) - Y(\tau-s)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \\ =&\|v^\rho(\tau) - v(\tau-s)\|^2 + (\delta^2+\lambda-\delta\alpha)\|u^\rho(\tau) - u(\tau-s)\|^2 + (1-\delta)\|\Delta(u^\rho(\tau) - u(\tau-s))\|^2 \\ \leq& 2\|v^\rho(\tau) - \psi\|^2 + 2\|v(\tau-s) - \psi\|^2 + 2(\delta^2+\lambda-\delta\alpha)(\|u^\rho(\tau) - \phi\|^2 + \|u(\tau-s) - \phi\|^2) \\ &+2(1-\delta)(\|\Delta(u^\rho(\tau) - \phi)\|^2 + \|\Delta(u(\tau-s) - \phi)\|^2), \end{aligned}$$

which along with (7.18) shows that there exists $\rho_5 \leq \rho_4$ such that for all $\rho \leq \rho_5$ and $s \in [-\rho, 0]$,

$$\|\widehat{Y}(\tau-s)\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq c \sup_{-\rho\leq s\leq 0} \|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + c\eta. \tag{7.23}$$

It follows from (7.22)-(7.23) that for all $\rho \leq \rho_5$, $t \in [\tau, \tau+T]$ with $t > \tau - s$ and $s \in [-\rho, 0]$,

$$\|Y^\rho(t+s, \tau, \omega, Y_0^\rho) - Y(t, \tau, \omega, Y_0)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq c \sup_{-\rho\leq s\leq 0} \|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + c\eta, \tag{7.24}$$

from which one can easily deduce the desired result (7.6). □

Lemma 7.2 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. If* $\rho_n \to 0$ *and* $Y_n \in \mathcal{A}^{\rho_n}(\tau, \omega)$, *then there exists a subsequence* $\{Y_{n_m}\}$ *of* $\{Y_n\}$ *and* $Y \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *such that*

$$\lim_{m\to\infty} \sup_{-\rho_{n_m}\leq s\leq 0} \|Y_{n_m}(s) - Y\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = 0, \tag{7.25}$$

where $Y = (u, v)^\top$.

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of numbers with $t_n \to \infty$ and $n \to \infty$. By the invariance of $\mathcal{A}^{\rho_n}$, there exists $\widehat{Y}_n \in \mathcal{A}^{\rho_n}(\tau - t_n, \theta_{-t_n}\omega)$ such that

$$Y_n = \Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n). \tag{7.26}$$

By (7.2), we have $\widehat{Y}_n \in K^{\rho_n}(\tau - t_n, \theta_{-t_n}\omega)$. Since all uniform estimates of solutions established in Section 5 are uniform with respect to $\rho \in (0,1]$, by the arguments of Lemma 5.2, we can verify the following:

(i) $\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0)$ is precompact in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

(ii) Given any $\eta > 0$, there exists $N_1 \geq 1$ such that for all $n \geq N_1$ and $s \in [-\rho_n, 0]$,

$$\|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(s) - \Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq \eta.$$

By (i) we find that there exists $Y \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that, up to a subsequence,

$$\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0) \to Y \quad \text{in} \quad H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n).$$

Therefore, there exists $N_2 \geq N_1$ such that for all $n \geq N_2$,

$$\|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0) - Y\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq \eta. \tag{7.27}$$

By (ii) and (7.27) we get, for all $n \geq N_2$ and $s \in [-\rho_n, 0]$,

$$\begin{aligned}
&\|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(s) - Y\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \\
\leq &\|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(s) - \Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \\
&+ \|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0) - Y\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq 2\eta.
\end{aligned} \tag{7.28}$$

which along with (7.27) implies that $\|Y_n - Y\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq 2\eta$ for all $n \geq N_2$ and $s \in [-\rho_n, 0]$ as desired. □

We are now in a position to prove the upper semicontinuity of attractors as $\rho \to 0$.

Theorem 7.1 *Assume that (3.1)-(3.5), (3.7), (3.14)-(3.16) hold. Then for every $\tau \in \mathbb{R}, \omega \in \Omega$,*

$$\lim_{\rho \to 0} d_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}\big(\mathcal{A}^{\rho}(\tau,\ \omega), \mathcal{A}^0(\tau)\big) = 0, \tag{7.29}$$

where $d_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$ is defined for any subsets $E \subseteq C_{V_2,V_0}(\mathbb{R}^n)$ and $S \subseteq H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ by

$$d_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}(E, S) = \sup_{\varphi \in E} \inf_{x \in S} \sup_{-\rho \leq s \leq 0} \|\varphi(s) - x\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}.$$

Proof. Let $\rho_n \to 0$ as $n \to \infty$, $Y_0^{\rho_n} \in C_{V_2,V_0}(\mathbb{R}^n)$ and $Y_0 \in H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ with $\sup\limits_{-\rho_n \leq s \leq 0} \|Y_0^{\rho_n}(s) - Y_0\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \to 0$ as $n \to \infty$. It follows from Lemma 7.1 that for any $\tau \in \mathbb{R}, \omega \in \Omega$ and $t \geq \tau$,

$$\sup_{-\rho_n \leq s \leq 0} \|\Phi^{\rho_n}(t, \tau, \omega, Y_0^{\rho_n})(s) - \Phi^0(t, \tau, \omega, Y_0)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad n \to \infty. \tag{7.30}$$

By (7.4), (7.5), (7.30) and Lemma 7.2 we get (7.29) from Theorem 2.1 in [18] immediately. □

Declarations

Availability of data and material
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Acknowledgements
The authors are grateful to anonymous reviewers for comments and remarks.

References

[1] L. Arnold. Random Dynamical Systems. *Springer-Verlag, New-York*, 1998.

[2] H. Crauel. Random Probability Measure on Polish Spaces. *Taylor and Francis, London*, 2002.

[3] Duan J., Lu K., Schmalfuss B. Invariant manifolds for stochastic partial differential equations. *Ann. Probab.*, 31: 2109–2135, 2003.

[4] M. J. Garrido-Atienza, A. Ogrowsky and B. Schmalfuss. Random differential equations with random delays. *Stoch. Dyn.*, 11: 369-388, 2011.

[5] J. K. Hale and S. M. V. Lunel. Introduction to Functional-Differential Equations. *Applied Mathematical Sciences*, 99: Springer-Verlag, New York, 1993.

[6] Kamache, Houria, Boumaza, Nouri, Gheraibia and Billel, General decay and blow up of solutions for the Kirchhoff plate equation with dynamic boundary conditions, delay and source terms. *Z. Angew. Math. Phys.*, 73: 1-23, 2022.

[7] K. Kinra and M. T. Mohan, Weak pullback mean random attractors for the stochastic convective Brinkman-Forchheimer equations and locally monotone stochastic partial differential equations. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 25: 2250005, 2022.

[8] P. E. Kloeden. Upper semicontinuity of attractors of delay differential equations in the delay. *Bull. Austral. Math. Soc.*, 73: 299–306, 2006.

[9] Qiao Zhen Ma, Xiaobin Yao, Tingting Liu. Existence and upper semi-continuity of random attractors for non-autonomous stochastic plate equations with multiplicative noise on $\mathbb{R}$. *Journal of Applied Analysis and Computation*, 11: 1422–1454, 2021.

[10] L. Montestruque and P. Antsaklis. Stability of model-based networked control systems with time-varying transmission times. *IEEE Transaction on Automatical Control*, 49: 1562–1572, 2004.

[11] J. Nilsson, B. Bernhardsson and B. Wittenmark. Stochastic analysis and control of real-time systems with random time delays. *Automatica J. IFAC*, 34: 57-64, 1998.

[12] Piskin, Erhan, Ekinci and Fatma. Global existence of solutions for a coupled viscoelastic plate equation with degenerate damping terms. *Tbilisi Math. J.*, 14: 195-206, 2021.

[13] L. Schenato. Optimal estimation in networked control systems subject to random delay and packet drop. *IEEE Transaction on Automatical Control*, 53: 1311–1317, 2008.

[14] B. Wang. Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems. *J. Differ. Equ.*, 253:1544 – 1583, 2012.

[15] B. Wang, X. Gao. Random attractors for wave equations on unbounded domains. *Discrete Contin. Dyn. Syst. Syst. Special*, 800 – 809, 2009.

[16] B. Wang. Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms, *Stoch. Dyn.*, 14, 1450009, 31 pp., (2014).

[17] S. P. Wang, Q. Z. Ma. Existence of pullback attractors for the non-autonomous suspension bridge equation with time delay. *Discrete and continuous dynamical systems series B*, 25:1299–1316, 2020.

[18] X. Wang, K. Lu and B. Wang. Random attractors for delay parabolic equations with additive noise and deterministic nonautonomous forcing. *SIAM J. Appl. Dyn. Syst.*, 14: 1018-1047, 2015.

[19] X. B. Yao. Dynamics of kirchhoff type plate equations with nonlinear damping driven by multiplicative noise. *Journal of Applied Analysis and Computation*, 14: 1148-1176, 2024.