Unidirectional Wave Propagation in a Nonlocal Dispersal Endemic Model with Nonlinear Incidence

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Abstract: This paper is concerned with existence and non-existence of traveling wave solutions in a nonlocal dispersal endemic model with nonlinear incidence. With the aid of upper-lower solutions method and Schauder's fixed point theorem together with Lyapunov functional technique, we derive the existence of super-critical and critical traveling wave solutions connecting disease-free equilibrium to endemic equilibrium. In a combination with the theory of two-sided Laplace transform and local skilled analysis, we obtain the non-existence of sub-critical traveling wave solutions. Our results illustrate that: (i) the existence and non-existence of traveling waves are determined by the basic reproduction number and the wave speed; (ii) the critical wave speed is equal to the minimal wave speed; (iii) the traveling waves only propagate along one direction.

Keywords: Traveling Waves; Nonlocal Dispersal; Endemic Model; Nonlinear Incidence.

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1 Introduction and main results

To model the transmission patterns of infectious disease, a great number of reaction-diffusion (Laplacian-operator-type) and nonlocal dispersal (convolution-operator-type) epidemic models have been proposed in the last several decades [3, 5, 7, 10, 12, 13, 21, 24, 27–30, 35–40, 42, 43]. From the viewpoint of mathematical epidemiology, the existence and non-existence of the traveling wave solutions with a constant speed for these model are important issues because they can predict whether or not the disease spread in the individuals and how fast a disease invades geographically. In the present paper, we shall consider these problems in the following two-component nonlocal dispersal endemic model

$$\begin{cases} S_t(x,t) = d_1 K[S](x,t) + b - \mu_1 S(x,t) - \beta S(x,t) g(I(x,t)), \\ I_t(x,t) = d_2 K[I](x,t) + \beta S(x,t) g(I(x,t)) - (\mu_2+\gamma) I(x,t), \end{cases} \tag{1.1}$$

where $S(x,t)$ and $I(x,t)$ stand for the densities of the susceptible and infected individuals in location x and at time t, respectively. The convolution operator

$$K[\phi](x,t) := \int_{\mathbb{R}} K(y)[\phi(x-y,t) - \phi(x,t)]dy \tag{1.2}$$

describes the probability that individuals in position y will jump to location x and it reflects that the movement of individuals can be in a large, random and free way. The positive constant b refers to the entering flux of the susceptible individuals. The parameters $d_j > 0$ and $\mu_j > 0$ $(j = 1, 2)$ denote the space diffusion rates and the natural death rates for the susceptible and infected individuals, respectively. The infection rate β and the removal rate γ are positive numbers. Note that the nonlinear incidence $Sg(I)$ in epidemic models has played a crucial role in giving a reasonable qualitative description for the disease dynamics [4,9,42,43]. Hereafter, the kernel function $K(x)$ and nonlinear function $g(I)$ satisfy the following hypotheses.

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(H1) $K(x) \in C(\mathbb{R})$, $K(x) = K(-x) \geq 0$, $\int_{\mathbb{R}} K(x)dx = 1$, $K(x)$ is compactly supported and $\text{supp}K = [-r, r]$ with the constant radius $r > 0$.

(H2) $g(I)$ is positive and continuous for $I > 0$ with $g(0) = 0$ and $g'(I) > 0$ for $I \geq 0$.

(H3) $g'(0) = \max_{I \in [0,\infty)} g'(I)$, $g(I)/I$ is continuous differential, non-increasing for $I > 0$ and $\lim_{I \to \infty} g(I)/I = 0$.

(H4) $g''(I) \leq 0$ for all $I > 0$.

It is not difficult to observe that a standard kernel function [17, 31]

$$K(x) = \begin{cases} C \exp\left(\dfrac{1}{|x|^2 - 1}\right), \ |x| < 1, \\ 0, \ |x| \geq 1, \end{cases}$$

where the constant $C > 0$ is chosen such that $\int_{\mathbb{R}} K(x)dx = 1$, satisfies (H1). As far as we know, the class of nonlinear functions $g(I)$ can include some types of functional responses, such as

(i) Holling type II functional response $g(I) = \frac{I}{1+\alpha I}$ with the constant $\alpha > 0$ [8, 11, 15, 20, 41];

(ii) Ivlev type functional response $g(I) = 1 - e^{-nI}$ with the constant $n > 0$ [1, 6, 14, 16, 19, 23, 25, 26, 34].

Note that the reaction system of (1.1) is given by

$$\begin{cases} \dot{S}(t) = b - \mu_1 S(t) - \beta S(t) g(I(t)), \\ \dot{I}(t) = \beta S(t) g(I(t)) - (\mu_2 + \gamma) I(t), \end{cases} \tag{1.3}$$

where the dot denotes the derivative with respect to t. System (1.3) always admits a disease-free equilibrium $(S_0, 0)$, where $S_0 := b/\mu_1$. By [18], one knows that the basic reproduction number of (1.3) is

$$R_0 := \frac{\beta S_0 g'(0)}{\mu_2 + \gamma}. \tag{1.4}$$

Then if $R_0 > 1$ and (H2)-(H4) hold, system (1.3) has a unique positive endemic equilibrium (S^*, I^*) satisfying

$$\begin{cases} b = \mu_1 S^* + \beta S^* g(I^*), \\ \beta S^* g(I^*) = (\mu_2 + \gamma) I^*. \end{cases} \tag{1.5}$$

Throughout this paper, we always assume that $R_0 > 1$. By a traveling wave solution of (1.1), we mean a solution in the form of

$$(S, I)(x, t) = (S, I)(z), \ z = x + ct, \tag{1.6}$$

where c is the wave speed. Inserting (1.6) into (1.1) yields

$$\begin{cases} cS'(z) = d_1 \displaystyle\int_{\mathbb{R}} K(y) S(z - y) dy + b - (d_1 + \mu_1) S(z) - \beta S(z) g(I(z)), & \text{(1.7a)} \\ cI'(z) = d_2 \displaystyle\int_{\mathbb{R}} K(y) I(z - y) dy + \beta S(z) g(I(z)) - (d_2 + \mu_2 + \gamma) I(z), & \text{(1.7b)} \end{cases}$$

where the prime denotes the derivative with respect to z. The aim of this paper is to establish the existence and non-existence of a positive solution $(S, I)(z)$ on the real line of system (1.7a)-(1.7b) satisfying the following asymptotic boundary conditions

$$(S, I)(-\infty) = (S_0, 0) \text{ and } (S, I)(\infty) = (S^*, I^*). \tag{1.8}$$

To this end, we define a function by

$$\Theta(\lambda, c) := d_2 \int_{\mathbb{R}} K(y) e^{-\lambda y} dy - c\lambda + \beta S_0 g'(0) - d_2 - \mu_2 - \gamma, \ (\lambda, c) \in [0, \infty) \times [0, \infty).$$

Using $R_0 > 1$ and (H1), we derive that $\Theta(0,c) = \beta S_0 g'(0) - \mu_2 - \gamma > 0$ and

$$\begin{aligned}\Theta(\lambda,0) &= d_2 \int_{\mathbb{R}} K(y)(e^{-\lambda y} - 1)dy + \beta S_0 g'(0) - \mu_2 - \gamma \\ &\geq -\lambda d_2 \int_{\mathbb{R}} yK(y)dy + \beta S_0 g'(0) - \mu_2 - \gamma \ \ (\text{since } e^x - 1 \geq x \text{ for } x \in \mathbb{R}) \\ &= \beta S_0 g'(0) - \mu_2 - \gamma > 0.\end{aligned}$$

It follows from $e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ and $K(-x) = K(x)$ for $x \in \mathbb{R}$ that

$$\int_{\mathbb{R}} K(y) \frac{e^{-\lambda y} - 1}{\lambda} dy = \int_{\mathbb{R}} K(y) \sum_{n=1}^{\infty} \frac{(-\lambda y)^n}{n!\lambda} dy = \sum_{n=1}^{\infty} \frac{\lambda^{2n-1}}{(2n)!} \int_{\mathbb{R}} K(y) y^{2n} dy \to \infty \text{ as } \lambda \to \infty.$$

Then for each $c > 0$, we deduce that $\lim_{\lambda\to\infty} \Theta(\lambda, c) = \infty$ and $\Theta_\lambda(0,c) = -\left[d_2 \int_{\mathbb{R}} yK(y)e^{-\lambda y}dy + c\right]\Big|_{\lambda=0} = -c < 0$. For any $(\lambda, c) \in \mathbb{R}^2$, it follows that $\Theta_{\lambda\lambda}(\lambda,c) = d_2 \int_{\mathbb{R}} y^2 K(y) e^{-\lambda y} dy > 0$. For each fixed $\lambda > 0$, we note that $\Theta_c(\lambda, c) = -\lambda < 0$ and $\lim_{c\to\infty} \Theta(\lambda,c) = -\infty$. Based on the above computations and $R_0 > 1$, we can define a positive value

$$c^* := \inf_{\lambda\in(0,\infty)} \frac{d_2[\int_{\mathbb{R}} K(y)e^{-\lambda y}dy - 1] + \beta S_0 g'(0) - \mu_2 - \gamma}{\lambda}$$

and obtain the following proposition.

Proposition 1.1 *Suppose that $R_0 > 1$. Then the following assertions are true.*

(i) There is a positive constant $\lambda^*(c^*) := \lambda^*$ such that $\Theta(\lambda^*, c^*) = 0$ and $\Theta_\lambda(\lambda^*, c^*) = 0$.

(ii) If $c > c^*$, then $\Theta(\lambda, c) = 0$ admits two positive roots $\lambda_1(c) := \lambda_1$ and $\lambda_2(c) := \lambda_2$ with $\lambda^* \in (\lambda_1, \lambda_2)$ such that $\Theta(\lambda, c) > 0$ for $\lambda \in [0, \lambda_1) \cup (\lambda_2, \infty)$ and $\Theta(\lambda, c) < 0$ for $\lambda \in (\lambda_1, \lambda_2)$.

(iii) If $0 < c < c^*$, then $\Theta(\lambda, c) > 0$ for $\lambda \in [0, \infty)$.

Using (H3) and $R_0 > 1$ yields that $\lim_{I\to 0^+} \beta S_0 g(I)/I = \beta S_0 g'(0) > \mu_2 + \gamma$ and $\lim_{I\to\infty} \beta S_0 g(I)/I = 0 < \mu_2 + \gamma$. Then there exists a unique constant $\bar{I} > 0$ such that

$$\beta S_0 g(\bar{I})/\bar{I} = \mu_2 + \gamma. \tag{1.9}$$

Now we are in a position to state our results.

Theorem 1.1 *If $R_0 > 1$ and $c \geq c^*$, then system (1.1) admits a non-trivial traveling wave solution $(S, I)(z)$, $z := x + ct$ satisfying $(S, I)(-\infty) = (S_0, 0)$ and $(S, I)(\infty) = (S^*, I^*)$. Moreover, $(S, I)(z)$ satisfies the following properties.*

(I) Positiveness and global boundedness of traveling wave solutions in (1.1). For $z \in \mathbb{R}$,

$$\underline{S} < S(z) < S_0 \text{ and } 0 < I(z) < \bar{I}$$

where $\underline{S} := b/[\mu_1 + \beta g'(0)\bar{I}]$ and $\bar{I} > 0$ is defined in (1.9).

(II) Limit behavior of I-component of traveling wave solutions in (1.1). If $z \to -\infty$, then

$$I(z) = \begin{cases} O(e^{\lambda_1 z}) \text{ for } c > c^*, \\ O(ze^{\lambda^* z}) \text{ for } c = c^*. \end{cases}$$

Theorem 1.2 *If $R_0 > 1$ and $c < c^*$, then system (1.1) has no traveling wave solutions satisfying $(S, I)(-\infty) = (S_0, 0)$ and $(S, I)(\infty) = (S^*, I^*)$, together with $\underline{S} < S(z) < S_0$ and $0 < I(z) < \bar{I}$ for $z \in \mathbb{R}$.*

Remark 1.1 *Theorems 1.1 and 1.2 assert that the existence and non-existence of traveling wave solutions of (1.1) depend on both the basic reproduction number and the critical wave speed. From the biomathematics point of view, the disease can transmit at the critical wave speed while can not spread for any wave speed smaller than the critical wave speed. These results mean that the critical wave speed is equal to the minimal wave speed.*

Remark 1.2 *Theorem 1.2 shows that system (1.1) has no non-trivial bounded traveling wave solutions with non-positive wave speed which implies that the traveling waves propagate in one direction. Chen et al. [5] and Yang et al. [38] also established the similar results for their models. However, our method adopted here is quite different from the work [5,38].*

Here we sketch our strategies. To prove the existence of traveling wave solutions of (1.1) with $c > c^*$, we first construct a pair of upper-lower solutions of (1.7a)-(1.7b) and an invariant cone of initial functions defined on a large bounded interval. Secondly, we apply the Schauder's fixed point theorem to prove the existence of solutions on the cone. Thirdly, we derive a uniform prior estimate of solutions on the large bounded domain and extend the existence of solutions on the bounded interval to the whole real line by a limiting argument. Finally, we study the asymptotic boundary of solutions at infinity via squeeze theorem coupled with Lyapunov functional techniques [20, 35, 44]. To show the existence of traveling wave solutions of (1.1) with $c = c^*$, we re-construct a pair of upper-lower solution of (1.7a)-(1.7b) to achieve our goal. Note that the used Lyapunov functional to get the convergence towards the endemic equilibrium point at plus infinity is independent of the wave speed c. So we still have $(S, I)(\infty) = (S^*, I^*)$ when $c = c^*$. To investigate the non-existence of traveling wave solutions of (1.1) with $c < c^*$, we shall apply the theory of two-sided Laplace transform and local skilled analysis to attain our aim. The remainder of this paper is organized as follows. In Section 2, we establish the existence of super-critical traveling waves. In Section 3, we obtain the existence of critical traveling waves. In Section 4, we investigate the non-existence of sub-critical traveling waves.

2 Existence of super-critical traveling wave solutions

2.1 Construction of the upper and lower solutions for (1.7a)-(1.7b)

Definition 2.1 *If $S_\pm(z)$ and $I_\pm(z)$ are of class $C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \mathcal{S})$ for some finite set $\mathcal{S}$ and if they satisfy*

$$
\begin{aligned}
&d_1 \int_{\mathbb{R}} K(y)S_+(z-y)dy - cS'_+(z) + b - (d_1+\mu_1)S_+(z) - \beta S_+(z)g(I_-(z)) \le 0,\\
&d_1 \int_{\mathbb{R}} K(y)S_-(z-y)dy - cS'_-(z) + b - (d_1+\mu_1)S_-(z) - \beta S_-(z)g(I_+(z)) \ge 0,\\
&d_2 \int_{\mathbb{R}} K(y)I_+(z-y)dy - cI'_+(z) + \beta S_+(z)g(I_+(z)) - (d_2+\mu_2+\gamma)I_+(z) \le 0\\
&d_2 \int_{\mathbb{R}} K(y)I_-(z-y)dy - cI'_-(z) + \beta S_-(z)g(I_-(z)) - (d_2+\mu_2+\gamma)I_-(z) \ge 0
\end{aligned}
$$

for any $z \in \mathbb{R} \setminus \mathcal{S}$, then the function pairs $(S_\pm, I_\pm)(z)$ are called a pair of upper and lower solutions for (1.7a)-(1.7b).

Now we construct four non-negative continuous functions on the real line, which are

$$S_+(z) := S_0, \tag{2.1}$$

$$I_+(z) := \min\{e^{\lambda_1 z}, \bar{I}\}, \tag{2.2}$$

$$S_-(z) := \max\{S_0 - \epsilon_1^{-1} e^{\epsilon_1 z}, \underline{S}\}, \tag{2.3}$$

$$I_-(z) := \max\{e^{\lambda_1 z} - M_1 e^{(\lambda_1+\epsilon_2)z}, 0\}. \tag{2.4}$$

In (2.1)-(2.4), $\lambda_1 > 0$ is defined in Proposition 1.1(ii), $\bar{I} > 0$ is given in (1.9),

$$\underline{S} := \frac{b}{\mu_1 + \beta g'(0)\bar{I}} < \frac{b}{\mu_1} = S_0, \tag{2.5}$$

and the constants $M_1, \epsilon_1, \epsilon_2 > 0$ will be determined later.

In the following lemmata of this subsection, we shall show that the function pairs $(S_\pm, I_\pm)(z)$ constructed in (2.1)-(2.4) are a pair of upper and lower solutions for (1.7a)-(1.7b).

Lemma 2.1 *The function $S_+(z)$ satisfies*

$$d_1 \int_{\mathbb{R}} K(y)S_+(z-y)dy - cS_+'(z) + b - (d_1+\mu_1)S_+(z) - \beta S_+(z)g(I_-(z)) \le 0 \tag{2.6}$$

for any $z \in \mathbb{R}$.

Proof. Since $S_+(z) = S_0 = b/\mu_1$ and $I_-(z) \ge 0$ for $z \in \mathbb{R}$, we have from (H1) and (H2) that

$$\begin{aligned}
&d_1 \int_{\mathbb{R}} K(y)S_+(z-y)dy - cS_+'(z) + b - (d_1+\mu_1)S_+(z) - \beta S_+(z)g(I_-(z)) \\
&= d_1 S_0 + b - (d_1+\mu_1)S_0 - \beta S_0 g(I_-(z)) \\
&= -\beta S_0 g(I_-(z)) \le 0 \text{ for } z \in \mathbb{R}.
\end{aligned}$$

Then inequality (2.6) holds and the proof is finished. ■

Lemma 2.2 *The function $I_+(z)$ satisfies*

$$d_2 \int_{\mathbb{R}} K(y)I_+(z-y)dy - cI_+'(z) + \beta S_+(z)g(I_+(z)) - (d_2+\mu_2+\gamma)I_+(z) \le 0 \tag{2.7}$$

for any $z \neq z_1 := \lambda_1^{-1}\log \bar{I}$.

Proof. By (2.2) and (H1)-(H3), we have for $z \in \mathbb{R}$ that

$$\int_{\mathbb{R}} K(y)I_+(z-y)dy \le \min\left\{e^{\lambda_1 z}\int_{\mathbb{R}} K(y)e^{-\lambda_1 y}dy,\ \bar{I}\right\} \tag{2.8}$$

and

$$g(I_+(z)) = g(I_+(z)) - g(0) \le g'(0)I_+(z). \tag{2.9}$$

If $z < z_1$, then $I_+(z) = e^{\lambda_1 z}$. Using (2.8), (2.9) and $\Theta(\lambda_1, c) = 0$, we obtain for $z < z_1$ that

$$\begin{aligned}
&d_2 \int_{\mathbb{R}} K(y)I_+(z-y)dy - cI_+'(z) + \beta S_+(z)g(I_+(z)) - (d_2+\mu_2+\gamma)I_+(z) \\
&\le d_2 e^{\lambda_1 z}\int_{\mathbb{R}} K(y)e^{-\lambda_1 y}dy - c\lambda_1 e^{\lambda_1 z} + \beta S_0 g'(0)e^{\lambda_1 z} - (d_2+\mu_2+\gamma)e^{\lambda_1 z} \\
&= e^{\lambda_1 z}\Theta(\lambda_1, c) = 0.
\end{aligned}$$

If $z > z_1$, then $I_+(z) = \bar{I}$. By (1.9) and (2.8), we get for $z > z_1$ that

$$\begin{aligned}
&d_2 \int_{\mathbb{R}} K(y)I_+(z-y)dy - cI_+'(z) + \beta S_+(z)g(I_+(z)) - (d_2+\mu_2+\gamma)I_+(z) \\
&\le d_2\bar{I} + \beta S_0 g(\bar{I}) - (d_2+\mu_2+\gamma)\bar{I} = 0.
\end{aligned}$$

Hence inequality (2.7) holds and the proof is completed. ■

Lemma 2.3 *Suppose that $\epsilon_1 \in (0, \lambda_1)$ is a sufficiently small constant. Then the function $S_-(z)$ satisfies*

$$d_1 \int_{\mathbb{R}} K(y)S_-(z-y)dy - cS_-'(z) + b - (d_1+\mu_1)S_-(z) - \beta S_-(z)g(I_+(z)) \ge 0$$

for any $z \neq z_2 := \epsilon_1^{-1}\log[\epsilon_1(S_0 - \underline{S})]$.

Proof. Utilizing (2.3) and (H1), we have for $z \in \mathbb{R}$ that

$$\int_{\mathbb{R}} K(y)S_-(z-y)dy \geq \max\left\{S_0 - \epsilon_1^{-1}e^{\epsilon_1 z}\int_{\mathbb{R}} K(y)e^{-\epsilon_1 y}dy,\ \underline{S}\right\}. \tag{2.10}$$

Noticing the fact that

$$\int_{\mathbb{R}} K(y)\frac{1-e^{-\epsilon_1 y}}{\epsilon_1}dy = \sum_{n=1}^{\infty}\frac{(-\epsilon_1)^{2n-1}}{(2n)!}\int_{\mathbb{R}} K(y)y^{2n}dy \to 0 \text{ as } \epsilon_1 \to 0^+,$$

one can select a sufficiently small constant $\epsilon_1 \in (0, \lambda_1)$ such that $z_2 < z_1$ and

$$d_1\int_{\mathbb{R}} K(y)\frac{1-e^{-\epsilon_1 y}}{\epsilon_1}dy + c + \frac{\mu_1}{\epsilon_1} - \beta S_0 g'(0)e^{(\lambda_1-\epsilon_1)z_2} \geq 0. \tag{2.11}$$

If $z < z_2$, we get that $S_-(z) = S_0 - \epsilon_1^{-1}e^{\epsilon_1 z}$ and $I_+(z) = e^{\lambda_1 z}$. Then it follows from (2.9)-(2.11) that

$$\begin{aligned}
&d_1\int_{\mathbb{R}} K(y)S_-(z-y)dy - cS_-'(z) + b - (d_1+\mu_1)S_-(z) - \beta S_-(z)g(I_+(z))\\
&\geq d_1\epsilon_1^{-1}e^{\epsilon_1 z}\int_{\mathbb{R}} K(y)(1-e^{-\epsilon_1 y})dy + ce^{\epsilon_1 z} + b - \mu_1 S_0 + \mu_1\epsilon_1^{-1}e^{\epsilon_1 z} - \beta(S_0 - \epsilon_1^{-1}e^{\epsilon_1 z})g'(0)e^{\lambda_1 z}\\
&\geq e^{\epsilon_1 z}\left[d_1\epsilon_1^{-1}\int_{\mathbb{R}} K(y)(1-e^{-\epsilon_1 y})dy + c + \mu_1\epsilon_1^{-1} - \beta S_0 g'(0)e^{(\lambda_1-\epsilon_1)z}\right]\\
&\geq e^{\epsilon_1 z}\left[d_1\int_{\mathbb{R}} K(y)\frac{1-e^{-\epsilon_1 y}}{\epsilon_1}dy + c + \frac{\mu_1}{\epsilon_1} - \beta S_0 g'(0)e^{(\lambda_1-\epsilon_1)z_2}\right]\\
&\geq 0 \text{ for } z < z_2.
\end{aligned}$$

If $z > z_2$, then $S_-(z) = \underline{S}$. By (2.5) and (2.10), we deduce for $z > z_2$ that

$$\begin{aligned}
&d_1\int_{\mathbb{R}} K(y)S_-(z-y)dy - cS_-'(z) + b - (d_1+\mu_1)S_-(z) - \beta S_-(z)g(I_+(z))\\
&\geq d_1\underline{S} + b - (d_1+\mu_1)\underline{S} - \beta\underline{S}g'(0)\bar{I} = 0.
\end{aligned}$$

The proof of this lemma is completed. ■

Lemma 2.4 *Assume that $M_1 \in (1,\infty)$ is a sufficiently large constant and $\epsilon_2 \in (0, \min\{\epsilon_1, \lambda_2 - \lambda_1\})$. Then the function $I_-(z)$ satisfies*

$$d_2\int_{\mathbb{R}} K(y)I_-(z-y)dy - cI_-'(z) + \beta S_-(z)g(I_-(z)) - (d_2+\mu_2+\gamma)I_-(z) \geq 0 \tag{2.12}$$

for any $z \neq z_3 := -\epsilon_2^{-1}\log M_1$.

Proof. Using (2.4) and (H1) lead to for $z \in \mathbb{R}$ that

$$\int_{\mathbb{R}} K(y)I_-(z-y)dy \geq \max\left\{e^{\lambda_1 z}\left[\int_{\mathbb{R}} K(y)e^{-\lambda_1 y}dy - M_1e^{\epsilon_2 z}\int_{\mathbb{R}} K(y)e^{-(\lambda_1+\epsilon_2)y}dy\right], 0\right\}. \tag{2.13}$$

Let $M_1 \in (1,\infty)$ be a sufficiently large constant and $\epsilon_2 \in (0, \min\{\epsilon_1, \lambda_2 - \lambda_1\})$ such that $-\epsilon_2^{-1}\log M_1 < \epsilon_1^{-1}\log[\epsilon_1(S_0 - \underline{S})]$, i.e., $z_3 < z_2$. Then if $z < z_3$, we deduce that

$$S_-(z) = S_0 - \epsilon_1^{-1}e^{\epsilon_1 z} \text{ and } I_-(z) = e^{\lambda_1 z}(1 - M_1e^{\epsilon_2 z}). \tag{2.14}$$

To show (2.12) with $z < z_3$ is equivalent to prove that

$$\begin{aligned}
&d_2\int_{\mathbb{R}} K(y)I_-(z-y)dy - cI_-'(z) + \left[\beta S_0 g'(0) - d_2 - \mu_2 - \gamma\right]I_-(z)\\
&\geq \beta S_0 g'(0)I_-(z) - \beta S_-(z)g(I_-(z)) \text{ for } z < z_3.
\end{aligned} \tag{2.15}$$

Then it follows from (2.13), (2.14), $\Theta(\lambda_1, c) = 0$ and the left-hand side of (2.15) that

$$\begin{aligned}
& d_2 \int_{\mathbb{R}} K(y) I_-(z-y)dy - cI_-'(z) + \big[\beta S_0 g'(0) - d_2 - \mu_2 - \gamma\big] I_-(z) \\
& \geq e^{\lambda_1 z}\left[d_2 \int_{\mathbb{R}} K(y) e^{-\lambda_1 y} dy - c\lambda_1 + \beta S_0 g'(0) - d_2 - \mu_2 - \gamma\right] \\
& \quad - M_1 e^{(\lambda_1+\epsilon_2)z}\left[d_2 \int_{\mathbb{R}} K(y) e^{-(\lambda_1+\epsilon_2)y} dy - c(\lambda_1+\epsilon_2) + \beta S_0 g'(0) - d_2 - \mu_2 - \gamma\right] \\
& = e^{\lambda_1 z}\Theta(\lambda_1, c) - M_1 e^{(\lambda_1+\epsilon_2)z}\Theta(\lambda_1+\epsilon_2, c) \\
& = -M_1 e^{(\lambda_1+\epsilon_2)z}\Theta(\lambda_1+\epsilon_2, c) \text{ for } z < z_3. \qquad (2.16)
\end{aligned}$$

Note from (H3) that $\lim_{I_-(z)\to 0^+} \frac{g(I_-(z))}{I_-(z)} = g'(0)$, that is, for any $\varepsilon \in (0, g'(0))$, there exists a small number $\tilde{\delta} > 0$ such that $\frac{g(I_-(z))}{I_-(z)} \geq g'(0) - \varepsilon$ for $I_-(z) \in (0, \tilde{\delta})$. Then we take $M_1 \in (1, \infty)$ to be a sufficiently large constant such that $I_-(z) \in (0, \tilde{\delta})$ and $S_-(z)$ is close to S_0 for $z < z_3$. Now choosing $\varepsilon \in (0, g'(0))$ to be sufficiently small, we have from the right-hand side of (2.15) that

$$\begin{aligned}
\beta S_0 g'(0) I_-(z) - \beta S_-(z) g(I_-(z)) &= \beta S_0 g'(0) I_-(z) - \beta S_0 g(I_-(z)) + \beta \epsilon_1^{-1} e^{\epsilon_1 z} g(I_-(z)) \\
&= \beta S_0 I_-(z)\left[g'(0) - \frac{g(I_-(z))}{I_-(z)}\right] + \beta \epsilon_1^{-1} e^{\epsilon_1 z} g(I_-(z)) \\
&\leq \beta S_0 \left[\frac{I_-(z) + g'(0) - \frac{g(I_-(z))}{I_-(z)}}{2}\right]^2 + \beta \epsilon_1^{-1} e^{\epsilon_1 z} g'(0) I_-(z) \\
&\leq \beta S_0 \left[\frac{I_-(z) + \varepsilon}{2}\right]^2 + \beta \epsilon_1^{-1} g'(0) e^{(\epsilon_1+\lambda_1)z} \\
&\leq \beta S_0 I_-^2(z) + \beta \epsilon_1^{-1} g'(0) e^{(\epsilon_1+\lambda_1)z} \\
&\leq \beta S_0 e^{2\lambda_1 z} + \beta \epsilon_1^{-1} g'(0) e^{(\epsilon_1+\lambda_1)z} \text{ for } z < z_3. \qquad (2.17)
\end{aligned}$$

Hence to prove (2.15) is sufficient to show

$$\beta S_0 e^{(\lambda_1-\epsilon_2)z} + \beta \epsilon_1^{-1} g'(0) e^{(\epsilon_1-\epsilon_2)z} \leq -M_1 \Theta(\lambda_1+\epsilon_2, c) \text{ for } z < z_3. \qquad (2.18)$$

Since $\lambda_1 + \epsilon_2 \in (\lambda_1, \lambda_2)$, we get from Proposition 1.1(ii) that $\Theta(\lambda_1 + \epsilon_2, c) < 0$. Then by the boundedness of the left-hand side of (2.18), we obtain that (2.18) holds for sufficiently large constant M_1.

If $z > z_3$, then $I_-(z) = 0$ and one can have from (2.13) that

$$\begin{aligned}
& d_2 \int_{\mathbb{R}} K(y) I_-(z-y)dy - cI_-'(z) + \beta S_-(z) g(I_-(z)) - (d_2 + \mu_2 + \gamma) I_-(z) \\
& = d_2 \int_{\mathbb{R}} K(y) I_-(z-y)dy \geq 0.
\end{aligned}$$

Consequently, inequality (2.12) holds and the proof is finished. ■

Remark 2.1 *Obviously, it follows from (2.1), (2.3) and (2.5) that $S_-(z) < S_+(z)$ for $z \in \mathbb{R}$. Moreover, by elementary computation, we have for $z \in \mathbb{R}$ that $I_-(z)$ attains its maximum at the point $\tilde{z} := \frac{1}{\epsilon_1}\log\frac{\lambda_1}{M_1(\lambda_1+\epsilon_1)}$. Then based on the choices of parameters M_1 and ϵ_1 (see Lemma 2.3 and Lemma 2.4), one can obtain that $\tilde{z} < z_1$, which implies that $I_-(z) < I_+(z)$ for $z \in \mathbb{R}$.*

2.2 Existence of solutions of (1.7a)-(1.7b) on a bounded interval

Now we define a set

$$\begin{aligned}
\Gamma_l := \Big\{ (\phi, \varphi)(z) \in C([-l, l], \mathbb{R}^2) \Big| & (\phi, \varphi)(-l) = (S_-, I_-)(-l),\ S_-(z) \leq \phi(z) \leq S_+(z), \\
& I_-(z) \leq \varphi(z) \leq I_+(z),\ \forall z \in [-l, l] \Big\},
\end{aligned}$$

where $l \gg \max\{|z_3|, r\}$ (r is the radius of supp$K$). It is not difficult to verify that Γ_l is a non-empty, bounded, closed and convex subset in $C([-l,l],\mathbb{R}^2)$. For any $(\phi,\varphi)(z) \in \Gamma_l$, we define

$$\hat{\phi}(z) := \begin{cases} \phi(l), & z > l, \\ \phi(z), & |z| \le l, \\ S_-(z), & z < -l, \end{cases} \qquad \hat{\varphi}(z) := \begin{cases} \varphi(l), & z > l, \\ \varphi(z), & |z| \le l, \\ I_-(z), & z < -l. \end{cases}$$

Consider the initial value problem

$$cS'(z) = d_1 \int_{\mathbb{R}} K(y)\hat{\phi}(z-y)dy + \alpha\phi(z) + b - (d_1+\mu_1+\alpha)S(z) - \beta\phi(z)g(\varphi(z)), \tag{2.19}$$

$$cI'(z) = d_2 \int_{\mathbb{R}} K(y)\hat{\varphi}(z-y)dy + \beta\phi(z)g(\varphi(z)) - (d_2+\mu_2+\gamma)I(z) \tag{2.20}$$

on $[-l,l]$ with

$$S(-l) = S_-(-l) \text{ and } I(-l) = I_-(-l), \text{ where } \alpha > \beta g(\bar{I}). \tag{2.21}$$

The theory of ODEs claims that (2.19)-(2.21) has a unique solution $(S_l, I_l)(z) \in C^1([-l,l],\mathbb{R}^2)$. Define an operator $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$: $\Gamma_l \mapsto C([-l,l],\mathbb{R}^2)$ as follows

$$\mathcal{O}_1(\phi,\varphi)(z) := S_l(z) \text{ and } \mathcal{O}_2(\phi,\varphi)(z) := I_l(z) \text{ for } z \in [-l,l].$$

Lemma 2.5 *The operator $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$ maps Γ_l into Γ_l.*

Proof. For any given $(\phi,\varphi)(z) \in \Gamma_l$, it suffices to prove that

$$S_-(z) \le \mathcal{O}_1(\phi,\varphi)(z) \le S_+(z) \text{ and } I_-(z) \le \mathcal{O}_2(\phi,\varphi)(z) \le I_+(z),$$

that is,

$$S_-(z) \le S_l(z) \le S_+(z) \text{ and } I_-(z) \le I_l(z) \le I_+(z).$$

Since $\alpha > \beta g(\bar{I})$ and $\varphi \le \bar{I}$, we have that $\alpha\phi - \beta\phi g(\varphi)$ is increasing with respect to ϕ. Then by Lemma 2.1 and Lemma 2.3, we obtain that

$$\begin{aligned} & d_1 \int_{\mathbb{R}} K(y)\hat{\phi}(z-y)dy - cS_+'(z) + \alpha\phi(z) + b - (d_1+\mu_1+\alpha)S_+(z) - \beta\phi(z)g(\varphi(z)) \\ & \le d_1 \int_{\mathbb{R}} K(y)S_+(z-y)dy - cS_+'(z) + \alpha S_+(z) + b - (d_1+\mu_1+\alpha)S_+(z) - \beta S_+(z)g(I_-(z)) \\ & \le 0 \text{ for } z \in [-l,l] \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} & d_1 \int_{\mathbb{R}} K(y)\hat{\phi}(z-y)dy - cS_-'(z) + \alpha\phi(z) + b - (d_1+\mu_1+\alpha)S_-(z) - \beta\phi(z)g(\varphi(z)) \\ & \ge d_1 \int_{\mathbb{R}} K(y)S_-(z-y)dy - cS_-'(z) + \alpha S_-(z) + b - (d_1+\mu_1+\alpha)S_-(z) - \beta S_-(z)g(I_+(z)) \\ & \ge 0 \text{ for } z \in [-l,z_2) \cup (z_2,l]. \end{aligned} \tag{2.23}$$

Using Lemma 2.2 and Lemma 2.4, we derive

$$\begin{aligned} & d_2 \int_{\mathbb{R}} K(y)\hat{\varphi}(z-y)dy - cI_+'(z) + \beta\phi(z)g(\varphi(z)) - (d_2+\mu_2+\gamma)I_+(z) \\ & \le d_2 \int_{\mathbb{R}} K(y)I_+(z-y)dy - cI_+'(z) + \beta S_+(z)g(I_+(z)) - (d_2+\mu_2+\gamma)I_+(z) \\ & \le 0 \text{ for } z \in [-l,z_1) \cup (z_1,l] \end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
& d_2 \int_{\mathbb{R}} K(y)\hat{\varphi}(z-y)dy - cI_-'(z) + \beta\phi(z)g(\varphi(z)) - (d_2+\mu_2+\gamma)I_-(z) \\
& \geq d_2 \int_{\mathbb{R}} K(y)I_-(z-y)dy - cI_-'(z) + \beta S_-(z)g(I_-(z)) - (d_2+\mu_2+\gamma)I_-(z) \\
& \geq 0 \text{ for } z \in [-l, z_3) \cup (z_3, l].
\end{aligned} \tag{2.25}$$

Noting that (2.21) gives that

$$S_-(-l) = S_l(-l) \leq S_+(-l) \text{ and } I_-(-l) = I_l(-l) \leq I_+(-l),$$

which together with (2.22)-(2.25), Comparison theorem and the continuity of $S_{\pm}(z)$, $I_{\pm}(z)$, $S_l(z)$, $I_l(z)$, we obtain

$$S_-(z) \leq S_l(z) \leq S_+(z) \text{ and } I_-(z) \leq I_l(z) \leq I_+(z) \text{ for } z \in [-l, l].$$

The proof of this lemma is completed. ■

Lemma 2.6 *The operator $\mathcal{O}$ is completely continuous with respect to the supremum norm in $C([-l,l], \mathbb{R}^2)$.*

Proof. Since $0 \leq S_l(z) \leq S_0$ and $0 \leq I_l(z) \leq \bar{I}$, we obtain from (2.19)-(2.21) that

$$|cS_l'(z)| \leq b + \big[2d_1 + 2\alpha + \mu_1 + \beta g(\bar{I})\big]S_0 \tag{2.26}$$

and

$$|cI_l'(z)| \leq \big(2d_2 + \mu_2 + \gamma\big)\bar{I} + \beta S_0 g(\bar{I}) \tag{2.27}$$

on $[-l, l]$. Hence $S_l'(z)$ and $I_l'(z)$ are uniformly bounded for any $z \in [-l, l]$. Then applying Arzelà-Ascoli theorem on $[-l, l]$ yields that the operator $\mathcal{O}$ is compact on Γ_l.

The unique solution $(S_l, I_l)(z)$ of initial value problem (2.19)-(2.21) can be given by

$$S_l(z) = S_-(-l)e^{-\frac{d_1+\mu_1+\alpha}{c}(z+l)} + \frac{1}{c}\int_{-l}^{z} e^{-\frac{d_1+\mu_1+\alpha}{c}(z-\eta)} v_{\phi,\varphi}(\eta)d\eta, \tag{2.28}$$

$$I_l(z) = I_-(-l)e^{-\frac{d_2+\mu_2+\gamma}{c}(z+l)} + \frac{1}{c}\int_{-l}^{z} e^{-\frac{d_2+\mu_2+\gamma}{c}(z-\eta)} w_{\phi,\varphi}(\eta)d\eta, \tag{2.29}$$

where

$$\begin{aligned}
v_{\phi,\varphi}(\eta) &= d_1 \int_{\mathbb{R}} K(y)\hat{\phi}(\eta-y)dy + \alpha\phi(\eta) + b - \beta\phi(\eta)g(\varphi(\eta)), \\
w_{\phi,\varphi}(\eta) &= d_2 \int_{\mathbb{R}} K(y)\hat{\varphi}(\eta-y)dy + \beta\phi(\eta)g(\varphi(\eta)).
\end{aligned}$$

Let $(\phi_j, \varphi_j) \in \Gamma_l$ $(j = 1, 2)$, then we have

$$\begin{aligned}
& |v_{\phi_1,\varphi_1}(\eta) - v_{\phi_2,\varphi_2}(\eta)| \\
& = \left| d_1 \int_{\mathbb{R}} K(\eta-y)[\hat{\phi}_1(y) - \hat{\phi}_2(y)]dy + \alpha[\phi_1(\eta) - \phi_2(\eta)] - \beta\big[\phi_1(\eta)g(\varphi_1(\eta)) - \phi_2(\eta)g(\varphi_2(\eta))\big] \right| \\
& \leq d_1 \left| \int_{-l}^{l} K(\eta-y)[\phi_1(y) - \phi_2(y)]dy \right| + d_1 \left| \int_{l}^{\infty} K(\eta-y)[\phi_1(l) - \phi_2(l)]dy \right| \\
& \quad + \beta\big|\phi_1(\eta)g(\varphi_1(\eta)) - \phi_1(\eta)g(\varphi_2(\eta)) + \phi_1(\eta)g(\varphi_2(\eta)) - \phi_2(\eta)g(\varphi_2(\eta))\big| + \alpha|\phi_1(\eta) - \phi_2(\eta)| \\
& \leq \big[2d_1 + \alpha + \beta g(\bar{I})\big] \max_{y\in[-l,l]} |\phi_1(y) - \phi_2(y)| + \beta S_0 g'(0) \max_{y\in[-l,l]} |\varphi_1(y) - \varphi_2(y)|
\end{aligned}$$

and

$$\begin{aligned}
|w_{\phi_1,\varphi_1}(\eta) - w_{\phi_2,\varphi_2}(\eta)| &= \left| d_2 \int_{\mathbb{R}} K(\eta-y)[\hat{\varphi}_1(y) - \hat{\varphi}_2(y)]dy + \beta\big[\phi_1(\eta)g(\varphi_1(\eta)) - \phi_2(\eta)g(\varphi_2(\eta))\big] \right| \\
&\leq \big[2d_2 + \beta S_0 g'(0)\big] \max_{y\in[-l,l]} |\varphi_1(y) - \varphi_2(y)| + \beta g(\bar{I}) \max_{y\in[-l,l]} |\phi_1(y) - \phi_2(y)|.
\end{aligned}$$

Then by (2.28) and (2.29), we conclude that $\mathcal{O}$ is continuous on Γ_l. Therefore, $\mathcal{O}$ is completely continuous with respect to the supremum norm. ■

Combining Lemma 2.5, Lemma 2.6 and Schauder's fixed point theorem, we obtain the following proposition.

Proposition 2.1 *The operator $\mathcal{O}$ admits a fixed point on Γ_l, that is, $(S_l, I_l)(z) = \mathcal{O}(S_l, I_l)(z)$, which satisfies*

$$S_-(z) \le S_l(z) \le S_+(z) \text{ and } I_-(z) \le I_l(z) \le I_+(z) \text{ for } z \in [-l, l]. \tag{2.30}$$

2.3 Existence of solutions of (1.7a)-(1.7b) on $\mathbb{R}$

Choose a positive increasing sequence $\{l_n\}_{n=1}^{\infty}$ such that $l_n \gg \max\{|z_3|, r\}$ and $\lim_{n\to\infty} l_n = \infty$. Then by Proposition 2.1, we have that there exists some $(S_{l_n}, I_{l_n})(z) \in \Gamma_{l_n}$ satisfying

$$\begin{cases} cS'_{l_n}(z) = d_1 \displaystyle\int_{\mathbb{R}} K(y)\hat{S}_{l_n}(z-y)dy + b - (d_1+\mu_1)S_{l_n}(z) - \beta S_{l_n}(z)g(I_{l_n}(z)), \\ cI'_{l_n}(z) = d_2 \displaystyle\int_{\mathbb{R}} K(y)\hat{I}_{l_n}(z-y)dy + \beta S_{l_n}(z)g(I_{l_n}(z)) - (d_2+\mu_2+\gamma)I_{l_n}(z) \end{cases} \tag{2.31}$$

for each $n \in \mathbb{N}^*$, where

$$\hat{S}_{l_n}(z) = \begin{cases} S_{l_n}(l_n), & z > l_n, \\ S_{l_n}(z), & |z| \le l_n, \\ S_-(z), & z < -l_n, \end{cases} \qquad \hat{I}_{l_n}(z) = \begin{cases} I_{l_n}(l_n), & z > l_n, \\ I_{l_n}(z), & |z| \le l_n, \\ I_-(z), & z < -l_n \end{cases}$$

with

$$S_-(z) \le S_{l_n}(z) \le S_+(z) \text{ and } I_-(z) \le I_{l_n}(z) \le I_+(z) \text{ for } z \in [-l_n, l_n]. \tag{2.32}$$

Inequalities (2.32) imply that $S_{l_n}(z)$ and $I_{l_n}(z)$ are all uniformly bounded on $[-l_n, l_n]$, which together with (2.31) implies that $S'_{l_n}(z)$ and $I'_{l_n}(z)$ are all uniformly bounded on $[-l_n + r, l_n - r]$. By differentiating system (2.31), one can infer that $S''_{l_n}(z)$ and $I''_{l_n}(z)$ are all uniformly bounded on $[-l_n + 2r, l_n - 2r]$. Utilizing the Arzelà-Ascoli theorem on $[-l_n + 2r, l_n - 2r]$ for every $n \in \mathbb{N}^*$ large enough, we obtain a subsequence which is still labeled l_n through the diagonal process such that $\lim_{n\to\infty} l_n = \infty$ and

$$S_{l_n} \to S,\ I_{l_n} \to I,\ S'_{l_n} \to S',\ I'_{l_n} \to I',\ S_{l_n}g(I_{l_n}) \to Sg(I) \text{ as } n \to \infty$$

uniformly in any compact subinterval of $\mathbb{R}$. Moreover, by Lebesgue dominated convergence theorem, we get that

$$\lim_{n\to\infty} \int_{\mathbb{R}} K(y)\hat{S}_{l_n}(z-y)dy = \int_{\mathbb{R}} K(y)S(z-y)dy$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} K(y)\hat{I}_{l_n}(z-y)dy = \int_{\mathbb{R}} K(y)I(z-y)dy.$$

Passing to the limits in (2.31) and (2.32) as $n \to \infty$ yields

$$\begin{cases} cS'(z) = d_1 \displaystyle\int_{\mathbb{R}} K(y)S(z-y)dy + b - (d_1+\mu_1)S(z) - \beta S(z)g(I(z)), \\ cI'(z) = d_2 \displaystyle\int_{\mathbb{R}} K(y)I(z-y)dy + \beta S(z)g(I(z)) - (d_2+\mu_2+\gamma)I(z) \end{cases} \tag{2.33}$$

with

$$S_-(z) \le S(z) \le S_+(z) \text{ and } I_-(z) \le I(z) \le I_+(z) \text{ for } z \in \mathbb{R}. \tag{2.34}$$

Therefore, we have proved the following results.

Theorem 2.1 *If $c > c^*$, then there exists some $(S, I)(z)$, $z \in \mathbb{R}$ satisfying (1.7a)-(1.7b) and (2.34). Furthermore,*

$$||S||_{C^2_{\rm loc}(\mathbb{R})} + ||I||_{C^2_{\rm loc}(\mathbb{R})} \le C_0 \tag{2.35}$$

for some positive constant C_0.

2.4 Positiveness and asymptotic boundary of solutions of (1.7a)-(1.7b)

Proposition 2.2 *The solution of (1.7a)-(1.7b) satisfies the following properties:*

(i) $(S, I)(-\infty) = (S_0, 0)$ *and* $\lim_{z\to-\infty} e^{-\lambda_1 z} I(z) = 1$*;*

(ii) $\underline{S} < S(z) < S_0$ *and* $0 < I(z) < \bar{I}$ *for* $z \in \mathbb{R}$*;*

(iii) $(S, I)(\infty) = (S^*, I^*)$.

Proof. (i) Using (2.34), we have for $z \in \mathbb{R}$ that

$$S_0 - \epsilon_1^{-1} e^{\epsilon_1 z} \le S(z) \le S_0 \text{ and } e^{\lambda_1 z}(1 - M_1 e^{\epsilon_2 z}) \le I(z) \le e^{\lambda_1 z},$$

which together with squeeze theorem implies that $(S, I)(-\infty) = (S_0, 0)$ and $\lim_{z\to-\infty} e^{-\lambda_1 z} I(z) = 1$.

(ii) Firstly, we show that $I(z) > 0$ for $z \in \mathbb{R}$. Assume that there exists some $z^* \in \mathbb{R}$ such that $I(z^*) = 0$ for the contrary. So $I'(z^*) = 0$. By (1.7b), we obtain $\int_{\mathbb{R}} K(y)I(z^* - y)dy = 0$, which yields $I(z) = 0$ for $z \in [z^* - r, z^* + r]$. Now, take some $z^{**} \in [z^* - r, z^* + r]$. It is obvious that $I(z^{**}) = 0$ and $I'(z^{**}) = 0$. Hence, it follows from (1.7b) that $\int_{\mathbb{R}} K(y)I(z^{**} - y)dy = 0$. Similarly, one can get that $I(z) = 0$ for $z \in [z^{**} - r, z^{**} + r]$. Repeating this process, one can deduce that $I(z) \equiv 0$ for $z \in \mathbb{R}$. This contradicts the fact that $I(z) \ge I_-(z) > 0$ for $z \in (-\infty, z_3)$ (see (2.4)). Thus, $I(z) > 0$ for $z \in \mathbb{R}$.

Secondly, we prove that $S(z) < S_0$ for $z \in \mathbb{R}$. Suppose that there is some $z_* \in \mathbb{R}$ such that $S(z_*) = S_0$. Thus $S'(z_*) = 0$. Using(1.7a), we have

$$\begin{aligned}
0 &= -cS'(z_*) + d_1 \int_{\mathbb{R}} K(y)S(z_* - y)dy + b - (d_1 + \mu_1)S(z_*) - \beta S(z_*)g(I(z_*)) \\
&= d_1 \int_{\mathbb{R}} K(y)S(z_* - y)dy + b - (d_1 + \mu_1)S_0 - \beta S_0 g(I(z_*)) \\
&= d_1 \left[\int_{\mathbb{R}} K(y)S(z_* - y)dy - S_0 \right] - \beta S_0 g(I(z_*)) \\
&\le -\beta S_0 g(I(z_*)) < 0,
\end{aligned}$$

since$\int_{\mathbb{R}} K(y)S(z_* - y)dy \le S_0$, $b = \mu_1 S_0$ and $g(I(z_*)) > 0$ for $I(z_*) > 0$. Then a contradiction appears. Thus $S(z) < S_0$ for $z \in \mathbb{R}$.

Thirdly, we demonstrate that $I(z) < \bar{I}$ for $z \in \mathbb{R}$. Suppose that there is a $\tilde{z} \in \mathbb{R}$ such that $I(\tilde{z}) = \bar{I}$. Hence $I'(\tilde{z}) = 0$. Utilizing (1.7b), we obtain

$$\begin{aligned}
0 &= -cI'(\tilde{z}) + d_2 \int_{\mathbb{R}} K(y)I(\tilde{z} - y)dy + \beta S(\tilde{z})g(I(\tilde{z})) - (d_2 + \mu_2 + \gamma)I(\tilde{z}) \\
&= d_2 \int_{\mathbb{R}} K(y)I(\tilde{z} - y)dy - d_2\bar{I} + \beta S(\tilde{z})g(\bar{I}) - (\mu_2 + \gamma)\bar{I} \\
&< d_2 \left[\int_{\mathbb{R}} K(y)I(\tilde{z} - y)dy - \bar{I} \right] + \beta S_0 g(\bar{I}) - (\mu_2 + \gamma)\bar{I} \\
&\le \beta S_0 g(\bar{I}) - (\mu_2 + \gamma)\bar{I} = 0,
\end{aligned}$$

due to $S(\tilde{z}) < S_0$, $\int_{\mathbb{R}} K(y)I(\tilde{z} - y)dy \le \bar{I}$ and $\beta S_0 g(\bar{I}) = (\mu_2 + \gamma)\bar{I}$ (see (1.9)). Then a contradiction occurs. So $I(z) < \bar{I}$ for $z \in \mathbb{R}$.

Finally, we prove that $S(z) > \underline{S}$ for $z \in \mathbb{R}$. Assume that there exists some $\hat{z} \in \mathbb{R}$ such that $S(\hat{z}) = \underline{S}$. Hence $S'(\hat{z}) = 0$. It follows from (1.7a) that

$$\begin{aligned}
0 &= -cS'(\hat{z}) + d_1 \int_{\mathbb{R}} K(y)S(\hat{z} - y)dy + b - (d_1 + \mu_1)S(\hat{z}) - \beta S(\hat{z})g(I(\hat{z})) \\
&= d_1 \int_{\mathbb{R}} K(y)S(\hat{z} - y)dy + b - (d_1 + \mu_1)\underline{S} - \beta \underline{S} g(I(\hat{z})) \\
&\ge b - [\mu_1 + \beta g(I(\hat{z}))]\underline{S}
\end{aligned}$$

$$\geq b - [\mu_1 + \beta g'(0) I(\hat{z})] \underline{S}$$
$$> b - [\mu_1 + \beta g'(0)\bar{I}] \underline{S} = 0,$$

where we have used $\int_{\mathbb{R}} K(y)S(\hat{z}-y)dy \geq \underline{S}$, $I(\hat{z}) < \bar{I}$ and $b = \left[\mu_1 + \beta g'(0)\bar{I}\right]\underline{S}$ (see (2.5)). Thus a contradiction appears. Then $S(z) > \underline{S}$ for $z \in \mathbb{R}$.

(iii) We shall use Lyapunov functional method to derive the asymptotic boundary of solution for (1.7a)-(1.7b) at plus infinity. Define four functions by

$$G(S,I)(z) := S(z)g(I)(z),\ h(y) := y - 1 - \log y,\ y > 0,$$
$$\alpha_1(y) := \int_y^{\infty} K(x)dx \text{ and } \alpha_2(y) := \int_{-\infty}^{y} K(x)dx.$$

It is obvious that the function $G(S,I)(z)$ is positive and bounded for $\underline{S} < S(z) < \bar{S}$ and $0 < I(z) < \bar{I}$. Meanwhile, the function $h(y)$ satisfies

$$\begin{cases} h(y) > 0,\ y \in (0,1) \cup (1,\infty), \\ h(y) = 0,\ y = 1. \end{cases} \tag{2.36}$$

Since $\int_{\mathbb{R}} K(x)dx = 1$, K is compactly supported and r is the radius of supp$K$, we have that

$$\begin{cases} \alpha_1(y) \equiv 0,\ y \geq r, \\ \alpha_2(y) \equiv 0,\ y \leq -r \end{cases} \tag{2.37}$$

with

$$\alpha_1(0) = \alpha_2(0) = \frac{1}{2} \text{ and } \frac{d}{dy}\alpha_2(y) = -\frac{d}{dy}\alpha_1(y) = K(y). \tag{2.38}$$

Define a Lyapunov functional by

$$V(S,I)(z) := V_1(S,I)(z) + d_1 S^* V_2(S)(z) + d_2 I^* V_3(I)(z), \tag{2.39}$$

where

$$V_1(S,I)(z) = c\left[S(z) - S^* - S^* \log\frac{S(z)}{S^*} + I(z) - I^* - I^* \log\frac{I(z)}{I^*}\right],$$
$$V_2(S)(z) = \int_0^{\infty} \alpha_1(y) h\left(\frac{S(z-y)}{S^*}\right)dy - \int_{-\infty}^{0} \alpha_2(y) h\left(\frac{S(z-y)}{S^*}\right)dy,$$
$$V_3(I)(z) = \int_0^{\infty} \alpha_1(y) h\left(\frac{I(z-y)}{I^*}\right)dy - \int_{-\infty}^{0} \alpha_2(y) h\left(\frac{I(z-y)}{I^*}\right)dy.$$

Obviously, the Lyapunov functional $V(S,I)(z)$ is bounded on $\mathbb{R}$. For convenience, we will drop some variables z in the sequel calculations. Differentiating the function $V_1(S,I)(z)$ with respect to z and using

$$\begin{cases} b = \mu_1 S^* + \beta G(S^*, I^*), \\ \beta G(S^*, I^*) = (\mu_2 + \gamma) I^*, \end{cases}$$

we derive

$$\begin{aligned}
&\frac{dV_1(S,I)}{dz} = cS'\left(1 - \frac{S^*}{S}\right) + cI'\left(1 - \frac{I^*}{I}\right) \\
&= \left(1 - \frac{S^*}{S}\right)\left[d_1 \int_{\mathbb{R}} K(y)S(z-y)dy - d_1 S + b - \mu_1 S - \beta G(S,I)\right] \\
&\quad + \left(1 - \frac{I^*}{I}\right)\left[d_2 \int_{\mathbb{R}} K(y)I(z-y)dy - d_2 I + \beta G(S,I) - (\mu_2+\gamma)I\right] \\
&= d_1\left(1 - \frac{S^*}{S}\right)\left[\int_{\mathbb{R}} K(y)S(z-y)dy - S\right] + d_2\left(1 - \frac{I^*}{I}\right)\left[\int_{\mathbb{R}} K(y)I(z-y)dy - I\right] \\
&\quad + \left(1 - \frac{S^*}{S}\right)\left[\mu_1 S^* - \mu_1 S + \beta G(S^*, I^*) - \beta G(S,I)\right] + \left(1 - \frac{I^*}{I}\right)\left[\beta G(S,I) - \beta G(S^*, I^*)\frac{I}{I^*}\right]
\end{aligned}$$

$$
\begin{aligned}
&= d_1\bigg(1-\frac{S^*}{S}\bigg)\bigg[\int_{\mathbb{R}} K(y)S(z-y)dy - S\bigg] + d_2\bigg(1-\frac{I^*}{I}\bigg)\bigg[\int_{\mathbb{R}} K(y)I(z-y)dy - I\bigg] \\
&\quad + \mu_1 S^*\bigg(1-\frac{S^*}{S}\bigg)\bigg(1-\frac{S}{S^*}\bigg) + \beta G(S^*,I^*)\bigg\{\bigg(1-\frac{S^*}{S}\bigg)\bigg[1-\frac{G(S,I)}{G(S^*,I^*)}\bigg] + \bigg(1-\frac{I^*}{I}\bigg)\bigg[\frac{G(S,I)}{G(S^*,I^*)} - \frac{I}{I^*}\bigg]\bigg\} \\
&= d_1\bigg(1-\frac{S^*}{S}\bigg)\bigg[\int_{\mathbb{R}} K(y)S(z-y)dy - S\bigg] + d_2\bigg(1-\frac{I^*}{I}\bigg)\bigg[\int_{\mathbb{R}} K(y)I(z-y)dy - I\bigg] \\
&\quad + \mu_1 S^*\bigg(1-\frac{S^*}{S}\bigg)\bigg(1-\frac{S}{S^*}\bigg) + \beta G(S^*,I^*)\bigg[\frac{G(S,I)S^*}{G(S^*,I^*)S} - \frac{I}{I^*} + \log\frac{G(S^*,I^*)SI}{G(S,I)S^*I^*}\bigg] \\
&\quad + \beta G(S^*,I^*)\bigg[1-\frac{S^*}{S} + \log\frac{S^*}{S} + 1 - \frac{G(S,I)I^*}{G(S^*,I^*)I} + \log\frac{G(S,I)I^*}{G(S^*,I^*)I}\bigg] \\
&= d_1\bigg(1-\frac{S^*}{S}\bigg)\bigg[\int_{\mathbb{R}} K(y)S(z-y)dy - S\bigg] + d_2\bigg(1-\frac{I^*}{I}\bigg)\bigg[\int_{\mathbb{R}} K(y)I(z-y)dy - I\bigg] \\
&\quad + \mu_1 S^*\bigg(1-\frac{S^*}{S}\bigg)\bigg(1-\frac{S}{S^*}\bigg) + \beta G(S^*,I^*)\bigg(1-\frac{S^*}{S} + \log\frac{S^*}{S}\bigg) \\
&\quad + \beta G(S^*,I^*)\bigg[1-\frac{G(S,I)I^*}{G(S^*,I^*)I} + \log\frac{G(S,I)I^*}{G(S^*,I^*)I}\bigg] \\
&\quad + \beta G(S^*,I^*)\bigg[-1-\frac{I}{I^*} + \frac{G(S,I)S^*}{G(S^*,I^*)S} + \frac{G(S^*,I^*)SI}{G(S,I)S^*I^*} + 1 - \frac{G(S^*,I^*)SI}{G(S,I)S^*I^*} + \log\frac{G(S^*,I^*)SI}{G(S,I)S^*I^*}\bigg] \\
&= d_1\bigg(1-\frac{S^*}{S}\bigg)\bigg[\int_{\mathbb{R}} K(y)S(z-y)dy - S\bigg] + d_2\bigg(1-\frac{I^*}{I}\bigg)\bigg[\int_{\mathbb{R}} K(y)I(z-y)dy - I\bigg] \\
&\quad + \mu_1 S^*\bigg(1-\frac{S^*}{S}\bigg)\bigg(1-\frac{S}{S^*}\bigg) + \beta G(S^*,S^*)\bigg(1-\frac{S^*}{S} + \log\frac{S^*}{S}\bigg) \\
&\quad + \beta G(S^*,I^*)\bigg[1-\frac{G(S,I)I^*}{G(S^*,I^*)I} + \log\frac{G(S,I)I^*}{G(S^*,I^*)I}\bigg] + \beta G(S^*,I^*)\bigg[1-\frac{G(S^*,I^*)SI}{G(S,I)S^*I^*} + \log\frac{G(S^*,I^*)SI}{G(S,I)S^*I^*}\bigg] \\
&\quad + \beta G(S^*,I^*)\bigg[\frac{I}{I^*} - \frac{G(S,I)S^*}{G(S^*,I^*)S}\bigg]\bigg[\frac{G(S^*,I^*)S}{G(S,I)S^*} - 1\bigg] = \sum_{i=1}^{7}\Phi_i, \qquad (2.40)
\end{aligned}
$$

where

$$
\begin{aligned}
&\Phi_1 = d_1\bigg(1-\frac{S^*}{S}\bigg)\bigg[\int_{\mathbb{R}} K(y)S(z-y)dy - S\bigg],\ \Phi_2 = d_2\bigg(1-\frac{I^*}{I}\bigg)\bigg[\int_{\mathbb{R}} K(y)I(z-y)dy - I\bigg], \\
&\Phi_3 = \mu_1 S^*\bigg(1-\frac{S^*}{S}\bigg)\bigg(1-\frac{S}{S^*}\bigg) = \mu_1 S^*\bigg(2-\frac{S^*}{S}-\frac{S}{S^*}\bigg) \le 0, \\
&\Phi_4 = \beta G(S^*,I^*)\bigg(1-\frac{S^*}{S} + \log\frac{S^*}{S}\bigg) = -\beta S^* g(I^*) h\bigg(\frac{S^*}{S}\bigg) \le 0, \\
&\Phi_5 = \beta G(S^*,I^*)\bigg[1-\frac{G(S,I)I^*}{G(S^*,I^*)I} + \log\frac{G(S,I)I^*}{G(S^*,I^*)I}\bigg] = -\beta G(S^*,I^*) h\bigg(\frac{G(S,I)I^*}{G(S^*,I^*)I}\bigg) \le 0, \\
&\Phi_6 = \beta G(S^*,I^*)\bigg[1-\frac{G(S^*,I^*)SI}{G(S,I)S^*I^*} + \log\frac{G(S^*,I^*)SI}{G(S,I)S^*I^*}\bigg] = -\beta G(S^*,I^*) h\bigg(\frac{G(S^*,I^*)SI}{G(S,I)S^*I^*}\bigg) \le 0, \\
&\Phi_7 = \beta G(S^*,I^*)\bigg[\frac{I}{I^*} - \frac{G(S,I)S^*}{G(S^*,I^*)S}\bigg]\bigg[\frac{G(S^*,I^*)S}{G(S,I)S^*} - 1\bigg] = \beta S^* g(I^*)\bigg[\frac{I}{I^*} - \frac{g(I)}{g(I^*)}\bigg]\bigg[\frac{g(I^*)}{g(I)} - 1\bigg].
\end{aligned}
$$

From (2.37) and (2.38), we obtain

$$
\begin{aligned}
\frac{dV_2(S)}{dz} &= \frac{d}{dz}\int_0^\infty \alpha_1(y) h\bigg(\frac{S(z-y)}{S^*}\bigg)dy - \frac{d}{dz}\int_{-\infty}^0 \alpha_2(y) h\bigg(\frac{S(z-y)}{S^*}\bigg)dy \\
&= \int_0^\infty \alpha_1(y)\frac{d}{dz} h\bigg(\frac{S(z-y)}{S^*}\bigg)dy - \int_{-\infty}^0 \alpha_2(y)\frac{d}{dz} h\bigg(\frac{S(z-y)}{S^*}\bigg)dy \\
&= -\int_0^\infty \alpha_1(y)\frac{d}{dy} h\bigg(\frac{S(z-y)}{S^*}\bigg)dy + \int_{-\infty}^0 \alpha_2(y)\frac{d}{dy} h\bigg(\frac{S(z-y)}{S^*}\bigg)dy \\
&= -\alpha_1(y) h\bigg(\frac{S(z-y)}{S^*}\bigg)\bigg|_{y=0}^{r} + \int_0^\infty \frac{d}{dy}\alpha_1(y) h\bigg(\frac{S(z-y)}{S^*}\bigg)dy \\
&\quad + \alpha_2(y) h\bigg(\frac{S(z-y)}{S^*}\bigg)\bigg|_{y=-r}^{0} - \int_{-\infty}^0 \frac{d}{dy}\alpha_2(y) h\bigg(\frac{S(z-y)}{S^*}\bigg)dy
\end{aligned}
$$

$$= h\left(\frac{S}{S^*}\right) - \int_{\mathbb{R}} K(y) h\left(\frac{S(z-y)}{S^*}\right) dy. \tag{2.41}$$

Then it follows from (2.36), (2.40) and (2.41) that

$$\begin{aligned}
\Phi_1 + d_1 S^* \frac{dV_2(S)}{dz} &= d_1\left(1 - \frac{S^*}{S}\right)\left[\int_{\mathbb{R}} K(y) S(z-y) dy - S\right] + d_1 S^* h\left(\frac{S}{S^*}\right) - d_1 S^* \int_{\mathbb{R}} K(y) h\left(\frac{S(z-y)}{S^*}\right) dy \\
&= d_1 \int_{\mathbb{R}} K(y) S(z-y) dy - d_1 S - d_1 S^* \int_{\mathbb{R}} K(y) \frac{S(z-y)}{S} dy + d_1 S^* \\
&\quad + d_1\left(S - S^* - S^* \log \frac{S}{S^*}\right) - d_1 S^* \int_{\mathbb{R}} K(y) h\left(\frac{S(z-y)}{S^*}\right) dy \\
&= d_1 S^* \int_{\mathbb{R}} K(y)\left[\frac{S(z-y)}{S^*} - \frac{S(z-y)}{S} - \log \frac{S}{S^*}\right] dy - d_1 S^* \int_{\mathbb{R}} K(y) h\left(\frac{S(z-y)}{S^*}\right) dy \\
&= d_1 S^* \int_{\mathbb{R}} K(y)\left[\frac{S(z-y)}{S^*} - 1 - \log \frac{S(z-y)}{S^*}\right] dy - d_1 S^* \int_{\mathbb{R}} K(y) h\left(\frac{S(z-y)}{S^*}\right) dy \\
&\quad - d_1 S^* \int_{\mathbb{R}} K(y)\left[\frac{S(z-y)}{S} - 1 - \log \frac{S(z-y)}{S}\right] dy \\
&= -d_1 S^* \int_{\mathbb{R}} K(y) h\left(\frac{S(z-y)}{S}\right) dy \le 0.
\end{aligned} \tag{2.42}$$

By the same calculations as (2.42), one can get

$$\Phi_2 + d_2 I^* \frac{dV_3(I)}{dz} = -d_2 I^* \int_{\mathbb{R}} K(y) h\left(\frac{I(z-y)}{I}\right) dy \le 0. \tag{2.43}$$

From (H2) and (H3), we know that $g(I)$ is strictly increasing and $g(I)/I$ is non-increasing for $I > 0$, which imply that

$$\begin{cases} \left[\dfrac{I}{I^*} - \dfrac{g(I)}{g(I^*)}\right]\left[\dfrac{g(I^*)}{g(I)} - 1\right] \le 0, \ 0 < I \le I^*, \\ \left[\dfrac{I}{I^*} - \dfrac{g(I)}{g(I^*)}\right]\left[\dfrac{g(I^*)}{g(I)} - 1\right] \le 0, \ I \ge I^*. \end{cases} \tag{2.44}$$

Utilizing (2.39)-(2.44), we obtain

$$\begin{aligned}
\frac{dV(S,I)}{dz} &= \frac{dV_1(S,I)}{dz} + d_1 S^* \frac{dV_2(S)}{dz} + d_2 I^* \frac{dV_3(I)}{dz} \\
&= \left[\Phi_1 + d_1 S^* \frac{dV_2(S)}{dz}\right] + \left[\Phi_2 + d_2 I^* \frac{dV_3(I)}{dz}\right] + \sum_{i=3}^{7} \Phi_i \le 0,
\end{aligned} \tag{2.45}$$

which yields that $V(S,I)(z)$ is non-increasing and

$$\frac{dV(S,I)(z)}{dz} = 0 \Leftrightarrow S(z) = S^* \text{ and } I(z) = I^* \text{ for } z \in \mathbb{R}. \tag{2.46}$$

Choose an increasing constant sequence $\{z_n\}$ satisfying $\lim_{n\to\infty} z_n = \infty$ and denote

$$\{S_n(z)\}_{n=1}^{\infty} = \{S(z+z_n)\}_{n=1}^{\infty} \text{ and } \{I_n(z)\}_{n=1}^{\infty} = \{I(z+z_n)\}_{n=1}^{\infty}.$$

Since $\{S_n(z)\}_{n=1}^{\infty}$ and $\{I_n(z)\}_{n=1}^{\infty}$ are uniformly bounded in $C^2_{\text{loc}}(\mathbb{R})$, there exists a subsequence of functions (still labeled by S_n and I_n) such that $\lim_{n\to\infty} S_n(z) = \tilde{S}(z)$ and $\lim_{n\to\infty} I_n(z) = \tilde{I}(z)$. Applying Lebesgue dominated convergence theorem yields $\lim_{n\to\infty} V(S_n, I_n)(z) = V(\tilde{S}, \tilde{I})(z)$. Note that $V(S,I)(z)$ is non-increasing and bounded from below, then for any $n \in \mathbb{N}^*$, there exists a constant C_1 such that

$$V(S_n, I_n)(z) = V(S,I)(z+z_n) \ge C_1,$$

which means that there is a constant $V_0 \in \mathbb{R}$ satisfying

$$\lim_{n\to\infty} V(S_n, I_n)(z) = \lim_{z+z_n\to\infty} V(S,I)(z+z_n) = V_0$$

for any $z \in \mathbb{R}$. So we obtain $V(\tilde{S}, \tilde{I})(z) = V_0$, which implies that

$$\frac{dV(\tilde{S}, \tilde{I})(z)}{dz} = 0. \tag{2.47}$$

Then it follows from (2.46) and (2.47) that $(\tilde{S}, \tilde{I})(z) = (S^*, I^*)$, that is, $(S,I)(\infty) = (S^*, I^*)$. ■

3 Existence of critical traveling wave solutions

In this section, we will establish the existence of traveling wave solution for $R_0 > 1$ and $c = c^*$. To this aim, we choose a constant $L_1 > \lambda^* e\bar{I}$ to be suitable large such that the equation $-L_1 z e^{\lambda^* z} = \bar{I}$ has two negative roots z_4 and z^* satisfying

$$z^* - z_4 > r, \tag{3.1}$$

where $r > 0$ is the radius of supp$K$. Now for $z \in \mathbb{R}$, we define the following non-negative continuous functions.

$$S_+^*(z) := S_0, \qquad I_+^*(z) := \begin{cases} -L_1 z e^{\lambda^* z}, \ z < z_4, \\ \bar{I}, \ z \geq z_4, \end{cases}$$

$$S_-^*(z) := \begin{cases} S_0 - \varepsilon_1^{-1} e^{\varepsilon_1 z}, \ z < z_5, \\ \underline{S}, \ z \geq z_5, \end{cases} \qquad I_-^*(z) := \begin{cases} -L_1 z e^{\lambda^* z} - L_2(-z)^{\frac{1}{2}} e^{\lambda^* z}, \ z < z_6, \\ 0, \ z \geq z_6, \end{cases}$$

where λ^* is defined in Proposition 1.1, $S_0 = b/\mu_1$, $\bar{I}$ is given in (1.9), z_4 is in (3.1), $z_5 = \varepsilon_1^{-1} \log[\varepsilon_1(S_0 - \underline{S})]$, $\underline{S} = \frac{b}{\mu_1 + \beta g'(0)\bar{I}}$, $z_6 = -\frac{L_2^2}{L_1^2}$, ε_1 and L_2 are positive constants to be determined later.

Lemma 3.1 *The function $S_+^*(z)$ satisfies*

$$d_1 \int_{\mathbb{R}} K(y) S_+^*(z-y)dy - c^*(S_+^*)'(z) + b - (d_1 + \mu_1)S_+^*(z) - \beta S_+^*(z) g(I_-^*(z)) \leq 0$$

for any $z \in \mathbb{R}$.

Proof. By $S_+^*(z) = S_0 = b/\mu_1$ and $I_-^*(z) \geq 0$ for $z \in \mathbb{R}$, we deduce from (H1) and (H2) that

$$\begin{aligned} & d_1 \int_{\mathbb{R}} K(y) S_+^*(z-y)dy - c^*(S_+^*)'(z) + b - (d_1 + \mu_1)S_+^*(z) - \beta S_+^*(z) g(I_-^*(z)) \\ &= d_1 S_0 + b - (d_1 + \mu_1)S_0 - \beta S_0 g(I_-^*(z)) \\ &= -\beta S_0 g(I_-^*(z)) \leq 0 \text{ for } z \in \mathbb{R}. \end{aligned}$$

This ends the proof. ■

Lemma 3.2 *The function $I_+^*(z)$ satisfies*

$$d_2 \int_{\mathbb{R}} K(y) I_+^*(z-y)dy - c^*(I_+^*)'(z) + \beta S_+^*(z) g(I_+^*(z)) - (d_2 + \mu_2 + \gamma) I_+^*(z) \leq 0$$

for any $z \neq z_4$.

Proof. By the definition of $I_+^*(z)$, we have

$$I_+^*(z) \leq -L_1 z e^{\lambda^* z} \text{ for } z \in (-\infty, z^*] \tag{3.2}$$

and

$$g(I_+^*(z)) = g(I_+^*(z)) - g(0) \leq g'(0) I_+^*(z) \text{ for } z \in \mathbb{R}. \tag{3.3}$$

If $z < z_4$, we obtain that

$$I_+^*(z) = -L_1 z e^{\lambda^* z}, \quad (I_+^*)'(z) = -L_1 e^{\lambda^* z}(1 + \lambda^* z) \tag{3.4}$$

and

$$\begin{aligned} \int_{\mathbb{R}} K(y) I_+^*(z-y)dy &= \int_{-\infty}^{z-z^*} K(y) I_+^*(z-y)dy + \int_{z-z^*}^{\infty} K(y) I_+^*(z-y)dy \\ &= \int_{z-z^*}^{\infty} K(y) I_+^*(z-y)dy \quad \text{[by (3.1) and (H1)]} \\ &\leq -L_1 \int_{z-z^*}^{\infty} K(y)(z-y) e^{\lambda^*(z-y)} dy \quad \text{[by (3.2)]} \\ &= -L_1 \int_{\mathbb{R}} K(y)(z-y) e^{\lambda^*(z-y)} dy \end{aligned}$$

$$
\begin{aligned}
&= -L_1 \int_{\mathbb{R}} K(y)(z+y)e^{\lambda^*(z+y)}dy \quad [\text{by } K(-y)=K(y)] \\
&= -L_1 z e^{\lambda^* z} \int_{\mathbb{R}} K(y)e^{\lambda^* y}dy - L_1 e^{\lambda^* z} \int_{\mathbb{R}} K(y)ye^{\lambda^* y}dy.
\end{aligned} \tag{3.5}
$$

Then by (3.3)-(3.5) and $\Theta(\lambda^*, c^*) = \Theta_\lambda(\lambda^*, c^*) = 0$, we derive for $z < z_4$ that

$$
\begin{aligned}
& d_2 \int_{\mathbb{R}} K(y)I_+^*(z-y)dy - c^*(I_+^*)'(z) + \beta S_+^*(z)g(I_+^*(z)) - (d_2+\mu_2+\gamma)I_+^*(z) \\
&\leq d_2\Big[-L_1 z e^{\lambda^* z}\int_{\mathbb{R}} K(y)e^{\lambda^* y}dy - L_1 e^{\lambda^* z}\int_{\mathbb{R}} K(y)ye^{\lambda^* y}dy\Big] - c^*\big[-L_1 e^{\lambda^* z}(1+\lambda^* z)\big] \\
&\quad + \beta S_0 g'(0)(-L_1 z e^{\lambda^* z}) - (d_2+\mu_2+\gamma)(-L_1 z e^{\lambda^* z}) \\
&= -L_1 z e^{\lambda^* z}\Big[d_2 \int_{\mathbb{R}} K(y)e^{\lambda^* y}dy - c^*\lambda^* + \beta S_0 g'(0) - d_2 - \mu_2 - \gamma\Big] \\
&\quad - L_1 e^{\lambda^* z}\Big[d_2 \int_{\mathbb{R}} K(y)ye^{\lambda^* y}dy - c^*\Big] \\
&= -L_1 z e^{\lambda^* z}\Theta(\lambda^*, c^*) - L_1 e^{\lambda^* z}\Theta_\lambda(\lambda^*, c^*) = 0.
\end{aligned}
$$

On the other hand, by (1.9) and $I_+^*(z) \leq \bar{I}$ for $z \in \mathbb{R}$, we have for $z > z_4$ that

$$
\begin{aligned}
& d_2 \int_{\mathbb{R}} K(y)I_+^*(z-y)dy - c^*(I_+^*)'(z) + \beta S_+^*(z)g(I_+^*(z)) - (d_2+\mu_2+\gamma)I_+^*(z) \\
&\leq d_2\bar{I} + \beta S_0 g(\bar{I}) - (d_2+\mu_2+\gamma)\bar{I} = 0.
\end{aligned}
$$

Thus the proof is finished. ■

Lemma 3.3 *Assume that $\varepsilon_1 \in (0, \lambda^*)$ is a small enough constant. Then the function $S_-^*(z)$ satisfies*

$$
d_1 \int_{\mathbb{R}} K(y)S_-^*(z-y)dy - c^*(S_-^*)'(z) + b - (d_1+\mu_1)S_-^*(z) - \beta S_-^*(z)g(I_+^*(z)) \geq 0
$$

for any $z \neq z_5 = \varepsilon_1^{-1}\log[\varepsilon_1(S_0 - \underline{S})]$.

Proof. Noting that $z_5 = \varepsilon_1^{-1}\log[\varepsilon_1(S_0 - \underline{S})] \to -\infty$ as $\varepsilon_1 \to 0^+$, we can choose a small enough constant $\varepsilon_1 \in (0, \lambda^*)$ such that $z_5 < z_4$. Then $I_+^*(z) = -L_1 z e^{\lambda^* z}$ for $z < z_5$. Since

$$
\int_{\mathbb{R}} K(y)\frac{1-e^{-\varepsilon_1 y}}{\varepsilon_1}dy = \sum_{n=1}^{\infty}\frac{(-\varepsilon_1)^{2n-1}}{(2n)!}\int_{\mathbb{R}} K(y)y^{2n}dy \to 0 \text{ as } \varepsilon_1 \to 0^+,
$$

we have

$$
d_1 \int_{\mathbb{R}} K(y)\frac{1-e^{-\varepsilon_1 y}}{\varepsilon_1}dy + c^* + \frac{\mu_1}{\varepsilon_1} + \beta S_0 g'(0)L_1 z e^{(\lambda^*-\varepsilon_1)z} \geq 0 \text{ for } z < z_5. \tag{3.6}
$$

By (H1) and the definition of $S_-^*(z)$, one has that

$$
\int_{\mathbb{R}} K(y)S_-^*(z-y)dy \geq S_0 - \varepsilon_1^{-1}e^{\varepsilon_1 z}\int_{\mathbb{R}} K(y)e^{-\varepsilon_1 y}dy \text{ for } z \in \mathbb{R}. \tag{3.7}
$$

For $z < z_5$, we obtain from (3.6) and (3.7) that

$$
\begin{aligned}
& d_1 \int_{\mathbb{R}} K(y)S_-^*(z-y)dy - c^*(S_-^*)'(z) + b - (d_1+\mu_1)S_-^*(z) - \beta S_-^*(z)g(I_+^*(z)) \\
&\geq d_1\varepsilon_1^{-1}e^{\varepsilon_1 z}\int_{\mathbb{R}} K(y)(1-e^{-\varepsilon_1 y})dy + c^* e^{\varepsilon_1 z} + b - \mu_1 S_0 + \mu_1\varepsilon_1^{-1}e^{\varepsilon_1 z} - \beta S_0 g'(0)I_+^*(z) \\
&= \Big[d_1 \int_{\mathbb{R}} K(y)\frac{1-e^{-\varepsilon_1 y}}{\varepsilon_1}dy + c^* + \frac{\mu_1}{\varepsilon_1} + \beta S_0 g'(0)L_1 z e^{(\lambda^*-\varepsilon_1)z}\Big]e^{\varepsilon_1 z} \geq 0.
\end{aligned}
$$

For $z > z_5$, it is easy to see that $S_-^*(z) = \underline{S}$, $I_+^*(z) \leq \bar{I}$ and $\int_{\mathbb{R}} K(y)S_-^*(z-y)dy \geq \underline{S}$. Then it follows that

$$
\begin{aligned}
& d_1 \int_{\mathbb{R}} K(y)S_-^*(z-y)dy - c^*(S_-^*)'(z) + b - (d_1+\mu_1)S_-^*(z) - \beta S_-^*(z)g(I_+^*(z)) \\
&\geq d_1\underline{S} + b - (d_1+\mu_1)\underline{S} - \beta\underline{S}g'(0)I_+^*(z) \\
&\geq b - \mu_1\underline{S} - \beta\underline{S}g'(0)\bar{I} = 0,
\end{aligned}
$$

where we have used the fact that $\underline{S} = \frac{b}{\mu_1+\beta g'(0)\bar{I}}$ in the last equality. Hence the claim of this lemma is shown. ■

Lemma 3.4 *Assume that $L_2 > 1$ is a large enough constant. Then the function $I_-^*(z)$ satisfies*

$$d_2 \int_{\mathbb{R}} K(y) I_-^*(z-y)dy - c^*(I_-^*)'(z) + \beta S_-^*(z) g(I_-^*(z)) - (d_2+\mu_2+\gamma) I_-^*(z) \geq 0$$

for any $z \neq z_6 = -\frac{L_2^2}{L_1^2}$.

Proof. Due to $L_1 > \lambda^* e \bar{I}$ is a fixed constant and $z_6 = -\frac{L_2^2}{L_1^2} \to -\infty$ as $L_2 \to \infty$, one can select a large enough constant $L_2 > 1$ such that

$$\frac{1}{16} d_2 L_2 \int_{-r}^{r} K(y) y^2 e^{-\lambda^* y} dy - \beta S_0 L_1^2 (-z)^{\frac{7}{2}} e^{\lambda^* z} - \beta \varepsilon_1^{-1} g'(0) L_1 (-z)^{\frac{5}{2}} e^{\varepsilon_1 z} \geq 0 \tag{3.8}$$

and

$$\int_{-r}^{r} K(y) y^2 e^{-\lambda^* y} dy + \frac{1}{z} \int_{-r}^{r} K(y) y^3 e^{-\lambda^* y} dy \geq 0 \text{ for } z < z_6. \tag{3.9}$$

A simple computation gives that

$$(I_-^*)'(z) = -L_1 e^{\lambda^* z} - L_1 \lambda^* z e^{\lambda^* z} + \frac{1}{2} L_2 (-z)^{-\frac{1}{2}} e^{\lambda^* z} - L_2 \lambda^* (-z)^{\frac{1}{2}} e^{\lambda^* z} \text{ for } z < z_6. \tag{3.10}$$

By Taylor's formula, we get for $z < z_6$ that

$$(-z+y)^{\frac{1}{2}} \leq (-z)^{\frac{1}{2}} + \frac{1}{2}(-z)^{-\frac{1}{2}} y - \frac{1}{8}(-z)^{-\frac{3}{2}} y^2 + \frac{1}{16}(-z)^{-\frac{5}{2}} y^3,$$

which implies that

$$\begin{aligned}
&\int_{\mathbb{R}} K(y) I_-^*(z-y)dy = \int_{-r}^{r} K(y) I_-^*(z-y)dy \\
&\geq \int_{-r}^{r} K(y) \big[-L_1(z-y) e^{\lambda^*(z-y)} - L_2(-z+y)^{\frac{1}{2}} e^{\lambda^*(z-y)} \big] dy \\
&\geq -L_1 \int_{-r}^{r} K(y)(z-y) e^{\lambda^*(z-y)} dy \\
&\quad - L_2 \int_{-r}^{r} K(y) \Big[(-z)^{\frac{1}{2}} + \frac{1}{2}(-z)^{-\frac{1}{2}} y - \frac{1}{8}(-z)^{-\frac{3}{2}} y^2 + \frac{1}{16}(-z)^{-\frac{5}{2}} y^3 \Big] e^{\lambda^*(z-y)} dy \\
&= -L_1 z e^{\lambda^* z} \int_{-r}^{r} K(y) e^{\lambda^* y} dy - L_1 e^{\lambda^* z} \int_{-r}^{r} K(y) y e^{\lambda^* y} dy - L_2(-z)^{\frac{1}{2}} e^{\lambda^* z} \int_{-r}^{r} K(y) e^{\lambda^* y} dy \\
&\quad + \frac{1}{2} L_2 (-z)^{-\frac{1}{2}} e^{\lambda^* z} \int_{-r}^{r} K(y) y e^{\lambda^* y} dy + \frac{1}{8} L_2 (-z)^{-\frac{3}{2}} e^{\lambda^* z} \int_{-r}^{r} K(y) y^2 e^{-\lambda^* y} dy \\
&\quad - \frac{1}{16} L_2 (-z)^{-\frac{5}{2}} e^{\lambda^* z} \int_{-r}^{r} K(y) y^3 e^{-\lambda^* y} dy.
\end{aligned} \tag{3.11}$$

The assumption $\lim_{I \to 0^+} g(I)/I = g'(0)$ yields that for any $\epsilon \in (0, g'(0))$, there exists a constant $\delta > 0$ such that

$$g(I)/I \geq g'(0) - \epsilon \text{ for } I \in (0, \delta). \tag{3.12}$$

Since $L_2 > 1$ is large enough, $I_-^*(z) \in (0, \delta)$ and $z_6 < z_5 < z_4$ for $z < z_6$. Thus we obtain from (3.12) that

$$\begin{aligned}
\beta S_0 g'(0) I_-^*(z) - \beta S_-^*(z) g(I_-^*(z)) &= \beta S_0 g'(0) I_-^*(z) - \beta S_0 g(I_-^*(z)) + \beta \varepsilon_1^{-1} e^{\varepsilon_1 z} g(I_-^*(z)) \\
&= \beta S_0 I_-^*(z) \left[g'(0) - \frac{g(I_-^*(z))}{I_-^*(z)} \right] + \beta \varepsilon_1^{-1} e^{\varepsilon_1 z} g(I_-^*(z)) \\
&\leq \beta S_0 \left[\frac{I_-^*(z) + g'(0) - \frac{g(I_-^*(z))}{I_-^*(z)}}{2} \right]^2 + \beta \varepsilon_1^{-1} e^{\varepsilon_1 z} g'(0) I_-^*(z) \\
&\leq \beta S_0 \left[\frac{I_-^*(z) + \epsilon}{2} \right]^2 + \beta \varepsilon_1^{-1} g'(0) e^{\varepsilon_1 z} I_-^*(z) \\
&\leq \beta S_0 (I_-^*)^2(z) + \beta \varepsilon_1^{-1} g'(0) e^{\varepsilon_1 z} I_-^*(z) \\
&\leq \beta S_0 L_1^2 z^2 e^{2\lambda^* z} - \beta \varepsilon_1^{-1} g'(0) L_1 z e^{(\varepsilon_1 + \lambda^*) z} \text{ for } z < z_6.
\end{aligned} \tag{3.13}$$

Using $\Theta(\lambda^*,c^*)=\Theta_\lambda(\lambda^*,c^*)=0$ and (3.8)-(3.13), we derive that

$$
\begin{aligned}
& d_2\int_{\mathbb{R}}K(y)I_-^*(z-y)dy-c^*(I_-^*)'(z)+\beta S_-^*(z)g(I_-^*(z))-(d_2+\mu_2+\gamma)I_-^*(z)\\
&=d_2\int_{\mathbb{R}}K(y)I_-^*(z-y)dy-c^*(I_-^*)'(z)+\big[\beta S_0g'(0)-d_2-\mu_2-\gamma\big]I_-^*(z)-\big[\beta S_0g'(0)I_-^*(z)-\beta S_-^*(z)g(I_-^*(z))\big]\\
&\geq d_2\bigg[-L_1ze^{\lambda^*z}\int_{-r}^{r}K(y)e^{\lambda^*y}dy-L_1e^{\lambda^*z}\int_{-r}^{r}K(y)ye^{\lambda^*y}dy-L_2(-z)^{\frac12}e^{\lambda^*z}\int_{-r}^{r}K(y)e^{\lambda^*y}dy\\
&\quad+\frac12L_2(-z)^{-\frac12}e^{\lambda^*z}\int_{-r}^{r}K(y)ye^{\lambda^*y}dy+\frac18L_2(-z)^{-\frac32}e^{\lambda^*z}\int_{-r}^{r}K(y)y^2e^{-\lambda^*y}dy\\
&\quad-\frac1{16}L_2(-z)^{-\frac52}e^{\lambda^*z}\int_{\mathbb{R}}K(y)y^3e^{-\lambda^*y}dy\bigg]\\
&\quad-c^*\big[-L_1e^{\lambda^*z}-L_1\lambda^*ze^{\lambda^*z}+\frac12L_2(-z)^{-\frac12}e^{\lambda^*z}-L_2\lambda^*(-z)^{\frac12}e^{\lambda^*z}\big]\\
&\quad+\big[\beta S_0g'(0)-d_2-\mu_2-\gamma\big]\big[-L_1ze^{\lambda^*z}-L_2(-z)^{\frac12}e^{\lambda^*z}\big]-\big[\beta S_0L_1^2z^2e^{2\lambda^*z}-\beta\varepsilon_1^{-1}g'(0)L_1ze^{(\varepsilon_1+\lambda^*)z}\big]\\
&=-L_1e^{\lambda^*z}\bigg\{d_2z\int_{-r}^{r}K(y)e^{\lambda^*y}dy-c^*\lambda^*z+\big[\beta S_0g'(0)-d_2-\mu_2-\gamma\big]z+d_2\int_{-r}^{r}K(y)ye^{\lambda^*y}dy-c^*\bigg\}\\
&\quad-L_2(-z)^{\frac12}e^{\lambda^*z}\bigg[d_2\int_{-r}^{r}K(y)e^{\lambda^*y}dy-c^*\lambda^*+\beta S_0g'(0)-d_2-\mu_2-\gamma\bigg]\\
&\quad+\frac12L_2(-z)^{-\frac12}e^{\lambda^*z}\bigg[d_2\int_{-r}^{r}K(y)ye^{\lambda^*y}dy-c^*\bigg]+\frac1{16}d_2L_2(-z)^{-\frac32}e^{\lambda^*z}\int_{-r}^{r}K(y)y^2e^{-\lambda^*y}dy\\
&\quad-\big[\beta S_0L_1^2z^2e^{2\lambda^*z}-\beta\varepsilon_1^{-1}g'(0)L_1ze^{(\varepsilon_1+\lambda^*)z}\big]\\
&\quad+\frac1{16}d_2L_2(-z)^{-\frac32}e^{\lambda^*z}\int_{-r}^{r}K(y)y^2e^{-\lambda^*y}dy-\frac1{16}d_2L_2(-z)^{-\frac52}e^{\lambda^*z}\int_{\mathbb{R}}K(y)y^3e^{-\lambda^*y}dy\\
&=-L_1e^{\lambda^*z}\big[z\Theta(\lambda^*,c^*)+\Theta_\lambda(\lambda^*,c^*)\big]-L_2(-z)^{\frac12}e^{\lambda^*z}\Theta(\lambda^*,c^*)+\frac12L_2(-z)^{-\frac12}e^{\lambda^*z}\Theta_\lambda(\lambda^*,c^*)\\
&\quad+(-z)^{-\frac32}e^{\lambda^*z}\bigg[\frac1{16}d_2L_2\int_{-r}^{r}K(y)y^2e^{-\lambda^*y}dy-\beta S_0L_1^2(-z)^{\frac72}e^{\lambda^*z}-\beta\varepsilon_1^{-1}g'(0)L_1(-z)^{\frac52}e^{\varepsilon_1z}\bigg]\\
&\quad+\frac1{16}d_2L_2(-z)^{-\frac32}e^{\lambda^*z}\bigg[\int_{-r}^{r}K(y)y^2e^{-\lambda^*y}dy+\frac1z\int_{-r}^{r}K(y)y^3e^{-\lambda^*y}dy\bigg]\\
&\geq 0 \text{ for } z<z_6.
\end{aligned}
$$

If $z>z_6$, then $I_-^*(z)=0$, which implies that

$$d_2\int_{\mathbb{R}}K(y)I_-^*(z-y)dy-c^*(I_-^*)'(z)+\beta S_-^*(z)g(I_-^*(z))-(d_2+\mu_2+\gamma)I_-^*(z)\geq 0 \text{ for } z>z_6.$$

The proof of this lemma is completed. ∎

Using Lemma 3.1-Lemma 3.4 yields that the continuous functions pairs $(S_-^*,I_-^*)(z)$ and $(S_+^*,I_+^*)(z)$ are a pair of upper and lower solutions of system (1.7a)-(1.7b) with $c=c^*$. Then by the analogous argument in Section 2, one can obtain that model (1.1) has a nontrivial, bounded and positive traveling wave solution with critical speed c^*, which satisfies (1.8). In particular, if $z\to-\infty$, $I^*(z)=O(ze^{\lambda^*z})$ for $R_0>1$ and $c=c^*$. In a combination with Section 2 and Section 3, we finish the proof of Theorem 1.1.

4 Non-existence of sub-critical traveling wave solutions

In this section, we will show the non-existence of traveling wave solutions with the wave speed $c\in(-\infty,c^*)$ for (1.1). To this end, we shall explore separately the cases $c\in(-\infty,0]$ and $c\in(0,c^*)$. By the way of contradiction, for $c\in(-\infty,c^*)$, we suppose that (1.7a)-(1.7b) possesses a nontrivial and positive solution $(S,I)(z)$ satisfying

$$(S,I)(-\infty)=(S_0,0) \text{ and } (S,I)(\infty)=(S^*,I^*), \tag{4.1}$$

together with

$$\underline{S} < S(z) < S_0 \text{ and } 0 < I(z) < \bar{I} \text{ for } z \in \mathbb{R}, \tag{4.2}$$

where $\underline{S}$ is defined in (2.5) and $\bar{I}$ is given in (1.9). From (4.1) and (H3), we have

$$\lim_{z\to-\infty} \beta S(z)\frac{g(I(z))}{I(z)} = \beta S_0 g'(0). \tag{4.3}$$

Then it follows from $R_0 = \frac{\beta S_0 g'(0)}{\mu_2+\gamma} > 1$ and (4.3) that there exists a number $\hat{z} \ll 0$ such that

$$\beta S(z)\frac{g(I(z))}{I(z)} > \frac{\beta S_0 g'(0)}{2} + \frac{\mu_2+\gamma}{2} \text{ for } z \le \hat{z}. \tag{4.4}$$

In view of (4.4), we obtain from (1.7b) that

$$\begin{aligned}
cI'(z) &= d_2 \int_{\mathbb{R}} K(y)[I(z-y) - I(z)]dy + \beta S(z)g(I(z)) - (\mu_2+\gamma)I(z) \\
&\ge d_2 \int_{\mathbb{R}} K(y)[I(z-y) - I(z)]dy + \frac{\beta S_0 g'(0) + \mu_2 + \gamma}{2} I(z) - (\mu_2+\gamma)I(z) \\
&= d_2 \int_{\mathbb{R}} K(y)[I(z-y) - I(z)]dy + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} I(z) \text{ for } z \le \hat{z}.
\end{aligned} \tag{4.5}$$

Noting the fact that

$$\begin{aligned}
\int_{-\infty}^{z} \int_{\mathbb{R}} K(y)[I(\eta-y) - I(\eta)]dyd\eta &= \lim_{s\to-\infty} \int_{s}^{z} \int_{-r}^{r} K(y)[I(\eta-y) - I(\eta)]dyd\eta \\
&= \lim_{s\to-\infty} \int_{-r}^{r} K(y) \int_{s}^{z} [I(\eta-y) - I(\eta)]d\eta dy \\
&= \lim_{s\to-\infty} \int_{-r}^{r} K(y) \int_{s}^{z} \int_{\eta}^{\eta-y} I'(t)dtd\eta dy \\
&= \lim_{s\to-\infty} \int_{-r}^{r} K(y) \int_{s}^{z} \int_{0}^{1} I'(\eta-\theta y)(-y)d\theta d\eta dy \\
&= \lim_{s\to-\infty} \int_{-r}^{r} (-y)K(y) \int_{0}^{1} [I(z-\theta y) - I(s-\theta y)]d\theta dy \\
&= \int_{-r}^{r} (-y)K(y) \int_{0}^{1} I(z-\theta y)d\theta dy.
\end{aligned} \tag{4.6}$$

Then integrating (4.5) over $(-\infty, z]$ with $z \le \hat{z}$ and using (4.1), (4.2) and (4.6) yield that

$$\begin{aligned}
cI(z) &\ge d_2 \int_{-\infty}^{z} \int_{\mathbb{R}} K(y)[I(\eta-y) - I(\eta)]dyd\eta + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} \int_{-\infty}^{z} I(\eta)d\eta \\
&= d_2 \int_{-r}^{r} (-y)K(y) \int_{0}^{1} I(z-\theta y)d\theta dy + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} \int_{-\infty}^{z} I(\eta)d\eta \\
&\ge -2d_2\bar{I} \int_{0}^{r} yK(y)dy + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} \int_{-\infty}^{z} I(\eta)d\eta,
\end{aligned}$$

which implies that the improper integral $J(z) := \int_{-\infty}^{z} I(\eta)d\eta$ is well-defined for any $z \le \hat{z}$. Obviously, $J(z)$ is a continuously differentiable, positive and strictly increasing function for $z \in (-\infty, \hat{z}]$.

Case I: The wave speed $c \in (-\infty, 0]$. Note that $J(z) > 0$ and $-yJ(z-\theta y)$ is non-decreasing with respect to $\theta \in [0,1]$. Then integrating (4.5) twice over $(-\infty, z]$ with $z \le \hat{z}$ and using (4.6) give that

$$\begin{aligned}
0 \ge cJ(z) &\ge d_2 \int_{-\infty}^{z} \int_{-r}^{r} (-y)K(y) \int_{0}^{1} I(\eta-\theta y)d\theta dyd\eta + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} \int_{-\infty}^{z} J(\eta)d\eta \\
&= \lim_{s\to-\infty} d_2 \int_{s}^{z} \int_{-r}^{r} (-y)K(y) \int_{0}^{1} J'(\eta-\theta y)d\theta dyd\eta + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} \int_{-\infty}^{z} J(\eta)d\eta \\
&= \lim_{s\to-\infty} d_2 \int_{-r}^{r} (-y)K(y) \int_{0}^{1} [J(z-\theta y) - J(s-\theta y)]d\theta dy + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} \int_{-\infty}^{z} J(\eta)d\eta \\
&= d_2 \int_{-r}^{r} (-y)K(y) \int_{0}^{1} J(z-\theta y)d\theta dy + \frac{\beta S_0 g'(0) - \mu_2 - \gamma}{2} \int_{-\infty}^{z} J(\eta)d\eta
\end{aligned}$$

$$\geq d_2 J(z)\int_{-r}^{r}(-y)K(y)dy + \frac{\beta S_0 g'(0)-\mu_2-\gamma}{2}\int_{-\infty}^{z} J(\eta)d\eta$$
$$= \frac{\beta S_0 g'(0)-\mu_2-\gamma}{2}\int_{-\infty}^{z} J(\eta)d\eta > 0, \tag{4.7}$$

which leads to a contradiction.

Case II: The wave speed $c \in (0, c^*)$**.** Note from (4.7) that

$$\frac{\beta S_0 g'(0)-\mu_2-\gamma}{2}\int_{-\infty}^{z} J(\eta)d\eta \leq cJ(z) \text{ for } z \leq \hat{z},$$

which implies that there exists a large enough constant $z_0 > 0$ such that

$$\frac{z_0[\beta S_0 g'(0)-\mu_2-\gamma]}{2} J(z-z_0) \leq cJ(z) \text{ for } z \leq \hat{z},$$

that is,

$$J(z-z_0) \leq \delta_0 J(z) \text{ for } z \leq \hat{z}, \tag{4.8}$$

where the constant $\delta_0 := \frac{2c}{z_0[\beta S_0 g'(0)-\mu_2-\gamma]}$. Define

$$\mu_0 := \frac{1}{z_0}\log\frac{1}{\delta_0} \text{ and } H(z) := J(z)e^{-\mu_0 z} \text{ for } z \leq \hat{z}. \tag{4.9}$$

Utilizing (4.8) and (4.9) gives that

$$H(z-z_0) = J(z-z_0)e^{-\mu_0(z-z_0)} \leq \delta_0 J(z)e^{-\mu_0 z}e^{\mu_0 z_0} = H(z) \text{ for } z \leq \hat{z},$$

which together with $H(z) > 0$ ensures that the limit value $\lim_{z\to-\infty} H(z)$ exists. This implies that

$$\sup_{z\in(-\infty,\hat{z}]}\{J(z)e^{-\mu_0 z}\} < \infty. \tag{4.10}$$

By (4.2), $g(I(z)) \leq g'(0)I(z)$ for $z \in \mathbb{R}$ and (1.7b), we have

$$cI'(z) \leq d_2\int_{\mathbb{R}} K(y)[I(z-y)-I(z)]dy + [\beta S_0 g'(0)-\mu_2-\gamma]I(z). \tag{4.11}$$

Integrating (4.11) over $(-\infty, z]$ with $z \leq \hat{z}-r$ and using $I(-\infty) = 0$ yield that

$$cI(z) \leq d_2\int_{\mathbb{R}} K(y)[J(z-y)-J(z)]dy + [\beta S_0 g'(0)-\mu_2-\gamma]J(z). \tag{4.12}$$

From (4.10) and (4.12), we obtain that

$$\sup_{z\in(-\infty,\hat{z}-r]}\{I(z)e^{-\mu_0 z}\} < \infty. \tag{4.13}$$

Using (4.13) and (4.2), we define the following two-sided Laplace transform of $I(z)$ by

$$\mathfrak{L}(\lambda) := \int_{\mathbb{R}} I(z)e^{-\lambda z}dz,$$

where $\lambda \in \mathbb{C}$ with $0 < \text{Re}\lambda < \mu_0$. Rewrite (1.7b) as follows

$$d_2\int_{\mathbb{R}} K(y)\big[I(z-y)-I(z)\big]dy - cI'(z) + \big[\beta S_0 g'(0)-\mu_2-\gamma\big]I(z) = \beta S_0 g'(0)I(z) - \beta S(z)g(I(z)). \tag{4.14}$$

Taking the two-sided Laplace transform on (4.14) and using $I(-\infty) = 0$, we deduce that

$$\Theta(\lambda,c)\mathfrak{L}(\lambda) = \int_{\mathbb{R}}\big[\beta S_0 g'(0)I(z) - \beta S(z)g(I(z))\big]e^{-\lambda z}dz, \tag{4.15}$$

where $0 < \text{Re}\lambda < \mu_0$ and $\Theta(\lambda,c) = d_2\int_{\mathbb{R}} K(y)e^{-\lambda y}dy - c\lambda + \beta S_0 g'(0) - d_2 - \mu_2 - \gamma$. Recall that $\lim_{I\to 0^+} g(I)/I = g'(0)$, which indicates that for any $\hat{\varepsilon} \in (0, g'(0))$, there exists a small positive constant $\hat{\delta}$ such that

$$\frac{g(I)}{I} \geq g'(0) - \hat{\varepsilon} \text{ when } 0 < I < \hat{\delta}.$$

Then if $0 < I(z) < \hat{\delta}$, it follows that

$$
\begin{aligned}
\beta S_0 g'(0) I(z) - \beta S(z) g(I(z)) &= \beta I(z) \left[S_0 g'(0) - S(z) \frac{g(I(z))}{I(z)} \right] \\
&\leq \beta \left[\frac{S_0 g'(0) - S(z) \frac{g(I(z))}{I(z)} + I(z)}{2} \right]^2 \\
&\leq \beta \left[\frac{S_0 g'(0) - S(z)(g'(0) - \hat{\varepsilon}) + I(z)}{2} \right]^2 .
\end{aligned} \tag{4.16}
$$

Since (4.16) holds for arbitrary small enough $\hat{\varepsilon} \in (0, g'(0))$ and $(S, I)(z) \to (S_0, 0)$ as $z \to -\infty$, one can infer from (4.16) that there exists a sufficient large number $Z > 0$ such that

$$
\beta S_0 g'(0) I(z) - \beta S(z) g(I(z)) \leq \beta I^2(z) \text{ for } z \leq -Z. \tag{4.17}
$$

Hence, we obtain from (4.17) and (4.13) that

$$
\sup_{z \in (-\infty, \min\{\hat{z} - r, -Z\}]} e^{-2\mu_0 z} \left[\beta S_0 g'(0) I(z) - \beta S(z) g(I(z)) \right] < \infty,
$$

which implies that

$$
\int_{\mathbb{R}} \left[\beta S_0 g'(0) I(z) - \beta S(z) g(I(z)) \right] e^{-\lambda z} dz < \infty \text{ for } 0 < \mathrm{Re}\lambda < 2\mu_0. \tag{4.18}
$$

In view of the property of Laplace transform [32], one can infer that one of the following two conclusions holds:

(i) $\mathfrak{L}(\lambda)$ is well-defined for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$;

(ii) There exists a positive constant μ_* such that $\mathfrak{L}(\lambda)$ is analytic for $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \mu_*$ and $\lambda = \mu_*$ is a singular point of $\mathfrak{L}(\lambda)$.

Notice from (4.15) that two Laplace integrals $\int_{\mathbb{R}} I(z) e^{-\lambda z} dz$ and $\int_{\mathbb{R}} [\beta S_0 g'(0) I(z) - \beta S(z) g(I(z))] e^{-\lambda z} dz$ must be analytically extended to the entire right half plane. If not, the Laplace $\int_{\mathbb{R}} I(z) e^{-\lambda z} dz$ in (4.15) is analytic for $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \mu_0$ and admits a singular point $\lambda = \mu_0$. However, it follows from (4.18) that $\int_{\mathbb{R}} [\beta S_0 g'(0) I(z) - \beta S(z) g(I(z))] e^{-\lambda z} dz$ in (4.15) is analytic for $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < 2\mu_0$, which yields a contradiction. Therefore, (4.15) holds for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$. Note that for each $c \in (0, c^*)$, $\Theta(\lambda, c) \to \infty$ as $\lambda \to \infty$. Then let $\lambda \to \infty$ in (4.15) lead to another contradiction. Based on the above arguments, we complete the proof of Theorem 1.2.

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