

# Solvability of nonlocal Hilfer fractional matrix boundary value problems with $p$ -Laplacian at resonance in $\mathbb{R}^n$

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**Abstract.** In this paper, the solvability of boundary value problems for a class of nonlinear Hilfer fractional differential equations at resonance in  $\mathbb{R}^n$  is studied. In the past, research on matrix boundary value problems has consistently been conducted within the context of linear differential equations. The main contribution of this paper is the extension of linear problems to nonlinear ones. We begin by defining two Banach spaces endowed with appropriate norms and constructing suitable operators in these Banach spaces. Subsequently, by using the extension for the continuous theorem, certain sufficient conditions for the solvability of the problem are obtained. Finally, an example is provided to verify the effectiveness of our main results.

**Keywords.** Matrix boundary value problem;  $p$ -Laplacian; Continuous theorem; Hilfer fractional derivative.

**MSC Classification.** 34A08, 34B15, 34A04, 34B10

## 1 Introduction

This work considers the solvability of the following nonlinear Hilfer fractional matrix boundary value problems in  $\mathbb{R}^n$ :

$$\begin{cases} D_{0+}^{\alpha_1, \beta_1} \varphi_p(D_{0+}^{\alpha_2, \beta_2} u(t)) = f(t, u(t), D_{0+}^{\alpha_2, \beta_2} u(t)), & 0 \leq t \leq 1, \\ D_{0+}^{\alpha_2, \beta_2} u(0) = D_{0+}^{\gamma_2-2} u(0) = \cdots = D_{0+}^{\gamma_2-m} u(0) = \theta, \\ u(1) = A \int_0^1 u(t) h(t) dt, \end{cases} \quad (1.1)$$

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where  $0 < \alpha_1 \leq 1$ ,  $m - 1 < \alpha_2 \leq m$ ,  $0 \leq \beta_1, \beta_2 \leq 1$ ,  $\gamma_2 = \alpha_2 + m\beta_2 - \alpha_2\beta_2$ ,  $p > 1$ ,  $\varphi_p(\vartheta) = |\vartheta|^{p-2}\vartheta$ ,  $u = (u_i)_{n \times 1}$ ,  $A = (a_{ii})_{n \times n}$ ,  $a_{ii} \leq 0$ ,  $m, n \in \mathbb{N}_+$ ,  $h(t) \geq 0$ ,  $f \in C([0, 1] \times \mathbb{R}^{2n}, \mathbb{R}^n)$ ,  $\theta$  is the zero vector in  $\mathbb{R}^n$  and  $D_{0+}^{\alpha, \beta}$  represents the Hilfer fractional derivative operator.

Fractional differential equations are widely used in physical and biological fields, such as elastomers, vibration and diffusion systems [1–14]. Fractional boundary value problems have been extensively studied, and numerous results regarding their solvability have been obtained. For example, Seal et al. [15] analyzed the convergence of solutions of fractional differential equations with integral boundary conditions by spline approximation method. In [16], Zaky discussed the existence, uniqueness and stability of solutions to nonlinear tempered fractional generalized boundary value problems. Furthermore, the method of singular spectrum collocation for obtaining the numerical solutions of these equations has been developed and analyzed. In [17], Azouzi et al. obtained the existence of solutions for generalized fractional boundary value problems by using the Mawhin continuation theorem. Moreover, Wang et al. [18] derived the existence of triple positive solutions for a class of fractional boundary value problems at resonance. Some new height functions and spectral theory are also used to solve the positive solutions. The main method used is the fixed point index theorem.

Mawhin's continuation theorem [19] is a classical method often used to study the existence of solutions for differential equations of the form  $Lx = Nx$  under resonance conditions, where the operator  $L$  is an irreversible linear operator. Ge et al. [20] first generalized the result of Mawhin in [19], in which the existence theory of solutions was obtained for the non-invertible nonlinear operator  $L$ . Furthermore, Jiang [21] considered the following nonlinear problem with integral boundary conditions in one-dimensional space:

$$\begin{cases} D_{0+}^{\theta}(\varphi_p(D_{0+}^{\gamma}x))(t) + f(t, x(t), D_{0+}^{\gamma-1}x(t), D_{0+}^{\gamma}x(t)) = 0, \\ x(0) = D_{0+}^{\gamma}x(0) = 0, \quad x(1) = \int_0^1 g(t)x(t)dt, \end{cases} \quad (1.2)$$

where  $p > 1$ ,  $0 < \theta \leq 1$ ,  $1 < \gamma \leq 2$ ,  $\varphi_p(\mu) = |\mu|^{p-2}\mu$ ,  $D_{0+}^{\alpha}$  denotes the Riemann-Liouville derivative operator. The author improved the results in [20] and proved the existence of the solution to the problem (1.2). Obviously, the problem (1.2) is a particular case of the problem (1.1) when  $n = 1$ ,  $m = 2$  and  $\beta_1 = \beta_2 = 0$ . Subsequently, Wang et al. [22] considered the solvability on the half-line at resonance for the case  $n = 1$  and  $\beta_1 = \beta_2 = 0$  in the problem (1.1). Baitiche et al. [23] also studied the boundary value problem similar to one of [22] by using upper and lower solution approximation. Recently, Feng et al. [24] have discussed the solvability of linear fractional boundary value problems

in  $\mathbb{R}^n$  without the  $p$ -Laplacian operator in the problem (1.1).

We should mention the main results obtained in [21, 24–26], which prompts us to consider the problem (1.1). In [25], Phung et al. first researched the following second-order linear boundary value problem:

$$\begin{cases} u''(t) = g(t, u, u'), & 0 < t < 1, \\ u'(0) = \theta, \quad u(1) = Au(\xi), \end{cases} \quad (1.3)$$

where  $\theta$  is a zero vector in  $\mathbb{R}^n$ ,  $0 < \xi < 1$  and  $A$  is an  $n$ -order square matrix satisfying one of the following two conditions:

$$\begin{cases} A^2 = I \text{ (} I \text{ stands for the unit matrix),} \\ A^2 = A. \end{cases} \quad (1.4)$$

By using Mawhin's continuation theorem, the solvability conditions of the problem (1.3) were obtained. Then, Phung et al. [26] studied the following Riemann-Liouville fractional linear boundary value problem:

$$\begin{cases} D^\mu u(t) = g(t, u(t), D^\mu u(t)), & a.e. \ 0 < t < 1, \\ u(0) = \theta, \quad D^{\mu-1}u(1) = AD^{\mu-1}u(\xi), \end{cases} \quad (1.5)$$

where  $1 < \mu \leq 2$ ,  $D^\mu$  is the Riemann-Liouville differential operator of order  $\mu$ .

In general, the highlights of this paper can be summarized as follows.

- On the one hand, compared with the linear problems (1.3) and (1.5), the nonlinear term  $\varphi_p$  is introduced in the problem (1.1), which makes it more complicated to study the existence of solutions. It is worth noting that we also extend the nonlinear boundary value problem (1.3) to  $n$ -dimensional Euclidean space. (To the best of the author's knowledge, this is the first study on nonlinear boundary value problems in  $\mathbb{R}^n$ ).
- On the other hand, the boundary condition of the problem (1.1) is presented as an integral form with a coefficient matrix, and the constraints on the coefficient matrix  $A$  have been weakened. It is no longer required the idempotent or involutory matrices in (1.4). This can be regarded as a generalization of the boundary conditions in the problem (1.3).
- In addition, the Hilfer fractional derivative in the problem (1.1) covers both Caputo and Riemann-Liouville derivatives, and can be regarded as a generalization of these two types of derivatives. Therefore, the research in this paper is not only an extension of the nonlinear boundary value problem but also provides an interesting case for the application of Hilfer fractional derivative in the field of calculus.

The rest of this paper includes the following sections. In Sect. 2, some definitions and lemmas are introduced, and two Banach spaces are constructed. In Sect. 3, we first give some preliminary results that are needed in the proof of our main theorem. Based on the extension for the continuous theorem, we then prove the existence of the solution of the problem (1.1). In Sect. 4, the main results are illustrated by an example. A conclusion is introduced in Sect. 5.

## 2 Preliminaries

**Definition 2.1.** [20] Suppose that  $Y$  and  $Z$  are two Banach spaces with norms of  $\|\cdot\|_Y$  and  $\|\cdot\|_Z$  respectively. If the continuous operator  $F: \text{dom}F \cap Y \rightarrow Z$  satisfies the following conditions :

- (a)  $\text{Ker}F := \{u \in \text{dom}F \cap Y : Fu = 0\}$  is linearly homeomorphic to  $\mathbb{R}^n$ ,
- (b)  $\text{Im}F := F(\text{dom}F \cap Y) \subset Z$  is a closed,

where  $n < \infty$ ,  $\text{dom}F$  is the domain of the operator  $F$ . Then the operator  $F$  is called quasi-linear.

**Definition 2.2.** [21] Assuming  $N_\kappa : \bar{\Omega} \rightarrow Z, \kappa \in [0, 1]$  is a bounded and continuous operator, let  $\Sigma_\kappa = \{x \in \bar{\Omega} : Fx = N_\kappa x\}$ ,  $\text{Ker}F = Y_1$ . Suppose furthermore that at least one vector space  $Z_1 \subset Z$  satisfies  $\dim Y_1 = \dim Z_1$ . If there exist operators  $P, R$  and  $Q$  satisfying the following conditions for any  $0 \leq \kappa \leq 1$ :

- (a)  $\text{Ker}Q = \text{Im}F$ ,
- (b)  $QNx = \theta \Leftrightarrow QN_\kappa x = \theta$ ,
- (c)  $R(\cdot, 0)$  is the zero operator, and  $R(\cdot, \kappa)|_{\Sigma_\kappa} = (I - P)|_{\Sigma_\kappa}$ ,
- (d)  $F[P + R(\cdot, \kappa)] = (I - Q)N_\kappa$ ,

where  $P : Y \rightarrow Y_1$  is a projector,  $R : \bar{\Omega} \times [0, 1] \rightarrow Y_2$  is a continuous compact operator, and  $Q : Z \rightarrow Z_1$  is a continuous bounded operator satisfying  $Q(I - Q) = 0$ . Then the operator  $N_\kappa$  is called  $F$ -quasi-compact in  $\bar{\Omega}$ .

**Definition 2.3.** [27] Suppose the function  $u(t)$  is defined on the interval  $(a, b)$ , and  $n - 1 < \mu \leq n$ ,  $n \in \mathbb{N}^*$ . The left Riemann-Liouville fractional derivative and integral of order  $\mu$  are defined as:

$$D_{a+}^\mu u(t) = \frac{d^n}{dt^n} (I_{a+}^{n-\mu} u)(t) \text{ and } I_{a+}^\mu u(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t - \xi)^{\mu-1} u(\xi) d\xi.$$

**Definition 2.4.** [28] Suppose the function  $u(t)$  is defined on the interval  $(a, b)$ , and  $n - 1 < \mu \leq n$ ,  $n \in \mathbb{N}^*$ ,  $0 \leq \delta \leq 1$ . The left/right Hilfer fractional derivative of order  $\mu$  and type  $\delta$  is defined as:

$$D_{a\pm}^{\mu, \delta} u(t) = (\pm)^n I_{a\pm}^{\delta(n-\mu)} \frac{d^n}{dt^n} (I_{a\pm}^{(1-\delta)(n-\mu)} u)(t).$$

**Remark 2.5.** [28]

- (1) The differential operator  $D_{a\pm}^{\mu,\delta}$  can be equivalently expressed as  $D_{a\pm}^{\mu,\delta} = I_{a\pm}^{\delta(n-\mu)} D_{a\pm}^{\gamma}$ ,  $\gamma = \mu + n\delta - \mu\delta$ .
- (2) The Riemann-Liouville derivative is equivalent to the Hilfer derivative when  $\delta = 0$ , that is,  $D_{a\pm}^{\mu} = D_{a\pm}^{\mu,0}$ .
- (3) The Caputo derivative is equivalent to the Hilfer derivative when  $\delta = 1$ , that is,  ${}^C D_{a\pm}^{\mu} = D_{a\pm}^{\mu,1}$ .

**Lemma 2.6.** [21] Assuming  $Y$  and  $Z$  are two Banach spaces with norms  $\|\cdot\|_Y$  and  $\|\cdot\|_Z$ , respectively, and  $\Omega$  is a bounded non-empty open subset of  $Y$ . Suppose furthermore that the operator  $F : \text{dom}F \cap Y \rightarrow Z$  is quasi-linear, and  $N_{\kappa} : \bar{\Omega} \rightarrow Z$ ,  $\kappa \in [0, 1]$  is  $F$ -quasi-compact. If

- (a)  $Fx \neq N_{\kappa}x$ , for all  $x \in \text{dom}F \cap \partial\Omega$  and  $\kappa \in (0, 1)$ ,
- (b)  $\text{deg}\{KQN, \Omega \cap \text{Ker}F, 0\} \neq 0$ ,

holds, where  $K : \text{Im}Q \rightarrow \text{Ker}F$  is a homeomorphism with  $K(\theta) = \theta$ . Then there exists at least one solution for the abstract equation  $Fx = Nx$  in  $\text{dom}F \cap \bar{\Omega}$ .

**Lemma 2.7.** [29] Assume  $m - 1 \leq \mu \leq m$ ,  $m \in \mathbb{N}^*$ , Suppose furthermore that  $u \in L^1(0, 1)$  and  $I_{0+}^{m-\mu}u \in AC^m[0, 1]$ , then

$$I_{0+}^{\mu} D_{0+}^{\mu} u(t) = u(t) - \sum_{j=1}^m \frac{(I_{0+}^{m-\mu} u(t))^{(m-j)}|_{t=0}}{\Gamma(\mu - j + 1)} t^{\mu-j}.$$

**Lemma 2.8.** [29] Suppose  $\mu > 0$  and  $\delta > 0$ , then

$$D_{0+}^{\mu} t^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\mu + \delta)} t^{\mu+\delta-1}.$$

**Lemma 2.9.** [30] For any  $x$  and  $y$  with  $x, y \geq 0$ , the following inequalities hold:

- (1)  $\varphi_p(x + y) \leq 2^{p-2}(\varphi_p(x) + \varphi_p(y))$ ,  $p \geq 2$ ,
- (2)  $\varphi_p(x + y) \leq \varphi_p(x) + \varphi_p(y)$ ,  $1 < p \leq 2$ ,

where  $\varphi_p(x) = |x|^{p-2}x$ .

Next, we define several Banach spaces and operators. By  $\|u\|_X = \max\{\|u\|_{\infty}, \|D_{0+}^{\alpha_2, \beta_2} u\|_{\infty}\}$  we denote the norm of  $u$  in the space  $X = \{u | u, D_{0+}^{\alpha_1, \beta_1} u \in C([0, 1]; \mathbb{R}^n)\}$ , where  $\|u\|_{\infty} = \max_{t \in [0, 1]} \max_{1 \leq i \leq n} |u_i(t)|$ . Furthermore, by  $\|y\|_{\infty}$  we denote the norm of  $u$  in the space  $Y = C([0, 1]; \mathbb{R}^n)$ . The operators  $L : \text{dom}L \cap X \rightarrow Y$  and  $N_{\lambda} : X \rightarrow Y$  are defined as follows

$$Lu(t) = D_{0+}^{\alpha_1, \beta_1} \varphi_p(D_{0+}^{\alpha_2, \beta_2} u(t)), \quad t \in [0, 1], \quad (2.1)$$

$$N_\lambda u(t) = \lambda f(t, u(t), D_{0+}^{\alpha_2, \beta_2} u(t)), \quad \lambda \in [0, 1], \quad (2.2)$$

where

$$\begin{aligned} \text{dom}L = \left\{ u \mid u \in X, D_{0+}^{\alpha_1, \beta_1} \varphi_p(D_{0+}^{\alpha_2, \beta_2} u) \in Y, D_{0+}^{\alpha_2, \beta_2} u(0) = D_{0+}^{\gamma_2-2} u(0) = \dots = D_{0+}^{\gamma_2-m} u(0) = \theta, \right. \\ \left. u(1) = A \int_0^1 u(t) h(t) dt \right\}. \end{aligned} \quad (2.3)$$

Therefore, we can write the problem (1.1) as  $Lu = Nu$ ,  $u \in \text{dom}L$ .

Let  $T = I - A \int_0^1 h(t) t^{\gamma_2-1} dt$  and  $T^+$  be the Moore-Penrose pseudoinverse matrix of  $T$ . It is necessary to give the following conclusions in [31] for our subsequent research:

- (a)  $\text{Im}(I - T^+T) = \text{Ker}T$ ;
- (b)  $\text{Im}T^+T = \text{Im}T$ ;
- (c)  $TT^+T = T$ ;
- (d)  $T^+TT^+ = T^+$ .

In addition, throughout this paper, we always suppose that  $\det(I - A \int_0^1 h(t) t^{\gamma_2-1} dt) = 0$ .

### 3 Main results

In this section, we will prove that the problem (1.1) has at least one solution. To make the proof process clearer, six lemmas and one theorem will be given respectively.

**Lemma 3.1.** *Suppose the condition  $\det(I - A \int_0^1 h(t) t^{\gamma_2-1} dt) = 0$  holds, then the operator  $L$  defined in (2.1) is quasi-linear.*

*Proof.* It is not difficult to obtain that

$$\text{Ker}L = \{u \in \text{dom}L \mid u(t) = ct^{\gamma_2-1}, c \in \text{Ker}T\}, \quad (3.1)$$

where  $T = I - A \int_0^1 h(t) t^{\gamma_2-1} dt$ . Now, we prove

$$\text{Im}L = \{y \in Y \mid \phi y \in \text{Im}T\}, \quad (3.2)$$

where  $\phi : Y \rightarrow \mathbb{R}^n$  is a linear operator defined by

$$\phi y = I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} y(t)) \Big|_{t=1} - A \int_0^1 h(t) I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} y(t)) dt, \quad \forall y \in Y. \quad (3.3)$$

In fact, for each  $y \in ImL$ , there exists a function vector  $u \in domL$  such that

$$D_{0+}^{\alpha_1, \beta_1} \varphi_p(D_{0+}^{\alpha_2, \beta_2} u(t)) = y(t).$$

By Lemma 2.7 and Remark 2.5, we obtain

$$D_{0+}^{\alpha_2, \beta_2} u(t) = \varphi_q(I_{0+}^{\alpha_1} y(t) + c_0 t^{\gamma_1 - 1}),$$

where  $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1 \beta_1$ ,  $q = \frac{p}{p-1}$ . Since  $D_{0+}^{\alpha_2, \beta_2} u(0) = D_{0+}^{\gamma_2 - 2} u(0) = \dots = D_{0+}^{\gamma_2 - m} u(0) = \theta$ , we can get

$$u(t) = I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} y(t)) + c_1 t^{\gamma_2 - 1}, \quad c_1 \in \mathbb{R}^n.$$

From  $u(1) = A \int_0^1 h(t) u(t) dt$ , it can be deduced that

$$I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} y(t))|_{t=1} - A \int_0^1 h(t) I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} y(t)) dt + \left( I - A \int_0^1 h(t) t^{\gamma_2 - 1} dt \right) c_1 = \theta. \quad (3.4)$$

Consequently,

$$ImL \subseteq \{y \in Y \mid \phi y \in ImL\}. \quad (3.5)$$

On the other hand, let  $u(t) = I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} y(t)) + \xi t^{\gamma_2 - 1}$ ,  $\xi \in \mathbb{R}^n$ , and assume that  $y \in Y$  satisfies (3.4). By simple calculation, we can infer that  $u(t)$  satisfies the boundary conditions of the problem (1.1) and

$$Lu(t) = D_{0+}^{\alpha_1, \beta_1} \varphi_p \left( D_{0+}^{\alpha_2, \beta_2} (I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} y(t)) + \xi t^{\gamma_2 - 1}) \right) = D_{0+}^{\alpha_1, \beta_1} \varphi_p(\varphi_q I_{0+}^{\alpha_1} y(t)) = y(t).$$

Thus,

$$ImL \supseteq \{y \in Y \mid \phi y \in ImL\}. \quad (3.6)$$

Combining (3.5) and (3.6), we can get

$$ImL = \{y \in Y \mid \phi y \in ImL\}. \quad (3.7)$$

Clearly,  $ImL \subset Y$  is closed. Thus, the operator  $L$  is called a quasi-linear operator.  $\square$

The operator  $P : X \rightarrow KerL$  is defined as

$$(Pu)(t) = (I - T^+ T) \frac{t^{\gamma_2 - 1}}{\Gamma(\gamma_2)} D_{0+}^{\gamma_2 - 1} u(0). \quad (3.8)$$

It can be derived by simple calculation that  $P^2 u = Pu$  and  $ImP = KerL$ , then  $KerP \oplus KerL = X$ . Hence,  $P : X \rightarrow KerL$  is a projector.

The operator  $Q : Y \rightarrow \mathbb{R}^n$  is defined as

$$Qy = c, \quad (3.9)$$

where  $c$  satisfies

$$\begin{aligned} & \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y(\tau) - c] d\tau \right) ds + T\xi \\ & - A \int_0^1 \frac{h(t)}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y(\tau) - c] d\tau \right) ds dt = \theta. \end{aligned} \quad (3.10)$$

It can be proved that  $c$  is the unique constant vector satisfying (3.10). In fact, let

$$\begin{aligned} F(c) = & \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y(\tau) - c] d\tau \right) ds + T\xi \\ & - A \int_0^1 \frac{h(t)}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y(\tau) - c] d\tau \right) ds dt \end{aligned} \quad (3.11)$$

for all  $y \in Y$ . Since

$$\begin{aligned} T &= I - A \int_0^1 h(t) t^{\gamma_2-1} dt \\ &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} a_{11} \int_0^1 h(t) t^{\gamma_2-1} dt & & & \\ & a_{22} \int_0^1 h(t) t^{\gamma_2-1} dt & & \\ & & \ddots & \\ & & & a_{nn} \int_0^1 h(t) t^{\gamma_2-1} dt \end{pmatrix} \\ &= \begin{pmatrix} 1 - a_{11} \int_0^1 h(t) t^{\gamma_2-1} dt & & & \\ & 1 - a_{22} \int_0^1 h(t) t^{\gamma_2-1} dt & & \\ & & \ddots & \\ & & & 1 - a_{nn} \int_0^1 h(t) t^{\gamma_2-1} dt \end{pmatrix}, \end{aligned}$$

we have

$$T\xi = \begin{pmatrix} 1 - ka_{11} & & & \\ & 1 - ka_{22} & & \\ & & \ddots & \\ & & & 1 - ka_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \xi_1(1 - ka_{11}) \\ \xi_2(1 - ka_{22}) \\ \vdots \\ \xi_n(1 - ka_{nn}) \end{pmatrix}, \quad (3.12)$$

where  $k = \int_0^1 h(t) t^{\gamma_2-1} dt$ . Substituting (3.12) into  $F(c) = (F_i(c))_{1 \times n}$  defined in (3.11), we can obtain

$$F_i(c) = \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y_i(\tau) - c_i] d\tau \right) ds + \xi_i(1 - ka_{ii})$$



$$- \frac{a_{ii}}{\Gamma(\alpha_2)} \int_0^1 h(t) \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y_i(\tau) - c_i] d\tau \right) ds dt.$$

Obviously,  $F_i(c)$  is continuous and strictly decreasing in  $\mathbb{R}$ . Define a cone  $\Lambda$  in  $\mathbb{R}^n$  as

$$\Lambda = \{(\Lambda_1, \Lambda_2, \dots, \Lambda_n)^\top, \Lambda_i \geq 0, \Lambda_i \in \mathbb{R}, i = 1, 2, \dots, n\}. \quad (3.13)$$

Take

$$b_i = \min_{t \in [0,1]} y_i(t) + m_i, \quad d_i = \max_{t \in [0,1]} y_i(t) + m_i, \quad i = 1, 2, \dots, n,$$

where  $m_i = \frac{\xi_i(1-ka_{ii})\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1q-\alpha_1+1)}{\Gamma(\alpha_1q-\alpha_1+1)(1-a_{ii}\int_0^1 h(t)t^{\alpha_2+\alpha_1q-\alpha_1} dt)}$ ,  $b_i, d_i \in \mathbb{R}$ . If  $b_i = \min_{t \in [0,1]} y_i(t) + m_i$ , then

$$\begin{aligned} F_i(b) &\geq 0 - \frac{m_i}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} d\tau \right) ds \\ &\quad + \frac{a_{ii}m_i}{\Gamma(\alpha_2)} \int_0^1 h(t) \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} d\tau \right) ds dt + \xi_i(1-ka_{ii}) \\ &= - \frac{m_i\Gamma(\alpha_1(q-1)+1)}{\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1(q-1)+1)} + \frac{m_ia_{ii}\Gamma(\alpha_1(q-1)+1)\int_0^1 h(t)t^{\alpha_2+\alpha_1q-\alpha_1} dt}{\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1(q-1)+1)} \\ &\quad + \xi_i(1-ka_{ii}) \\ &= - \frac{m_i\Gamma(\alpha_1(q-1)+1)(1-a_{ii}\int_0^1 h(t)t^{\alpha_2+\alpha_1q-\alpha_1} dt)}{\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1(q-1)+1)} + \xi_i(1-ka_{ii}) \\ &= - \xi_i(1-ka_{ii}) + \xi_i(1-ka_{ii}) = 0. \end{aligned}$$

Similarly, if  $d_i = \max_{t \in [0,1]} y_i(t) + m_i$ , then  $F_i(d) \leq 0$ . It is not difficult to see that  $F(b) \in \Lambda$  and  $-F(d) \in \Lambda$ , where  $b = (b_1, b_2, \dots, b_n)^\top$ ,  $d = (d_1, d_2, \dots, d_n)^\top$ . Hence, there must be a unique  $c$  satisfying  $c - b \in \Lambda$  and  $d - c \in \Lambda$ , such that  $F(c) = \theta$ . In addition, the boundedness of  $Q(\Omega)$  can be deduced from the fact that space  $\Omega \subset Y$  is bounded.

**Remark 3.2.** By the definition of  $Q$  in (3.9), it is not difficult to conclude that  $Q$  is not a projector but satisfies  $Q(I - Q)y = \theta$  for all  $y \in Y$ .

**Lemma 3.3.** *The operator  $Q$  by (3.9) is continuous in  $Y$ .*

*Proof.* For any  $g, y \in Y$ , suppose  $Qg = d$ ,  $Qy = b$ , where  $b, d \in \mathbb{R}^n$ . Since  $\varphi_q$  is strictly increasing, if  $d_i - b_i > \max_{t \in [0,1]} (g_i(t) - y_i(t))$ ,  $i = 1, 2, \dots, n$ , then

$$\begin{aligned} 0 &= \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [g_i(\tau) - d_i] d\tau \right) ds dt + \xi_i(1-ka_{ii}) \\ &\quad - a_{ii} \int_0^1 h(t) \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [g_i(\tau) - d_i] d\tau \right) ds dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [(y_i(\tau) - b_i) + (g_i(\tau) - y_i(\tau)) - (d_i - b_i)] d\tau \right) ds dt \\
&\quad + \xi_i(1 - ka_{ii}) - a_{ii} \int_0^1 h(t) \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [(y_i(\tau) - b_i) \right. \\
&\quad \left. + (g_i(\tau) - y_i(\tau)) - (d_i - b_i)] d\tau \right) ds dt \\
&< \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y_i(\tau) - b_i] d\tau \right) ds + \xi_i(1 - ka_{ii}) \\
&\quad - a_{ii} \int_0^1 h(t) \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y_i(\tau) - b_i] d\tau \right) ds dt = 0.
\end{aligned}$$

This is a contradiction. Conversely, if  $d_i - b_i < \min_{t \in [0,1]} (g_i(t) - y_i(t))$ , then

$$\begin{aligned}
0 &= \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [g_i(\tau) - d_i] d\tau \right) ds dt + \xi_i(1 - ka_{ii}) \\
&\quad - a_{ii} \int_0^1 h(t) \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [g_i(\tau) - d_i] d\tau \right) ds dt \\
&> \frac{1}{\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y_i(\tau) - b_i] d\tau \right) ds dt + \xi_i(1 - ka_{ii}) \\
&\quad - a_{ii} \int_0^1 h(t) \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} [y_i(\tau) - b_i] d\tau \right) ds dt = 0,
\end{aligned}$$

the contradiction appears. Consequently,

$$\min_{t \in [0,1]} (g_i(t) - y_i(t)) \leq d_i - b_i \leq \max_{t \in [0,1]} (g_i(t) - y_i(t)).$$

Then, it can be concluded that  $Q : Y \rightarrow \mathbb{R}^n$  is continuous.  $\square$

**Lemma 3.4.** *The definition of the operator  $R : X \times [0, 1] \rightarrow X_2$  is*

$$R(u, \lambda)(t) = I_{0+}^{\alpha_2} \varphi_q (I_{0+}^{\alpha_1} (N_\lambda u(t) - QN_\lambda u(t))) - T^+ \phi (N_\lambda u(t) - QN_\lambda u(t)) t^{\gamma_2-1}, \quad (3.14)$$

where  $\phi$  is defined in (3.3),  $\text{Ker} L \oplus X_2 = X$ . Then the operator  $R : \bar{\Omega} \times [0, 1] \rightarrow X_2$  is continuous and compact, where  $\Omega \subset X$  is an open bounded set.

*Proof.* Obviously,  $R$  is continuous. Next, we show that  $R$  is compact. In fact, for any  $u \in \bar{\Omega}$ , by the boundedness of  $f$  on a bounded closed domain and the boundedness of  $Q$ , we obtain that there exist constants  $k_1 > 0$ ,  $k_2 > 0$  such that  $\max_{(t,u) \in [0,1] \times \bar{\Omega}} |f(t, u(t), D_{0+}^{\alpha_2, \beta_2} u(t))| \leq k_1$ ,  $|Qf(t, u(t), D_{0+}^{\alpha_2, \beta_2} u(t))| \leq k_2$ , then

$$|R(u, \lambda)(t)| = \left| \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1-1} [N_\lambda u(r) - QN_\lambda u(r)] dr \right) ds \right|$$

$$\begin{aligned}
& \left| -T^+ \phi(N_\lambda u(t) - QN_\lambda u(t)) t^{\gamma_2-1} \right| \\
& \leq \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \varphi_q \left( \frac{k_1+k_2}{\Gamma(\alpha_1+1)} \right) ds + \|T^+\|_* |\phi(N_\lambda u(t) - QN_\lambda u(t))| \\
& \leq \frac{1}{\Gamma(\alpha_2+1)} \varphi_q \left( \frac{k_1+k_2}{\Gamma(\alpha_1+1)} \right) + \|T^+\|_* \left| I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1}[N_\lambda u(t) - QN_\lambda u(t)]) \right|_{t=1} \\
& \quad - A \int_0^1 h(t) I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1}[N_\lambda u(t) - QN_\lambda u(t)]) dt \Big| \\
& \leq \frac{1}{\Gamma(\alpha_2+1)} \varphi_q \left( \frac{k_1+k_2}{\Gamma(\alpha_1+1)} \right) + \frac{\|T^+\|_*}{\Gamma(\alpha_2+1)} \varphi_q \left( \frac{k_1+k_2}{\Gamma(\alpha_1+1)} \right) \\
& \quad + \frac{\|T^+\|_* \|A\|_*}{\Gamma(\alpha_2+1)} \varphi_q \left( \frac{k_1+k_2}{\Gamma(\alpha_1+1)} \right) \int_0^1 h(t) dt \\
& \leq \left( 1 + \|T^+\|_* + \|T^+\|_* \|A\|_* \int_0^1 h(t) dt \right) \frac{\varphi_q \left( \frac{k_1+k_2}{\Gamma(\alpha_1+1)} \right)}{\Gamma(\alpha_2+1)}
\end{aligned}$$

and

$$|D_{0+}^{\alpha_2, \beta_2} R(u, \lambda)(t)| = |I_{0+}^{\beta_2(n-\alpha_2)} D_{0+}^{\gamma_2} R(u, \lambda)(t)| = |\varphi_q(I_{0+}^{\alpha_1}[N_\lambda u(t) - QN_\lambda u(t)])| \leq \varphi_q \left( \frac{k_1+k_2}{\Gamma(\alpha_1+1)} \right),$$

where  $\|\cdot\|_*$  stand for the max-norm of matrices,  $|x| = \max\{|x_i|, i = 1, 2, \dots, n\}$ . Therefore,  $R$  is bounded.

For any  $u \in \bar{\Omega}$ ,  $0 \leq \lambda \leq 1$  and  $0 \leq t_1 < t_2 \leq 1$ , there are

$$\begin{aligned}
& \left| R(u, \lambda)(t_2) - R(u, \lambda)(t_1) \right| \\
& = \left| \frac{1}{\Gamma(\alpha_2)} \int_0^{t_2} (t_2-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1-1} (N_\lambda u(r) - QN_\lambda u(r)) dr \right) ds \right. \\
& \quad - \frac{1}{\Gamma(\alpha_2)} \int_0^{t_1} (t_1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1-1} (N_\lambda u(r) - QN_\lambda u(r)) dr \right) ds \\
& \quad - T^+ \left[ I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1}[N_\lambda u(t) - QN_\lambda u(t)]) \right]_{t=1} \\
& \quad - A \int_0^1 h(t_2) \frac{1}{\Gamma(\alpha_2)} \int_0^{t_2} (t_2-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1-1} [N_\lambda u(r) - QN_\lambda u(r)] dr \right) ds dt_2 \Big| t_2^{\gamma_2-1} \\
& \quad + T^+ \left[ I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1}[N_\lambda u(t) - QN_\lambda u(t)]) \right]_{t=1} \\
& \quad - A \int_0^1 h(t_1) \frac{1}{\Gamma(\alpha_2)} \int_0^{t_1} (t_1-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1-1} [N_\lambda u(r) - QN_\lambda u(r)] dr \right) ds dt_1 \Big| t_1^{\gamma_2-1} \Big| \\
& \leq \frac{1}{\Gamma(\alpha_2)} \left| \int_0^{t_1} [(t_2-s)^{\alpha_2-1} - (t_1-s)^{\alpha_2-1}] \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1-1} [N_\lambda u(r) - QN_\lambda u(r)] dr \right) ds \right. \\
& \quad \left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha_2-1} \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-r)^{\alpha_1-1} [N_\lambda u(r) - QN_\lambda u(r)] dr \right) ds \right|
\end{aligned}$$

$$\begin{aligned}
& + \frac{\|T^+\|_*}{\Gamma(\alpha_2 + 1)} \varphi_q \left( \frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)} \right) (t_2^{\gamma_2 - 1} - t_1^{\gamma_2 - 1}) + \frac{\|T^+ A\|_* \int_0^1 h(t) dt}{\Gamma(\alpha_2 + 1)} (t_2^{\gamma_2 - 1} - t_1^{\gamma_2 - 1}) \varphi_q \left( \frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)} \right) \\
& \leq \frac{1}{\Gamma(\alpha_2 + 1)} \varphi_q \left( \frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)} \right) (t_2^{\alpha_2} - t_1^{\alpha_2}) \\
& + \left( \|T^*\|_* + \|T^* A\|_* \int_0^1 h(t) dt \right) \frac{1}{\Gamma(\alpha_2 + 1)} \varphi_q \left( \frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)} \right) (t_2^{\gamma_2 - 1} - t_1^{\gamma_2 - 1})
\end{aligned}$$

and

$$\begin{aligned}
\left| D_{0+}^{\alpha_2, \beta_2} R(u, \lambda)(t_2) - D_{0+}^{\alpha_2, \beta_2} R(u, \lambda)(t_1) \right| & = \left| \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^{t_2} (t_2 - s)^{\alpha_1 - 1} (N_\lambda u(s) - Q N_\lambda u(s)) ds \right) \right. \\
& \quad \left. - \varphi_q \left( \frac{1}{\Gamma(\alpha_1)} \int_0^{t_1} (t_1 - s)^{\alpha_1 - 1} (N_\lambda u(s) - Q N_\lambda u(s)) ds \right) \right|.
\end{aligned}$$

Since

$$\begin{aligned}
& \left| \frac{1}{\Gamma(\alpha_1)} \int_0^{t_2} (t_2 - s)^{\alpha_1 - 1} (N_\lambda u(s) - Q N_\lambda u(s)) ds - \frac{1}{\Gamma(\alpha_1)} \int_0^{t_1} (t_1 - s)^{\alpha_1 - 1} (N_\lambda u(s) - Q N_\lambda u(s)) ds \right| \\
& = \frac{1}{\Gamma(\alpha_1)} \left| \int_0^{t_1} [(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}] (N_\lambda u(s) - Q N_\lambda u(s)) ds \right. \\
& \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_1 - 1} (N_\lambda u(s) - Q N_\lambda u(s)) ds \right| \\
& \leq \frac{k_1 + k_2}{\Gamma(\alpha_1)} \left| \int_0^{t_1} [(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_1 - 1} ds \right| \\
& \leq \frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)} (t_2^{\alpha_1} - t_1^{\alpha_1}),
\end{aligned}$$

$$\left| \int_0^t \frac{(t - s)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (N_\lambda u(s) - Q N_\lambda u(s)) ds \right| \leq \frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)}$$

and  $\varphi_q(\vartheta)$  is uniformly continuous on  $\left[ -\frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)}, \frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)} \right]$ . Consequently,  $\{R(u, \lambda) \mid (u, \lambda) \in \bar{\Omega} \times [0, 1]\}$  and  $\{D_{0+}^{\alpha_2, \beta_2} R(u, \lambda) \mid (u, \lambda) \in \bar{\Omega} \times [0, 1]\}$  are equicontinuous. In view of the Arzela-Ascoli Theorem, it yields that  $R : \bar{\Omega} \times [0, 1] \rightarrow X_2$  is compact.  $\square$

**Lemma 3.5.** *Suppose that  $\Omega$  is a bounded, open subset of  $X$ . Then the operator  $N_\lambda$  defined in (2.2) is  $L$ -quasi-compact in  $\bar{\Omega}$ .*

*Proof.* It is not difficult to deduce that  $\dim \text{Ker} L = \dim \text{Im} Q$ ,  $\text{Ker} Q = \text{Im} L$ ,  $R(\cdot, 0) = \theta$  and  $Q N_\lambda u(t) = \theta \Leftrightarrow Q N u(t) = \theta$ . Then (a) and (b) of Definition 2.2 hold.

For each  $u \in \Sigma_\lambda = \{u \in \overline{\Omega} \mid Lu = N_\lambda u\}$ , there is  $N_\lambda u \in \text{Im}L = \text{Ker}Q$ , so  $QN_\lambda u = \theta$ . It follows from  $N_\lambda u = Lu(t) = D_{0+}^{\alpha_1, \beta_1} \varphi_p(D_{0+}^{\alpha_2, \beta_2} u(t))$  and  $R(u, 0)(t) = D_{0+}^{\alpha_2, \beta_2} R(u, 0)(t) = \theta$  that

$$\begin{aligned}
R(u, \lambda)(t) &= I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} (N_\lambda u(t) - QN_\lambda u(t)) - T^+ \phi(N_\lambda u(t) - QN_\lambda u(t)) t^{\gamma_2-1}) \\
&= I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} (N_\lambda u(t)) - T^+ \phi(N_\lambda u(t)) t^{\gamma_2-1}) \\
&= I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} I_{0+}^{\beta_1(1-\alpha_1)} D_{0+}^{\gamma_1} \varphi_p(D_{0+}^{\alpha_2, \beta_2} u(t)) - T^+ \phi(N_\lambda u(t)) t^{\gamma_2-1}) \\
&= u(t) - \frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)} D_{0+}^{\gamma_2-1} u(0) + T^+ T \frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)} D_{0+}^{\gamma_2-1} u(0) \\
&= u(t) - (I - T^+ T) \frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)} D_{0+}^{\gamma_2-1} u(0) \\
&= (I - P)u.
\end{aligned}$$

Consequently, (c) in Definition 2.2 is satisfied.

For any  $u \in \overline{\Omega}$ , there is

$$\begin{aligned}
L[Pu + R(u, \lambda)](t) &= D_{0+}^{\alpha_1, \beta_1} \varphi_p\left(D_{0+}^{\alpha_2, \beta_2} (pu(t) + R(u, \lambda)(t))\right) \\
&= D_{0+}^{\alpha_1, \beta_1} \varphi_p\left[D_{0+}^{\alpha_2, \beta_2} (I - T^+ T) \frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)} D_{0+}^{\gamma_2-1} u(0) \right. \\
&\quad \left. + D_{0+}^{\alpha_2, \beta_2} \left(I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} (N_\lambda u(t) - QN_\lambda u(t))) - T^+ \phi(N_\lambda u(t) - QN_\lambda u(t)) t^{\gamma_2-1}\right)\right] \\
&= D_{0+}^{\alpha_1, \beta_1} \varphi_p\left(D_{0+}^{\alpha_2, \beta_2} I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} (N_\lambda u(t) - QN_\lambda u(t)))\right) \\
&= (I - Q)N_\lambda u(t),
\end{aligned}$$

then (d) of Definition 2.2 holds. Thus, the operator  $N_\lambda$  is L-quasi-compact in  $\overline{\Omega}$ .  $\square$

Next, we will give the main theorem.

**Theorem 3.6.** *Suppose the following conditions hold:*

(H<sub>1</sub>) *There exists a constant  $M > 0$  such that for every  $u \in \text{dom}L$ , if  $|t^{1-\gamma_2} u(t)| > M$ ,  $t \in [0, 1]$ , then either*

$$(1) \langle t^{1-\gamma_2} u, Qf \rangle > 0 \quad \text{or} \quad (2) \langle t^{1-\gamma_2} u, Qf \rangle < 0, \quad \forall t \in [0, 1]$$

*holds, where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbb{R}^n$ .*

(H<sub>2</sub>) *There exist three non-negative functions  $a, \psi, \varsigma \in C[0, 1]$  such that*

$$\left| f(t, \omega, \varpi) \right| \leq a(t) \varphi_p(|\omega|) + \psi(t) \varphi_p(|\varpi|) + \varsigma(t), \quad 0 \leq t \leq 1,$$

*where  $\max\{1, 2^{q-2}\} [\varphi_q(\|\psi\|_\infty) + \frac{2\varphi_q(\|a\|_\infty)}{\Gamma(\alpha_2+1)}] < \varphi_q(\Gamma(\alpha_1 + 1))$ .*

*Then there exists at least one solution in  $X$  for the problem (1.1).*

To prove Theorem 3.6, the following lemmas are first established.

**Lemma 3.7.** *Assume that  $(H_1)$  and  $(H_2)$  hold. Let  $\Omega_1 = \{u \in \text{dom}L \mid Lu = N_\lambda u, \lambda \in (0, 1)\}$ , then  $\Omega_1$  is bounded in  $X$ .*

*Proof.* For any  $u \in \Omega_1$ , we have  $Lu = N_\lambda u$ ,  $N_\lambda u \in \text{Im}L = \text{Ker}Q$ , then  $QN_\lambda u(t) = \theta$ . It is known from  $(H_1)$  that there exists  $t_0 \in [0, 1]$  such that  $|t_0^{1-\gamma_2} u(t_0)| \leq M$ . Since  $Lu = N_\lambda u$ , there is

$$u(t) = I_{0+}^{\alpha_2} \varphi_q(\lambda I_{0+}^{\alpha_1} Nu(t)) + \xi t^{\gamma_2-1}, \quad (3.15)$$

and then by  $(H_2)$ , it follows that

$$\begin{aligned} |\xi| &\leq |t_0^{1-\gamma_2} u(t)| + |t_0^{1-\gamma_2} I_{0+}^{\alpha_2} \varphi_q(I_{0+}^{\alpha_1} Nu(t))| \\ &\leq M + \frac{t_0^{1-\gamma_2}}{\Gamma(\alpha_2)} \int_0^{t_0} (t_0 - s)^{\alpha_2-1} \varphi_q\left(\frac{1}{\Gamma(\alpha_1)} \int_0^s (s - \varrho)^{\alpha_1-1} |f(\varrho, u(\varrho), D_{0+}^{\alpha_2, \beta_2} u(\varrho))| d\varrho\right) ds \\ &\leq M + \frac{t_0^{1-\gamma_2}}{\Gamma(\alpha_2)} \int_0^{t_0} (t_0 - s)^{\alpha_2-1} \varphi_q\left(\frac{\|a\|_\infty \varphi_p(\|u\|_\infty) + \|\psi\|_\infty \varphi_p(\|D_{0+}^{\alpha_2, \beta_2} u\|_\infty) + \|\varsigma\|_\infty}{\Gamma(\alpha_1 + 1)}\right) ds \\ &\leq M + \frac{\max\{1, 2^{q-2}\} [\varphi_q(\|a\|_\infty) \|u\|_\infty + \varphi_q(\|\psi\|_\infty) \|D_{0+}^{\alpha_2, \beta_2} u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\Gamma(\alpha_2 + 1) \varphi_q(\Gamma(\alpha_1 + 1))}. \end{aligned}$$

Since

$$\begin{aligned} |D_{0+}^{\alpha_2, \beta_2} u(t)| &= |D_{0+}^{\alpha_2, \beta_2} I_{0+}^{\alpha_2} \varphi_q(\lambda I_{0+}^{\alpha_1} Nu(t)) + D_{0+}^{\alpha_2, \beta_2} \xi t^{\gamma_2-1}| \\ &\leq \frac{\max\{1, 2^{q-2}\} [\varphi_q(\|a\|_\infty) \|u\|_\infty + \varphi_q(\|\varsigma\|_\infty) + \varphi_q(\|\psi\|_\infty) \|D_{0+}^{\alpha_2, \beta_2} u\|_\infty]}{\varphi_q(\Gamma(\alpha_1 + 1))}, \end{aligned}$$

we can get

$$\|D_{0+}^{\alpha_2, \beta_2} u\|_\infty \leq \frac{\max\{1, 2^{q-2}\} [\varphi_q(\|a\|_\infty) \|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\} \varphi_q(\|\psi\|_\infty)}. \quad (3.16)$$

Therefore,

$$\begin{aligned} |\xi| &\leq M + \frac{\max\{1, 2^{q-2}\} [\varphi_q(\|a\|_\infty) \|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\Gamma(\alpha_2 + 1) \varphi_q(\Gamma(\alpha_1 + 1))} \\ &\quad + \frac{\max\{1, 2^{q-2}\} \varphi_q(\|\psi\|_\infty) [\varphi_q(\|a\|_\infty) \|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\Gamma(\alpha_2 + 1) \varphi_q(\Gamma(\alpha_1 + 1)) [\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\} \varphi_q(\|\psi\|_\infty)]} \\ &= M + \frac{\max\{1, 2^{q-2}\} \varphi_q(\Gamma(\alpha_1 + 1)) [\varphi_q(\|a\|_\infty) \|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\Gamma(\alpha_2 + 1) \varphi_q(\Gamma(\alpha_1 + 1)) [\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\} \varphi_q(\|\psi\|_\infty)]}. \end{aligned}$$

Bring the above inequality into (3.15) to get

$$|u(t)| \leq |I_{0+}^{\alpha_2} \varphi_q(\lambda I_{0+}^{\alpha_1} Nu(t))| + |\xi t^{\gamma_2-1}|$$

$$\leq \frac{\max\{2, 2^{q-1}\} \varphi_q(\Gamma(\alpha_1 + 1)) [\varphi_q(\|a\|_\infty) \|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\Gamma(\alpha_2 + 1) \varphi_q(\Gamma(\alpha_1 + 1)) [\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\} \varphi_q(\|\psi\|_\infty)]} + M.$$

Then,

$$\|u\|_\infty \leq \frac{\max\{2, 2^{q-1}\} \varphi_q(\|\varsigma\|_\infty) + M \Gamma(\alpha_2 + 1) [\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\} \varphi_q(\|\psi\|_\infty)]}{\Gamma(\alpha_2 + 1) [\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\} \varphi_q(\|\psi\|_\infty)] - \max\{2, 2^{q-1}\} \varphi_q(\|a\|_\infty)}. \quad (3.17)$$

Hence, together with (3.16) and (3.17), it can be deduced that  $\Omega_1$  is bounded in  $X$ .  $\square$

**Lemma 3.8.** *Assume that  $(H_1)$  holds, then  $\Omega_2 = \{u | u \in \text{Ker}L, QNu = \theta\}$  is bounded in  $X$ .*

*Proof.* Let  $u \in \Omega_2$ , we have  $QNu(t) = \theta$  and  $u(t) = ct^{\gamma_2-1}$ ,  $c \in \mathbb{R}^n$ . According to  $(H_1)$ , there exists  $t_0 \in [0, 1]$  such that  $|t_0^{1-\gamma_2} u(t_0)| \leq M$ . Thus we get that  $|c| \leq M$ , then  $\Omega_2$  is bounded in  $X$ .  $\square$

**The following is the proof of Theorem 3.6.**

*Proof.* Let  $\Omega \supset (\bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \{x | x \in X, \|x\| \leq M\})$  be a bounded open subset of  $X$ . Lemma 3.7 implies that  $Lu \neq N_\lambda u$ ,  $u \in \text{dom}L \cap \partial\Omega$ , while Lemma 3.8 leads to the conclusion that  $QNu \neq \theta$ ,  $u \in \text{Ker}L \cap \partial\Omega$ . Let  $H(u, \zeta) = \rho\zeta u + (1-\zeta)JQNu$ , where  $u \in \text{Ker}L \cap \bar{\Omega}$ ,  $\zeta \in [0, 1]$ ,  $J : \text{Im}Q \rightarrow \text{Ker}L$  is a homeomorphism with  $J\eta = \eta t^{\gamma_2-1}$ , and

$$\rho = \begin{cases} 1, & \text{if } (H_1) \text{ (1) holds,} \\ -1, & \text{if } (H_1) \text{ (2) holds.} \end{cases}$$

Given any  $u \in \text{Ker}L \cap \partial\Omega$ , there are  $u(t) = \eta_0 t^{\gamma_2-1}$  and  $H(u, \zeta) = \rho\zeta\eta_0 t^{\gamma_2-1} + (1-\zeta)(Qf)t^{\gamma_2-1}$ .

If  $\zeta = 1$ , then  $H(u, 1) = \rho\eta_0 t^{\gamma_2-1} \neq \theta$ .

If  $\zeta = 0$ , then  $H(u, 0) = (Qf)t^{\gamma_2-1} \neq \theta$ .

If  $0 < \zeta < 1$ , suppose  $H(u, \zeta) = \theta$ , then  $\rho\zeta\eta_0 t^{\gamma_2-1} = -(1-\zeta)(Qf)t^{\gamma_2-1}$ . So there is  $\eta_0 = -\frac{(1-\zeta)(Qf)}{\zeta\rho}$ .

It follows from  $(H_1)$  and  $|\eta_0| = |t^{1-\gamma_2} u(t)| > M$  that

$$\langle \eta_0, \eta_0 \rangle = -\frac{1-\zeta}{\zeta} \frac{\langle \eta_0, Qf \rangle}{\rho} < 0.$$

This is a contradiction. Hence,  $H(u, \zeta) \neq \theta$ , for all  $u \in \text{Ker}L \cap \partial\Omega$ ,  $\zeta \in [0, 1]$ . The homotopy property of degree yields the result that

$$\begin{aligned} \deg(JQN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, \theta) &= \deg(H(\cdot, 0), \Omega \cap \text{Ker}L, \theta) \\ &= \deg(H(\cdot, 1), \Omega \cap \text{Ker}L, \theta) \\ &= \deg(\rho I, \Omega \cap \text{Ker}L, \theta) \neq 0. \end{aligned}$$

Combining Lemmas 3.1-3.5 and applying Lemma 2.6, we conclude that the problem (1.1) has at least one solution in  $X$ . The proof is completed.  $\square$

## 4 Example

**Example 1.** We consider the following boundary value problem at resonance in  $\mathbb{R}^2$ :

$$\begin{cases} D_{0+}^{\frac{1}{2}, \frac{1}{2}} \varphi_{\frac{5}{2}}(D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_1(t)) = f_1(t, u_1(t), u_2(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_1(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_2(t)), & 0 \leq t \leq 1, \\ D_{0+}^{\frac{1}{2}, \frac{1}{2}} \varphi_{\frac{5}{2}}(D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_2(t)) = f_2(t, u_1(t), u_2(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_1(t), D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_2(t)), & 0 \leq t \leq 1, \\ D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_1(0) = D_{0+}^{\frac{5}{2}, \frac{1}{2}} u_2(0) = 0, \quad D_{0+}^{\frac{3}{4}} u_1(0) = D_{0+}^{\frac{3}{4}} u_2(0) = 0, \\ u_1(1) = -2 \int_0^1 t^{-\frac{3}{4}} u_1(t) dt, \quad u_2(1) = -3 \int_0^1 t^{-\frac{3}{4}} u_2(t) dt, \end{cases} \quad (4.1)$$

where  $\alpha_1 = \frac{1}{2}$ ,  $\beta_1 = \frac{1}{2}$ ,  $\alpha_2 = \frac{5}{2}$ ,  $\beta_2 = \frac{1}{2}$ ,  $\gamma_2 = \frac{11}{4}$ ,  $p = \frac{5}{2}$ ,  $h(t) = t^{-\frac{3}{4}}$ ,  $f : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}^2$  are defined as

$$f(t, u, z) = (f_1(t, u_1, u_2, z_1, z_2), f_2(t, u_1, u_2, z_1, z_2))^{\top} = \left( -\frac{u_1 + z_1 - e^5}{20}, \frac{|u_2| + |z_2| + e^3}{40} \right)^{\top}$$

for any  $t \in [0, 1]$  and  $u = (u_1, u_2)^{\top}$ ,  $z = (z_1, z_2)^{\top} \in \mathbb{R}^2$ .

Clearly,  $A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$ ,  $T = \int_0^1 h(t) t^{\gamma_2 - 1} dt = \begin{pmatrix} 2 & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$ . Let  $\xi = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ , then  $T\xi = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ .

Now we prove that the conditions of Theorem 3.6 hold. Choose nonnegative integrable functions  $a = \psi = \frac{1}{20}$  and  $\varsigma = \frac{e^5}{20}$ , then there is

$$|f(t, u, z)| \leq a(t) \varphi_p(|u|) + \psi(t) \varphi_p(|z|) + \varsigma(t).$$

After some simple calculations,

$$\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\} [\varphi_q(\|\psi\|_{\infty}) + \frac{2}{\Gamma(\alpha_2 + 1)} \varphi_q(\|a\|_{\infty})] \approx 0.6505 > 0$$

can be obtained. Therefore,  $(H_2)$  is satisfied.

In order to check  $(H_1)$ , let  $M = 3$ ,  $c = (\|f_1\|_{\infty} + 5.8997, \|f_2\|_{\infty} + 12.6422)^{\top}$ , then  $c$  satisfies (3.10). If  $|t_0^{1-\gamma_2} u(t_0)| > M = 3$  hold for any  $t \in [0, 1]$ , then  $\langle t^{1-\gamma_2} u, Qf \rangle = \langle t^{1-\gamma_2} u, c \rangle > 0$ . Hence, the condition  $(H_1)$  holds. From Theorem 3.6, it can be obtained that the problem (4.1) has at least one solution.

To intuitively illustrate the existence of solutions for the problem (4.1), we conducted numerical simulations using MATLAB. Figures 1 and 2 depict the cases for  $p = 2.5$  and  $p = 1.5$ , respectively.



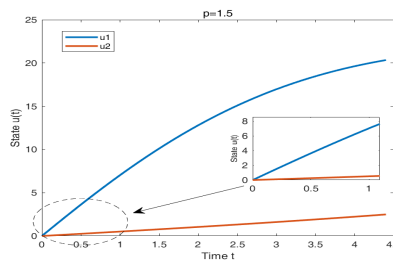
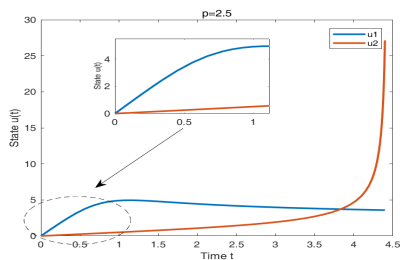


Fig. 1: State  $u(t)$  of the system (4.1) when  $p = 2.5$ . Fig. 2: State  $u(t)$  of the system (4.1) when  $p = 1.5$ .

## 5 Conclusion

In this paper, we investigated the nonlinear Hilfer fractional boundary value problem at resonance in  $\mathbb{R}^n$ . By using the extension for the continuous theorem, the conclusion that the problem (1.1) has at least one solution in  $X$  was obtained. To achieve our main results, we defined two Banach spaces with specified norms and construct the appropriate operators  $P$ ,  $Q$  and  $R$  within these Banach spaces. Subsequently, we proved the necessary requirements before applying Lemma 2.6. It is worth noting that the variables in the  $n$ -dimensional Euclidean space are represented as vectors or matrices, and we cannot assume a direct size relationship. The cone in (3.13) is skillfully defined, effectively resolving existing issues. Finally, we provided an example to verify the validity of our conclusion.

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### Availability of data and material

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

All authors read and approved the final manuscript.

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