Ordering graphs with fixed size and girth by their A_α-spectral radius*

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Abstract: For a graph G and real number $\alpha \in [0,1]$, the A_α-spectral radius of G is the largest eigenvalue of $A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal degree matrix of G, respectively. Recently, for $\alpha \in [\frac{1}{2}, 1]$, Chen, Li and Huang [*Discrete Appl. Math.*, 340(2023), 350-362], as well as Ye, Guo and Zhang [Discrete Appl. Math., 342(2024), 286-294] independently identified the graph with maximum A_α-spectral radius among all graphs in $\mathcal{G}(m,g)$, the class of connected graphs on m edges with girth g. In this paper, we further determine the second to the $\left(\lfloor \frac{g}{2} \rfloor + 2\right)$th largest A_α-spectral radius of graphs in $\mathcal{G}(m,g)$. Moreover, for $\alpha \in [\frac{1}{2}, 1]$, we also determine the first to the $\left(\lfloor \frac{g}{2} \rfloor + 3\right)$th largest A_α-spectral radius of graphs in $\mathcal{G}(m, \geq g)$, the class of connected graphs on m edges with girth no less than g, which generalizes the recent result of Hu, Lou and Huang (2022) on $\alpha = \frac{1}{2}$.

Keywords: A_α-spectral radius, size, girth, order.

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1 Introduction

All graphs considered in this paper are undirected and simple (*i.e.*, without loops or multiple edges). Let $G = (V(G), E(G))$ be a graph of order n and size m. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ (or $d(v)$ and $N(v)$ for short) be the degree and the set of neighbors of v, respectively. And let $N_G[v] = N_G(v) \cup \{v\}$. The maximum degree of G is denoted by $\Delta(G)$ (or Δ for short). The girth of a graph G, denoted by g, is the length of the shortest cycle in G. Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal degree matrix of a graph G, respectively. For $\alpha \in [0,1]$, Nikiforov [14] defined the $A_\alpha(G)$-matrix of G as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G).$$

Clearly, $A_0(G) = A(G)$, $A_1(G) = D(G)$ and $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$, where $Q(G)$ is known as the signless Laplacian matrix of G. Note that $A_\alpha(G)$ is a non-negative and real symmetric matrix. Then the eigenvalues of $A_\alpha(G)$ (also called the A_α-*eigenvalues* of G) are real. The largest

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eigenvalue of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$, is called the A_α-*spectral radius* of G. Obviously, $2\rho_{\frac{1}{2}}(G)$ is the signless Laplacian spectral radius of G (the largest eigenvalue of $Q(G)$).

Cvetković et al. [5] proposed some possible directions for further investigations on graph spectra. One of which is how to order graphs according to their spectral invariants. Hence ordering graphs with various properties by their spectra, specially by their spectral radius (the largest eigenvalues of various matrices associated with graphs), becomes an attractive topic and has received a lot of attentions in recent years (see [1, 3, 6–11, 13, 15] for details).

Let $\mathcal{G}(m,g)$ ($\mathcal{G}(m,\geq g)$) be the set of connected graphs with size m and girth g (girth at least g). Moreover, let $C_g = u_0u_1\ldots u_{g-2}u_{g-1}u_o$ be a cycle of length g, and denote by $G_i \in \mathcal{G}(m,g)$ the graph obtained from C_g by attaching $m-g-1$ pendent vertices to a vertex u_0 and a pendent vertex w being adjacent to u_i in C_g, where $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$. Clearly, G_0 is the graph obtained by identifying a vertex u_0 of C_g and the central vertex of $K_{1,m-g}$. Let G^* be the graph obtained from C_g by attaching $m-g-2$ pendent edges and a P_3, respectively, to u_0, where $P_3 = u_0w_1v_1$. The above mentioned three graphs are shown in Figure 1.

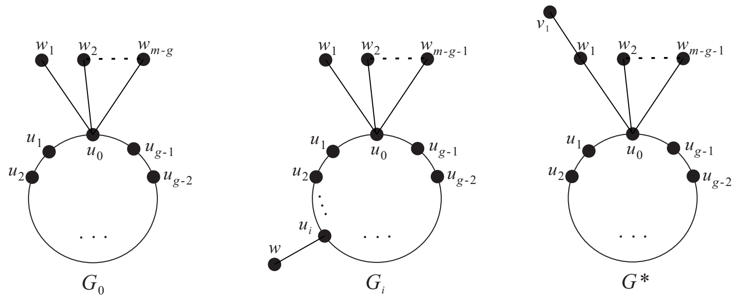

Figure 1: The graphs G_0, G_i and G^*.

Among all graphs in $\mathcal{G}(m,g)$, Chen, Wang and Zhai [4] identified the graph with maximum signless Laplacian spectral radius. Very recently, Hu, Lou and Huang [8] further determined the second to the $(\lfloor \frac{g}{2} \rfloor + 2)$th largest graphs according to their signless Laplacian spectral radius. Their results can be combined into the following theorem.

Theorem 1.1 **([4, 8])** *Among all graphs in $\mathcal{G}(m,g)$ with $m \geq 3g \geq 12$, the order of the first $(\lfloor \frac{g}{2} \rfloor + 2)$th largest graphs according to their signless Laplacian spectral radius is given by:*

$$G_0,\ G_1,\ G^*,\ G_2,\ G_3,\ \ldots,\ G_{\lfloor \frac{g}{2} \rfloor}.$$

Let G_g^i and G_g^* instead of G_i ($0 \leq i \leq \lfloor \frac{g}{2} \rfloor$) and G^*, respectively. Hu, Lou and Huang [8] also determined the first to the $(\lfloor \frac{g}{2} \rfloor + 3)$th largest graphs according to their signless Laplacian spectral radius among all graphs in $\mathcal{G}(m,\geq g)$ as follows.

Theorem 1.2 **([8])** *Among all graphs in $\mathcal{G}(m,\geq g)$ with $m \geq 3g \geq 12$, the order of the first $(\lfloor \frac{g}{2} \rfloor + 3)$th largest graphs according to their signless Laplacian spectral radius is given by:*

$$G_g^0,\ G_g^1,\ G_g^*,\ G_g^2,\ G_g^3,\ \ldots,\ G_g^{\lfloor \frac{g}{2} \rfloor},\ G_{g+1}^0.$$

Very recently, for $\alpha \in [\frac{1}{2}, 1)$, Chen, Li and Huang [2], as well as Ye, Guo and Zhang [18] independently determined the following graph with maximum A_α-spectral radius among all graphs in $\mathcal{G}(m,g)$, which generalizes the result of Chen, Wang and Zhai [4] on $\alpha = \frac{1}{2}$.

Theorem 1.3 **([2, 18])** *For any $G \in \mathcal{G}(m,g)$ and $\alpha \in [\frac{1}{2}, 1)$, we have $\rho_\alpha(G) \leq \rho_\alpha(G_0)$. Moreover, the equality holds if and only if $G \cong G_0$.*

In this paper, for $\alpha \in [\frac{1}{2}, 1)$, we further determine the second to the $\left(\lfloor \frac{g}{2} \rfloor + 2\right)$th largest graphs in $\mathcal{G}(m, g)$ according to their A_α-spectral radius as follows.

Theorem 1.4 *For $\alpha \in [\frac{1}{2}, 1)$, the first $\lfloor \frac{g}{2} \rfloor + 1$ graphs in $\mathcal{G}(m, g) \backslash \{G_0\}$ with $m \geq 3g \geq 12$ according to their A_α-spectral radius are as follows:*

$$G_1,\ G^*,\ G_2,\ G_3,\ \ldots,\ G_{\lfloor \frac{g}{2} \rfloor}.$$

Remark 1.1 Combining Theorem 1.3 and Theorem 1.4, we know that among all graphs in $\mathcal{G}(m, g)$, the first to $\left(\lfloor \frac{g}{2} \rfloor + 2\right)$th largest graphs according to their A_α-spectral radius for $\alpha \in [\frac{1}{2}, 1)$ are determined, which generalizes Theorem 1.1.

Moreover, we further consider the first to the $(\lfloor \frac{g}{2} \rfloor + 3)$th largest graphs according to their A_α-spectral radius among all graphs in $\mathcal{G}(m, \geq g)$ and extend Theorem 1.2 as follows.

Theorem 1.5 *For $\alpha \in [\frac{1}{2}, 1)$, the first $\lfloor \frac{g}{2} \rfloor + 3$ graphs in $\mathcal{G}(m, \geq g)$ with $m \geq 3g \geq 12$ according to their A_α-spectral radius are as follows:*

$$G_g^0,\ G_g^1,\ G_g^*,\ G_g^2,\ G_g^3,\ \ldots,\ G_g^{\lfloor \frac{g}{2} \rfloor},\ G_{g+1}^0.$$

Clearly, Theorem 1.2 follow from Theorem 1.5 if we let $\alpha = \frac{1}{2}$.

2 Preliminaries

In this section, we present some preliminary results and lemmas which are useful.

Recall that $A_\alpha(G)$ is a nonnegative and real symmetric matrix. Then there is a non-negative unit eigenvector $\boldsymbol{x}$ of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$ such that

$$\rho_\alpha(G) = \boldsymbol{x}^T A_\alpha(G)\boldsymbol{x} = (2\alpha - 1) \sum_{u \in V(G)} d(u)x_u^2 + (1 - \alpha) \sum_{uv \in E(G)} (x_u + x_v)^2, \tag{1}$$

where x_u is the entry of $\boldsymbol{x}$ corresponding to the vertex u. We call such eigenvector $\boldsymbol{x}$ the *Perron vector* of $A_\alpha(G)$. In addition, if G is connected, then $A_\alpha(G)$ is irreducible and thus its Perron vector is a positive unit eigenvector. Clearly, the Perron vector of $A_\alpha(G)$ satisfies the *eigenvalue equation* $A_\alpha(G)\boldsymbol{x} = \rho_\alpha(G)\boldsymbol{x}$, that is

$$\rho_\alpha(G)x_u = \alpha d(u)x_u + (1 - \alpha) \sum_{v \in N(u)} x_v. \tag{2}$$

Consider an $n \times n$ real symmetric matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix}$$

whose rows and columns are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The *quotient matrix* $\mathcal{B}$ of the matrix M is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The partition is *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.1 ([19]) *Let M be a square matrix with an equitable partition π and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative and M_π is irreducible, then the largest eigenvalues of M and M_π are equal.*

Let $S_{n,3}$ be a graph on n vertices obtained from $K_{1,n-7}$ by attaching three pendant paths of length 2 at the center vertex of $K_{1,n-7}$, and let H_0 be an unicycle graph of order n and girth 4. The graphs $S_{n,3}$ and H_0 are shown in Figure 2.

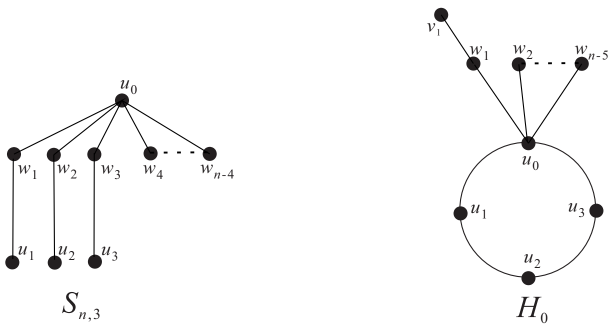

Figure 2: The graphs $S_{n,3}$ and H_0.

Lemma 2.2 *For $\alpha \in [0,1]$, $\rho_\alpha(S_{n,3})$ and $\rho_\alpha(H_0)$ are the largest root of $f(\lambda,n)=0$ and $g(\lambda,n)=0$, respectively, where*

$$\begin{aligned} f(\lambda,n)=&\lambda^4-\alpha n\lambda^3+\left[(3\alpha^2+2\alpha-1)n-6\alpha-8\alpha^2+3\right]\lambda^2-\left[(\alpha^3+8\alpha^2-4\alpha)n-36\alpha^2+18\alpha\right]\lambda \\ &+2\alpha^3n+3\alpha^2n-4\alpha n+n-2\alpha^3-27\alpha^2+28\alpha-7 \end{aligned}$$

and

$$\begin{aligned} &g(\lambda,n) \\ =&x^6-(n+5)\alpha x^5+[(7\alpha^2+2\alpha-1)n+\alpha^2]x^4-[(15\alpha^3+20\alpha^2-10\alpha)n-17\alpha^3-40\alpha^2+20\alpha]x^3 \\ &+[(10\alpha^4+50\alpha^3-13\alpha^2-12\alpha+3)n-8\alpha^4-132\alpha^3+18\alpha^2+48\alpha-12]x^2 \\ &-[(2\alpha^5+28\alpha^4+34\alpha^3-48\alpha^2+12\alpha)n-44\alpha^4-194\alpha^3+216\alpha^2-54\alpha]x \\ &+(4\alpha^5+14\alpha^4-20\alpha^2+12\alpha-2)n-48\alpha^4-48\alpha^3+132\alpha^2-72\alpha+12. \end{aligned}$$

Proof. We partition $V(S_{n,3})$ as $V(S_{n,3})=\{u_1,u_2,u_3\}\cup\{w_1,w_2,w_3\}\cup\{u_0\}\cup\{w_4,\dots,w_{n-4}\}$. Then the corresponding quotient matrix of $A_\alpha(S_{n,3})$ is

$$\mathcal{B}_1=\begin{pmatrix} \alpha & 1-\alpha & 0 & 0 \\ 1-\alpha & 2\alpha & 1-\alpha & 0 \\ 0 & 3(1-\alpha) & (n-4)\alpha & (n-7)(1-\alpha) \\ 0 & 0 & 1-\alpha & \alpha \end{pmatrix}.$$

It is easy to versify that the characteristic polynomial of $\mathcal{B}_1$ is $f(\lambda,n)$. Note that the partition is equitable. Then Lemma 2.1 implies that $\rho_\alpha(S_{n,3})$ is the largest root of $f(\lambda,n)=0$.

Similarly, we partition $V(H_0)$ as $V(H_0)=\{v_1\}\cup\{w_1\}\cup\{w_2,\dots,w_{n-5}\}\cup\{u_0\}\cup\{u_1,u_3\}\cup\{u_2\}$. Then the corresponding quotient matrix of $A_\alpha(H_0)$ is

$$\mathcal{B}_2=\begin{pmatrix} \alpha & 1-\alpha & 0 & 0 & 0 & 0 \\ 1-\alpha & 2\alpha & 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & \alpha & 1-\alpha & 0 & 0 \\ 0 & 1-\alpha & (n-6)(1-\alpha) & (n-3)\alpha & 2(1-\alpha) & 0 \\ 0 & 0 & 0 & 1-\alpha & 2\alpha & 1-\alpha \\ 0 & 0 & 0 & 0 & 2(1-\alpha) & 2\alpha \end{pmatrix}.$$

It is easy to versify that the characteristic polynomial of $\mathcal{B}_2$ is $g(\lambda, n)$. Note that the partition is equitable. Then Lemma 2.1 implies that $\rho_\alpha(H_0)$ is the largest root of $g(\lambda, n) = 0$. □

Lemma 2.3 ([14]) *If G is a connected graph and H is a proper subgraph of G, then we have $\rho_\alpha(H) < \rho_\alpha(G)$.*

Lemma 2.4 ([17]) *For a connected graph G and $u, v \in V(G)$, let $W \subseteq N(v)\backslash(N(u) \cup \{u\})$. Let $G' = G - \{vw : w \in W\} + \{uw : w \in W\}$ and $\boldsymbol{x}$ be the Perron vector of $A_\alpha(G)$. If $x_u \geq x_v$ and $W \neq \emptyset$, then $\rho_\alpha(G') > \rho_\alpha(G)$ for $\alpha \in [0, 1)$.*

The following lemma can be derived directly from lemma 2.3 in Ref. [12].

Lemma 2.5 ([12]) *Let G' be a graph obtained from a connected graph G by a local switching of edges u_1v_1 and u_2v_2 to the positions of non-edges u_1v_2 and v_1u_2. Let $\boldsymbol{x}$ be the Perron vector of $A_\alpha(G)$. If $(u_1 - u_2)(v_1 - v_2) \geq 0$, then $\rho_\alpha(G') \geq \rho_\alpha(G)$ for $\alpha \in [0, 1)$, with equality if and only if $u_1 = u_2$ and $v_1 = v_2$.*

Lemma 2.6 ([14, 16]) *Let G be a graph of order n with maximum degree Δ. Then*

$$\rho_\alpha(G) \geq \begin{cases} \alpha(\Delta + 1), & \text{for } \alpha \in [0, \frac{1}{2}], \\ \alpha\Delta + 1 - \alpha, & \text{for } \alpha \in [\frac{1}{2}, 1). \end{cases}$$

Lemma 2.7 ([14]) *Let G be a graph without isolated vertices. Then for $\alpha \in [0, 1)$, we have*

$$\rho_\alpha(G) \leq \max_{u \in V(G)} \{\alpha d(u) + (1 - \alpha)m(u)\},$$

where $m(u) = m_G(u) = \frac{1}{d(u)} \sum\limits_{v \in N(u)} d(v)$. If $\alpha \in (\frac{1}{2}, 1)$ and G is connected, then the equality holds if and only if G is regular.

Lemma 2.8 *For any connected graph G with size $m \geq 5$ and maximum degree Δ. If $\Delta \leq s$ and $s \geq \frac{2m}{3}$, then for $\alpha \in [\frac{1}{2}, 1)$, we have*

$$\rho_\alpha(G) \leq \alpha s + 2(1 - \alpha).$$

Proof. Let $z \in V(G)$ such that

$$\alpha d(z) + (1 - \alpha)\frac{\sum_{v \in N(z)} d(v)}{d(z)} = \max_{u \in V(G)} \left\{\alpha d(u) + (1 - \alpha)\frac{\sum_{v \in N(u)} d(v)}{d(u)}\right\}.$$

If $d(z) = 1$, then by Lemma 2.7, $\rho_\alpha(G) \leq \alpha d(z) + (1-\alpha)\frac{\sum_{v \in N(z)} d(v)}{d(z)} \leq \alpha + (1-\alpha)\Delta \leq \alpha + (1-\alpha)s$. If $d(z) = 2$, then by Lemma 2.7, $\rho_\alpha(G) \leq \alpha d(z) + (1 - \alpha)\frac{\sum_{v \in N(z)} d(v)}{d(z)} \leq 2\alpha + (1 - \alpha)\Delta \leq 2\alpha + (1 - \alpha)s$. Next, we only need to consider $d(z) \geq 3$. Then by Lemma 2.7, we have

$$\rho_\alpha(G) \leq \alpha d(z) + (1-\alpha)\frac{\sum_{v \in N(z)} d(v)}{d(z)} \leq \alpha d(z) + (1-\alpha)\frac{2m - d(z)}{d(z)} = \alpha d(z) + (1-\alpha)\frac{2m}{d(z)} - 1 + \alpha.$$

Let $f(x) = \alpha x + \frac{2m(1-\alpha)}{x}$. Clearly, $f(x) \geq f_{min} = f\left(\sqrt{\frac{2m(1-\alpha)}{\alpha}}\right)$ for $x > 0$. Note that $3 \leq x \leq \Delta \leq s$, $\sqrt{\frac{2m(1-\alpha)}{\alpha}} \leq \sqrt{2m} < \frac{2m}{3}$ and

$$f\left(\frac{2m}{3}\right) - f(3) = \frac{2m}{3}(2\alpha - 1) + 3(1 - 2\alpha) \geq \frac{10}{3}(2\alpha - 1) + 3(1 - 2\alpha) = \frac{1}{3}(2\alpha - 1) \geq 0.$$

Then for any $s \geq \frac{2m}{3}$, we have

$$\rho_\alpha(G) \leq f(d(z)) - 1 + \alpha \leq \alpha s + (1-\alpha)\frac{2m}{s} - 1 + \alpha \leq \alpha s + (1-\alpha)\frac{2m}{\frac{2m}{3}} - 1 + \alpha = \alpha s + 2(1-\alpha).$$

On the other hand, note that $\max\{2\alpha + (1-\alpha)s, \alpha + (1-\alpha)s\} \leq \alpha s + 2(1-\alpha)$. Thus, $\rho_\alpha(G) \leq \alpha s + 2(1-\alpha)$, as desired. $\square$

Note that if $m = g$, then $\mathcal{G}(m,g) = C_g$. If $m = g+1$, then $\mathcal{G}(m,g) = C_g^+$, where C_g^+ is a graph obtained from C_g by attaching a pendant edge at some vertex of C_g. In what follows, we consider $m \geq g+2$ and the corresponding $|\mathcal{G}(m,g)| \geq 2$. For $g \geq 3$ and $m \geq g+2$, let $\mathcal{H}(m,g)$ be the set of graphs in $\mathcal{G}(m,g)$ with maximum degree $\Delta = m-g+1$. Hu, Lou and Huang [8] obtained the following result.

Lemma 2.9 **([8])** *$\mathcal{H}(m,g) = \{G_1, G_2, \ldots, G_{\lfloor \frac{g}{2} \rfloor}, G^*\}$, where $g \geq 4$ and $m \geq g+2$.*

By simple observation, we see that G_0 is the unique graph among $\mathcal{G}(m, \geq g)$ with maximum degree $\Delta(G_0) = m-g+2$. Then from Lemma 2.8, we have the following result.

Corollary 2.10 *Let $G \in \mathcal{G}(m, \geq g)$ with $m \geq 3g-3$. Then $\alpha \in [\frac{1}{2}, 1)$, we have $\rho_\alpha(G) \leq \rho_\alpha(G_0)$, with equality holds if and only if $G \cong G_0$.*

Proof. For any graph $G \in \mathcal{G}(m, \geq g) \backslash \{G_0\}$, we have $\Delta(G) \leq m-g+1$. Note that $m-g+1 \geq \frac{2m}{3}$ since $m \geq 3g-3$. Then by Lemma 2.8, we have $\rho_\alpha(G) \leq \alpha(m-g+1)+2(1-\alpha) = \alpha(m-g-1)+2$. On the other hand, since $\Delta(G_0) = m-g+2$ and $K_{1,\Delta(G_0)}$ is a proper subgraph of G_0, then by Lemmas 2.3 and 2.6, we have

$$\rho_\alpha(G_0) > \alpha(m-g+2) + 1 - \alpha = \alpha(m-g+1) + 1 \geq \alpha(m-g-1) + 2 \geq \rho_\alpha(G),$$

as desired. $\square$

By Lemma 2.8, we can compare the A_α-spectral radius of graphs with distinct girths and maximum degrees, respectively.

Corollary 2.11 *Let G and H respectively be graph with the maximum A_α-spectral radius in $\mathcal{G}(m,g)$ and $\mathcal{G}(m,g')$. If $g < g'$ and $m \geq 3g'-3$, then $\rho_\alpha(G) > \rho_\alpha(H)$.*

Proof. Since $m \geq 3g'-3$ and $g' > g$, we have $m \geq 3g-3$. Then Corollary 2.10 implies that $\Delta(G) = m-g+2$ and $\Delta(H) = m-g'+2$. Note that $\Delta(H) = m-g'+2 \geq \frac{2m}{3}$ since $m \geq 3g'-6$. Then by Lemma 2.8, we have $\rho_\alpha(H) \leq \alpha(m-g'+2)+2(1-\alpha) = \alpha(m-g')+2$. On the other hand, since $\Delta(G) = m-g+2$ and $K_{1,\Delta(G)}$ is a proper subgraph of G, then by Lemmas 2.3 and 2.6, we have

$$\rho_\alpha(G) > \alpha(m-g+2)+1-\alpha = \alpha(m-g+1)+1 \geq \alpha(m-g'+2)+1 \geq \alpha(m-g')+2 \geq \rho_\alpha(H),$$

as desired. $\square$

Corollary 2.12 *Let G and H be graphs with size $m \geq 5$ and maximum degree $\Delta(G)$ and $\Delta(H)$, respectively. If $\Delta(G) > \Delta(H) \geq \frac{2m}{3}$, then $\rho_\alpha(G) > \rho_\alpha(H)$.*

Proof. Since $K_{1,\Delta(G)}$ is a proper subgraph of G, then by Lemmas 2.3 and 2.6, we have $\rho_\alpha(G) > \alpha\Delta(G) + 1 - \alpha \geq \alpha\Delta(H) + 1$. On the other hand, since $\Delta(H) \geq \frac{2m}{3}$, then by Lemma 2.6, we have $\rho_\alpha(H) \leq \alpha\Delta(H) + 2(1-\alpha) \leq \alpha\Delta(H) + 1 < \rho_\alpha(G)$, as desired. $\square$

3 Proofs of Theroems 1.4 and 1.5

Before giving the proofs of Theorems 1.4 and 1.5, we need some necessary lemmas.

Lemma 3.1 *Let $G_i \in \mathcal{H}(m, g)$ be the graph as shown in Figure 1, where $1 \le i \le \lfloor \frac{g}{2} \rfloor$. Then*

$$\rho_\alpha(G_1) > \rho_\alpha(G_2) > \cdots > \rho_\alpha(G_{\lfloor \frac{g}{2} \rfloor}).$$

Proof. For any $2 \le i \le \lfloor \frac{g}{2} \rfloor$, let $\boldsymbol{x}$ be the Perron vector of $\rho_\alpha(G_i)$. It suffices to show $\rho_\alpha(G_i) < \rho_\alpha(G_{i-1})$. To prove our result, first we give the following claim.

Claim 1. If there exists $1 \le j \le i-1$ such that $x_{u_{i-j}} < x_{u_{i+j-1}}$ and $x_{u_{i-j-1}} \ge x_{u_{i+j}}$, then $\rho_\alpha(G_i) < \rho_\alpha(G_{i-1})$.

Proof. Let $G' = G_i - \{u_{i-j-1}u_{i-j}, u_{i+j-1}, u_{i+j}\} + \{u_{i-j-1}u_{i+j-1}, u_{i-j}u_{i+j}\}$, where G_i and G' are shown in Figure 3. Clearly, $G' \cong G_{i-1}$. Note that $x_{u_{i-j}} < x_{u_{i+j-1}}$ and $x_{u_{i-j-1}} \ge x_{u_{i+j}}$. Then by Lemma 2.5, we have $\rho_\alpha(G_i) < \rho_\alpha(G') = \rho_\alpha(G_{i-1})$. □

We start to prove by firstly assuming $x_{u_{i-1}} \ge x_{u_i}$. Now we construct

$$G'' = G_i - \{wu_i\} + \{wu_{i-1}\}$$

from G_i, where G'' is shown in Figure 3. Clearly, $G'' \cong G_{i-1}$. By Lemma 2.4, we have $\rho_\alpha(G_i) < \rho_\alpha(G'') = \rho_\alpha(G_{i-1})$. Otherwise $x_{u_{i-1}} < x_{u_i}$, if $x_{u_{i-2}} \ge x_{u_{i+1}}$ then from Claim 1 we get $\rho_\alpha(G_i) < \rho_\alpha(G_{i-1})$ by taking $j = 1$. Otherwise $x_{u_{i-2}} < x_{u_{i+1}}$, if $x_{u_{i-3}} \ge x_{u_{i+2}}$ then from Claim 1 we get $\rho_\alpha(G_i) < \rho_\alpha(G_{i-1})$ by taking $j = 2$. Repeating i steps we come to the assumption $x_{u_0} < x_{u_{2i-1}}$ for $j = i$. Note that $N_{G_i}(u_0) = \{u_1, u_{g-1}, w_1, \dots, w_{m-g-1}\}$. Let

$$G''' = G_i - \{u_0 w_s | 1 \le s \le m-g-1\} + \{u_{2i-1} w_t | 1 \le t \le m-g-1\}.$$

Clearly, $G''' \cong G_{i-1}$. By Lemma 2.4, we have $\rho_\alpha(G_i) < \rho_\alpha(G''') = \rho_\alpha(G_{i-1})$, as desired. □

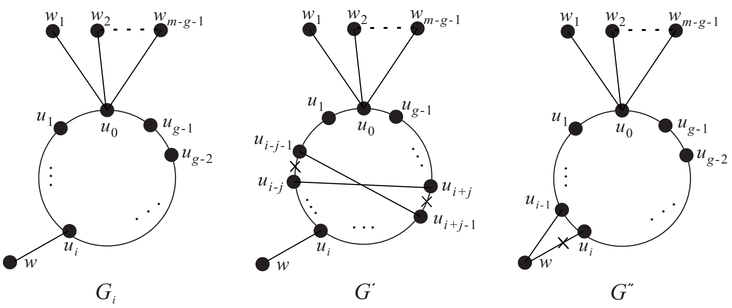

Figure 3: The graphs G_i, G' and G'', where the edge with "**X**" represents it is deleted.

Lemma 3.2 *Let $G_i \in \mathcal{H}(m, g)$ be graphs as shown in Figure 1, where $1 \le i \le \lfloor \frac{g}{2} \rfloor$. If $m \ge g+3$ and $\frac{1}{2} \le \alpha < 1$, then*

$$\alpha(m-g) + 1 < \rho_\alpha(G_i) \le \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1} < \alpha(m-g) + 2 - \alpha.$$

Proof. Since $\Delta(G_i) = m - g + 1$ and $K_{1,\Delta(G_i)}$ is a proper subgraph of G_i, then by Lemmas 2.3 and 2.6, we have $\rho_\alpha(G_i) > \rho_\alpha(K_{1,\Delta(G_i)}) = \alpha(m-g+1) + 1 - \alpha = \alpha(m-g) + 1$. On the other hand, let $z \in V(G_i)$ such that

$$\alpha d(z) + (1-\alpha)\frac{\sum_{v \in N_{G_i}(z)} d(v)}{d(z)} = \max_{u \in V(G_i)} \left\{ \alpha d(u) + (1-\alpha)\frac{\sum_{v \in N_{G_i}(u)} d(v)}{d(u)} \right\}.$$

If $z = u_0$, then by Lemma 2.7, we have

$$\rho_\alpha(G_i) \le \alpha(m-g+1) + (1-\alpha)\frac{m-g+3}{m-g+1} = \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1} := f(\alpha).$$

If $z = u_i$, then by Lemma 2.7 and Wolfram Mathematica, we have

$$\rho_\alpha(G_i) \le \alpha d(u_i) + (1-\alpha)\frac{d(w)+d(u_{i-1})+d(u_0)}{d(u_i)} = 3\alpha + (1-\alpha)\frac{m-g+4}{3} < \alpha(m-g)+1+\frac{2(1-\alpha)}{m-g+1}.$$

If $d(z) = 1$, then by Lemma 2.7, we have

$$\rho_\alpha(G_i) \le \alpha d(z) + (1-\alpha)d(u_0) = \alpha + (1-\alpha)(m-g+1) := g(\alpha).$$

Now let $\varphi(\alpha) = f(\alpha) - g(\alpha) = (2\alpha - 1)(m-g) + \frac{2(1-\alpha)}{m-g+1}$. Then we have

$$\varphi'(\alpha) = 2(m-g) - \frac{2}{m-g+1} = 2\left(m - g - \frac{1}{m-g+1}\right) > 0.$$

Thus $\varphi(\alpha)$ is a monotonically increasing function on $\alpha \ge \frac{1}{2}$. Hence, $\varphi(\alpha) \ge \varphi\left(\frac{1}{2}\right) = \frac{1}{m-g+1} > 0$. It follows that

$$\rho_\alpha(G_i) \le g(\alpha) < f(\alpha) = \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1}.$$

If $z \in V(G)\backslash\{u_0, u_i\}$ is not a pendent vertex, then $d(z) = 2$ and by Lemma 2.7, we have

$$\rho_\alpha(G_i) \le 2\alpha + (1-\alpha)\frac{d(u_0)+d(u_i)}{2} = 2\alpha + (1-\alpha)\frac{m-g+4}{2} := \phi(\alpha).$$

Now let $\psi(\alpha) = f(\alpha) - \phi(\alpha)$, it follows that

$$\begin{aligned}\psi(\alpha) = f(\alpha) - \phi(\alpha) &= \alpha(m-g-1) + (1-\alpha)\frac{m-g+3}{m-g+1} - (1-\alpha)\frac{m-g+4}{2} \\ &= \frac{1}{2}\left((3\alpha-1)(m-g) + \frac{4(1-\alpha)}{m-g+1} - 2\right).\end{aligned}$$

Then we have

$$\psi'(\alpha) = \frac{1}{2}\left(3(m-g) - \frac{4}{m-g+1}\right) = \frac{3(m-g)(m-g+1)-4}{2(m-g+1)} > 0.$$

Thus $\psi(\alpha)$ is a monotonically increasing function on $\alpha \ge \frac{1}{2}$. Hence

$$\psi(\alpha) \ge \psi\left(\frac{1}{2}\right) = \frac{m-g}{4} + \frac{1}{m-g+1} - 1 \ge 0.$$

It follows that

$$\rho_\alpha(G_i) \le \psi(\alpha) \le f(\alpha) = \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1},$$

as desired. This completes the proof of Lemma 3.2. □

Lemma 3.3 *For $1 \le i \le \lfloor \frac{g}{2} \rfloor$ and $m \ge g+4$, let $\boldsymbol{x}$ be the Perron vector of $\rho_\alpha(G_i)$. Then for $\frac{1}{2} \le \alpha < 1$, we have $x_{u_0} = \max\limits_{u \in V(G_i)} x_u$.*

Proof. By Lemma 3.2, we have $\rho_\alpha(G_i) > \alpha(m-g)+1 \ge 3$. On the other hand, let $x_{u^*} = \max\limits_{u\in V(G_i)} x_u$. We then assert that $d_{G_i}(u^*) \ge 4$. Suppose to the contrary that $d_{G_i}(u^*) \le 3$. Then, by (2), we have

$$\begin{aligned}\rho_\alpha(G_i)x_{u^*} &= \alpha d_{G_i}(u^*)x_{u^*} + (1-\alpha)\sum_{u \in N_{G_i}(u^*)} x_u \\ &\le \alpha d_{G_i}(u^*)x_{u^*} + (1-\alpha)d_{G_i}(u^*)x_{u^*} \le 3x_{u^*},\end{aligned}$$

which implies that $\rho_\alpha(G_i) \le 3$. However, recall that $\rho_\alpha(G_i) > 3$, a contradiction. Thus, $d_{G_i}(u^*) \ge 4$. It follows that $u^* = u_0$, as desired. □

Lemma 3.4 *If $g \geq 4$, $\frac{1}{2} \leq \alpha < 1$ and $m \geq g+7$, then $\rho_\alpha(G_1) > \rho_\alpha(G^*) > \rho_\alpha(G_2)$.*

Proof. We first prove $\rho_\alpha(G_1) > \rho_\alpha(G^*)$. Let $\boldsymbol{x}$ be the Perron vector of $A_\alpha(G^*)$. The vertices w_1 and v_1 of G^* are shown in Fig. 1. By (2), we have

$$\rho_\alpha\left(G^*\right)x_{v_1} = \alpha x_{v_1} + (1-\alpha)x_{w_1}, \quad \rho_\alpha\left(G^*\right)x_{w_1} = 2\alpha x_{w_1} + (1-\alpha)(x_{v_1}+x_{u_0}),$$
$$\rho_\alpha\left(G^*\right)x_{u_0} = \alpha(m-g+1)x_{u_0} + (1-\alpha)\left(\frac{(1-\alpha)(m-g-2)}{\rho_\alpha\left(G^*\right)-\alpha}x_{u_0} + x_{w_1} + x_{u_1} + x_{u_{g-1}}\right).$$

Note that $x_{u_1} = x_{u_{g-1}}$ due to the symmetry of G^*. From the above equalities, we have

$$\begin{cases} x_{w_1} = & \frac{(1-\alpha)(\rho_\alpha(G^*)-\alpha)}{(\rho_\alpha(G^*)-2\alpha)(\rho_\alpha(G^*)-\alpha)-(1-\alpha)^2}x_{u_0}, \\ x_{u_1} = & \frac{1}{2}\left(\frac{\rho_\alpha(G^*)-\alpha(m-g+1)}{1-\alpha} - \frac{(1-\alpha)(m-g-2)}{\rho_\alpha(G^*)-\alpha} - \frac{(\rho_\alpha(G^*)-\alpha)(1-\alpha)}{(\rho_\alpha(G^*)-2\alpha)(\rho_\alpha(G^*)-\alpha)-(1-\alpha)^2}\right)x_{u_0}. \end{cases}$$

Let

$$\begin{aligned} f(x) &= \frac{1}{2}\left(\frac{x-\alpha(m-g+1)}{1-\alpha} - \frac{(1-\alpha)(m-g-2)}{x-\alpha}\right) - \frac{3(x-\alpha)(1-\alpha)}{2\left(x-2\alpha\right)\left(x-\alpha\right)-(1-\alpha)^2} \\ &= \frac{(x^2-3\alpha x+\alpha^2-1+2\alpha)((x-\alpha(m-g+1))(x-\alpha)-(1-\alpha)^2(m-g-2))-3(x-\alpha)^2(1-\alpha)^2}{2\left(x^2-3\alpha x+\alpha^2-1+2\alpha\right)(x-\alpha)(1-\alpha)}. \end{aligned}$$

Then $x_{u_1} - x_{w_1} = f\left(\rho_\alpha\left(G^*\right)\right)x_{u_0}$. On the other hand, by Wolfram Mathematica, we have

$$\begin{aligned} & f(x, m-g+5) \\ =& x^4 - (m-g+5)\alpha x^3 + ((3m-3g+7)\alpha^2 + 2(m-g+2)\alpha - m + g - 2)x^2 \\ & + \alpha(4m-(m-g+5)\alpha^2 - 4(2m-2g+1)\alpha - 4g+2)x + (m-g+5)(2\alpha^3+3\alpha^2-4\alpha) + m \\ & - g - 2\alpha^3 - 27\alpha^2 + 28\alpha - 2 \\ =& (x^2-3\alpha x+\alpha^2-1+2\alpha)((x-\alpha(m-g+1))(x-\alpha)-(1-\alpha)^2(m-g-2)) - 3(x-\alpha)^2(1-\alpha)^2, \end{aligned}$$

where $f(x, m-g+5)$ is defined in Lemma 2.2. Then

$$f(x) = \frac{f(x, m-g+5)}{2\left(x^2-3\alpha x+\alpha^2-1+2\alpha\right)(x-\alpha)(1-\alpha)}, \tag{3}$$

If $g \geq 5$, then $S_{m-g+5,3}$ is a proper subgraph of G^*, where $S_{m-g+5,3}$ is shown in Figure 2. Therefore, $\rho_\alpha\left(G^*\right) > \rho_\alpha\left(S_{m-g+5,3}\right)$. Note that $\Delta(G^*) = m-g+1$ and $K_{1,(G^*)}$ is a proper subgraph of G^*. By Lemma 2.6, we have $\rho_\alpha(G^*) > \alpha(m-g)+1$. It is easy to versify that $x^2-3\alpha x+\alpha^2-1+2\alpha > 0$ for $x > 7\alpha+1$. Thus $f\left(\rho_\alpha\left(G^*\right)\right) > 0$, which implies that $x_{u_1} > x_{w_1}$. If $g = 4$, then $G^* \cong H_0$, where H_0 is shown in Fig.2. Thus, (3) becomes

$$f(x) = \frac{1}{2\left(x^2-3\alpha x+\alpha^2-1+2\alpha\right)(x-\alpha)(1-\alpha)} \cdot \frac{g(x,m)+2(\alpha-1)^4(x-\alpha)x}{x^2-4\alpha x+2(\alpha^2+2\alpha-1)},$$

where $g(x,m)$ is defined by Lemma 2.2. Clearly, $g(\rho_\alpha(G^*), m) = 0$. On the other hand, we have $x^2-3\alpha x+\alpha^2-1+2\alpha > 0$ and $x^2-4\alpha x+2(\alpha^2+2\alpha-1) > 0$ for $x > 7\alpha+1$. Recall that $\rho_\alpha(G^*) > 7\alpha+1$. Thus $f(\rho_\alpha(G^*)) > 0$, it follows that $x_{u_1} > x_{w_1}$. Let $G' = G^* - \{w_1v_1\} + \{u_1v_1\}$. Clearly, $G' \cong G_1$. By Lemma 2.4, we have $\rho_\alpha(G^*) < \rho_\alpha(G') = \rho_\alpha(G_1)$.

Next we will prove $\rho_\alpha(G^*) > \rho_\alpha(G_2)$. Let $\boldsymbol{y}$ be the Perron vector of $A_\alpha(G_2)$. By Lemma 3.3, we have $y_{u_0} = \max\limits_{u \in V(G_2)} y_u$. Let $u \in V(G_2)$ with $d_{G_2}(u) = 2$, by (2), we have

$$\rho_\alpha(G_2)y_u = 2\alpha y_u + (1-\alpha)\sum_{v \in N_{G_2}(u)} y_v \leq 2\alpha y_u + 2(1-\alpha)y_{u_0}.$$

It follows that $y_u \leq \frac{2(1-\alpha)}{\rho_\alpha(G_2)-2\alpha} y_{u_0}$. Using (2) again, we have

$$\rho_\alpha(G_2)y_w = \alpha y_w + (1-\alpha)y_{u_2}, \quad \rho_\alpha(G_2)y_{u_2} = 3\alpha y_{u_2} + (1-\alpha)(y_w + y_{u_1} + y_{u_3}).$$

Then

$$\rho_\alpha(G_2)(y_{u_2} - y_w) = (4\alpha - 1)y_{u_2} + (1-2\alpha)y_w + (1-\alpha)(y_{u_1} + y_{u_3}).$$

Thus

$$\frac{\rho_\alpha(G_2)(\rho_\alpha(G_2)-1)}{\rho_\alpha(G_2)-\alpha} y_{u_2} = \frac{(4\alpha-1)(\rho_\alpha(G_2)-\alpha) + (1-2\alpha)(1-\alpha)}{\rho_\alpha(G_2)-\alpha} y_{u_2} + (1-\alpha)(y_{u_1} + y_{u_3}),$$

which implies that

$$y_{u_2} \leq \frac{\rho_\alpha(G_2)-\alpha}{\rho_\alpha^2(G_2) - 4\alpha\rho_\alpha(G_2) + 2\alpha(\alpha+1) - 1} \cdot \frac{4(1-\alpha)^2}{\rho_\alpha(G_2)-2\alpha} y_{u_0}. \tag{4}$$

Now let $f_1(x) = 4(1-\alpha)^2(x-a)^2$, $g_1(x) = (1-\alpha)(x-2\alpha)(x^2 - 4\alpha x + 2\alpha(\alpha+1) - 1)$, $h(x) = g_1(x) - f_1(x)$ and $x > 7\alpha + 1$. Then we have

$$h(x) = (1-a)\left[x^3 - 2(\alpha+2)x^2 + (2\alpha(\alpha+5) - 1)x + 2\alpha(1-4\alpha)\right] := (1-\alpha)\zeta(x).$$

It follows that

$$\begin{aligned}\zeta'(x) &= 3x^2 - 4(\alpha+2)x + 2\alpha(\alpha+5) - 1 \\ &> 3(7\alpha+1)^2 - 4(\alpha+2)(7\alpha+1) + 2\alpha(\alpha+5) - 1 = 121\alpha^2 - 8\alpha - 6 > 0.\end{aligned}$$

Thus $\zeta(x)$ is a monotonically increasing function on $x > 7\alpha + 1$. Hence, $h(x) = (1-\alpha)\zeta(x) > (1-\alpha)\zeta(7\alpha+1) = (1-\alpha)(259\alpha^3 - 13\alpha^2 - 32\alpha - 4) > 0$. It follows that $g_1(x) > f_1(x)$ and

$$\frac{x-\alpha}{x^2 - 4\alpha x + 2\alpha(\alpha+1) - 1} \cdot \frac{4(1-\alpha)^2}{x-2\alpha} < \frac{1-\alpha}{x-\alpha}.$$

Note that $\rho_\alpha(G_2) > 7\alpha + 1$ and $y_v = \frac{1-\alpha}{\rho_\alpha(G_2)-\alpha} y_{u_0}$ for any $v \in \{w_1, \ldots, w_{m-g-1}\}$, then by (4), we have

$$y_{u_2} \leq \frac{\rho_\alpha(G_2)-\alpha}{\rho_\alpha^2(G_2) - 4\alpha\rho_\alpha(G_2) + 2\alpha(\alpha+1) - 1} \cdot \frac{4(1-\alpha)^2}{\rho_\alpha(G_2)-2\alpha} y_{u_0} < \frac{1-\alpha}{\rho_\alpha(G_2)-\alpha} y_{u_0} = y_{w_1}.$$

Hence, $y_{u_2} < y_{w_1}$. Let $G'' = G_2 - \{wu_2\} + \{w_1 w\}$. Clearly, $G'' \cong G^*$. By Lemma 2.4, we have $\rho_\alpha(G_2) < \rho_\alpha(G'') = \rho_\alpha(G^*)$. □

Now we can give the proof of Theorem 1.4.

Proof of Theorem 1.4: Notice that $g \geq 4$ and $m \geq g + 7$ since $m \geq 3g \geq 12$. Then by Lemmas 3.1 and 3.4, we have

$$\rho_\alpha(G_1) > \rho_\alpha(G^*) > \rho_\alpha(G_2) > \cdots > \rho_\alpha(G_{\lfloor \frac{g}{2} \rfloor}).$$

Set $\mathcal{G}_{m-g}(m,g) = \mathcal{G}(m,g) \backslash (\{G_0\} \cup \mathcal{H}(m,g))$. Note that G_0 is a unique graph with maximum degree $m-g+2$ among $\mathcal{G}(m,g)$. By Lemma 2.9, $\mathcal{H}(m,g) = \{G_1, G_2, \ldots, G_{\lfloor \frac{g}{2} \rfloor}, G^*\}$ is the set of graphs in $\mathcal{G}(m,g)$ with maximum degree $m-g+1$. Then for any $G' \in \mathcal{G}_{m-g}(m,g)$, we have $\Delta(G') \leq m-g$. On the other hand, we have $\Delta(G_{\lfloor \frac{g}{2} \rfloor}) = m-g+1 > m-g \geq \Delta(G') \geq \frac{2m}{3}$ since $m \geq 3g$. By Corollary 2.12, we obtain $\rho_\alpha(G_{\lfloor \frac{g}{2} \rfloor}) > \rho_\alpha(G')$. Thus, the second to the $\left(\lfloor \frac{g}{2} \rfloor + 2\right)$th largest graphs in $\mathcal{G}(m,g) \backslash \{G_0\}$ by their A_α-spectral radius is given by

$$\rho_\alpha(G_1) > \rho_\alpha(G^*) > \rho_\alpha(G_2) > \cdots > \rho_\alpha(G_{\lfloor \frac{g}{2} \rfloor}).$$

This completes the proof of Theorem 1.4. $\square$

Moreover, we further consider the first to the $(\lfloor \frac{g}{2} \rfloor + 3)$th largest graphs according to their A_α-spectral radius among all graphs in $\mathcal{G}(m, \geq g)$. For $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$, we use G_g^i and G_g^* instead of G_i and G^* to distinguish the girth of the graphs in the following proof. Now, we give the proof of Theorem 1.5.

Proof of Theorem 1.5: Denote by M and N are the sets of all graphs in $\mathcal{G}(m, \geq g)$ with maximum degree at least $m-g+1$ and at most $m-g$, respectively. It follows that $\mathcal{G}(m, \geq g) = M \cup N$. It is easy to check that $M = \{G_g^0, G_g^1, G_g^*, G_g^2, \ldots, G_g^{\lfloor \frac{g}{2} \rfloor}, G_{g+1}^0\}$, where the maximum degree of $G_g^1, G_g^*, G_g^2, \ldots, G_g^{\lfloor \frac{g}{2} \rfloor}, G_{g+1}^0$ is $m-g+1$ and $\Delta(G_g^0) = m-g+2$. For any $G \in M$, note that $\Delta(G) \geq m-g+1$ and $K_{1,(G)}$ is a proper subgraph of G. By Lemma 2.6, we have $\rho_\alpha(G) > \alpha(m-g)+1$. For any $G' \in N$, since $\Delta(G') \leq m-g$ and $m-g \geq \frac{2m}{3}$, then by Lemma 2.8, we have

$$\rho_\alpha(G') \leq \alpha(m-g) + 2(1-\alpha).$$

Since $\alpha(m-g) + 2(1-\alpha) \leq \alpha(m-g)+1$, each A_α-spectral radius of the graph in M is more than that of the graph in N. By Theorem 1.4, to complete the proof it remains to show $\rho_\alpha(G_g^{\lfloor \frac{g}{2} \rfloor}) > \rho_\alpha(G_{g+1}^0)$.

Let $\boldsymbol{x}$ be the Perron vector of $\rho_\alpha(G_{g+1}^0)$. Note that

$$\rho_\alpha(G_{g+1}^0) > \alpha(m-g) + 1 = 8\alpha + 1$$

since $m \geq 3g \geq 12$. If g is even, by the symmetry of G_{g+1}^0, then $x_{u_{\lfloor \frac{g}{2} \rfloor}} = x_{u_{\lfloor \frac{g}{2} \rfloor+1}}$ (see Figure 4). If g is odd, by the symmetry of G_{g+1}^0, then $x_{u_{\lfloor \frac{g}{2} \rfloor}} = x_{u_{\lfloor \frac{g}{2} \rfloor+2}}$ (see Figure 4). By (2), we have

$$\rho_\alpha(G_{g+1}^0)x_{u_{\lfloor \frac{g}{2} \rfloor+1}} = 2\alpha x_{u_{\lfloor \frac{g}{2} \rfloor+1}} + (1-\alpha)(x_{u_{\lfloor \frac{g}{2} \rfloor}} + x_{u_{\lfloor \frac{g}{2} \rfloor+2}}) = 2\alpha x_{u_{\lfloor \frac{g}{2} \rfloor+1}} + 2(1-\alpha)x_{u_{\lfloor \frac{g}{2} \rfloor}},$$

which implies that

$$x_{u_{\lfloor \frac{g}{2} \rfloor}} = \frac{\rho_\alpha(G_{g+1}^0) - 2\alpha}{2(1-\alpha)} x_{u_{\lfloor \frac{g}{2} \rfloor+1}} > \frac{6\alpha+1}{2(1-\alpha)} x_{u_{\lfloor \frac{g}{2} \rfloor+1}} \geq 4x_{u_{\lfloor \frac{g}{2} \rfloor+1}}.$$

It follows that $x_{u_{\lfloor \frac{g}{2} \rfloor}} > x_{u_{\lfloor \frac{g}{2} \rfloor+1}}$. Thus, we have $x_{u_{\lfloor \frac{g}{2} \rfloor}} \geq x_{u_{\lfloor \frac{g}{2} \rfloor+1}}$. Let

$$G' = G_{g+1}^0 - \{u_{\lfloor \frac{g}{2} \rfloor+1}u_{\lfloor \frac{g}{2} \rfloor+2}\} + \{u_{\lfloor \frac{g}{2} \rfloor}u_{\lfloor \frac{g}{2} \rfloor+2}\}.$$

Clearly, $G' \cong G_g^{\lfloor \frac{g}{2} \rfloor}$ (see Figure 4). By Lemma 2.4, we have $\rho_\alpha(G_{g+1}^0) < \rho_\alpha(G') = \rho_\alpha(G_g^{\lfloor \frac{g}{2} \rfloor})$. This completes the proof of Theorem 1.5. $\square$

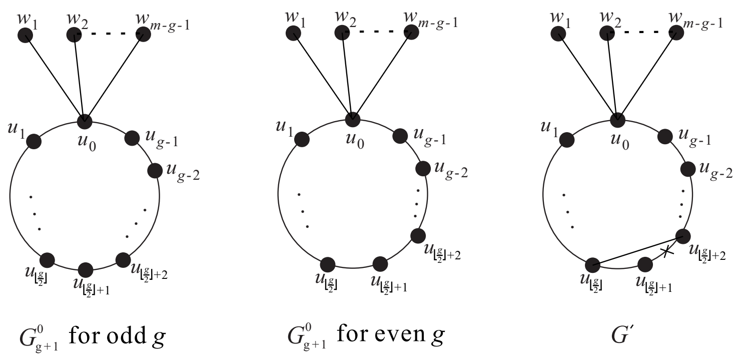

Figure 4: The graphs G_{g+1}^0 and G', where the edge with "**X**" represents it is deleted.

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