General decay of solutions of a weakly coupled abstract evolution equations with one finite memory control

Aissa Guesmia[1], Salim A. Messaoudi[2,] and Mostafa Zahri[2*]

[1]Institut Elie Cartan de Lorraine, UMR 7502,
Université de Lorraine, 3 Rue Augustin Fresnel
BP 45112, 57073 Metz Cedex 03, France
E-mail: aissa.guesmia@univ-lorraine.fr

[2]Department of Mathematics, College of Sciences,
University of Sharjah, P. O. Box 27272
Sharjah, United Arab Emirates
E-mail: smessaoudi@sharjah.ac.ae
E-mail: mzahri@sharjah.ac.ae

Abstract

In this work, we consider the following abstract evolution system:

$$\begin{cases} u_{tt}(t) + Au(t) - \displaystyle\int_0^t g(t-s)A^\theta u(s)ds + \alpha v(t) = 0, & t > 0 \\ v_{tt}(t) + Av(t) + \alpha u(t) = 0, & t > 0 \\ u(0) = u_0,\ u_t(0) = u_1,\ v(0) = v_0,\ v_t(0) = v_1, & \end{cases}$$

where $A : \mathcal{D}(A) \subset H \longrightarrow H$ is a linear positive definite self-adjoint operator, H is a Hilbert space, g is a positive nonincreasing function with some general decay rate, $\theta \in [0,1]$, α is a positive constant and u_0, u_1, v_0 and v_1 are fixed initial data. Under appropriate conditions on g, α and the regularity of the initial data, we establish a general decay rate of the solution energy which generalizes some earlier results in the literature. We, also, illustrated our results by performing several numerical tests.

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*Corresponding author.

1 Introduction

Given a real separable Hilbert space H with associated inner product and norm denoted, respectively, by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. The subject of this paper is to study the following problem:

$$\begin{cases} u_{tt}(t) + Au(t) - \displaystyle\int_0^t g(t-s)A^{\theta}u(s)ds + \alpha v(t) = 0, & t>0 \\ v_{tt}(t) + Av(t) + \alpha u(t) = 0, & t>0 \\ u(0)=u_0,\ u_t(0)=u_1,\ v(0)=v_0,\ v_t(0)=v_1, \end{cases} \tag{1.1}$$

where α is a positive constant, $\theta \in [0,1]$, u_0, u_1, v_0 and v_1 are fixed initial data, and

$$A : \mathcal{D}(A) \subset H \longrightarrow H$$

is a linear positive definite self-adjoint operator on H such that the embedding $\mathcal{D}(A^{\beta}) \hookrightarrow \mathcal{D}(A^{\sigma})$ is compact, for any $\beta > \sigma \geq 0$. This embedding guarantees the existence of positive constants a_0, a_1 and a_2 such that

$$\|w\|^2 \leq a_0 \left\|A^{1/2}w\right\|^2 \text{ and } \left\|A^{\theta/2}w\right\|^2 \leq a_1 \left\|A^{1/2}w\right\|^2, \qquad \forall\, w \in \mathcal{D}\left(A^{1/2}\right) \tag{1.2}$$

and

$$\left\|A^{1/2}w\right\|^2 \leq a_2 \left\|A^{1-\theta/2}w\right\|^2 \qquad \forall\, w \in \mathcal{D}\left(A^{1-\theta/2}\right). \tag{1.3}$$

Problem (1.1) describes the evolution of two interacting elastic membranes (or two plates) through a force that attracts one membrane (or a plate) to the other one with the coefficient α. Whereas, the integral term in the first equation acts as a stabilizer.

Coupled wave systems have been considered in various contexts. In [12], the authors studied a system of two compactly coupled wave equations, where the boundary damping are effective on both equations, and obtained the exponential stability in the linear-damping case and a polynomial stability for the case of polynomial-like damping. Similar results were also established by Aassila [1] and [2]. More related works, could be found in [21, 22]. In our context the authors in [13], looked into the following weakly two coupled wave equations with one linear frictional damping u_t:

$$\begin{cases} u_{tt} - \Delta u + u_t + \alpha v = 0 & \text{in } \Omega \times (0,\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.4}$$

together with initial and Dirichlet-boundary conditions, where Ω is a bounded domain of $\mathbb{R}^n$ with $n \in \mathbb{N}^*$ and α is a positive and small parameter. They showed that the solution of (1.4) decays polynomially and the decay rate is optimal. They also gave some computational experiments in the one-dimensional setting. In [4], the authors considered a coupled system in an abstract setting and showed that, for sufficiently smooth initial conditions and for $|\alpha| > 0$ and small enough, the energy of the solution decays polynomially to zero as $t \to \infty$. Najafi [19] studied the following one-dimensional weakly two coupled wave equations with two linear frictional dampings βu_t and βv_t:

$$\begin{cases} u_{tt} - c^2 u_{xx} = \alpha(v-u) + \beta(v_t - u_t) & \text{in } (0,1)\times(0,\infty) \\ v_{tt} - c^2 v_{xx} = \alpha(u-v) + \beta(u_t - v_t) & \text{in } (0,1)\times(0,\infty) \end{cases} \tag{1.5}$$

together with initial and mixed boundary conditions, and established, by using the frequency-domain method, the exponential decay rate of solutions of (1.5).

In [5], the authors investigated the following weakly coupled two wave equations with one infinite memory:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)ds + \alpha v = 0 & \text{in } (0,1)\times(0,\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } (0,1)\times(0,\infty) \end{cases} \tag{1.6}$$

together with initial and Dirichlet-boundary conditions, where Ω is a bounded domain of $\mathbb{R}^n$ with $n \in \mathbb{N}^*$, α is a positive and small parameter and the kernel decay exponentially. They proved that the solution of (1.6) has a polynomial rate of decay, but it does not have exponential decay. The optimality of the polynomial decay has been recently established by Cordeiro et al. [7]. Guesmia [9] considered the following coupled system of two linear abstract evolution equations with one infinite memory:

$$\begin{cases} u_{tt}(t) + Au(t) - \displaystyle\int_0^\infty g(s)Bu(t-s)ds + \widetilde{B}v(t) = 0, & t > 0 \\ v_{tt}(t) + \widetilde{A}v(t) + \widetilde{B}u(t) = 0 & t > 0 \end{cases} \tag{1.7}$$

together with initial conditions, where

$$A : \mathcal{D}(A) \subset H \longrightarrow H, \quad B : \mathcal{D}(B) \subset H \longrightarrow H \quad \text{and} \quad \widetilde{A} : \mathcal{D}(\widetilde{A}) \subset H \longrightarrow H$$

are positive definite self-adjoint operators on H such that the embedding $\mathcal{D}(A) \subset \mathcal{D}(B) \subset H$ and $\mathcal{D}(\widetilde{A}) \subset H$ are dense. In addition, $\widetilde{B}$ is a self-adjoint linear bounded operator. Under appropriate conditions on the kernel, the operators and the history data, he proved that the stability of (1.7) holds for kernels with decay rates that are much weaker than the exponential decay. In addition, he presented applications to various distributed coupled systems of second-order with one infinite memory acting only on the first equation. Some results, similar to the those of [9], were proved in [6] in case of complementary frictional damping and infinite memory

$$Du_t(t) - \int_0^\infty g(s)Bu(t-s)ds,$$

where D is a bounded operator. The results of [6] and [9] generalized those of [11] and [14], where systems of coupled wave equations and coupled abstract equations, respectively, have been discussed with kernels g having a negligible flat zone; i.e.

$$\mu\left(\{s \geq 0 : \, g'(s) = 0\}\right) = 0, \tag{1.8}$$

where μ is the Lebesgue measure, or satsifying $g' \leq -kg$, for some positive constant k (which implies that g converges exponentially to zero).

Jin et al. [10] treated the stability of the following coupled two abstract evolution equations with one finite memory:

$$\begin{cases} u_{tt}(t) + Au(t) - \displaystyle\int_0^t g(s)Au(t-s)ds + \alpha u + \beta Bv(t) = f(u), & t > 0 \\ v_{tt}(t) + Av(t) + \beta Bu(t) = 0, & t > 0, \end{cases} \tag{1.9}$$

where A and B are given operators, α is a nonnegative constant, β is a positive constant, $f : \mathbb{R} \to \mathbb{R}$ is a fixed function and g satisfies (1.8) and some smallness and regularity conditions. Under specific conditions on A, B and f, and some smallness conditions on the initial data, they proved that (1.9) is polynomially stable, where the decay rate of energy at infinity is of order t^{-1}. For recent works in this direction, we cite [3] and [20].

In this present work, we discuss the longtime behavior of solutions of (1.1) under general conditions on g and establish a general decay result. Our result generalises the ones of [10] to the case of presence of a fractional power θ. Moreover, in the particular case where g converges exponentially to zero, our general decay result covers the decay rate t^{-1} of [10].

Our paper contains five sections, in addition to the introduction and the conclusion. In Section 2, we set our hypotheses on g and α, introduce some "energy" functionals, state an existence result and prove few related lemmas. Section 3 gives certain necessary lemmas and in Section 4, we establish our main decay result. Section 5 is devoted to some general comments and illustrating examples. In Section 6, we present a numerical analysis and perform six numerical tests.

2 Preliminaries

In this section, our hypotheses on g and α are given, some "energy" functionals are introduced and related lemmas and the existence result are established. So, in addition to (1.2) and (1.3), we consider the following conditions:

(H1) The strictly decreasing differentiable relaxation function $g : [0, \infty) \longrightarrow (0, \infty)$ satisfies

$$g(0) > 0 \qquad \text{and} \qquad \widetilde{g} := \int_0^{+\infty} g(s)ds < \frac{1}{a_1}, \tag{2.1}$$

and there exist $1 \leq p < 3/2$ and a non-increasing function $\xi : [0, \infty) \longrightarrow (0, \infty)$ such that

$$g'(t) \leq -\xi(t) g^p(t), \quad \forall t \geq 0. \tag{2.2}$$

(H2) The constant α is such that

$$0 < \alpha < \frac{\sqrt{1 - a_1 \widetilde{g}}}{a_0}. \tag{2.3}$$

where a_0 and a_1 are defined in (1.2).

We state the existence, regularity and uniqueness theorem whose proof can be established similarly to that in [18]. See also [17].

Theorem 2.1. Suppose that

$$(u_0, u_1), (v_0, v_1) \in \mathcal{D}\left(A^{\sigma+1/2}\right) \times \mathcal{D}\left(A^{\sigma}\right), \tag{2.4}$$

for $\sigma \geq 0$, and (**H1**) and (**H2**) hold. Then problem (1.1) has a unique global strong solution satisfying

$$u, v \in \boldsymbol{C}\left(\mathbb{R}_+; \mathcal{D}\left(A^{\sigma+1/2}\right)\right) \cap \boldsymbol{C}^1\left(\mathbb{R}_+; \mathcal{D}\left(A^{\sigma}\right)\right) \cap \boldsymbol{C}^2\left(\mathbb{R}_+; \mathcal{D}\left(A^{\sigma-1/2}\right)\right). \tag{2.5}$$

For $(u_0, u_1), (v_0, v_1) \in \mathcal{D}\left(A^{\sigma+1/2}\right) \times \mathcal{D}\left(A^{\sigma}\right)$ such that

$$\sigma \geq \max\left\{\frac{\theta}{2}, \frac{1-\theta}{2}\right\}, \tag{2.6}$$

we define the following energy functionals associated with the solution (u, v) of (1.1):

$$E(t) := \frac{1}{2}\left[\begin{array}{c} \|u_t(t)\|^2 + \|v_t(t)\|^2 + \left\|A^{1/2}u(t)\right\|^2 + \left\|A^{1/2}v(t)\right\|^2 \\ -\left(\int_0^t g(s)ds\right)\left\|A^{\theta/2}u(t)\right\|^2 + 2\alpha\langle u, v\rangle + \left(g \circ A^{\theta/2}u\right)(t) \end{array}\right], \tag{2.7}$$

$$\mathcal{E}(t) := \frac{1}{2}\left[\begin{array}{c} \left\|A^{\theta/2}u_t(t)\right\|^2 + \left\|A^{\theta/2}v_t(t)\right\|^2 + \left\|A^{(1+\theta)/2}u(t)\right\|^2 \\ + \left\|A^{(1+\theta)/2}v(t)\right\|^2 - \left(\int_0^t g(s)ds\right)\left\|A^{\theta}u(t)\right\|^2 \\ +2\alpha\langle A^{\theta/2}u, A^{\theta/2}v\rangle + \left(g \circ A^{\theta}u\right)(t) \end{array}\right] \tag{2.8}$$

and

$$\mathcal{E}_*(t) := \frac{1}{2}\left[\begin{array}{c} \left\|A^{(1-\theta)/2}u_t(t)\right\|^2 + \left\|A^{(1-\theta)/2}v_t(t)\right\|^2 + \left\|A^{1-\theta/2}u(t)\right\|^2 \\ + \left\|A^{1-\theta/2}v(t)\right\|^2 - \left(\int_0^t g(s)ds\right)\left\|A^{1/2}u(t)\right\|^2 \\ +2\alpha\langle A^{(1-\theta)/2}u, A^{(1-\theta)/2}v\rangle + \left(g \circ A^{1/2}u\right)(t) \end{array}\right], \tag{2.9}$$

where, for any $t \geq 0$ and for any $w \in L^2_{loc}(\mathbb{R}_+; H)$,

$$(g \circ w)(t) := \int_0^t g(t-s)\|w(t) - w(s)\|^2 ds.$$

Observe that the regularity (2.5) and condition (2.6) guarantee that all terms in E, $\mathcal{E}$ and $\mathcal{E}_*$ make sense.

Lemma 2.2. Under the conditions (**H1**), (**H2**), (2.4) and (2.6), there exists a constant $c_0 > 0$ such that, for any $t \geq 0$,

$$E(t) \geq c_0\left[\begin{array}{c} \|u_t(t)\|^2 + \|v_t(t)\|^2 + \left\|A^{1/2}u(t)\right\|^2 + \left\|A^{1/2}v(t)\right\|^2 \\ + \left(g \circ A^{\theta/2}u\right)(t) \end{array}\right], \tag{2.10}$$

$$\mathcal{E}(t) \geq c_0 \left[\begin{array}{c} \left\|A^{\theta/2}u_t(t)\right\|^2 + \left\|A^{\theta/2}v_t(t)\right\|^2 + \left\|A^{(1+\theta)/2}u(t)\right\|^2 \\ + \left\|A^{(1+\theta)/2}v(t)\right\|^2 + \left(g \circ A^{\theta}u\right)(t) \end{array} \right] \tag{2.11}$$

and

$$\mathcal{E}_*(t) \geq c_0 \left[\begin{array}{c} \left\|A^{(1-\theta)/2}u_t(t)\right\|^2 + \left\|A^{(1-\theta)/2}v_t(t)\right\|^2 + \left\|A^{1-\theta/2}u(t)\right\|^2 \\ + \left\|A^{1-\theta/2}v(t)\right\|^2 + \left(g \circ A^{1/2}u\right)(t) \end{array} \right]. \tag{2.12}$$

Proof. By using the Cauchy-Schwarz and Young's inequality and (1.2), we have, for any $\varepsilon > 0$,

$$E(t) \geq \frac{1}{2} \left[\begin{array}{c} \|u_t(t)\|^2 + \|v_t(t)\|^2 + (1 - \widetilde{g}a_1 - \alpha a_0 \varepsilon) \left\|A^{1/2}u(t)\right\|^2 \\ +(1 - \frac{\alpha a_0}{\varepsilon}) \left\|A^{1/2}v(t)\right\|^2 + \left(g \circ A^{\theta/2}u\right)(t) \end{array} \right].$$

Recalling (2.1) and (2.3), we can choose ε so that

$$\alpha a_0 < \varepsilon < \frac{1 - a_1 \widetilde{g}}{a_0 \alpha},$$

and, hence, we obtain (2.10) with $c_0 = \frac{1}{2} \min \left\{1 - \frac{\alpha a_0}{\varepsilon}, 1 - \widetilde{g}a_1 - \alpha a_0 \varepsilon\right\}$. The same aruguments can be used to establish (2.11) and (2.12) with the same constant c_0.

Lemma 2.3. Under the conditions of Lemma 2.2, the energy functionals (2.7), (2.8) and (2.9) satisfy, for any $t \geq 0$,

$$E'(t) = -\frac{1}{2} g(t) \left\|A^{\theta/2}u(t)\right\|^2 + \frac{1}{2} \left(g' \circ A^{\theta/2}u\right)(t), \tag{2.13}$$

$$\mathcal{E}'(t) = -\frac{1}{2} g(t) \left\|A^{\theta}u(t)\right\|^2 + \frac{1}{2} \left(g' \circ A^{\theta}u\right)(t) \tag{2.14}$$

and

$$\mathcal{E}_*'(t) = -\frac{1}{2} g(t) \left\|A^{1/2}u(t)\right\|^2 + \frac{1}{2} \left(g' \circ A^{1/2}u\right)(t). \tag{2.15}$$

Proof. By performing the H-inner product of u_t and v_t with $(1.1)_1$ and $(1.1)_2$, respectively, and do routine calculations as in [18], we obtain (2.13).

Similarly, the H-inner product of $A^{\theta}u_t$ and $A^{\theta}v_t$ with $(1.1)_1$ and $(1.1)_2$, respectively, we get (2.14), for sufficiently regular solutions; that is, for

$$\sigma \geq \max\{\theta, 1 - \theta\}.$$

By density arguments, the estimate (2.14) remains valid for solutions established in Theorem 2.1.

Also, the H-inner product of $A^{1-\theta}u_t$ and $A^{1-\theta}v_t$ with $(1.1)_1$ and $(1.1)_2$, respectively, then routine calculations, together with density arguments as in the case of (2.14), yield (2.15).

Next, for the sake of completeness, we report here a result from [15].

Lemma 2.4. Under the conditions of Theorem 2.1, we have, for $1 \leq p < 3/2$,

$$\xi(t)(g \circ A^{1/2}u)(t) \leq C\left[-\mathcal{E}_*'(t)\right]^{\frac{1}{2p-1}}, \quad \forall t \geq 0, \tag{2.16}$$

where the positive constant C depends continuously on $\mathcal{E}_*(0)$.

Proof. For the proof, we refer the reader to Corollary 2.1 of [15].

3 Technical Lemmas

In this section, we state and prove some lemmas necessary for the proof of our main result. We use C to denote generic positive constants which may change from step to step and depend on α, a_0, a_1, a_2, $g(0)$ and $\widetilde{g}$.

Lemma 3.1. Let (**H1**) and (**H2**) hold. Then the functional

$$\Phi(t) := \langle u(t), u_t(t)\rangle + \langle v(t), v_t(t)\rangle$$

satisfies, along the solutions of (1.1) and for any $\varepsilon > 0$,

$$\begin{aligned} \Phi'(t) \leq\ & \|u_t(t)\|^2 + \|v_t(t)\|^2 - (1-\varepsilon)\left\|A^{1/2}u(t)\right\|^2 - \left\|A^{1/2}v(t)\right\|^2 \\ & -2\alpha\langle u(t), v(t)\rangle + \frac{C}{\varepsilon}\left(g \circ A^{\theta/2}u\right)(t) + \left(\int_0^t g(s)ds\right)\left\|A^{\theta/2}u(t)\right\|^2. \end{aligned} \tag{3.1}$$

Proof. Direct differentiations, using (1.1), give

$$\begin{aligned} \Phi'(t) =\ & \|u_t(t)\|^2 + \|v_t(t)\|^2 - \left\|A^{1/2}u(t)\right\|^2 - \left\|A^{1/2}v(t)\right\|^2 \\ & + \left(\int_0^t g(s)ds\right)\left\|A^{\theta/2}u(t)\right\|^2 - 2\alpha\langle u(t), v(t)\rangle \\ & + \langle A^{\theta/2}u(t), \int_0^t g(t-s)(A^{\theta/2}u(s) - A^{\theta/2}u(t))ds\rangle. \end{aligned} \tag{3.2}$$

Now, we estimate the last term of (3.2), using the Cauchy-Schwarz and Young's inequalities, then Hölder's inequality and (1.2) as follows:

$$\begin{aligned} & \langle A^{\theta/2}u(t), \int_0^t g(t-s)(A^{\theta/2}u(s) - A^{\theta/2}u(t))ds\rangle \\ \leq\ & ||A^{\theta/2}u(t)|| \left|\left|\int_0^t g(t-s)(A^{\theta/2}u(s) - A^{\theta/2}u(t))ds\right|\right| \\ \leq\ & \frac{\varepsilon}{a_1}||A^{\theta/2}u(t)||^2 + \frac{C}{\varepsilon}\left|\left|\int_0^t g(t-s)(A^{\theta/2}u(s) - A^{\theta/2}u(t))ds\right|\right|^2 \\ \leq\ & \varepsilon||A^{1/2}u(t)||^2 + \frac{C}{\varepsilon}\left(g \circ A^{\theta/2}u\right)(t). \end{aligned} \tag{3.3}$$

Inserting (3.3) in (3.2), we obtain (3.1).

Lemma 3.2. Let (**H1**) and (**H2**) hold. Then the functional

$$\Psi(t) := -\langle u_t(t), \int_0^t g(t-s)(u(t)-u(s))ds\rangle$$

satisfies, along the solutions of (1.1) and for any $\varepsilon > 0$,

$$\begin{aligned}\Psi'(t) \;\; \leq \;\; & -\left(\int_0^t g(s)ds - \varepsilon\right) \|u_t(t)\|^2 + \varepsilon \left\|A^{1/2}u(t)\right\|^2 + \varepsilon \left\|A^{1/2}v(t)\right\|^2 \\ & +C\left(1+\frac{1}{\varepsilon}\right)\left(g \circ A^{1/2}u\right)(t) - \frac{C}{\varepsilon}\left(g' \circ A^{1/2}u\right)(t).\end{aligned} \quad (3.4)$$

Proof. Direct differentiations, using (1.1), give

$$\begin{aligned}\Psi'(t) \;\; = \;\; & -\left(\int_0^t g(s)ds\right)\|u_t(t)\|^2 - \langle u_t(t), \int_0^t g'(t-s)(u(t)-u(s))ds\rangle \\ & +\langle A^{1/2}u(t), \int_0^t g(t-s)A^{1/2}(u(t)-u(s))ds\rangle \\ & +\left\|\int_0^t g(t-s)A^{\theta/2}(u(t)-u(s))ds\right\|^2 + \alpha\langle v(t), \int_0^t g(t-s)(u(t)-u(s))ds\rangle \\ & -\left(\int_0^t g(s)ds\right)\langle A^{\theta/2}u(t), \int_0^t g(t-s)A^{\theta/2}(u(t)-u(s))ds\rangle.\end{aligned}$$

By exploiting the Cauchy-Schwarz, Young's and Hölder's inequalities and (1.2), we arrive at (3.4).

Lemma 3.3. Let (**H1**) and (**H2**) hold. Then the functional

$$\chi(t) := \langle u_{tt}(t), v_t\rangle - \langle v_{tt}(t), u_t\rangle$$

satisfies, along the solutions of (1.1) and for any $\varepsilon > 0$,

$$\chi'(t) \leq -(\alpha-\varepsilon)\|v_t(t)\|^2 + \alpha\|u_t(t)\|^2 + \frac{C}{\varepsilon}g(t)\left\|A^\theta u(t)\right\|^2 - \frac{C}{\varepsilon}\left(g' \circ A^\theta u\right)(t). \quad (3.5)$$

Proof. Differentiate the first equation of (1.1), with respect to t then perform its H-inner product with $v_t(t)$, to obtain

$$\begin{aligned}\frac{d}{dt}\langle u_{tt}(t), v_t(t)\rangle \;\; = \;\; & \langle u_{tt}(t), v_{tt}(t)\rangle - \langle Au_t(t), v_t(t)\rangle + g(t)\langle A^\theta u(t), v_t(t)\rangle \\ & -\alpha\|v_t(t)\|^2 - \langle v_t(t), \int_0^t g'(t-s)A^\theta(u(t)-u(s))ds\rangle.\end{aligned}$$

Similarly, a differentiation of the second equation of (1.1), with respect to t then performing its H-inner product with $u_t(t)$ lead to

$$-\frac{d}{dt}\langle v_{tt}(t), u_t(t)\rangle = -\langle v_{tt}(t), u_{tt}(t)\rangle + \langle Av_t(t), u_t(t)\rangle + \alpha\|u_t(t)\|^2.$$

Addition of the last two identities gives

$$\begin{aligned}\chi'(t) &= g(t)\langle A^{\theta}u(t), v_t(t)\rangle - \alpha \|v_t(t)\|^2 \\ &\quad +\alpha \|u_t(t)\|^2 - \langle v_t(t), \int_0^t g'(t-s)A^{\theta}(u(t)-u(s))ds\rangle.\end{aligned}$$

Then Young's inequality yields (3.5).

4 Decay Results

In this section we state and prove our main decay result. For this purpose we introduce the following functional:

$$\mathcal{L}(t) := N[E(t) + \mathcal{E}(t) + \mathcal{E}_*(t)] + N_1\Phi(t) + N_2\Psi(t) + \chi(t). \tag{4.1}$$

Lemma 4.1 Let (**H1**) and (**H2**) hold. Then, for any $t_0 > 0$, there exist positive constants N, N_1, N_2 and ε such that the functional $\mathcal{L}$ satisfies, along the solutions of (1.1),

$$\mathcal{L}'(t) \leq -\frac{\alpha}{2}E(t) + C(g \circ A^{1/2}u)(t), \quad \forall t \geq t_0 \tag{4.2}$$

and

$$\mathcal{L} \sim E + \mathcal{E} + \mathcal{E}_*. \tag{4.3}$$

Proof. Combining (2.13), (2.15), (2.14), (3.1), (3.4) and (3.5), and using the second inequality in (1.2), we easily get

$$\begin{aligned}\mathcal{L}'(t) &\leq -[\alpha - \varepsilon - N_1]\|v_t(t)\|^2 - \left[\left(\int_0^t g(s)ds - \varepsilon\right)N_2 - N_1 - \alpha\right]\|u_t(t)\|^2 \\ &\quad - [N_1(1-\varepsilon) - \varepsilon N_2]\left\|A^{1/2}u(t)\right\|^2 - [N_1 - \varepsilon N_2]\left\|A^{1/2}v(t)\right\|^2 \\ &\quad -2N_1\alpha\langle u(t), v(t)\rangle + N_1\left(\int_0^t g(s)ds\right)\left\|A^{\theta/2}u(t)\right\|^2 \\ &\quad -\left[\frac{N}{2} - \frac{C}{\varepsilon}\right]g(t)\left\|A^{\theta}u(t)\right\|^2 - \frac{N}{2}g(t)\left[\left\|A^{\theta/2}u(t)\right\|^2 + \left\|A^{1/2}u(t)\right\|^2\right] \\ &\quad +\left[\frac{N}{2} - \frac{N_2C}{\varepsilon}\right]\left(g' \circ A^{1/2}u\right)(t) + \left[\frac{N}{2} - \frac{C}{\varepsilon}\right]\left(g' \circ A^{\theta}u\right)(t) \\ &\quad +\frac{N}{2}\left(g' \circ A^{\theta/2}u\right)(t) + \left[\frac{N_1C}{\varepsilon} + N_2\left(C + \frac{C}{\varepsilon}\right)\right]\left(g \circ A^{1/2}u\right)(t).\end{aligned}$$

Let $t_0 > 0$ fixed and $g_0 := \int_0^{t_0} g(s)ds$, take $N_1 = \alpha/2$ and $N_2 = 2\alpha/g_0$. Hence, using (2.7) and (2.10) and noticing that $g' \leq 0$ and $g_0 \leq \int_0^t g(s)ds$ for any $t \geq t_0$, we get, for some positive constants c_1 and c_2,

$$\begin{aligned}\mathcal{L}'(t) &\leq -(\alpha - c_1\varepsilon) E(t) - \left[\frac{N}{2} - \frac{c_2}{\varepsilon}\right] g(t) \left\|A^\theta u(t)\right\|^2 \\ &\quad + \left[\frac{N}{2} - \frac{c_2}{\varepsilon}\right] \left(g' \circ A^{1/2}u\right)(t) + \left[\frac{N}{2} - \frac{c_2}{\varepsilon}\right] \left(g' \circ A^\theta u\right)(t) \\ &\quad + \left(C + \frac{C}{\varepsilon}\right) \left(g \circ A^{1/2}u\right)(t), \quad \forall t \geq t_0.\end{aligned}$$

We choose $\varepsilon = \alpha/2c_1$ and notice again that $g' \leq 0$, we obtain (4.2), for any $N \geq 2c_2/\varepsilon$.

On the other hand, using (1.2), (2.10), (2.11) and (2.12), and exploiting the Young's and the Cauchy-Schwarz inequalities, we see that

$$|\Phi| \leq CE, \quad |\Psi| \leq C(E + \mathcal{E}_*) \quad \text{and} \quad |\chi| \leq C(E + \mathcal{E} + \mathcal{E}_*),$$

therefore, we find, for some positive constant c_3 (which does not depend on N),

$$(N - c_3)(E + \mathcal{E} + \mathcal{E}_*) \leq \mathcal{L} \leq (N + c_3)(E + \mathcal{E} + \mathcal{E}_*),$$

so, by choosing $N > \max\{2c_2/\varepsilon, c_3\}$, we get also (4.3).

Theorem 4.2 Assume that (**H1**) and (**H2**) hold. Let

$$(u_0, u_1), (v_0, v_1) \in \mathcal{D}\left(A^{\sigma+1/2}\right) \times \mathcal{D}\left(A^\sigma\right)$$

such that

$$\sigma \geq \max\left\{\frac{\theta}{2}, \frac{1-\theta}{2}\right\}.$$

Then, for any $t_0 > 0$, there exists a positive constant K such that the energy E of the soution of (1.1) satisfies

$$E(t) \leq K \left[\frac{\left(E^{2p-2}(E + \mathcal{E} + \mathcal{E}_*) + \mathcal{E}_*\right)(t_0)}{\displaystyle\int_{t_0}^t \xi^{2p-1}(s)ds}\right]^{1/(2p-1)}, \quad \forall t > t_0. \tag{4.4}$$

Proof. **Case** $p = 1$. We multiply (4.2) by $\xi(t)$ and exploit (2.2) and (2.15), to get

$$\begin{aligned}\xi(t)\mathcal{L}'(t) &\leq -\frac{\alpha}{2}\xi(t)E(t) - C(g' \circ A^{1/2}u)(t) \\ &\leq -\frac{\alpha}{2}\xi(t)E(t) - C\mathcal{E}_*'(t), \quad \forall t \geq t_0.\end{aligned}$$

Recalling that $\xi' \leq 0$ and $\mathcal{L} \geq 0$, we easily see that

$$(\xi(t)\mathcal{L}(t))' \leq \xi(t)\mathcal{L}'(t) \leq -\frac{\alpha}{2}\xi(t)E(t) - C\mathcal{E}_*'(t), \quad \forall t \geq t_0$$

which yields

$$\xi(t)E(t) \leq -\frac{2}{\alpha}\left(\xi\mathcal{L} + C\mathcal{E}_*\right)'(t), \quad \forall t \geq t_0.$$

Simple integration over (t_0, t) leads to

$$\begin{aligned} E(t)\int_{t_0}^{t}\xi(s)ds &\leq \int_{t_0}^{t}\xi(s)E(s)ds \leq -\frac{2}{\alpha}\int_{t_0}^{t}\left(\xi\mathcal{L} + C\mathcal{E}_*\right)'(s)\,ds \\ &\leq \frac{2}{\alpha}\left(\xi\mathcal{L} + C\mathcal{E}_*\right)(t_0), \quad \forall t \geq t_0. \end{aligned}$$

Using (4.3), estimate (4.4), with $p = 1$, is established.

Case $1 < p < 3/2$. We multiply (4.2) by $\xi(t)$ and use (2.2), (2.15) and (2.16) to get

$$\begin{aligned} \xi(t)\mathcal{L}'(t) &\leq -\frac{\alpha}{2}\xi(t)E(t) + C\xi(t)(g \circ A^{1/2}u)(t) \\ &\leq -\frac{\alpha}{2}\xi(t)E(t) + C\left(-\mathcal{E}_*'(t)\right)^{1/(2p-1)}, \quad \forall t \geq t_0. \end{aligned} \tag{4.5}$$

We then multiply this last inequality by $\xi^{\beta}(t)E^{\beta}(t)$, for $\beta > 0$ to be chosen properly. Thus, (4.5) becomes

$$\begin{aligned} E^{\beta}(t)\xi^{\beta+1}(t)\mathcal{L}'(t) &\leq -\frac{\alpha}{2}\xi^{\beta+1}(t)E^{\beta+1}(t) \\ &\quad +C\xi^{\beta}(t)E^{\beta}(t)\left(-\mathcal{E}_*'(t)\right)^{1/(2p-1)}, \quad \forall t \geq t_0. \end{aligned} \tag{4.6}$$

Applying Young's inequality for the last term of (4.6) we obtain, for any $\varepsilon > 0$,

$$\begin{aligned} E^{\beta}(t)\xi^{\beta+1}(t)\mathcal{L}'(t) &\leq -\frac{\alpha}{2}\xi^{\beta+1}(t)E^{\beta+1}(t) + \varepsilon\xi^{\beta+1}(t)E^{\beta+1}(t) \\ &\quad +C_{\varepsilon}\left(-\mathcal{E}_*'(t)\right)^{(\beta+1)/(2p-1)}, \quad \forall t \geq t_0. \end{aligned}$$

By choosing $\varepsilon = \alpha/4$ and taking $\beta = 2p - 2$, we arrive at

$$E^{2p-2}(t)\xi^{2p-1}(t)\mathcal{L}'(t) \leq -\frac{\alpha}{4}\xi^{2p-1}(t)E^{2p-1}(t) - C\mathcal{E}_*'(t), \quad \forall t \geq t_0.$$

This, in turns, yields

$$\xi^{2p-1}(t)E^{2p-1}(t) \leq -\frac{4}{\alpha}\left(E^{2p-2}\xi^{2p-1}\mathcal{L} + C\mathcal{E}_*\right)'(t), \quad \forall t \geq t_0.$$

Again, a simple integration over (t_0, t) and use of (4.3) lead to

$$\begin{aligned} E^{2p-1}(t)\int_{t_0}^{t}\xi^{2p-1}(s)ds &\leq \int_{t_0}^{t}\xi^{2p-1}(s)E^{2p-1}(s)ds \\ &\leq -\frac{4}{\alpha}\int_{t_0}^{t}\left(E^{2p-2}\xi^{2p-1}\mathcal{L} + C\mathcal{E}_*\right)'(s)ds \\ &\leq \frac{4}{\alpha}\left(E^{2p-2}\xi^{2p-1}\mathcal{L} + C\mathcal{E}_*\right)(t_0) \\ &\leq K^{2p-1}\left(E^{2p-2}(E + \mathcal{E} + \mathcal{E}_*) + \mathcal{E}_*\right)(t_0), \end{aligned}$$

for some $K > 0$. Therefore, (4.4) follows for $1 < p < 3/2$.

5 General comments and examples

1. If $\xi \equiv 1$, then we have

$$E(t) \le K \left[\frac{\left(E^{2p-2}(E + \mathcal{E} + \mathcal{E}_*) + \mathcal{E}_*\right)(t_0)}{t - t_0} \right]^{1/(2p-1)}, \quad \forall t > t_0. \tag{5.1}$$

If, in addition, $p = 1$ (which implies that g converges exponentially to zero), we get from (5.1) the decay rate t^{-1} obtained in [10] for (1.9). The following examples illustrate our result.

Example 5.1. Let $g(t) = \frac{a}{(1+t)^\nu}$, $\nu > 2$, where $a > 0$ is a constant so that

$$\int_0^\infty g(t)dt < \frac{1}{a_1}. \tag{5.2}$$

We have

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^\nu}\right)^{\frac{\nu+1}{\nu}} = -bg^p(t), \quad p = \frac{\nu+1}{\nu} < \frac{3}{2}, \quad b = \nu a^{-1/\nu}.$$

Therefore (4.4), with $\xi(t) = b$, yields

$$E(t) \le \frac{C}{(t-t_0)^{\frac{1}{2p-1}}} = \frac{C}{(t-t_0)^{\frac{\nu}{\nu+2}}}, \quad \forall t > t_0.$$

Example 5.2. Let $g(t) = ae^{-(1+t)^\nu}$, $0 < \nu \le 1$, where $0 < a < 1$ is chosen so that (5.2) holds. Then

$$g'(t) = -a\nu(1+t)^{\nu-1}e^{-(1+t)^\nu} = -\xi(t)g(t),$$

where $\xi(t) = \nu(1+t)^{\nu-1}$, so ξ is a decreasing function, $\xi(0) > 0$ and (2.2) holds with $p = 1$. Therefore we can use (4.4) to deduce

$$E(t) \le \frac{C}{(1+t)^\nu - (1+t_0)^\nu}, \quad \forall t > t_0.$$

Notice that, for $\nu = 1$, we again get the decay rate obtained in [10] for (1.9).

2. Our results hold true if $-\frac{\sqrt{1-a_1\tilde{g}}}{a_0} < \alpha < 0$. Indeed, it is enought to replace v by $w := -v$ to get (1.1) with $(u, w, -\alpha, u_0, u_1, -v_0, -v_1)$ instead of $(u, v, \alpha, u_0, u_1, v_0, v_1)$.

3. Similar results to ours can be proved for the more general system than (1.1), where the operator A in $(1.1)_2$ and αId are, respectively, replaced by operators $\tilde{A}$ and $\tilde{B}$ satisfying similar conditions to those considered in [9].

4. Our abstract system (1.1) includes several practical applications such as, for example, coupled wave-wave equations (i.e. $A = -a\Delta$) and coupled plate-plate equations (i.e. $A = a\Delta^2$), where a is a positive constant.

6 Numerical Tests

In the following section, we examine the computational behavior of the system (1.1) using the finite volume method by discretizing the system on the space-time domain $[0, 1] \times [0, 60]$ using second order finite difference method. We implement Lax-Wendroff method. For a similar construction, we refer to [16, 8]. The choice of the function g will be based on examples 1. and 2. We implement these two cases by choosing the exponential function $g(t) = e^{-10(1+t)}$ and the polynomial function $g(t) = \frac{1}{10(2+t)^{\frac{5}{2}}}$. At any grid point (x_i, t_j) for $i = 1, \dots, n$ and $j = 1, \dots, m$, the temporal evolution of the waves u and v are approximated using the first order forward finite difference method:

$$u_t \approx \frac{u(x_i, t_j + \Delta t_j) - u(x_i, t_j)}{\Delta t_j} \quad \text{and} \quad v_t \approx \frac{v(x_i, t_j + \Delta t_j) - v(x_i, t_j)}{\Delta t_j}. \tag{6.1}$$

At any grid point (x_i, t_j) for $i = 1, \dots, n$ and $j = 1, \dots, m$, the spacial evolution of the waves u and v are approximated using the second order finite difference method:

$$u_x \approx \frac{u(x_i + \Delta x_i, t_j) - u(x_i, t_j)}{(\Delta x_i)^2} \quad \text{and} \quad v_x \approx \frac{v(x_i + \Delta x_i, t_j) - v(x_i, t_j)}{(\Delta x_i)^2}. \tag{6.2}$$

The temporal Laplacian operators are approximated using the second order centred method

$$u_{tt} \approx \frac{u(x_i, t_j + \Delta t_j) - 2u(x_i, t_j) + u(x_i, t_j - \Delta t_j)}{(\Delta t_j)^2} \tag{6.3}$$

and

$$v_{tt} \approx \frac{v(x_i, t_j + \Delta t_j) - 2v(x_i, t_j) + v(x_i, t_j - \Delta t_j)}{(\Delta t_j)^2}. \tag{6.4}$$

Similarly, we approximate the spacial using the following second order finite difference method:

$$\Delta u = u_{xx} \approx \frac{u(x_i + \Delta x_i, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x_i, t_j)}{(\Delta x_i)^2} \tag{6.5}$$

and

$$\Delta v = v_{xx} \approx \frac{v(x_i + \Delta x_i, t_j) - 2v(x_i, t_j) + v(x_i - \Delta x_i, t_j)}{(\Delta x_i)^2}, \tag{6.6}$$

and for the bi-laplacian, we use the following recurrence.

$$\Delta^2 u = \Delta(\Delta u) \quad \text{and} \quad \Delta^2 v = \Delta(\Delta v). \tag{6.7}$$

For the memory term

$$\int_0^t g(t-s) A^\theta u(s) ds, \tag{6.8}$$

we use the accurate Simpson's Rule over each time interval $[t_j, t_j + \Delta t_j]$. First, we use the following notation for the sub-inegrals

$$I_j = \int_{t_j}^{t_j+\Delta t_j} g(t_j + \Delta t_j - s) A^\theta(u(s))\, ds.$$

The Simpson's approximation of I_j is given by,

$$I_j \approx \frac{\Delta t_j}{6}\left[g(\Delta t_j) A^\theta(u(t_j)) + 4g\left(\frac{\Delta t_j}{2}\right) A^\theta \left(u\left(t_j + \frac{\Delta t_j}{2}\right)\right) + g(0) A^\theta(u(t_j + \Delta t_j))\right].$$

Thus, we obtain from the cumulative sum the numerical approximation of (6.8):

$$\int_0^{t_n} g(t-s) A^\theta u(s) ds \approx \sum_{j=0}^{j=n} I_j. \tag{6.9}$$

Finally, for updating the numerical solutions of the waves, we combine (6.1)-(6.9) in the conservative Lax-Wendroff Schemes,

$$\begin{aligned} u(x_i, t_j + \Delta t_j) &= \frac{C_u}{2}(1 + C_u) u(x_i, t_j - \Delta t_j) + (1 - C_u^2) u(x_i, t_j) + \frac{C_u}{2}(C_u - 1) u(x_i + \Delta x_i, t_j) \\ &= \alpha_1 u(x_i, t_j - \Delta t_j) + \alpha_1 u(x_i, t_j) + \alpha_1 u(x_i + \Delta x_i, t_j) \end{aligned}$$

and

$$\begin{aligned} v(x_i, t_j + \Delta t_j) &= \frac{C_v}{2}(1 + C_v) v(x_i, t_j - \Delta t_j) + (1 - C_v^2) v(x_i, t_j) + \frac{C_v}{2}(C_v - 1) u(x_i + \Delta x_i, t_j) \\ &= \beta_1 v(x_i, t_j - \Delta t_j) + \beta_1 v(x_i, t_j) + \beta_1 u(x_i + \Delta x_i, t_j), \end{aligned}$$

where C is the so-called diffusion and dispersion coefficient, which results from the numerical discretization of the differential system. Moreover, it should be stressed that at each step the convex combination of the parameters above satisfy the following equality:

$$\sum_{s=1}^{3} \alpha_s = 1 = \sum_{s=1}^{3} \beta_s. \tag{6.10}$$

Now, we present six tests for our numerical simulations:

TEST 1. In the first test, we examine the case $A = \Delta^2$ for $g(t) = \dfrac{1}{10(1+t)^{\frac{5}{2}}}$.

TEST 2. In the second test, we examine the case $A = \Delta^2$ for $g(t) = e^{-10(1+t)}$.

TEST 3. In the third test, we examine the case $A = -\Delta$ for $g(t) = \dfrac{1}{(1+t)^{\frac{5}{2}}}$.

TEST 4. In the fourth test, we examine the case $A = -\Delta$ for $g(t) = e^{-10(1+t)}$.

TEST 5. In the fifth test, we examine the case $A = Id$ for $g(t) = \dfrac{1}{(1+t)^{\frac{5}{2}}}$.

TEST 6. In the sixth test, we examine the case $A = Id$ for $g(t) = e^{-10(1+t)}$.

In order to ensure the stability of the numerical scheme, we use a temporal and spatial steps satisfying the Courant-Friedrichs-Lewy (CFL) inequality $\Delta t = 0.0075 < \Delta x = 0.02$, where Δt represents the time step and Δx is the spatial step. The spatial interval $[0, 1]$ is subdivided into 200 subintervals and the temporal interval $[0, 1]$ is deduced from the stability condition above. We run our code for $8000 = 60/\Delta t$ time steps taking the following initial conditions:

$$\begin{aligned} u(x,0) &= u_t(x,0) = \sin(\pi x);\ \text{in } [0,1]. \\ v(x,0) &= v_t(x,0) = x(1-x);\ \text{in } [0,1]. \end{aligned} \tag{6.11}$$

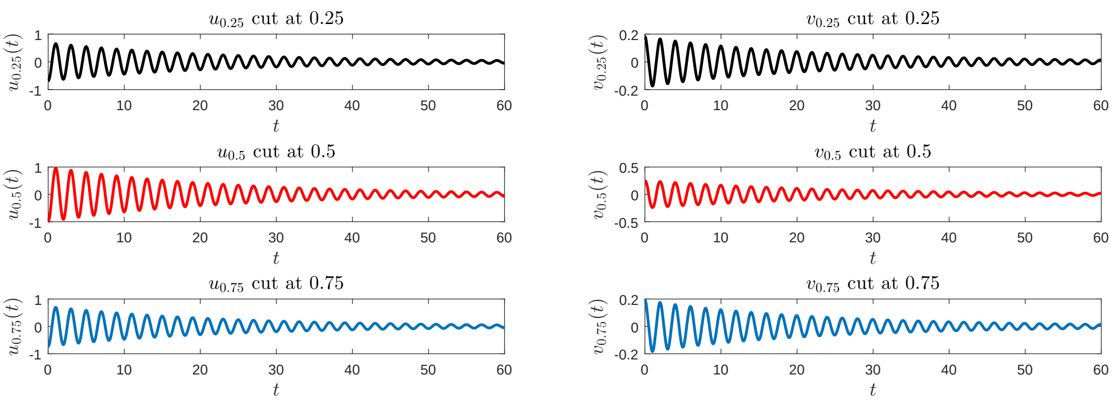

Figure 1: Damping behavior of the waves u and v for Test 1.

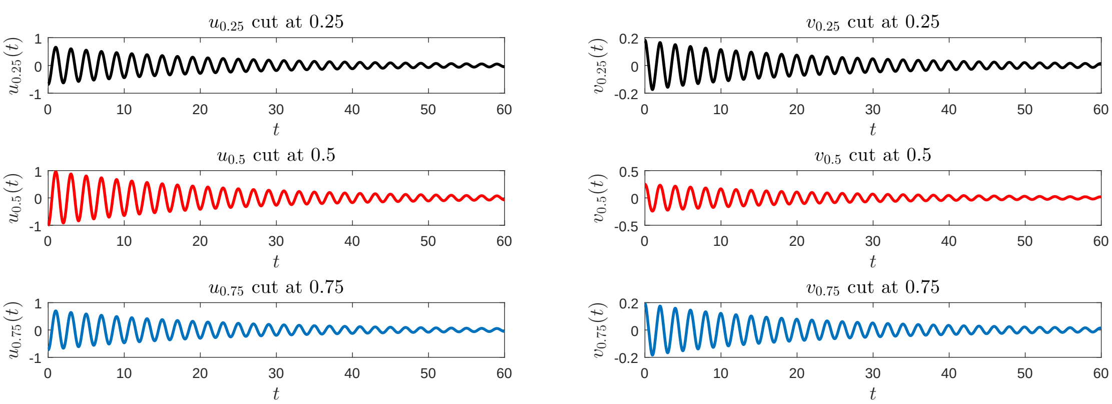

Figure 2: Damping behavior of the waves u and v for Test 2:

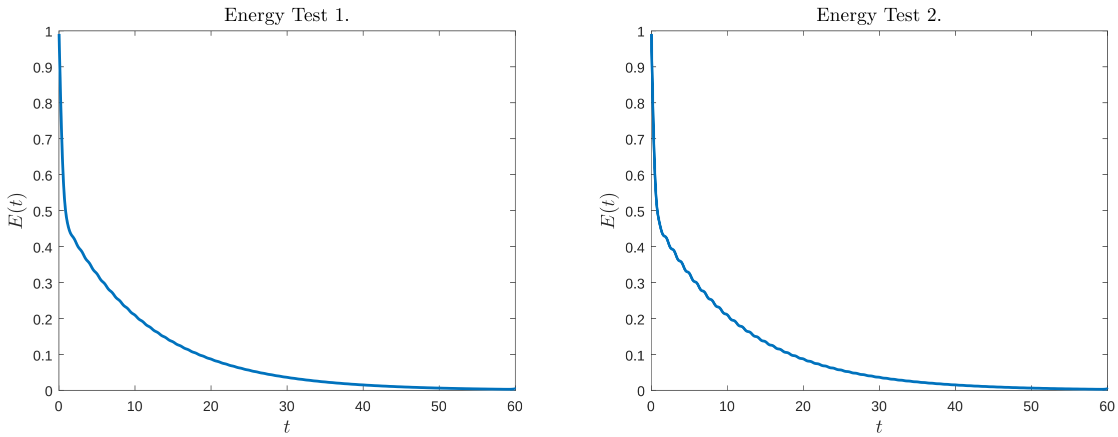

Figure 3: Energy functions for Tests 1. and 2.

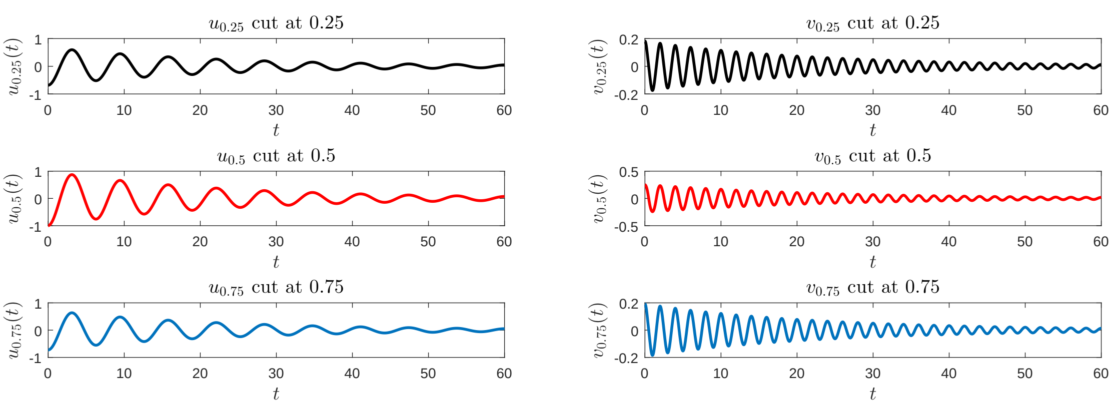

Figure 4: Damping behavior of the waves u and v for Test 3:

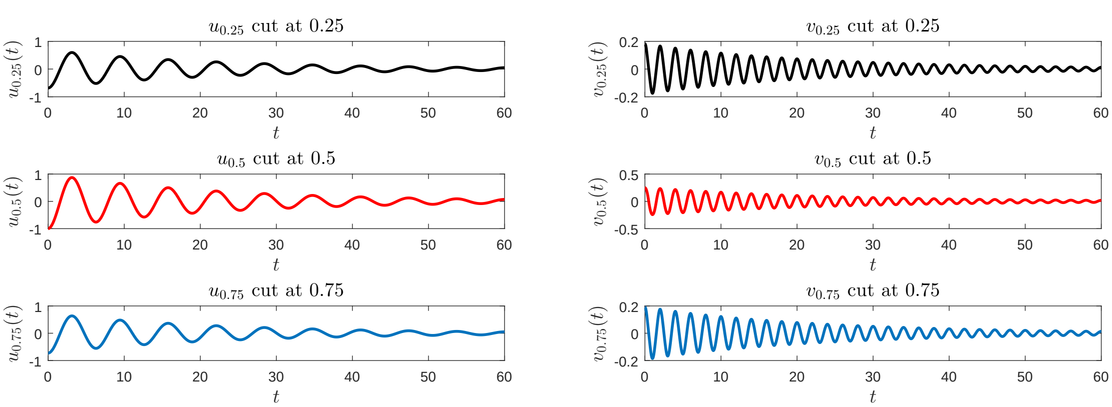

Figure 5: Damping behavior of the waves u and v for Test 4:

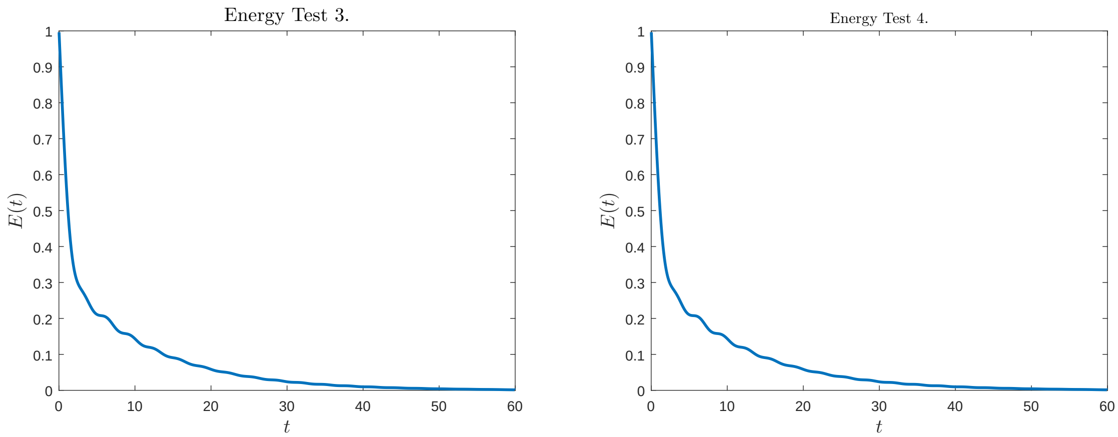

Figure 6: Energy functions for Tests 3. and4.

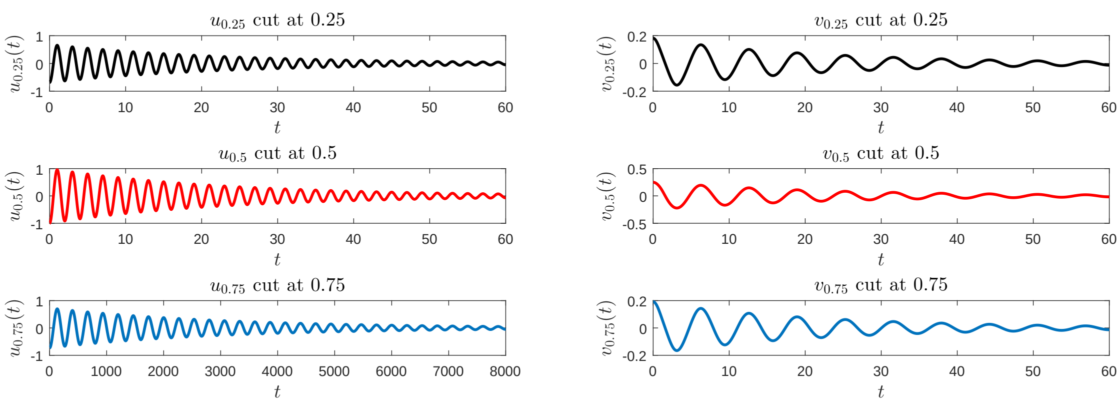

Figure 7: Damping behavior of the waves u and v for Test 5.

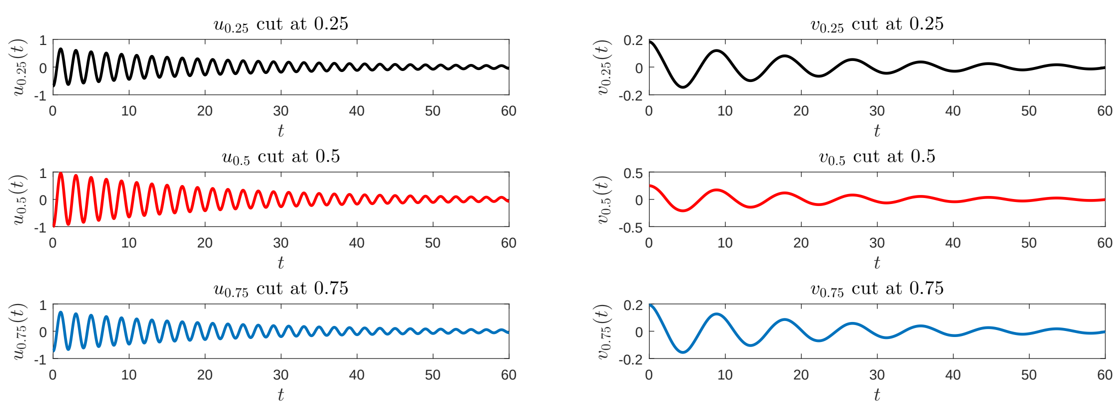

Figure 8: Damping behavior of the waves u and v for Test 6.

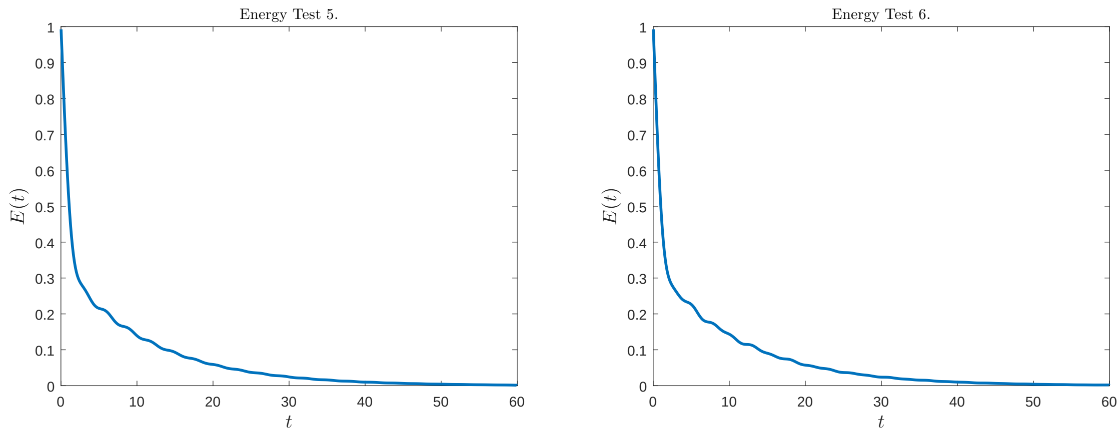

Figure 9: Energy functions for Tests 5. and 6.

In Figures 1, 2, 4, 5, 7 and 8, we plot the damping behavior of the two waves u and v for the cross sections $x = 0.25$, $x = 0.5$ and $x = 1$. In Figures 3, 6 and 9, we plot and compare the decay behavior of resulting energies. As a final conclusion, it should be stressed that, we performed several tests for two types of decaying relaxation functions; namely the exponential decay and the polynomial decay types. We noticed that the energy, in each test, decays at least in a polynomial rate. this comes in agreement with our theoretical results.

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7 Conclusions

In this work, we discussed the longtime behavior of solutions of a viscoelatic system of two linear wave or two linear plate equations, where only one equation is controlled by the presence of a viscoelastic term. We showed that the decay of energy is weaker than that of the relaxation function. Our decay result is of a general type and gives as particular case, for instance, the results of [10], when $f = 0$. It would be very interesting to establish the same result for the nonlinear system treated in [10]. In addition, we gave several illustrative numerical examples to support our theoretical findings. These numerical graphs came in agreement with the theoretical results, however, they did not show any significant difference for different values of fractional power θ.

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This work does not have any conflicts of interest.

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