Journal of Applied Analysis and Computation
Volume *, Number *, * *, 1–19
Website:http://www.jaac-online.com
DOI:10.11948/***

PROPERTIES OF A NEW GENERALIZED CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

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Abstract We investigate properties of a new fractional derivative recently introduced in the literature, which aims at generalizing the well-known Caputo-Fabrizio operator. We study the null space of the generalized derivative, the associated fractional integral operator, the null space of this integral, the validity of a fundamental theorem of calculus, the equivalence of integral problems with ordinary differential equations, the existence and uniqueness of solution for integral problems, and the form the nonsingular kernel should have to ensure consistency with the fractional order. A complete example with power input function is analyzed, which gives rise to a novel non-elementary solution and new dynamics in terms of the famous Lambert function.

Keywords Fractional calculus, fractional derivative, fractional integral operator, nonsingular kernel

MSC(2010) 34A08, 26A33, 34A05

1. Introduction

There exist many fractional operators of real or even complex order, that aim at extending the traditional derivative and thus give more flexibility when modeling dynamic processes [1,7,20,30,37]. By modifying the singular kernel of the standard Caputo fractional derivative, a new fractional operator with nonsingular kernel (of exponential type) was defined by Caputo and Fabrizio in [10]. There are several contributions in the literature on finding explicit solutions to Caputo-Fabrizio models; for example, see [19,27] for fractional logistic growth, [6] for quadratic-power input function, and [15] for other equations such as Riccati. There are also works on modeling and numerical simulation, consult for instance [2,8,18,24,28]. Analogous articles exist for other fractional derivatives, such as [5,17,26] for theory on explicit solutions and [3,4,25] for applications.

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[*]The research of Juan J. Nieto was partially supported by the Agencia Estatal de Investigación (AEI) of Spain Grant PID2020-113275GB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe", by the "European Union" and Xunta de Galicia, grant ED431C 2023/12 for Competitive Reference Research Groups (2023–2026).
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The generalization of previous definitions and results is at the core of mathematics. In this paper, we investigate a general Caputo-Fabrizio operator recently proposed in the literature [29], which extends the class of nonsingular kernels with an abstraction of the multiplicative property of the exponential function. Our approach follows the methodology from [21, 22], which examined the Losada-Nieto integral operator associated to the standard Caputo-Fabrizio derivative. For the new generalized Caputo-Fabrizio operator, we find for the first time the corresponding integral operator and investigate its properties, such as null space, a fundamental theorem for this calculus, and the relationship with ordinary differential equations. The null space is formed by the functions that are vanished by the operators, and this is connected with the fundamental theorem of calculus, which gives the compositions of the derivative and the integral operators and relates differential and Volterra integral problems. We fully characterize the form of the new kernel function, and a related integral equation is solved as an example to extend [6].

The article is organized as follows. In Section 2, we present the main concepts and definitions regarding the generalized Caputo-Fabrizio fractional operator. Then, in Section 3, we introduce the corresponding integral operator. We study the null space and prove a fundamental theorem of calculus. The equivalence between generalized Caputo-Fabrizio integral equations and suitable ordinary differential equations is established, and existence and uniqueness results are shown. A study on the kernel function is conducted. In Section 4, a concrete example that exhibits interesting dynamics is given, to illustrate the results. Finally, Section 5 is devoted to final remarks (summary and extensions of the paper).

2. The generalized Caputo-Fabrizio fractional derivative

2.1. Definitions

Fix a real number $0 < \alpha < 1$. We consider *fractional derivatives* of functions, namely operators $\mathcal{D}^\alpha$ with non-local behavior such that $\mathcal{D}^{1^-}$ is the ordinary derivative.

The well-known Caputo-Fabrizio derivative is recalled in the following definition [10]. The kernel suggested is nonsingular, as opposed to the Caputo and Riemann-Liouville fractional derivatives, for example [20]. Hence it does not blow up at any point, which simplifies the treatment and mathematical analysis of the operator.

Definition 2.1. The Caputo-Fabrizio derivative of the continuously differentiable function $\phi : [0, b] \to \mathbb{R}$ is

$$ {}^{CF}\mathcal{D}^\alpha \phi(x) = \frac{1}{1-\alpha} \int_0^x \mathrm{e}^{-\frac{\alpha}{1-\alpha}(x-\xi)} \phi'(\xi) \mathrm{d}\xi, \tag{2.1} $$

for $x \in [0, b]$, where ϕ' denotes the ordinary derivative.

In [29], the authors recently introduced a generalization of the Caputo-Fabrizio derivative, by expanding the class of kernels with absence of singularity. Let $\kappa_\alpha : [0, b] \to \mathbb{R}$ be a positive continuously differentiable function.

Definition 2.2. The generalized Caputo-Fabrizio derivative of the continuously differentiable function $\phi : [0, b] \to \mathbb{R}$ with respect to κ_α is

$$ {}^{G}\mathcal{D}^{\alpha}\phi(x) = \frac{1}{1-\alpha}\int_0^x \frac{\kappa_\alpha(\xi)}{\kappa_\alpha(x)}\phi'(\xi)\mathrm{d}\xi, \tag{2.2} $$

for $x \in [0, b]$.

In the particular case

$$ \kappa_\alpha(x) = \mathrm{e}^{\frac{\alpha}{1-\alpha}x} \tag{2.3} $$

in (2.2), one has ${}^{G}\mathcal{D}^{\alpha} = {}^{CF}\mathcal{D}^{\alpha}$. The authors in [29] did not pose more conditions on the kernel

$$ {}^{G}\mathcal{K}_\alpha(x, \xi) = \frac{\kappa_\alpha(\xi)}{\kappa_\alpha(x)} \tag{2.4} $$

in (2.2). Among other results, we will characterize the generalized Caputo-Fabrizio fractional derivative by imposing specific constraints on the nonsingular kernel (2.4).

2.2. Null space of the generalized Caputo-Fabrizio operator

Consider the equation

$$ {}^{G}\mathcal{D}^{\alpha}\phi(x) = 0. $$

We will see that ϕ is constant. That is, the null space or kernel of the Caputo-Fabrizio fractional derivative operator is the one-dimensional subspace of constant functions. The same occurs for the ordinary and Caputo derivatives. The Riemann-Liouville derivative, by contrast, possesses a null space generated by $t^{\alpha-1} \neq$ constant, which is often seen as a disadvantage.

Theorem 2.1. *The null space of the generalized Caputo-Fabrizio operator is*

$$ \mathrm{Ker}({}^{G}\mathcal{D}^{\alpha}) = \{c : c \in \mathbb{R}\}. $$

Proof. If $\phi \in \mathrm{Ker}({}^{G}\mathcal{D}^{\alpha})$ and ${}^{G}\mathcal{D}^{\alpha}\phi = 0$, then

$$ \int_0^x \kappa_\alpha(\xi)\phi'(\xi)\mathrm{d}\xi = 0. $$

If we differentiate,

$$ \kappa_\alpha(x)\phi'(x) = 0. $$

Since κ_α is positive,

$$ \phi'(x) = 0 $$

and ϕ is constant, as wanted. □

3. Fractional integral operator associated to the generalized Caputo-Fabrizio operator

3.1. Definition

We start with the specific definition. Later we will prove that it is indeed valid, similarly to [21].

Definition 3.1. The integral operator associated to the generalized Caputo-Fabrizio derivative is

$$ {}^{G}\mathcal{I}^{\alpha}\phi(x) = (1-\alpha)[\phi(x)-\phi(0)] + (1-\alpha)\int_0^x \frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)}\phi(\xi)\mathrm{d}\xi, \tag{3.1} $$

where $\phi : [0,b] \to \mathbb{R}$ is a continuous function and $x \in [0,b]$.

In (3.1), we have a combination of an increment of ϕ,

$$ \Delta\phi(x) = \phi(x) - \phi(0), $$

which is related to the mean value of ϕ',

$$ \int_0^x \phi'(\xi)\mathrm{d}\xi, $$

and a weighted integral of ϕ,

$$ \int_0^x \frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)}\phi(\xi)\mathrm{d}\xi = \int_0^x \phi(\xi)\mathrm{d}(\log\kappa_\alpha(\xi)). $$

For the standard Caputo-Fabrizio integral (Losada-Nieto integral), the combination is convex and the weight for the resulting integral is 1.

3.2. Null space of the associated integral operator

For the classical Riemann integral, the kernel is trivial. The same happens with the Riemann-Liouville fractional integral, with null space $\{0\}$. We will see that this is not the case for the generalized Caputo-Fabrizio operator, as it occurs with the standard Caputo-Fabrizio operator [22].

Theorem 3.1. *The null space of the integral operator associated to the generalized Caputo-Fabrizio derivative is*

$$ \mathrm{Ker}({}^{G}\mathcal{I}^{\alpha}) = \langle \frac{1}{\kappa_\alpha} \rangle. \tag{3.2} $$

Proof. If $\phi \in \mathrm{Ker}({}^{G}\mathcal{I}^{\alpha})$, then we take derivatives in (3.1):

$$ (1-\alpha)\phi'(x) + (1-\alpha)\frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)}\phi(x) = 0. \tag{3.3} $$

We obtain the differential equation

$$ \phi'(x) = -\frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)}\phi(x), \tag{3.4} $$

whose solution is, by the method of separation of variables,

$$ \phi(x) = C\mathrm{e}^{-\int \frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)}\mathrm{d}x} = C\mathrm{e}^{-\log\kappa_\alpha(x)} = \frac{C}{\kappa_\alpha(x)}, $$

for $C \in \mathbb{R}$.

Reciprocally, if

$$ \phi(x) = \frac{C}{\kappa_\alpha(x)}, $$

then (3.4) holds by differentiation. That is, (3.3). By integration on $[0,x]$ and Barrow's rule, one concludes that ${}^{G}\mathcal{I}^{\alpha}\phi = 0$. □

3.3. A fundamental theorem of calculus

We present and prove the relationship between the derivative operator (2.2) and the integral operator (3.1). This is a necessary step when dealing with a new fractional derivative, to generalize standard calculus and understand and characterize the associated differintegral equations. The first identity, (3.5), is analogous to Barrow's rule in standard calculus. However, the second identity, (3.6), contains an extra term; this fact is related to the non-triviality of the null space of ${}^{G}\mathcal{I}^{\alpha}$, see (3.2). This aspect of the theory that the derivative of the integral is not the same function has to be considered in applications of differential equations, and it is applicable for the broad class of fractional operators with nonsingular kernels. For Caputo and Riemann-Liouville operators, of singular kernels, the fundamental theorem of calculus holds.

Theorem 3.2. *The relations*

$$
{}^{G}\mathcal{I}^{\alpha} \circ {}^{G}\mathcal{D}^{\alpha}\phi(x) = \phi(x) - \phi(0) \tag{3.5}
$$

and

$$
{}^{G}\mathcal{D}^{\alpha} \circ {}^{G}\mathcal{I}^{\alpha}\phi(x) = \phi(x) - \frac{\kappa_\alpha(0)}{\kappa_\alpha(x)}\phi(0) \tag{3.6}
$$

hold.

Proof. For (3.5), let

$$
u(x) = {}^{G}\mathcal{D}^{\alpha}\phi(x). \tag{3.7}
$$

By definition (2.2),

$$
u(x) = \frac{1}{1-\alpha}\int_0^x \frac{\kappa_\alpha(\xi)}{\kappa_\alpha(x)}\phi'(\xi)\mathrm{d}\xi.
$$

Rewrite the expression as

$$
\int_0^x \kappa_\alpha(\xi)\phi'(\xi)\mathrm{d}\xi = (1-\alpha)\kappa_\alpha(x)u(x).
$$

Now differentiate:

$$
\kappa_\alpha(x)\phi'(x) = (1-\alpha)\kappa'_\alpha(x)u(x) + (1-\alpha)\kappa_\alpha(x)u'(x).
$$

That is,

$$
\phi'(x) = (1-\alpha)\frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)}u(x) + (1-\alpha)u'(x).
$$

Now integrate, with Barrow's rule, while recalling (3.1) and (3.7):

$$
\begin{aligned}
\phi(x) - \phi(0) &= (1-\alpha)[u(x) - u(0)] + (1-\alpha)\int_0^x \frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)}u(\xi)\mathrm{d}\xi \\
&= {}^{G}\mathcal{I}^{\alpha}u(x) = {}^{G}\mathcal{I}^{\alpha} \circ {}^{G}\mathcal{D}^{\alpha}\phi(x).
\end{aligned}
$$

For (3.6), first notice that

$$
\frac{\mathrm{d}}{\mathrm{d}x}{}^{G}\mathcal{I}^{\alpha}\phi(x) = (1-\alpha)\phi'(x) + (1-\alpha)\frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)}\phi(x).
$$

Then, by (2.2),

$$\begin{aligned}{}^{G}\mathcal{D}^{\alpha}\circ{}^{G}\mathcal{I}^{\alpha}\phi(x)&=\frac{1}{1-\alpha}\int_0^x\frac{\kappa_\alpha(\xi)}{\kappa_\alpha(x)}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}{}^{G}\mathcal{I}^{\alpha}\phi(\xi)\right]\mathrm{d}\xi\\&=\frac{1}{1-\alpha}\int_0^x\frac{\kappa_\alpha(\xi)}{\kappa_\alpha(x)}\left[(1-\alpha)\phi'(\xi)+(1-\alpha)\frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)}\phi(\xi)\right]\mathrm{d}\xi\\&=\frac{1}{\kappa_\alpha(x)}\int_0^x\kappa_\alpha(\xi)\phi'(\xi)\mathrm{d}\xi+\frac{1}{\kappa_\alpha(x)}\int_0^x\kappa'_\alpha(\xi)\phi(\xi)\mathrm{d}\xi\\&=\phi(x)-\frac{\kappa_\alpha(0)}{\kappa_\alpha(x)}\phi(0),\end{aligned}$$

where in the last equality we applied integration by parts in the term corresponding to ϕ'. □

3.4. Equivalence with ordinary differential equations

Differential and integral equations model dynamic states. This is why the study of these problems belongs to the theoretical framework of fractional calculus. It is necessary to establish results that ensure that fractional models are well-posed, in terms of existence and uniqueness of solution. In this sense, we establish a connection between Caputo-Fabrizio equations and ordinary differential equations. Since ${}^{G}\mathcal{D}^{\alpha}\phi(0)=0$ is always true, which may certainly limit the selection of vector fields in contrast to singular kernels (this fact is also related to (3.6)), we will work with integral equations based on (3.1), i.e., Volterra integral equations [33, 34]. Such a procedure is analogous to the use of the Losada-Nieto fractional integral operator when dealing with the standard Caputo-Fabrizio derivative, see [21, 22]. For examples of application, the fractional logistic equation has been analyzed in terms of the Losada-Nieto operator, which yields an implicit solution [19, 27]. Other fractional equations, such as Riccati, have been studied [15, 36].

The following theorem was partially derived in the recent contribution [29]. However, in our proof, we rely on the new integral operator (3.1). The theorem is a distinctive property of Caputo-Fabrizio derivatives, by the multiplicative property of the kernel that is not shared by other operators.

Theorem 3.3. *Based on* (3.1), *consider the functional equation*

$$y(x)=y_0+{}^{G}\mathcal{I}^{\alpha}f(x,y(x)),\tag{3.8}$$

where $y_0\in\mathbb{R}$ *and* f *is a continuously differentiable function around* $(0,y_0)$, *with partial derivatives* f_x *and* f_y, *such that*

$$1-(1-\alpha)f_y(0,y_0)\neq 0.\tag{3.9}$$

Then (3.8) *is equivalent to the ordinary differential equation*

$$y'=\frac{(1-\alpha)\left[f_x(x,y)+\frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)}f(x,y)\right]}{1-(1-\alpha)f_y(x,y)},\tag{3.10}$$

with initial state $y(0)=y_0$. *Under* (3.9), *one can guarantee existence and uniqueness of solution when* f *is twice continuously differentiable (standard Picard-Lindelöf theorem).*

If $f(x,y)=f(y)$ is independent of x ($f_x=0$), then

$$y' = \frac{(1-\alpha)\frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)} f(y)}{1-(1-\alpha)f'(y)}. \tag{3.11}$$

Moreover, if $f(y_0)\neq 0$, then the implicit solution is given by

$$\frac{\kappa_\alpha(x)|f(y)|}{\kappa_\alpha(0)|f(y_0)|} \mathrm{e}^{\frac{F(y)-F(y_0)}{1-\alpha}} = 1, \tag{3.12}$$

where

$$F(\xi) = \int \frac{1}{f(\xi)} \mathrm{d}\xi$$

is a specific primitive.

Proof. By differentiation in (3.8), we have that (3.8) is equivalent to

$$y' = (1-\alpha)\left[f_x(x,y) + f_y(x,y)y' + \frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)} f(x,y)\right],$$

that is, (3.10). In particular, if $f_x = 0$, then (3.11) is obtained. With the method of separation of variables, the following steps are performed:

$$\frac{1-(1-\alpha)f'(y)}{f(y)} y' = (1-\alpha)\frac{\kappa'_\alpha(x)}{\kappa_\alpha(x)},$$

$$F(y) - (1-\alpha)\log|f(y)| = (1-\alpha)\log\kappa_\alpha(x) + C,$$

$$C = F(y_0) - (1-\alpha)\log|f(y_0)| - (1-\alpha)\log\kappa_\alpha(0),$$

and

$$\frac{F(y)-F(y_0)}{1-\alpha} - (\log|f(y)| - \log|f(y_0)|) = \log\kappa_\alpha(x) - \log\kappa_\alpha(0).$$

By applying exponential, one gets the solution (3.12).

We notice that the used condition $f(y_0)\neq 0$ ensures that $f(y)\neq 0$ on a neighborhood of y_0, hence (3.12) and F make sense in there. In general, the interval of definition of the solution y is contained in $\{t\geq 0 : f(y(t))\neq 0\}$. By continuity, there is a $t_1>0$ such that $[0,t_1)\subseteq\{t\geq 0 : f(y(t))\neq 0\}$. □

Remark 3.1. Theorem 3.2 also holds with the integral operator

$$\phi(x) \mapsto (1-\alpha)\phi(x) + (1-\alpha)\int_0^x \frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)}\phi(\xi)\mathrm{d}\xi,$$

instead of our definition (3.1), because the function u in (3.7) satisfies $u(0)=0$ by definition (2.2). This type of operator was pointed out for the standard Caputo-Fabrizio derivative in [21,22]. However, in that case, there would be no equivalence between (3.8) and (3.10) in Theorem 3.3. Only the implication (3.8) ⇒ (3.10) would be true. The reverse implication would not hold because Barrow's rule would fail to meet (3.8). Analogously, the statement (3.2) in Theorem 3.1 about the null space would no longer be satisfied. Therefore, to sum up and conclude the remark, (3.1) is the only acceptable definition for the integral operator.

Example 3.1. As (3.8), we consider the integral problem $y(x) = y_0 + {}^{G}\mathcal{I}^{\alpha}(\lambda y)(x)$, $\lambda \in \mathbb{R}\backslash\{0, 1/(1-\alpha)\}$. By (3.12), if $y_0 \neq 0$, then

$$\frac{\kappa_\alpha(x)}{\kappa_\alpha(0)} \frac{y}{y_0} \mathrm{e}^{\frac{\log(y/y_0)}{\lambda(1-\alpha)}} = 1.$$

The solution is

$$y(x) = y_0 \left(\frac{\kappa_\alpha(0)}{\kappa_\alpha(x)}\right)^{\frac{1}{1+1/(\lambda(1-\alpha))}}.$$

If $y_0 = 0$, then $y(x) = 0$ is the solution, because it is the root of the right-hand side of (3.11).

Example 3.2. Consider the Riccati integral problem $y(x) = y_0 + {}^{G}\mathcal{I}^{\alpha}(y^2+1)(x)$. By (3.12), if $y_0 \neq 1/(2(1-\alpha))$, then

$$\frac{\kappa_\alpha(x)}{\kappa_\alpha(0)} \frac{y^2+1}{y_0^2+1} \mathrm{e}^{\frac{\arctan y - \arctan y_0}{1-\alpha}} = 1.$$

This is the solution, in implicit form, as it often occurs with Caputo-Fabrizio operators [27]. In general, as $\int \frac{1}{f(\xi)}\mathrm{d}\xi$ needs to be computed, solutions may not be given with simple formulae, and even when a primitive is known, an implicit curve may be obtained.

In the statement of Theorem 3.3, we commented that, if the input function f is twice continuously differentiable around $(0, y_0)$ and (3.9) holds, then the integral problem (3.8) has a unique solution. Such a result is due to the relation between (3.8) and the ordinary differential equation (3.10), involving the partial derivatives of the nonlinearity. Now, we aim at proving that those two assumptions on f can be relaxed for existence and uniqueness, by supposing that f is Lipschitz continuous with respect to y on a vicinity of $(0, y_0)$ and that α is sufficiently near 1. Banach's fixed-point theorem plays a key role in the proof, as in the standard theory and other fractional operators, such as Caputo. The disadvantage of this new result is that it gives no guide on how the form of the concrete solution is.

We fix some notation. For $\epsilon_1, \epsilon_2 > 0$, consider a neighborhood

$$\mathcal{E} = [0, \epsilon_1] \times [y_0 - \epsilon_2, y_0 + \epsilon_2]$$

of $(0, y_0)$. Let

$$\|f\|_\infty = \sup_{(x,\phi)\in\mathcal{E}} |f(x, \phi)| < \infty \tag{3.13}$$

and

$$N = \sup_{\xi\in[0,\tau^*]} \frac{|\kappa_\alpha'(\xi)|}{\kappa_\alpha(\xi)} < \infty. \tag{3.14}$$

Theorem 3.4. *Based on (3.1), consider the functional equation (3.8), where $y_0 \in \mathbb{R}$. Suppose that f is continuous and there exists $M > 0$ such that*

$$|f(x, \phi_1) - f(x, \phi_2)| \leq M|\phi_1 - \phi_2| \tag{3.15}$$

on $\mathcal{E}$, with

$$M < \frac{1}{1-\alpha}, \quad \|f\|_\infty < \frac{\epsilon_2}{2(1-\alpha)}. \tag{3.16}$$

Then, there exists a unique continuous solution for (3.8) *on a local interval of* 0, *given by*

$$[0,\tau^*) = \left[0, \min\left\{\epsilon_1, \frac{1}{N}\left(\frac{\epsilon_2}{(1-\alpha)\|f\|_\infty} - 2\right), \frac{1}{N}\left(\frac{1}{(1-\alpha)M} - 1\right)\right\}\right). \quad (3.17)$$

Proof. Let $\mathcal{C}[0,\tau]$, for $\tau < \tau^*$, be the set of continuous functions on $[0,\tau]$, endowed with the supremum norm $\|\cdot\|_\infty$:

$$\|\phi\|_\infty = \sup_{x\in[0,\tau]} |\phi(x)|,$$

for $\phi \in \mathcal{C}[0,\tau]$. We work in the subspace

$$\mathcal{B}[0,\tau] = \{\phi \in \mathcal{C}[0,\tau] :\, \phi(0) = y_0,\, \|\phi - y_0\|_\infty \leq \epsilon_2\}, \quad (3.18)$$

which is Banach because it is closed.

Consider the operator

$$\Lambda :\, \mathcal{B}[0,\tau] \to \mathcal{B}[0,\tau],$$

$$\Lambda\phi(x) = y_0 + (1-\alpha)[f(x,\phi(x)) - f(0,\phi(0))] + (1-\alpha)\int_0^x \frac{\kappa_\alpha'(\xi)}{\kappa_\alpha(\xi)} f(\xi,\phi(\xi))\mathrm{d}\xi.$$

Let us see that it is well defined. If $\phi \in \mathcal{B}[0,\tau]$, then it is clear that $\Lambda\phi \in \mathcal{C}[0,\tau]$ by the continuity of f on the region $\mathcal{E}$. Also, $\Lambda\phi(0) = y_0$. On the other hand,

$$\begin{aligned}
\|\Lambda\phi - y_0\|_\infty &\leq (1-\alpha)[|f(x,\phi(x))| + |f(0,\phi(0))|] \\
&\quad + (1-\alpha)\int_0^x \frac{|\kappa_\alpha'(\xi)|}{\kappa_\alpha(\xi)} |f(\xi,\phi(\xi))|\mathrm{d}\xi \\
&\leq 2(1-\alpha)\|f\|_\infty + (1-\alpha)N\tau\|f\|_\infty \quad (3.19)\\
&= (1-\alpha)\|f\|_\infty(2+N\tau) \\
&\leq \epsilon_2. \quad (3.20)
\end{aligned}$$

In (3.19), we use the supremum (3.13) and the value N in (3.14). It is clear that $N < \infty$, because κ_α is positive and continuously differentiable. In the last step (3.20), we employ $\tau < \tau^*$ and (3.17). Thus, considering (3.18), $\Lambda\phi \in \mathcal{B}[0,\tau]$.

Equation (3.8) is equivalent to the fixed-point problem $y = \Lambda y$. For $\phi_1, \phi_2 \in \mathcal{B}[0,\tau]$ and $x \in [0,\tau]$, we have:

$$\begin{aligned}
&|\Lambda\phi_1(x) - \Lambda\phi_2(x)| \leq (1-\alpha)|f(x,\phi_1(x)) - f(x,\phi_2(x))| \\
&\quad + (1-\alpha)\int_0^x \frac{|\kappa_\alpha'(\xi)|}{\kappa_\alpha(\xi)} |f(\xi,\phi_1(\xi)) - f(\xi,\phi_2(\xi))|\mathrm{d}\xi \\
&\leq (1-\alpha)M|\phi_1(x) - \phi_2(x)| + (1-\alpha)M\int_0^x \frac{|\kappa_\alpha'(\xi)|}{\kappa_\alpha(\xi)} |\phi_1(\xi) - \phi_2(\xi)|\mathrm{d}\xi \quad (3.21)\\
&\leq (1-\alpha)M|\phi_1(x) - \phi_2(x)| + (1-\alpha)MN\int_0^x |\phi_1(\xi) - \phi_2(\xi)|\mathrm{d}\xi \quad (3.22)\\
&\leq (1-\alpha)M\|\phi_1 - \phi_2\|_\infty + (1-\alpha)MNx\|\phi_1 - \phi_2\|_\infty \\
&\leq (1-\alpha)M(1+N\tau)\|\phi_1 - \phi_2\|_\infty. \quad (3.23)
\end{aligned}$$

For (3.21), the Lipschitz condition (3.15) has been used. For (3.22), the value N in (3.14) has been employed. The term $(1-\alpha)M(1+N\tau)$ in (3.23) is strictly less than 1, by $\tau < \tau^*$ and (3.17). Notice that the interval (3.17) is well defined by (3.16). The conclusion is that the operator Λ is a contraction. By Banach's fixed-point theorem [16, Chapter 1], there exists a unique $y \in \mathcal{B}[0,\tau]$ such that $y = \Lambda y$. As $\tau < \tau^*$ is arbitrary, we have the solution y on $[0,\tau^*)$. □

Remark 3.2. Under the conditions of Theorem 3.4, the solution of (3.8) is given as the limit of the sequence $\{\phi_k\}_{k=0}^{\infty}$, where

$$\phi_0 = y_0$$

and

$$\begin{aligned}\phi_{k+1}(x) = y_0 &+ (1-\alpha)[f(x,\phi_k(x)) - f(0,\phi_k(0))] \\ &+ (1-\alpha)\int_0^x \frac{\kappa_\alpha'(\xi)}{\kappa_\alpha(\xi)} f(\xi,\phi_k(\xi))\mathrm{d}\xi,\end{aligned}$$

for $k \geq 0$. This is a Picard's iteration. It converges on $[0,\tau^*)$. Nevertheless, the solution cannot be obtained in explicit form.

3.5. Characterization of the nonsingular kernel

The definition (2.2) is quite broad with respect to the integration kernel (2.4). Should we impose more conditions on it? At the end of [29], the authors suggested the following line of research: "*It's worth noting that further research could potentially lead to characterizing the generalized Caputo-Fabrizio fractional derivative by imposing specific constraints on the nonsingular kernel*". This is the problem that we solve in this part of the paper. We find that the consistency of the operators with respect to the fractional order α characterizes the form of the kernel. The specific criteria in determining the form of the nonsingular kernel rely on obtaining the ordinary derivative when α tends to 1, and the translated identity operator when α tends to 0. These are not actually restrictions, because any suitable fractional derivative should satisfy such requirements (Caputo, Riemann-Liouville, etc.).

Theorem 3.5. *To ensure that*

$$\lim_{\alpha\to 1^-} {}^{G}\mathcal{D}^{\alpha}\phi(x) = \phi'(x)$$

and

$$\lim_{\alpha\to 0^+} {}^{G}\mathcal{D}^{\alpha}\phi(x) = \phi(x) - \phi(0)$$

for $b \geq x > 0$, one must have

$$\kappa_\alpha(x) = C\mathrm{e}^{\frac{1}{1-\alpha}\int_0^x \theta_\alpha(\xi)\mathrm{d}\xi}, \tag{3.24}$$

where $C = \kappa_\alpha(0) > 0$ is a constant and $\theta_\alpha : [0,b] \to \mathbb{R}$ is a continuous function such that

$$\lim_{\alpha\to 1^-} \theta_\alpha(\xi) = 1 \tag{3.25}$$

and

$$\lim_{\alpha\to 0^+} \theta_\alpha(\xi) = 0 \tag{3.26}$$

for all $\xi \in [0,b]$.

Proof. Indeed, to ensure that (3.1) becomes $\int_0^x \phi(\xi)\mathrm{d}\xi$ when $\alpha \to 1^-$, one must have

$$\lim_{\alpha\to 1^-}(1-\alpha)\frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)} = 1. \tag{3.27}$$

Analogously, (3.1) is $\phi(x)-\phi(0)$ when $\alpha \to 0^+$ if

$$\lim_{\alpha\to 0^+}(1-\alpha)\frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)} = 0. \tag{3.28}$$

If we define

$$\theta_\alpha(\xi) = (1-\alpha)\frac{\kappa'_\alpha(\xi)}{\kappa_\alpha(\xi)}, \tag{3.29}$$

we obtain (3.24) by the method of separation of variables. The properties (3.27) and (3.28) are equivalent to (3.25) and (3.26), respectively. □

Notice that, for the standard Caputo-Fabrizio operator (2.1), the function (3.29) is simply α for every argument ξ, and it satisfies the restrictions (3.25) and (3.26). In this sense, the conventional kernel (2.3) seems to be a natural choice, which additionally endows the operator with a convolution structure.

For example, the kernel employed in [29, Example 4.2],

$$\kappa_\alpha(x) = (x+1)\mathrm{e}^{\frac{\alpha}{1-\alpha}x},$$

is not totally valid in the sense of Theorem 3.5. By (3.29), we compute

$$\theta_\alpha(\xi) = \frac{1-\alpha}{\xi+1}+\alpha. \tag{3.30}$$

When $\alpha \to 1^-$, there is convergence of (3.30) towards 1, which agrees with (3.25). Nonetheless, (3.26) is not fulfilled, because (3.30) goes to $1/(\xi+1)$ when $\alpha \to 0^+$. In summary, one does not have absolute freedom when selecting the integration kernel (2.4).

An example of valid kernel (2.4) can be obtained from a constant θ_α,

$$\theta_\alpha(\xi) = c = \mathrm{e}^{1-\frac{1}{\alpha}} \in (0,1),$$

$$\alpha = \frac{1}{1-\log c},$$

which satisfies (3.25) and (3.26),

$$\kappa_\alpha(x) = C\exp\left(\frac{x}{1-\alpha}\mathrm{e}^{1-\frac{1}{\alpha}}\right) = \mathrm{e}^{\frac{c(\log c-1)}{\log c}x} \tag{3.31}$$

(see (3.24)), and

$${}^{G}\mathcal{K}_\alpha(x,\xi) = \exp\left(-\frac{x-\xi}{1-\alpha}\mathrm{e}^{1-\frac{1}{\alpha}}\right) = \mathrm{e}^{-\frac{c(\log c-1)}{\log c}(x-\xi)}, \tag{3.32}$$

for $C>0$ and $0<c<1$.

Similarly, other admissible kernels (2.4) may be built. For example, with

$$\theta_\alpha(\xi) = \sin\left(\frac{\pi}{2}(\alpha+(1-\alpha)\xi)\right) - (1-\alpha)\sin\left(\frac{\pi}{2}\xi\right),$$

which meets (3.25) and (3.26).

4. Lambert-function example

In [29], there are some examples of differintegral equations with the generalized Caputo-Fabrizio operator, for example, a logistic model. The fractional logistic solution exhibits dynamics that are similar to the S-shaped forms of the ordinary case $\alpha = 1$, although with higher flexibility.

Here, we solve another problem, with alternative and interesting features,

$$y(x) = y_0 + {}^{G}\mathcal{I}^{\alpha}(y^p)(x), \tag{4.1}$$

for a strictly increasing function κ_α. We suppose $p > 1$ is any real number and $y_0 > 0$. Problem (4.1) generalizes [6] to power $p \neq 2$ and the generalized Caputo-Fabrizio operator.

Theorem 3.3 applies. By (3.12),

$$\frac{\kappa_\alpha(x) y^p}{\kappa_\alpha(0) y_0^p} \mathrm{e}^{\frac{1}{1-\alpha}\left[\frac{1}{(p-1)y^{p-1}} - \frac{1}{(p-1)y_0^{p-1}}\right]} = 1. \tag{4.2}$$

The absolute values are omitted by the assumption $y_0 > 0$. This identity (4.2) is equivalent to

$$\frac{\kappa_\alpha(x)^{\frac{p-1}{p}} y^{p-1}}{\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1}} \mathrm{e}^{\frac{1}{p(1-\alpha)}\left[\frac{1}{y^{p-1}} - \frac{1}{y_0^{p-1}}\right]} = 1.$$

Rewrite it as

$$\frac{-1}{p(1-\alpha)y^{p-1}} \mathrm{e}^{\frac{-1}{p(1-\alpha)y^{p-1}}} = -\frac{\kappa_\alpha(x)^{\frac{p-1}{p}}}{p(1-\alpha)\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1}} \mathrm{e}^{-\frac{1}{p(1-\alpha)y_0^{p-1}}}.$$

We consider the solution of the equation $w\mathrm{e}^w = z$, which defines the Lambert or product-logarithm function $\mathcal{W}(z) = w$ [12, 13]. The Lambert function has been recently applied to study dynamical models [11, 23, 31]. Here, we have

$$\frac{-1}{p(1-\alpha)y^{p-1}} = \mathcal{W}\left(-\frac{\kappa_\alpha(x)^{\frac{p-1}{p}}}{p(1-\alpha)\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1}} \mathrm{e}^{-\frac{1}{p(1-\alpha)y_0^{p-1}}}\right).$$

Observe that the real function $w \in \mathbb{R} \mapsto w\mathrm{e}^w$ has a minimum at $w = -1$ and that minimum is $-\mathrm{e}^{-1} = -0.367879\ldots$. In other words, for $w, x \in \mathbb{R}$, the equation $w\mathrm{e}^w = x$ is solvable only for $x \geq -\mathrm{e}^{-1}$, with a unique solution for $x \geq 0$ and two values for $x \in (-\mathrm{e}^{-1}, 0)$. Both values become the same for $x = -\mathrm{e}^{-1}$. Later we will specify the branch of $\mathcal{W}$, since this function may be multivalued. Hence

$$y(x) = \left(\frac{-1}{p(1-\alpha)\mathcal{W}\left(-\frac{\kappa_\alpha(x)^{\frac{p-1}{p}}}{p(1-\alpha)\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1}} \mathrm{e}^{-\frac{1}{p(1-\alpha)y_0^{p-1}}}\right)}\right)^{\frac{1}{p-1}}. \tag{4.3}$$

If

$$1 - (1-\alpha)py_0^{p-1} < 0,$$

then one selects the principal branch $\mathcal{W} = \mathcal{W}_0 : (-\mathrm{e}^{-1}, 0) \to (-1, 0)$ and the solution (4.3) is decreasing. If, by contrast,

$$1 - (1-\alpha)py_0^{p-1} > 0,$$

then $\mathcal{W} = \mathcal{W}_{-1} : (-\mathrm{e}^{-1}, 0) \to (-\infty, -1)$ and the solution (4.3) is increasing. In Figure 1, we plot $\mathcal{W}_{-1}$ and $\mathcal{W}_0$. Since these branches are considered on the domain $(-\mathrm{e}^{-1}, 0)$, we impose

$$-\mathrm{e}^{-1} < -\frac{\kappa_\alpha(x)^{\frac{p-1}{p}}}{p(1-\alpha)\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1}} \mathrm{e}^{-\frac{1}{p(1-\alpha)y_0^{p-1}}} < 0.$$

Such a condition holds if and only if

$$\kappa_\alpha(x) < \left(\mathrm{e}^{-1}(1-\alpha)p\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1} \mathrm{e}^{\frac{1}{(1-\alpha)py_0^{p-1}}}\right)^{\frac{p}{p-1}},$$

that is,

$$x < \kappa_\alpha^{-1}\left(\left(\mathrm{e}^{-1}(1-\alpha)p\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1} \mathrm{e}^{\frac{1}{(1-\alpha)py_0^{p-1}}}\right)^{\frac{p}{p-1}}\right).$$

The inverse κ_α^{-1} exists because we assumed that κ_α is strictly increasing. In consequence, the interval of definition of (4.3) with both branches $\mathcal{W}_0$ and $\mathcal{W}_{-1}$ is

$$[0, x_\infty) = \left[0, \kappa_\alpha^{-1}\left(\left(\mathrm{e}^{-1}(1-\alpha)p\kappa_\alpha(0)^{\frac{p-1}{p}} y_0^{p-1} \mathrm{e}^{\frac{1}{(1-\alpha)py_0^{p-1}}}\right)^{\frac{p}{p-1}}\right)\right).$$

The two solutions (4.3), corresponding to $\mathcal{W}_0$ and $\mathcal{W}_{-1}$, are not simple functions due to the non-elementary nature of the Lambert function [9]. A function is called non-elementary if it cannot be written by addition, multiplication, division and composition from rational, trigonometric, logarithm and exponential functions and their inverses. Another distinctive property of the two solutions and their dynamics is that they collide at x_∞, from above and below, respectively, with value

$$y(x_\infty^-) = \left(\frac{1}{(1-\alpha)p}\right)^{\frac{1}{p-1}}$$

and infinite derivative. This is a phenomenon that does not appear when $\alpha = 1$ and $y' = y^p$, for which the solution is elementary and grows towards infinity.

The development is illustrated in Figures 2–4. Fix $\alpha = 0.11$, $y_0 = 2.1$ and $p = \pi$. In Figure 2, we plot the fractional solution of (4.1) that starts at $y_0 = 2.1$, with the generalized kernel defined by (3.31), (3.32).

We also draw the other solution that collapses with it at $x_\infty = 7203.80$. The asymptotic value is 0.618716.

On the other hand, in Figure 3, we conduct the same experiment with the standard Caputo-Fabrizio kernel defined through (2.3). The two solutions touch at $x_\infty = 20.0601$. Observe that the solutions are qualitatively similar, but the scales are totally different.

Finally, in Figure 4, we plot the ordinary solution, for $\alpha = 1$; there is only one solution on $[0, 0.0953236)$ that grows towards infinity and is elementary. For these computations, the software Mathematica® [35] is used.

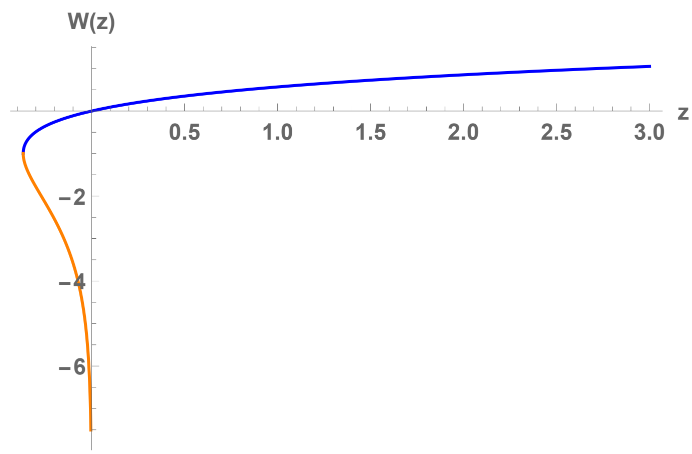

Figure 1. Branches $\mathcal{W}_{-1}$ (orange color) and $\mathcal{W}_0$ (blue color) of the Lambert function (real solutions of $w\mathrm{e}^w = z$). Both functions collide at $z = -\mathrm{e}^{-1} = -0.367879\ldots$.

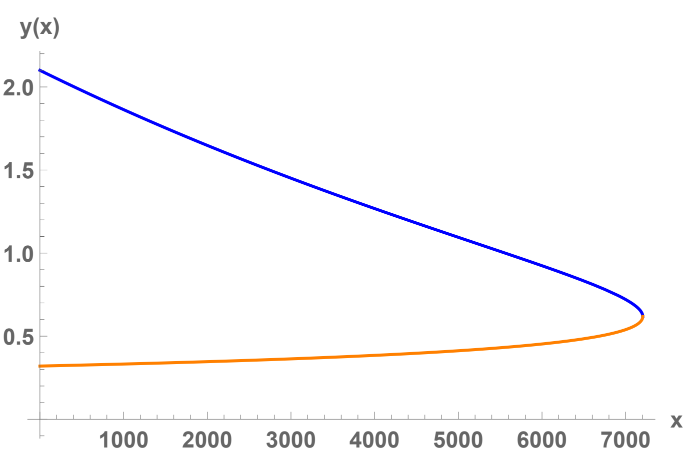

Figure 2. Collapsing solutions of (4.1) for $\alpha = 0.11$, $y_0 = 2.1$ and $p = \pi$, with the generalized kernel defined by (3.31), (3.32).

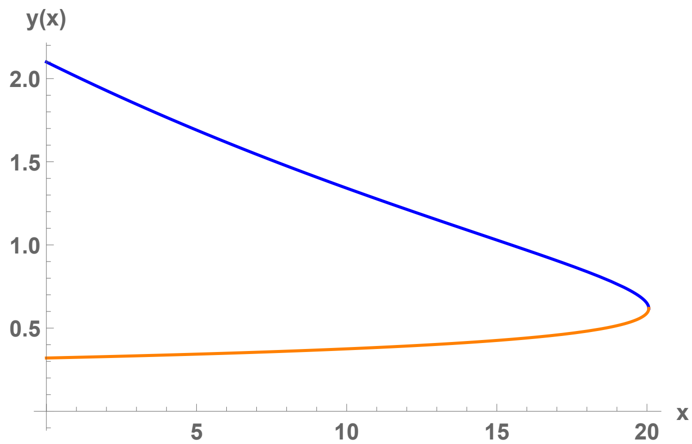

Figure 3. Collapsing solutions of (4.1) for $\alpha = 0.11$, $y_0 = 2.1$ and $p = \pi$, with the standard kernel defined through (2.3).

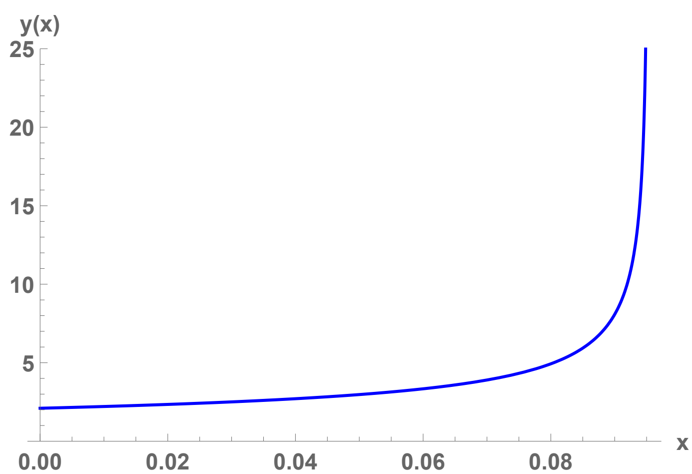

Figure 4. Ordinary solution of (4.1) for $\alpha = 1$, $y_0 = 2.1$ and $p = \pi$.

5. Final remarks

5.1. Summary of results

The new fractional derivative (2.2) extends the Caputo-Fabrizio operator (2.1) by enlarging the class of integration kernels. The null space of the generalized derivative is formed by the constant functions, as the ordinary derivative. The associated fractional integral operator (3.1) is a combination of an increment and a weighted integral (for the standard Caputo-Fabrizio or Losada-Nieto integral, it is a convex combination with weight equal to one). In contrast to the classical Riemann integral, the null space of the integral operator is not trivial, but the set spanned by a function. The fundamental theorem of calculus, concerning the composition of the derivative and the integral and vice versa, see (3.5) and (3.6), is partially true, due to the non-triviality of the null space. Generalized Caputo-Fabrizio problems are equivalent to ordinary differential equations, and for autonomous vector fields the solution can be obtained in implicit form, by separation of variables. There is an alternative result on existence and uniqueness of solution, with other hypotheses and with a determined interval of existence, by using Banach's fixed-point theorem. The nonsingular kernel in the definition of the generalized operator has to meet certain conditions, regarding the boundary values of the fractional index. The example with power input function is solved with the non-elementary Lambert function, for which a new collapsing phenomenon of solutions emerges. The numerical computations support the theoretical findings.

5.2. Comments on extensions

Our paper contributes to advancing in the theoretical aspects of fractional calculus, by investigating the properties and gaps of a new fractional operator in a complete manner. Some comments and possible future research directions from the work are described next.

The new Caputo-Fabrizio operator (2.2) generalizes the exponential kernel of the standard operator, with (2.4). It does not seem possible to extend these ideas to Prabhakar operators [5], because the Mittag-Leffler function cannot be decomposed with products as opposed to the exponential function. This is a key difference between the new Caputo-Fabrizio operator and the other operators with bounded kernels, that simplifies its treatment.

Considering (3.6), the fundamental theorem of calculus would be analogous to standard calculus if $\kappa_\alpha(0) = 0$ and $\kappa_\alpha(x) > 0$ for $x > 0$. In such a case, however, the ordinary differential equation (3.10) would not be equivalent. It seems that, if $\kappa_\alpha(0) = 0$, then the kernel in the operator would be singular and a different theory would be developed.

Section 4 could be explored when the input function is $y^p + \delta$, where $p > 1$ and $\delta \neq 0$ is a constant. We do not know if the Lambert function or some extension of it would emerge.

The role of the Laplace-transform or power-series methods for this generalized Caputo-Fabrizio operator shall be investigated [14, 17, 21]. Also, computational aspects have not been addressed in the study.

The work lies in the theoretical area. Adequate kernels (2.4) satisfying the compatibility conditions (3.27) and (3.28) might be found for modeling purposes and practitioners, if any, in mathematical physics, engineering, and applied mathematics.

Despite the simpler treatment, an inherent limitation of fractional operators with nonsingular kernel is the zero value at $t = 0$, which is inconsistent with most of the vector fields in differential equations. This fact is also related to the form of the fundamental theorem of calculus and the null space. Thus, one needs to work with the associated Volterra integral equation, which is mathematically admissible. It appears legitimate to directly treat with Volterra equations, as all types of fractional models are particular cases of these equations with certain kernels [33, 34]. However, in the generalized Caputo-Fabrizio context, further studies are needed to deal with this issue and explore the appropriateness and utility of the integral form in applications, beyond pure work, specially considering the equivalence with an ordinary differential equation, of local behavior [32].

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