Dynamical investigation and numerical modelling of a fractional mixed nonlinear partial integro-differential problem in time and space

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Abstract

In the current study, a novel and effective method for solving the nonlinear fractional mixed partial integro-differential equation (NfrPIo-DE) based on a continuous kernel is presented and discussed. The NfrPIo-DE is transformed into the nonlinear Fredholm integral equation (NFIE) through the utilisation of the separation of variables. The NFIE reduction was then transformed into a system of nonlinear algebraic equations (SNAE) with the application of Chebyshev polynomials of the sixth type (CP6K). By utilising the Banach fixed point theorem, we can describe the existence of the solution of NfrPIo-DE as well as its uniqueness. Furthermore, the convergence and the stability of the reduced error have been described. Finally, numerical example is presented to illustrate the theoretical results. The Maple 18 software is responsible for getting all of the computational outcomes.

keywords: NfrPIo-DE; Caputo fractional derivative; Existence and uniqueness; Convergence and stability; Separation of variables, CP6K.

MSC: 65R20, 45M10, 35R09.

1. Introduction

There are types of math equations called nonlinear integral equations that use both integrals and nonlinear functions [1]. In many areas of science and engineering, these equations are very important for explaining things where the link between variables is not linear [2, 3]. Over the past few years, many of the authors' attention has been drawn to the mixed integral equations (MIE) in time and position. Mahdy et al. in [4], presented computational methods for solving mixed-difference integro-differential. Using the Lerch polynomial approach, the technique for responding to MIE was presented by Alhazmi and colleagues [5], and it employed a solitary kernel that was essentially symmetric. A numerical algorithm for generating MIEs with singular kernels was presented by Mahdy et al. in the publication [6]. Abdou and Sharifah in [7], used Toeplitz matrix method for solving NfrMIo-DE. The Toeplitz matrix approach was applied by Jan in [8] to determine the MIE. To discuss quadratic nonlinear MIE, Basseem and Alalyani presented the Toeplitz and Nystrom algorithms in [9]. Abdou and Youssef [10] proposed analytical and numerical approaches to the resolution of three MIE problems. In [11], Abdou and Basseem used separation of variables and Toeplitz matrix algorithm for solving nonlinear MIEs.

Most of the researchers focused their attention on the first, second, third, and fourth types of CPs, but they did not address the sixth type of CPs, as an illustration, Youssri et al. suggested the numerical to hyperbolic telegraph question using applying shifted CP6K [12]. The fractional partial differential equations were solved by Sadri and Aminikhah using CP6K, as described in [13]. In [14] Yaghoubi et al. introduce shifted CP6K to obtain the solution of frIo-DEs.

Several benefits of the suggested method include superior accuracy in approximations and reduced computation time achieved by selecting CP6K as basic functions and extracting a few terms from the retained modes. Furthermore, the resulting inaccuracies are negligible.

Fractional calculus has been a popular topic for academics in a variety of scientific and technical domains in recent years. The ability of fractional derivatives to offer a better method of characterizing the memory and genetic characteristics of different materials and processes is one of the main benefits of fractional calculus. For instance, there are many benefits to employing fractional derivatives: Fractional derivatives can be used to model systems with memory. Fractional order differential equations readily relate to systems with memory, which are present in most physical problems, models in extended thermoelasticity theory, and biological systems (FDEs). FDEs show the true biphasic decline behaviour of diseases and elastic media, albeit more slowly. FDEs fit better than integer-order models when it comes to modelling complex systems, like biological systems, extended thermoelasticity theory models, and physical challenges.

It is difficult to find efficient numerical solutions for most of these equations since it is not possible to find exact solutions for them. Different approaches to solving these kinds of equations have been discussed and studied, for example, Abdou et al. [15] used Bernoulli matrix approach for solving NfrIo-DE. Alharbi et al. in [16], introduced Haar-Wavelet method for solving FrODEs. In [17], Hamdan et al. used Haar Wavelet method for solving LFrIEs. In [18], Bekkouche and colleagues conducted research on the computational approach to the fractional boundary value question. Khalouta in [19], discussed the solution of FrODEs by utilizing new general integral transform. Euler wavelets were introduced by Mohammad et al. in [20] in order to solve the fractional diffusion wave equation. Panda and Mohapatra in [21] used semi- analytical methods for solving FrPDEs. Youssri and Atta in [22] presented Fejr-quadrature collocation algorithm for solving frIo-DEs. Also in [23], Youssri and Hafez used Chebyshev collocation treatment to obtain the solution of VFIE. While Youssri and Hafez in [24] introduced spectral Legendre-Chebyshev treatment for solving 2D nonlinear MVFIE. While dealing with FrMIEs are considered modern for this purpose. To solve the continuous kernels mixed nonlinear partial integro-differential equations, Al-Bugami et al. introduced a sixth type of Chebyshev and Bernoulli polynomial numerical approaches [25].

This paper is devoted to investigating a new technique to demonstrate the computational approach for dealing with the study of the numerical solution of NfrMIo-DE based on a continuous kernel.

Consider the following NfrPIo-DE:

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}\right] = F(t)\int_{-1}^{1} k(x,y)\gamma(\Phi(y,t))dy, \quad 0<\alpha<1. \tag{1.1}$$

Under the condition

$$\Phi(x,0) = u(x), \tag{1.2}$$

where these continuous functions are known: $k(x,t), f(x,t), G(t)$, and $F(t)$; $\Phi(x,t)$ is unknown.

This type of equation in this case or in special cases of it can appear widely in genetic engineering, nanotechnology and many nonlinear disciplines based on the study of fractional time and its results related to materials. In many cases, the primary problem may be treated as it is presented. Therefore, we will transform the problem from fractional differential-integral equations to an equivalent integral equation, and we will then shed light on how to solve this integral equation.

Integrating (1.1) and using (1.2), we have

$$[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}]_0^t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F(\tau)\int_{-1}^1 k(x,y)\gamma(\Phi(y,\tau))dyd\tau,$$

$$[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}] - [\frac{\Phi(x,0)-f(x,0)}{\delta(G(0),\Phi(x,0))}] = \frac{1}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau,$$

$$[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}] = [\frac{\Phi(x,0)-f(x,0)}{\delta(G(0),\Phi(x,0))}] + \frac{1}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau.$$

Then

$$\Phi(x,t) = f(x,t)+\delta(G(t),\Phi(x,t))[\frac{u(x)-f(x,0)}{\delta(G(0),\Phi(x,0))}]+\frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau,$$

$$\Phi(x,t) = f(x,t) + D(x)\delta(G(t),\Phi(x,t)) + \frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau, \tag{1.3}$$

where $D(x) = [\frac{u(x)-f(x,0)}{\delta(G(0),\Phi(x,0))}]$.

The formula (1.3) represents a nonlinear mixed integral equation.

Special Situations

(I) In the case where $u(x) = f(x,0)$, the quadratic nonlinear integral equation is as described below:

$$\Phi(x,t) = f(x,t) + \frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau. \tag{1.4}$$

(II) Assuming that $f(x,t) = 0, \Phi(x,0) = u(x)$, we obtain

$$\Phi(x,t) = D^*(x)\delta(G(t),\Phi(x,t)) + \frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau, \tag{1.5}$$

where $D^*(x) = [\frac{u(x)}{\delta(G(0),\Phi(x,0))}]$.

(III) Assuming that $f(x,t) = 0, \Phi(x,0) = 0$, we obtain

$$\Phi(x,t) = \frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau. \tag{1.6}$$

The rest of this study's parts are organized as follows. In section 2, we will discuss several characteristics that apply to fractional derivatives. In section 3, we will discuss the existence of the nonlinear fractional MIE as well as the unique solution to it. The proposed solution's convergence will be studied in Section 4, which is dedicated to doing so. Furthermore taken into account is the stability of the generated error in section 5. The nonlinear fractional MIE is converted into the NFIE in section 6, which can be completed by the use of the algorithm of separation of variables. The NFIE was solved using CP6K in section 7. Illustrative example presented in section 8. Finally, Characterization of fractional time discussed in section 9.

2. Definitions of fractional calculus in basic terms

This section provides an overview of fundamental concepts and characteristics dealing with the theory of fractional calculus.

Definition 2.1. [26, 27] Fractional integral of function based on the Riemann-Liouville formula $f : (0,\infty) \longrightarrow R$, for $\alpha \in R^+$ consists of the definition that

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2. [26, 27] The fractional derivative of the Caputo function $f : (0,\infty) \longrightarrow R$, consists of a description that

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t f^{(m)}(\tau)(t-\tau)^{m-\alpha-1}\, d\tau, & \alpha \neq m \in R - M, \\ f^{(m)}(t), & \alpha = m \in M. \end{cases} \tag{2.2}$$

Therefore, we have the next characteristics:

1. $I^\alpha\ I^\nu g = I^{\alpha+\nu} g\ , \qquad \alpha, \nu > 0\ , \mu > 0\ .$ (2.3)

2. $I^\alpha\ x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\ x^{\alpha+\gamma}, \qquad \alpha > 0\ , \gamma > -1,\ x > 0\ .$ (2.4)

3. $I^\alpha\ D^\alpha g(x) = g(x) - \sum_{k=0}^{m-1} g^{(k)}(0^+)\ \dfrac{x^k}{k!}, \qquad x > 0\ ,\ m-1 < \alpha \leq m.$ (2.5)

4. $D^\alpha\ I^\alpha\ g(x) = g(x)\ , \qquad x > 0\ ,\ m-1 < \alpha \leq m.$ (2.6)

We must state Banach fixed point theorem.

3. Equation (1.3) existence and unique solution

In order to have a discussion on whether or not Equation (1.3) has a single, unique solution in light using the theorem of Banach fixed points, according to the integral operator form, we shall express (1.3) as follows:

$$\Phi(x,t) = \overline{W}\Phi(x,t) = f(x,t) + D(x)\delta(G(t),\Phi(x,t)) + \delta(G(t),\Phi(x,t))W\Phi(x,t), \tag{3.1}$$

$$W\Phi(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y)\gamma(\Phi(y,\tau))dyd\tau. \tag{3.2}$$

Thereafter, let us suppose the conditions that are listed below:

$(i)\forall C > C_1, C_2, C_3, (\int_a^b \int_a^b k^2(x,y)dxdy)^{1/2} \leq C_1, \quad | F(t) | \leq C_2, \quad | G(t) | \leq C_3.$

$(ii) f(x,t) \in L_2[-1,1] \times C[-1,1]$ and it's norm is $\| f(x,t) \| = \max_t \int_0^t \{\int_a^b f^2(x,\tau)dx\}^{\frac{1}{2}} d\tau = E.$

$(iii)\forall A > A_1, A_2,$ we have

$(iii-a)\ | \gamma(\Phi(x,t)) | < A_1 \mid \Phi(x,t) |, \qquad (iii-b)\ | \gamma(\Phi_1(x,y)) - \gamma(\Phi_2(x,y)) | < A_2 \mid \Phi_1(x,t) - \Phi_2(x,t) |\ .$

$(iv)\forall B > B_1, B_2,$ we have

$(iv-a)\ | \delta(G(t),\Phi(x,t)) | < B_1 \mid \Phi(x,t) |,$

$(iv-b) \mid \delta(G(t),\Phi_1(x,t)) - \delta(G(t),\Phi_2(x,t)) \mid < B_2 \mid \Phi_1(x,t) - \Phi_1(x,t) \mid$.

(v) If the researchers follow the way of integration with respect to time, they must consider that: the kernel $(t-s)^{\alpha-1} \forall t,\tau \in [0,T], 0 \leq \tau \leq T < 1$, is fulfilling for each continuous function $F(t), \forall 0 \leq \tau_1 \leq \tau_2 \leq t$ and each type of constant $M > 0$, the integrals $\int_{\tau_1}^{\tau_2} (t-\tau)^{\alpha-1} F(\tau) d\tau$, $\int_0^t (t-\tau)^{\alpha-1} F(\tau) d\tau$, $\max_{0\leq t\leq T} \int_0^t (t-\tau)^{\alpha-1} d\tau$, are continuous functions of t. In general, we let $\| \int_0^t (t-\tau)^{\alpha-1} F(\tau) d\tau \| \leq H$.

The integral operator's normality and continuity are both discussed.

Lemma 3.1. We are going to demonstrate that the integral operator is normal. The operator, which translates the space $L_2[-1,1] \times C[0,T]$ into itself, must be implemented in order to take into account the constraints (i)(iii-a) $\overline{W}\Phi(x,t)$ described using (3.1).

Proof:

Since

$$\overline{W}\Phi(x,t) = f(x,t) + D(x)\delta(G(t),\Phi(x,t)) + \delta(G(t),\Phi(x,t))W\Phi(x,t).$$

Then

$$\| \overline{W}\Phi(x,t) \| \leq \| f(x,t) \| + \| D(x)\delta(G(t),\Phi(x,t)) \| + \| \delta(G(t),\Phi(x,t)) \| \| W\Phi(x,t) \|,$$

$$\| f(x,t) \| = \max_{0\leq t\leq T} \int_0^t \{ \int_{-1}^1 f^2(x,\tau) dx \}^{\frac{1}{2}} d\tau = E, \| D(x)\delta(G(t),\Phi(x,t)) \| \leq DB \| \Phi(x,t)) \|,$$

$$\| \delta(G(t),\Phi(x,t)) \| \leq B \| \Phi(x,t)) \|,$$

$$\| W\Phi(x,t) \| = \frac{1}{\Gamma(\alpha)} \| \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y,\tau)) dy d\tau \|$$

$$\leq \frac{A}{\Gamma(\alpha)} \{ \max_{0\leq t\leq T} \int_0^t (t-\tau)^{\alpha-1} d\tau \mid F(\tau) \mid \} \{ (\int_{-1}^1 \int_{-1}^1 k^2(x,y) dx dy)^{1/2} \} (\max_{0\leq t\leq T} \int_0^t [\int_{-1}^1 \Phi^2(x,\tau) dx]^{\frac{1}{2}} d\tau).$$

We have, by applying the conditions (i) through (iii),

$$\| W\Phi(x,t) \| \leq \frac{ACHB}{\Gamma(\alpha)} \| \Phi(x,t)) \|,$$

$$\| \overline{W}\Phi(x,t) \| \leq E + DB \| \Phi(x,t)) \| + \frac{ACHB}{\Gamma(\alpha)} \| \Phi(x,t)) \| . \tag{3.3}$$

In the final inequality (3.3), it is demonstrated that the operator $\overline{W}$ is responsible for mapping the ball S_r into itself, where

$$r = \frac{E\Gamma(\alpha)}{\Gamma(\alpha)(1-DB) - ACHB}.$$

Since $r > 0$, therefore we have $\sigma < 1, \sigma = (DB + \frac{ACHB}{\Gamma(\alpha)})$.

Lemma 3.2. We shall establish the integral operator's continuity. The operator in the space $L_2[-1,1] \times C[0,T]$ is continuous and must take into account the requirements (i)-(iii-a) $\overline{W}\Phi(x,t)$ given by (3.1).

Proof.

For the continuity, we assume

$$\overline{W}(\Phi_1(x,t) - \Phi_2(x,t)) = D(x)\{\delta(G(t),\Phi_1(x,t)) - \delta(G(t),\Phi_2(x,t))\}$$

$$+\frac{(\delta(G(t),\Phi_1(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_1(y,\tau))dyd\tau$$

$$-\frac{\delta(G(t),\Phi_2(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_2(y,\tau))dyd\tau\}.$$

$$\mid \overline{W}(\Phi_1(x,t)-\Phi_2(x,t)) \mid\leq\mid D(x)\mid\times\mid\{\delta(G(t),\Phi_1(x,t))-\delta(G(t),\Phi_2(x,t))\}\mid$$

$$+\frac{1}{\Gamma(\alpha)}\mid(\delta(G(t),\Phi_1(x,t))\mid\times\mid\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\{\gamma(\Phi_1(y,\tau))-\gamma(\Phi_2(y,\tau))\}dyd\tau\mid$$

$$+\frac{1}{\Gamma(\alpha)}\mid(\delta(G(t),\Phi_1(x,t))-(\delta(G(t),\Phi_2(x,t))\mid\times\mid\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_2(y,\tau))dyd\tau\mid$$

$$\parallel \overline{W}(\Phi_1(x,t)-\Phi_2(x,t))\parallel\leq DB\parallel\Phi_1(x,t)-\Phi_2(x,t)\parallel+\frac{ACHB}{\Gamma(\alpha)}\parallel\Phi_1(x,t)-\Phi_2(x,t)\parallel\parallel\Phi_1(x,t)\parallel$$

$$+\frac{ACHB}{\Gamma(\alpha)}\parallel\Phi_1(x,t)-\Phi_2(x,t)\parallel\parallel\Phi_2(x,t)\parallel.$$

Consider $\parallel\Phi_1(x,t)\parallel=Q_1,\parallel\Phi_2(x,t)\parallel=Q_2$, hence, we have

$$\parallel \overline{W}(\Phi_1(x,t)-\Phi_2(x,t))\parallel\leq(DB+\frac{ACHBQ_1+ACHBQ_2}{\Gamma(\alpha)})\parallel\Phi_1(x,t)-\Phi_2(x,t)\parallel.$$

Hence the operator $\overline{W}$ is continuous and under the condition $\sigma=(DB+\frac{ACHB}{\Gamma(\alpha)})$, we have a contraction mapping.

4. Convergence of solutions

The form of the family of solutions should be supposed $\Phi(x,t)=\{\Phi_0(x,t),\Phi_1(x,t),\ldots,\Phi_{n-1}(x,t),\Phi_n(x,t),\ldots\}=\{\Phi_i(x,t)\}_{i=0}^{\infty}$,

$\Phi_0(x,t)=f(x,t)+D(x)\times\delta(G(t),\Phi_0(x,t))$,

$\mid\Phi_0(x,t)\mid\leq\mid f(x,t)\mid+B\mid D(x)\mid\parallel\Phi_0(x,t)\parallel\leq E+BD\parallel\Phi_0(x,t)\parallel\leq\frac{E}{1-BD},\ (1-BD\neq0),\ \mid D(x)\mid=D.$

Pick up two functions $\{\Phi_n(x,t),\Phi_{n-1}(x,t)\}\in\{\Phi_i(x,t)\}_{i=0}^{\infty}$

$$\Phi_n(x,t)=f(x,t)+D(x)\times\delta(G(t),\Phi_n(x,t))+\frac{\delta(G(t),\Phi_n(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_{n-1}(y,\tau))dyd\tau, \tag{4.1}$$

$$\Phi_{n-1}(x,t)=f(x,t)+D(x)\times\delta(G(t),\Phi_{n-1}(x,t))+\frac{\delta(G(t),\Phi_n(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_{n-2}(y,\tau))dyd\tau. \tag{4.2}$$

$$\mid\Phi_n(x,t)-\Phi_{n-1}(x,t)\mid\leq BD\mid\Phi_n(x,t)-\Phi_{n-1}(x,t)\mid+$$
$$\frac{AB}{\Gamma(\alpha)}\mid\Phi_n(x,t)\mid\times\mid\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\mid\Phi_{n-1}(y,\tau)-\Phi_{n-2}(y,\tau)\mid dyd\tau\mid$$
$$+\frac{AB}{\Gamma(\alpha)}\mid\Phi_n(x,t)-\Phi_{n-1}(x,t)\mid\times\mid\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\mid\Phi_{n-2}(y,\tau)\mid dyd\tau\mid. \tag{4.3}$$

Hence,

$$\mid\Psi_n(x,t)\mid\leq BD\mid\Psi_n(x,t)\mid+\frac{AB}{\Gamma(\alpha)}\mid\Phi_n(x,t)\mid\times\mid\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\mid\Psi_{n-1}(y,\tau)\mid dyd\tau\mid$$

$$+\frac{AB}{\Gamma(\alpha)} \mid \Psi_n(x,t) \mid \times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \mid \Phi_{n-2}(y,\tau) \mid dy d\tau \mid . \tag{4.4}$$

Let

$$\mid L \mid = \mid \int_0^t \int_{-1}^1 F(\tau) k(x,y) \mid \Phi_{n-2}(y,\tau) \mid dy d\tau \mid .$$

Since

$$\mid \Phi_{n-2}(y,\tau) \mid \leq \mid \Psi_{n-2}(y,\tau) \mid + \mid \Psi_{n-3}(y,\tau) \mid + \ldots + \mid \Psi_1(y,\tau) \mid + \mid \Psi_0(y,\tau) \mid,$$

i.e

$$\mid \Phi_{n-2}(y,\tau) \mid \leq \sum_{i=0}^{n-2} \mid \Psi_i(y,\tau) \mid . \tag{4.5}$$

Then

$$\mid L \mid \leq \mid \int_0^t \int_{-1}^1 F(\tau) k(x,y) \sum_{i=0}^{n-2} \mid \Psi_i(y,\tau) \mid dy d\tau \mid .$$

Hence, we have

$$\parallel L \parallel \leq \frac{C_1 C_2 T^\alpha}{\Gamma(1+\alpha)} \sum_{i=0}^{n-2} \parallel \Psi_i(y,\tau) \parallel . \tag{4.6}$$

Using (4.5) and (4.6)into (4.4), we follow

$$(1-BD) \parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} \sum_{i=0}^{n} \parallel \Psi_i(x,t) \parallel \parallel \Psi_{n-1}(x,t) \parallel + \frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} \parallel \Psi_n(x,t) \parallel [\sum_{i=0}^{n-2} \parallel \Psi_i(x,t) \parallel].$$

This equation can be modified to have the following format:

$$(1-BD) \parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} \parallel \Psi_n(x,t) \parallel [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel] + \frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} \sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel \parallel \Psi_{n-1}(x,t) \parallel$$

$$(1-BD) \parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} [\parallel \Psi_n(x,t) \parallel + \parallel \Psi_{n-1}(x,t) \parallel] \times [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel].$$

To obtain the format, rewrite the equation used previously.

$$(1-BD-\frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel]) \parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} \parallel \Psi_{n-1}(x,t) \parallel] \times [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel].$$

Lastly, we have

$$\parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 ABT^\alpha \parallel \Psi_{n-1}(x,t) \parallel] \times [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel]}{(1-BD)\Gamma(1+\alpha) - C_1 C_2 ABT^\alpha [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel]}, L \geq [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel].$$

Let $n = 0$,

$$\parallel \Psi_0(x,t) \parallel \leq \frac{E}{1-BD}, (1-BD \neq 0).$$

Let $n = 1$,

$$\parallel \Psi_1(x,t) \parallel \leq \frac{C^2 T^\alpha BA\{\parallel \Psi_0(x,t) \parallel\} \parallel \Psi_0(x,t) \parallel}{\{(1-BD)\Gamma(1+\alpha) - C^2 T^\alpha BA\{\parallel \Psi_0(x,t) \parallel\}\}}, \parallel \Psi_0(x,t) \parallel \leq \frac{E}{1-BD},$$

$$\parallel \Psi_1(x,t) \parallel \leq \frac{(C^2 T^\alpha BAE^2)}{(1-BD)\Delta_1}, \Delta_1 = \{(1-BD)^2 \Gamma(1+\alpha) - C^2 T^\alpha BAE\}.$$

Let $n = 2$,

$$\| \Psi_2(x,t) \| \leq \frac{C^2T^\alpha BA\{\sum_{i=0}^{1} \| \Psi_i(x,t) \|\} \| \Psi_1(x,t) \|}{\{(1-BD)\Gamma(1+\alpha) - C^2BAT^\alpha[\sum_{i=0}^{1} \| \Psi_i(x,t) \|]\}}$$

$$\| \Psi_2(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^2K_2}{[(1-BD)\Delta_1]\Delta_2},\ where\ K_2 = \{E\Delta_1 + C^2T^\alpha BAE^2\},$$

$$\Delta_1 = \{(1-BD)^2\Gamma(1+\alpha) - C^2T^\alpha BAE\}, \quad \Delta_2 = \{(1-BD)^2\Gamma(1+\alpha)\Delta_1 - C^2T^\alpha BAK_2\}.$$

Let $n = 3$,

$$\| \Psi_3(x,t) \| \leq \frac{C^2T^\alpha BA \| \Psi_2(x,t) \| \{\| \Psi_0(x,t) \| + \| \Psi_1(x,t) \| + \| \Psi_2(x,t) \|\}}{\{(1-BD)\Gamma(1+\alpha) - C^2BAT^\alpha[\| \Psi_0(x,t) \| + \| \Psi_1(x,t) \| + \| \Psi_2(x,t) \|\}}.$$

Since

$$\{\| \Psi_0(x,t) \| + \| \Psi_1(x,t) \| + \| \Psi_2(x,t) \|\} = \frac{E}{1-BD} + \frac{(C^2T^\alpha BAE^2)}{(1-BD)\Delta_1} + \frac{E^2(C^2T^\alpha BA)^2K_2}{(1-BD)\Delta_1\Delta_2}$$

$$C^2T^\alpha BA\{\| \Psi_0(x,t) \| + \| \Psi_1(x,t) \| + \| \Psi_2(x,t) \|\} = \frac{(C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2]}{(1-BD)\Delta_1\Delta_2},$$

$$\begin{aligned}\{(1-BD)\Gamma(1+\alpha) - C^2BAT^\alpha[\| \Psi_0(x,t) \| + \| \Psi_1(x,t) \| + \| \Psi_2(x,t) \|\} &= (1-BD)\Gamma(1+\alpha) \\ &- \frac{(C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2]}{(1-BD)\Delta_1\Delta_2} \\ = \frac{(1-BD)^2\Gamma(1+\alpha)\Delta_1\Delta_2 - (C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2]}{(1-BD)\Delta_1\Delta_2}.\end{aligned}$$

Then

$$\| \Psi_3(x,t) \| \leq \frac{C^2T^\alpha BA \| \Psi_2(x,t) \| \{\| \Psi_0(x,t) \| + \| \Psi_1(x,t) \| + \| \Psi_2(x,t) \|\}}{\{(1-BD)\Gamma(1+\alpha) - C^2BAT^\alpha[\| \Psi_0(x,t) \| + \| \Psi_1(x,t) \| + \| \Psi_2(x,t) \|\}}$$

$$= \frac{\frac{(C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2]}{(1-BD)\Delta_1\Delta_2}}{\frac{(1-BD)^2\Gamma(1+\alpha)\Delta_1\Delta_2 - (C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2]}{(1-BD)\Delta_1\Delta_2}}.$$

Hence,

$$\| \Psi_3(x,t) \|$$

$$\leq \frac{E^2(C^2T^\alpha BA)^3K_2[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2]}{(1-BD)\Delta_1\Delta_2\{(1-BD)^2\Gamma(1+\alpha)\Delta_1\Delta_2 - (C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2]\}}$$

$$\| \Psi_3(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^3K_2K_3}{(1-BD)\Gamma(1+\alpha)[(1-BD)\Delta_1\Delta_2]^2 - (C^2T^\alpha BA)(1-BD)\Delta_1\Delta_2K_3}.$$

Finally, we have

$$\| \Psi_3(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^3K_2K_3}{(1-BD)\Gamma(1+\alpha)\Delta_1\Delta_2[(1-BD)^2\Delta_1\Delta_2 - (C^2T^\alpha BA)K_3]},$$

$$\| \Psi_3(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^3K_2K_3}{(1-BD)\Gamma(1+\alpha)\Delta_1\Delta_2\Delta_3}, \quad \Delta_3 = [(1-BD)^2\Delta_1\Delta_2 - (C^2T^\alpha BA)K_3],$$

where

$$K_3 = [E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2K_2], \quad \Delta_3 = [(1-BD)^2\Delta_1\Delta_2 - (C^2T^\alpha BA)K_3].$$

By induction, we have

$$\| \Psi_n(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^3K_2K_3\dots K_n}{(1-BD)\Gamma(1+\alpha)\Delta_1\Delta_2\Delta_3\dots\Delta_n},$$

$$K_n = [E\Delta_1\Delta_2\Delta_3\dots\Delta_{n-1} + E^2(C^2T^\alpha BA)\Delta_2\dots\Delta_{n-1} + E^2(C^2T^\alpha BA)^2K_2\Delta_3\dots\Delta_{n-1} + \dots + E^2(C^2T^\alpha BA)^{n-1}K_{n-1}],$$

$$\Delta_{n-1} = [(1-BD)^2]\Delta_1\Delta_2\Delta_3\dots\Delta_{n-1} - (\alpha\beta TBA)K_{n-1}].$$

Then, under the condition $\frac{C^2T^\alpha BA}{\Gamma(1+\alpha)} < (1-BD)$, the function $\Psi_n(x,t), \forall n$ convergence

$$C^2T^\alpha BA + \Gamma(1+\alpha)(1-DB) < \Gamma(1+\alpha).$$

5. The stability of error

The numerical solution $\Phi_n(x,t)$, of Equation (1.3) becomes

$$\Phi_n(x,t) = f_n(x,t) + D(x)\delta(G(t),\Phi_n(x,t)) + \frac{\delta(G(t),\Phi_n(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_n(y,\tau))dyd\tau. \tag{5.1}$$

Therefore, we have the error $R_n(x,t)$ after subtracting (1.3) and (5.1) to have

$$R_n(x,t) = h_n(x,t) + D(x)\{\delta(G(t),\Phi(x,t)) - \delta(G(t),\Phi_n(x,t))\}$$
$$+\{\frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau$$
$$-\frac{\delta(G(t),\Phi_n(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_n(y,\tau))dyd\tau\},$$
$$(R_n(x,t) = \Phi(x,t) - \Phi_n(x,t), \quad h_n(x,t) = f(x,t) - f_n(x,t)).$$

The equation above can be modified to take the form

$$R_n(x,t) = h_n(x,t) + D(x)\{\delta(G(t),\Phi(x,t)) - \delta(G(t),\Phi_n(x,t))\}$$
$$+\{\frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)[\gamma(\Phi(y,\tau)) - \gamma(\Phi_n(y,\tau)]dyd\tau\}$$
$$+\{\frac{\delta(G(t),\Phi(x,t)) - \delta(G(t),\Phi_n(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_n(y,\tau)dyd\tau\}$$
$$| R_n(x,t) | \leq | h_n(x,t) | + | D(x) || \delta(G(t),\Phi(x,t)) - \delta(G(t),\Phi_n(x,t)) |$$
$$+\frac{| \delta(G(t),\Phi(x,t)) |}{\Gamma(\alpha)} | \int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)[\gamma(\Phi(y,\tau)) - \gamma(\Phi_n(y,\tau)]dyd\tau |$$
$$+\frac{| \delta(G(t),\Phi(x,t)) - \delta(G(t),\Phi_n(x,t)) |}{\Gamma(\alpha)} | \int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_n(y,\tau)dyd\tau | .$$

Using the conditions (i)-(v), we follow

$$| R_n(x,t) | \leq | h_n(x,t) | + DB_2 | R_n(x,t) | + \frac{B_1}{\Gamma(\alpha)} | \Phi(x,t) || \int_0^t\int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)[\gamma(\Phi(y,\tau)) - \gamma(\Phi_n(y,\tau)]dy |$$

$$+\frac{B_1}{\Gamma(\alpha)} \mid R_n(x,t) \mid\mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi_n(y,\tau) dy d\tau \mid .$$

Hence, we have

$$\| R_n(x,t) \| \leq \| h_n(x,t) \| + DB \| R_n(x,t) \| + \frac{BCHA}{\Gamma(\alpha)} \| \Phi(x,t) \| \| \| R_n(x,t) \| \frac{BCHA}{\Gamma(\alpha)} \| \Phi_n(x,t) \| \| \| R_n(x,t) \| .$$

Assuming that $\| \Phi(x,t) \| = Q, \| \Phi_n(x,t) \| = Q_n$, hence we have

$$\| R_n(x,t) \| \leq \| h_n(x,t) \| + \frac{BD\Gamma(\alpha) + BCHAQ + BCHAQ_n}{\Gamma(\alpha)} \| R_n(x,t) \| .$$

Hence, the error is stable under the condition

$$\Gamma(\alpha) > \frac{BCHA(Q+Q_n)}{1-BD}, (BD < 1).$$

6. The separation of the variables

It is in this section that the approach of separation of variables is applied the fractional MIE (1.3) needs to be transformed into the FIE in position. Suppose that the functions in equation (1.3) that are unknown and known are, in turn, the following:

$$\Phi(x,t) = \Phi(x)\Psi(t), \quad f(x,t) = f(x)\Psi(t), \quad \Psi(0) \neq 0, \tag{6.1}$$

where $\Psi(t), f(x)$ are known functions and $\Phi(x)$ has unknown function.

We get the following result by substituting (6.1) into (1.3):

$$\Phi(x)\Psi(t) = f(x)\Psi(t) + D(x)\delta(G(t), \Phi(x)\Psi(t)) + \frac{\delta(G(t), \Phi(x)\Psi(t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y)\Psi(\tau)) dy d\tau. \tag{6.2}$$

By dividing both sides of Equation (6.2) via $\Psi(t)$, we are able to generate the following result:

$$\Phi(x) = f(x) + D(x)\frac{\delta(G(t), \Phi(x)\Psi(t))}{\Psi(t)} + \frac{\delta(G(t), \Phi(x)\Psi(t))}{\Gamma(\alpha)\Psi(t)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y)\Psi(\tau)) dy d\tau. \tag{6.3}$$

The expression for formula (6.3) is

$$\Phi(x) = f(x) + D(x)\delta(\Phi(x))\delta_1(t) + \delta(\Phi(x))\delta_1(t)\lambda(t) \int_{-1}^1 k(x,y)\gamma(\Phi(y)) dy, \tag{6.4}$$

where

$$\delta_1(t) = \frac{\delta(G(t), \Phi(x)}{\Psi(t))},$$

$$\lambda(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(\tau) \gamma(\Psi(\tau)) d\tau.$$

Equation (6.4) represent NFIE of the second type.

7. Chebyshev polynomials of the sixth type

This section presents an application of CP6K for the purpose of solving NFIE (6.4).

There are orthogonal functions that are Chebyshev polynomials of the sixth type, denoted by $Y_j(\xi)$ on the interval $[-1,1]$ described using the next recurrence relation:

$$Y_j(\xi) = \xi Y_{j-1} - \frac{j(j+1)+(-1)^j(2j+1)+1}{4j(j+1)} Y_{j-2}(\xi), \qquad Y_0(\xi)=1, \qquad Y_1(\xi)=\xi, \qquad j \geq 2. \tag{7.1}$$

On the interval $[-1,1]$, the following information polynomials are orthogonal to the weight function according to the weight function $\omega_2(\xi) = \xi^2\sqrt{1-\xi^2}$ and satisfies:

$$\int_{-1}^{1} \xi^2\sqrt{1-\xi^2}\, Y_i(\xi) Y_j(\xi) d\xi = \begin{cases} \Theta_i, & i = j, \\ 0, & i \neq j, \end{cases}$$

where

$$\Theta_i = \frac{\pi}{2^{2i+3}} \begin{cases} 1, & i \text{ even}, \\ \frac{i+3}{i+1}, & i \text{ odd}. \end{cases}$$

The trigonometric representation of CP6K was demonstrated by Abd-Elhameed and Youssri in [28] as follows:

$$Y_j(\cos\theta) = \begin{cases} \frac{\sin((j+2)\theta)}{2^j \sin(2\theta)}, & j \text{ even}, \\ \frac{\sin((j+1)\theta)+(j+1)\cos(\theta)\sin((j+2)\theta)}{2^{j+1}(j+1)\cos^2(\theta)\sin(\theta)}, & j \text{ odd}. \end{cases} \tag{7.2}$$

Lemma 7.1. [29] Let m be any non-negative integer. Therefore, the power form that can be used to describe CP6K is as explained below:

$$Y_{2m}(\xi) = \frac{\Gamma(m+\frac{3}{2})}{(2m+1)!} \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i} (2m-i+1)!}{\Gamma(m-i+\frac{3}{2})}\, \xi^{2m-2i}, \tag{7.3}$$

and

$$Y_{2m+1}(\xi) = \frac{\Gamma(m+\frac{5}{2})}{(2m+2)!} \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i} (2m-i+2)!}{\Gamma(m-i+\frac{5}{2})}\, \xi^{2m-2i+1}. \tag{7.4}$$

Lemma 7.2. [29] Let p be any non-negative integer. Following that, the formulas for the inversion of CP6K are as follows:

$$\xi^{2p} = (2p+1)! \sum_{i=0}^{p} \frac{2^{1-2i}(p-i+1)!}{i!(2p-i+2)!}\, Y^{2p-2i}(\xi), \tag{7.5}$$

and

$$\xi^{2p+1} = (2p+1)!(2p+3) \sum_{i=0}^{p} \frac{2^{1-2i}(p-i+1)!}{i!(2p-i+3)!}\, Y^{2p-2i+1}(\xi). \tag{7.6}$$

Let $\Phi(x) \in L_2[-1,1]$, hence the following $Y_n(x)$ approximation for $\Phi(x)$ can be made:

$$\Phi(x) \approx \Phi_N(x) = \sum_{n=0}^{N} a_n\, Y_n(x), \tag{7.7}$$

where the constants a_n are $n = 0, 1, \ldots, N$.

Substituting from (7.7) into (6.4) we get

$$\sum_{n=0}^{N} a_n\, Y_n(x) - D(x)\delta(\sum_{n=0}^{N} a_n\, Y_n(x))\delta_1(t) + \delta(\sum_{n=0}^{N} a_n\, Y_n(x))\delta_1(t)\lambda(t)\int_{-1}^{1} k(x,y)\gamma(\sum_{n=0}^{N} a_n\, Y_n(y))dy = f(x). \tag{7.8}$$

Now, by employing the collection of points

$$x_r = a + \frac{(b-a)r}{N}, \quad r = 0, 1, 2, \ldots, N, \tag{7.9}$$

For $N+1$ unknowns, we get the following $N+1$ SNAE:

$$\sum_{n=0}^{N} a_n\, Y_n(x_r) - D(x)\delta(\sum_{n=0}^{N} a_n\, Y_n(x_r))\delta_1(t) + \delta(\sum_{n=0}^{N} a_n\, Y_n(x_r))\delta_1(t)\lambda(t)\int_{-1}^{1} k(x_r, y)\gamma(\sum_{n=0}^{N} a_n\, Y_n(y))dy = f(x_r). \tag{7.10}$$

As a result of solving the system described above, we present the approximate solution and employing the use of (6.1) through numerical analysis, we are able to get the solution to equation (1.3), which corresponds to the NfrPIo-DE equation (1.1).

8. Illustrative Example

This section will provide an illustration of the outcomes presented above by the following example given several kinds of values of time and various values of α.

Example 1. Suppose that the NfrPIo-DE:

$$\Phi(x,t) = f(x,y) + (e^{-t}\Phi(x,t))[\frac{(u(x) - f(x,0))}{\delta(G(0), u(x))}] + \frac{1}{\Gamma(\alpha)}(e^{-t}\Phi(x,t))\int_0^t \int_{-1}^{1} (t-\tau)^{\alpha-1}(0.1+\tau)x^2y^2\Phi^2(y,\tau)dyd\tau, \tag{8.1}$$

with exact solution $\Phi(x,t) = x^4(0.01+t^2), G(t) = e^{-t}, \delta(G(t), \Phi(x,t)) = e^{-t}\Phi(x,t)$.

At $N = 5$ and different values of T and α are used with separation of variables and CP6K for Equation (8.1), the numbers in tables 1-4 and figures 1-4 show the results.

Table 1: The error of absolute value of example 1, $T = 0.001$

x	Error, $\alpha = 0.0004$,	Error, $\alpha = \frac{1}{5}$	Error, $\alpha = \frac{1}{3}$	Error, $\alpha = \frac{1}{2}$	Error,$\alpha = 0.8$	Error, $\alpha = 0.9$
-1	2.7×10^{-13}	1.9×10^{-12}	4.61×10^{-26}	1.9×10^{-12}	1.01×10^{-29}	2.2×10^{-13}
-0.8	9.7×10^{-14}	1.2×10^{-12}	3.6×10^{-26}	1.2×10^{-12}	2.9×10^{-30}	7.3×10^{-14}
-0.6	5.8×10^{-14}	7.1×10^{-13}	2.6×10^{-26}	7.1×10^{-13}	6.1×10^{-32}	1.3×10^{-14}
-0.4	5.2×10^{-14}	3.2×10^{-13}	1.7×10^{-26}	3.2×10^{-13}	4.2×10^{-30}	6.2×10^{-15}
-0.2	3.4×10^{-14}	8.6×10^{-14}	8.6×10^{-27}	8.6×10^{-14}	4.01×10^{-30}	6.9×10^{-15}
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	3.4×10^{-14}	7.3×10^{-14}	8.6×10^{-27}	7.3×10^{-14}	4.01×10^{-30}	6.9×10^{-15}
0.4	5.2×10^{-14}	3.1×10^{-13}	1.7×10^{-26}	3.1×10^{-13}	4.2×10^{-30}	7.10×10^{-15}
0.6	5.8×10^{-14}	7.2×10^{-13}	2.6×10^{-26}	7.2×10^{-13}	6.1×10^{-32}	3.1×10^{-15}
0.8	9.7×10^{-14}	1.2×10^{-12}	3.6×10^{-26}	1.2×10^{-12}	2.9×10^{-30}	1.6×10^{-14}
1	2.7×10^{-13}	2.01×10^{-12}	4.6×10^{-26}	2.01×10^{-12}	1.01×10^{-29}	1.6×10^{-15}

Table 2: At $T = 0.2$, clarification of the absolute error for example 1

x	Error, $\alpha = 0.0004$,	Error, $\alpha = \frac{1}{5}$	Error, $\alpha = \frac{1}{3}$	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$	Error, $\alpha = 0.9$
-1	1.7×10^{-13}	5.3×10^{-14}	4.6×10^{-14}	5.09×10^{-12}	2.1×10^{-13}	5.03×10^{-12}
-0.8	4.9×10^{-14}	1.5×10^{-14}	3.3×10^{-14}	3.2×10^{-12}	8.1×10^{-14}	3.2×10^{-12}
-0.6	2.4×10^{-14}	5.8×10^{-14}	1.3×10^{-14}	1.7×10^{-12}	2.6×10^{-14}	1.8×10^{-12}
-0.4	2.4×10^{-14}	6.6×10^{-14}	1.52×10^{-14}	7.87×10^{-13}	6.9×10^{-15}	8.1×10^{-13}
-0.2	1.7×10^{-14}	4.2×10^{-14}	1.8×10^{-14}	1.9×10^{-13}	1.6×10^{-15}	2.09×10^{-13}
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	1.7×10^{-14}	4.2×10^{-14}	1.8×10^{-14}	2.08×10^{-13}	1.6×10^{-15}	1.9×10^{-13}
0.4	2.4×10^{-14}	6.6×10^{-14}	1.5×10^{-14}	8.1×10^{-13}	6.9×10^{-15}	7.8×10^{-13}
0.6	2.4×10^{-14}	5.8×10^{-14}	1.3×10^{-14}	1.8×10^{-12}	2.6×10^{-14}	1.7×10^{-12}
0.8	4.9×10^{-14}	1.5×10^{-14}	3.3×10^{-14}	3.1×10^{-12}	8.1×10^{-14}	3.1×10^{-12}
1	1.7×10^{-13}	5.3×10^{-14}	4.6×10^{-14}	4.9×10^{-12}	2.1×10^{-13}	4.9×10^{-12}

When $T = 0.4, \alpha = 0.0004, \frac{1}{5}, \frac{1}{3}$ we obtain the approximate solution as the exact solution, while at the same value of $T = 0.4$ and $\alpha = \frac{1}{2}, 0.8, 0.9$ the numerical results are shown in table 3.

Table 3: The error of absolute value for example 1, $T = 0.4$

x	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$	Error, $\alpha = 0.9$
-1	5.09×10^{-25}	5.005×10^{-13}	6.5×10^{-13}
-0.8	6.3×10^{-25}	1.4×10^{-15}	1.4×10^{-14}
-0.6	6.04×10^{-25}	7.3×10^{-15}	4.2×10^{-15}
-0.4	4.6×10^{-25}	9.3×10^{-14}	1.03×10^{-13}
-0.2	2.5×10^{-25}	9.01×10^{-14}	1.03×10^{-13}
0	0.000000	0.000000	0.000000
0.2	2.5×10^{-25}	9.01×10^{-14}	1.03×10^{-13}
0.4	4.6×10^{-25}	9.3×10^{-14}	1.03×10^{-13}
0.6	6.04×10^{-25}	7.3×10^{-15}	4.24×10^{-15}
0.8	4.24×10^{-25}	1.4×10^{-15}	1.4×10^{-14}
1	5.09×10^{-25}	5.005×10^{-13}	6.5×10^{-12}

When $T = 0.6, \alpha = \frac{1}{3}, 0.9$ we obtain the approximate solution as the exact solution, while at the same value of $T = 0.4$ and some different values of $\alpha = 0.0004, \frac{1}{5}, \frac{1}{2}, 0.8$ the numerical results are shown in table 4.

Table 4: The error of absolute value for example 1, $T = 0.6$

x	Error, $\alpha = 0.0004$	Error, $\alpha = \frac{1}{5}$	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$
-1	3.7×10^{-11}	3.4×10^{-11}	5.5×10^{-12}	1.04×10^{-12}
-0.8	2.3×10^{-11}	2.3×10^{-11}	1.4×10^{-12}	1.2×10^{-12}
-0.6	1.3×10^{-11}	1.3×10^{-11}	4.6×10^{-13}	4.6×10^{-13}
-0.4	5.9×10^{-14}	6.1×10^{-12}	8.4×10^{-14}	1.4×10^{-13}
-0.2	1.4×10^{-12}	1.5×10^{-12}	2.9×10^{-14}	2.5×10^{-13}
0	0.000000	0.000000	0.000000	0.000000
0.2	1.4×10^{-12}	1.3×10^{-12}	2.9×10^{-14}	2.5×10^{-13}
0.4	5.9×10^{-12}	5.7×10^{-12}	8.4×10^{-14}	1.4×10^{-13}
0.6	1.3×10^{-11}	1.3×10^{-11}	1.08×10^{-13}	4.6×10^{-13}
0.8	2.3×10^{-11}	2.4×10^{-11}	1.4×10^{-12}	1.2×10^{-12}
1	3.7×10^{-11}	3.9×10^{-11}	5.5×10^{-12}	1.04×10^{-12}

Table 5: Absolute Error of example 1, $T = 0.8$

x	Error, $\alpha = 0.0004$,	Error, $\alpha = \frac{1}{4}$	Error, $\alpha = \frac{1}{3}$	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$	Error, $\alpha = 0.9$
-1	2.4×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.05×10^{-10}	1.7×10^{-10}
-0.8	3.8×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	2.5×10^{-11}	3.2×10^{-12}
-0.6	2.8×10^{-11}	7.9×10^{-12}	7.9×10^{-12}	7.9×10^{-12}	7.2×10^{-11}	5.2×10^{-11}
-0.4	2.8×10^{-14}	1.3×10^{-11}	1.3×10^{-11}	1.3×10^{-11}	7.8×10^{-11}	3.9×10^{-11}
-0.2	9.6×10^{-12}	4.7×10^{-12}	4.7×10^{-12}	4.7×10^{-14}	6.9×10^{-11}	1.2×10^{-11}
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	9.6×10^{-12}	4.7×10^{-12}	4.7×10^{-12}	4.7×10^{-12}	6.9×10^{-11}	1.2×10^{-11}
0.4	2.8×10^{-11}	1.3×10^{-11}	1.3×10^{-11}	1.3×10^{-11}	7.8×10^{-11}	3.9×10^{-11}
0.6	2.8×10^{-11}	7.9×10^{-12}	7.9×10^{-12}	7.9×10^{-12}	7.2×10^{-11}	5.2×10^{-11}
0.8	3.8×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	2.5×10^{-11}	3.20×10^{-12}
1	2.4×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.05×10^{-10}	1.7×10^{-10}

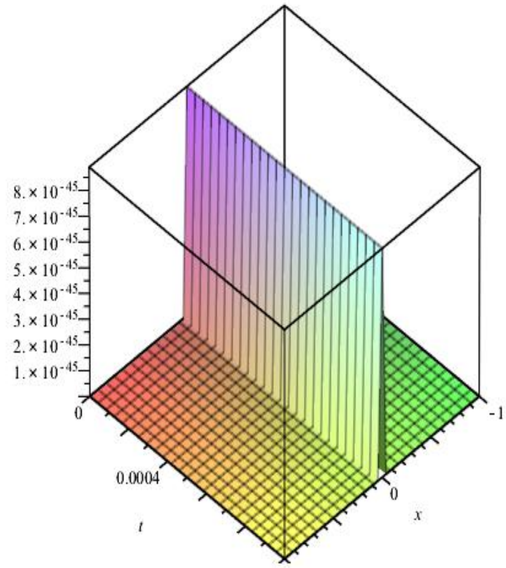

Figure 1: Clarification of the error of absolute value of example 1, $T = 0.001, \alpha = 0.8$

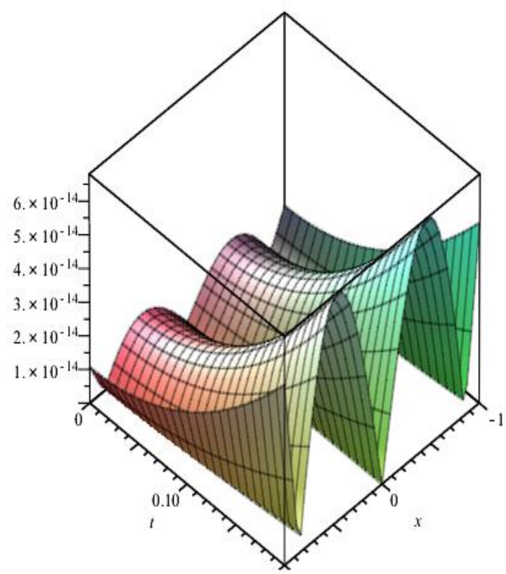

Figure 2: Clarification of the error of absolute value example 1 at, $T = 0.2, \alpha = \frac{1}{5}$

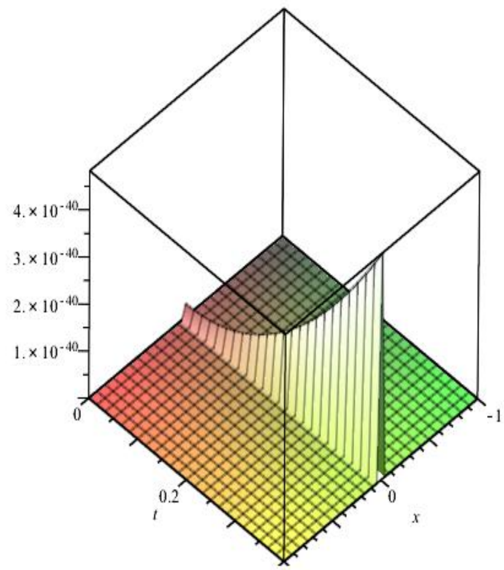

Figure 3: At $T = 0.4, \alpha = \frac{1}{2}$, clarification of the absolute error of example 1.

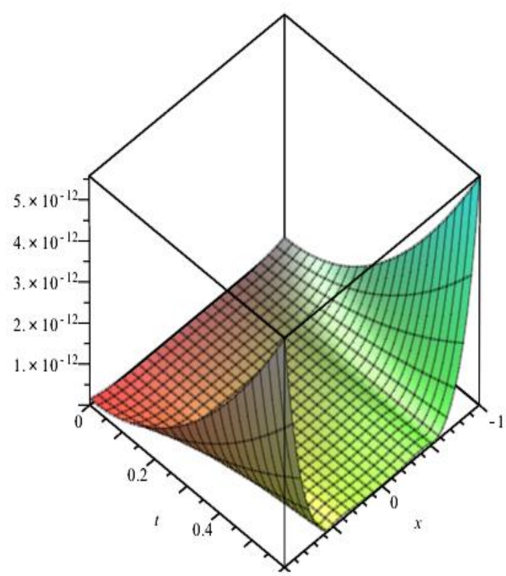

Figure 4: At $T = 0.6, \alpha = \frac{1}{2}$, clarification of the error of absolute value of example 1.

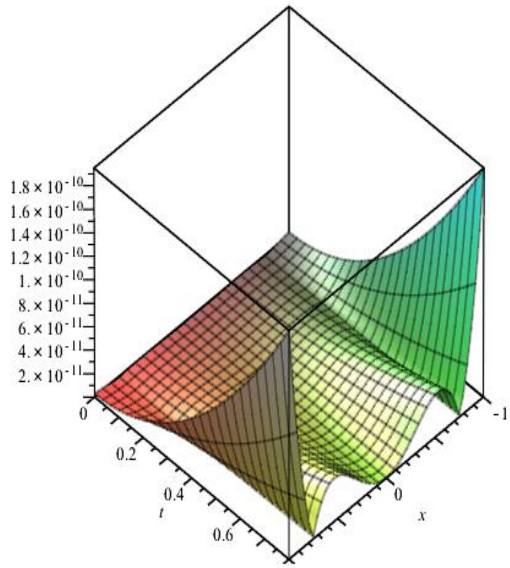

Figure 5: At $T = 0.8, \alpha = 0.0004$, clarification of the error of absolute value of example 1.

9. Conclusion and Characterization of fractional time

In the present study, CP6K is used to solve NfrPIo-DE with a continuous kernel with the assistance of variable separation. We illustrate the theoretical results by numerical example for different values of time and α. The results were analyzed as follows:
The fractional tense is related to the kernel of Apple on the form $F(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$.

This nucleus has only one anomalous point on the path of integration when $t = \tau$. Knowing the form of the time function, researchers can convert this abnormal integral into a form of the beta function.

Therefore, researchers prefer to convert FDEs into fractional integral equations, and in the end, the situation will be, knowing the time function in one of the forms of the beta function. Also, the results depend on our first choice of the form of the time function. In economics in general, it means the time required to reach the best product. The authors indicated that the best general form of the time function is as follows $G(t) = e^{-t}$. This choice is often used by researchers in the science of thermoelasticity or in the science of atmospheric differential equations emerging in the science of mathematical physics, in general, and is known as the normal mode method. As for the second aspect in solving the problems of fractional equations in general, it is represented in our choice of the numerical method. It has become clear through the applications and the choice of methods that the traditional methods lead to the accumulation of error and its increase with the increase in the time of the experiment. But it turns out that the best methods to use are the methods derived from the polynomials.

It is noted that the value of α plays a critical role in the description and flow of the line of integrations, and the following tables shows the relationship of α, time and absolute error.

Table 6.

T	α	Less value for error
0.001	0.8	6.1×10^{-32}
0.2	0.8	1.6×10^{-15}
0.4	$\frac{1}{2}$	2.8×10^{-25}
0.6	$\frac{1}{2}$	2.9×10^{-14}
0.8	0.0004	2.8×10^{-14}

Table 7.

T	α	The largest error value
0.001	$\frac{1}{2}$	1.2×10^{-12}
0.2	0.9	1.8×10^{-12}
0.4	0.9	6.5×10^{-12}
0.6	$\frac{1}{5}$	1.3×10^{-11}
0.8	0.0004	2.4×10^{-10}

It is known that the authors, during their discussion of the research, dealt with the time function as follows:

If $G(t) = e^{-t}, t \in [0, T], T < 1$. This choice was based on the fact that the time function should be decreasing and not increasing. And so we find on for example
$G(t) = e^{t} \Longrightarrow G(0.5) = e^{0.5} = 1.652271164185831, \quad G(0.25) = e^{0.25} = 1.285407003320672.$
Meaning that the positive exponential function when the fractional time is greater than zero is the time function Often greater than the correct one, which makes the error increasing in cumulative proportions.

As for when the function $G(t) = e^{-t}, t \in [0, T], T < 1$ the highest value at zero is the correct one and at the fractional time we find that $G(0.5) = e^{-0.5} = 0.6052275326688024, \quad G(0) = 1, \quad G(0.24) = 0.7779637090949696$. These results show that the most appropriate choice of time is the negative exponential function or choosing a part of it.

Future work: The technique described above can be expanded in further work to solve NfrPIo-DE with a discontinuous kernel.

Conflict of interest statement

Not Applicable.

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