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LEVINSON'S CONJECTURE TO NEWTONIAN SYSTEMS WITH JUMPING NONLINEARITY

Yingying Li[1,†]

Abstract This paper concerns the jumping nonlinear Newtonian systems with friction. We show the existence of periodic solutions by using Lyapunov's methods and the modular degree theory. Furthermore, we apply our main result to find periodic solutions in a suspension bridge model.

Keywords Jumping nonlinearity, periodic solutions, Newtonian systems with friction, Lyapunov's methods.

1. Introduction

At the end of the 19th century, Poincaré and Lyapunov established the qualitative theory of differential equations, in which the study of periodic solutions is fundamental, see [1, 5–7, 10, 13, 15, 16, 18–20, 22] for some developments. Fixed-point theorems play an significant role in the study of periodic solutions, and remarkable progress has been done in this field. For example, Furumochi and Naito [10] discussed the autonomous difference equations and showed the existence of periodic solutions by the Schauder fixed point theorem. Wang [20] established a second-order non-autonomous singular dynamic systems and proved the existence of positive periodic solutions by the Krasnoselskii fixed point theorem. For a periodic system, Fink [9] showed that if systems are exponential uniform asymptotic stable, then a periodic solution exists. Li et al. [16] studied the existence of affine-periodic solutions for Newtonian systems with friction, and proved Levinson's conjecture by Lyapunov's methods.

In physical applications, differential systems with piecewise linearity may characterization various vibration processes, such as engineering [3,14], neural networks ones [21], and in particular in mechanics [17], among others. Fonda et al. [8] proved the existence of large-scale subharmonic solutions by means of a method known as the Poincaré-Birkhoff theorem. Humphreys and Mackenna [12] proved multiple periodic solutions by Leray-Schauder degree theory. Aravinth et al. [2] were concerned with the time periodic piecewise systems and obtained the criterion for the stability of the system by constructing a Lyapunov function. To the best of our knowledge, the bridge will appear large vibration in a large storm. Thereby, the stability analysis for the nonlinear suspension bridge model will be explored later as an application of the main results of this paper. The primary goal of this paper is to study the periodic solutions under jumping nonlinearity for Newtonian systems with friction.

[†]The corresponding author.
[1]School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
Email: yingyingli0306@nenu.edu.cn(Y. Li),

Based on the theoretical underpinning of the Lyapunov function approach in [16], we are now in a position to prove the theorem associated with periodic solutions. To establish the result, we need the modular degree theorem [23].

This article is organized as follows. In Section 2, we state the main result and give its proof. Finally, we apply our main result to the classical example of the suspension bridge model.

2. Newtionian systems with jump

2.1. Preliminaries

In this section, we study the following Newtonian equation

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + \nabla V(x) = p(t), \tag{2.1}$$

where $D : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p : \mathbb{R} \to \mathbb{R}$ are continuous and $V \in C^1(\mathbb{R}, \mathbb{R})$.

When

$$V(x) = \begin{cases} \frac{1}{2}bx^2, & x \geq A, \\ \frac{1}{2}ax^2, & x < -A, \end{cases}$$

where $a, b > 0$ are constant, $|x| \geq A > 0$, then (2.1) becomes

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + bx^+ - ax^- = p(t), \tag{2.2}$$

where $x^+ = max\{x, 0\}$, $x^- = max\{-x, 0\}$.

If $a \neq b$, such an equation is usually called the jumping nonlinear Newtonian one. The above equation is equivalent to

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + bx = p(t), \qquad x \geq A, \tag{2.3}$$

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + ax = p(t), \qquad x < -A. \tag{2.4}$$

2.2. Theorem and Proof

Theorem 2.1. *Suppose D and p are continuous and satisfy the following T-periodicity:*

$$D(t + T, x, \dot{x}) = D(t, x, \dot{x}),$$
$$p(t + T) = p(t),$$

and $D \geq \sigma_0 > 0$. Also assume the solution of (2.2) with respect to initial values is unique. Then system (2.2) has T-periodic solutions.

Proof. Equation (2.3) is equivalent to

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -Dy - bx + p(t). \end{cases}$$

Let

$$U(x, y) = \lambda y^2 + (x + y)^2, \quad \lambda \gg 1,$$

then

$$
\begin{aligned}
\dot{U}_{(2.3)} =& 2\lambda y(-Dy - bx + p) + 2(x+y)(y - Dy - bx + p) \\
=& -2\lambda Dy^2 - 2\lambda bxy + 2\lambda py + 2xy - 2Dxy - 2bx^2 + 2px + 2y^2 \\
& -2Dy^2 - 2bxy + 2py \\
=& -(2\lambda+2)Dy^2 - 2bx^2 + 2px + (-2\lambda bxy + 2\lambda py + 2xy - 2Dxy \\
& + 2y^2 - 2bxy + 2py) \\
=& -(\lambda+2)Dy^2 - bx^2 - b(x^2 - 2\frac{p}{b}x) - \lambda\Big[(\sqrt{D}y)^2 + 2(\sqrt{D}y)\frac{bx}{\sqrt{D}} \\
& -2(\sqrt{D}y)\frac{p}{\sqrt{D}} - 2(\sqrt{D}y)\frac{x}{\lambda\sqrt{D}} + 2(\sqrt{D}y)\frac{\sqrt{D}x}{\lambda} - 2(\sqrt{D}y)\frac{y}{\lambda\sqrt{D}} \\
& + 2(\sqrt{D}y)\frac{bx}{\lambda\sqrt{D}} - 2(\sqrt{D}y)\frac{p}{\lambda\sqrt{D}}\Big] \\
=& -(\lambda+2)Dy^2 - bx^2 - b\big(x - \frac{p}{b}\big)^2 + b\big(\frac{p}{b}\big)^2 - \lambda\Big[\sqrt{D}y - \Big(-\frac{b}{\sqrt{D}}x \\
& + \frac{p}{\sqrt{D}} + \frac{x}{\lambda\sqrt{D}} - \frac{\sqrt{D}x}{\lambda} + \frac{y}{\lambda\sqrt{D}} - \frac{bx}{\lambda\sqrt{D}} + \frac{p}{\lambda\sqrt{D}}\Big)\Big]^2 \\
& + \lambda\Big[-\frac{bx}{\sqrt{D}} + \frac{p}{\sqrt{D}} + \frac{x}{\lambda\sqrt{D}} - \frac{\sqrt{D}x}{\lambda} + \frac{y}{\lambda\sqrt{D}} - \frac{bx}{\lambda\sqrt{D}} + \frac{p}{\lambda\sqrt{D}}\Big]^2 \\
\le& -(\lambda+2)Dy^2 - bx^2 + \frac{p^2}{b} + \lambda\Big[-\frac{bx}{\sqrt{D}} + \frac{p}{\sqrt{D}} + \frac{x}{\lambda\sqrt{D}} - \frac{\sqrt{D}x}{\lambda} \\
& + \frac{y}{\lambda\sqrt{D}} - \frac{bx}{\lambda\sqrt{D}} + \frac{p}{\lambda\sqrt{D}}\Big]^2 \\
=& -(\lambda+2)Dy^2 - bx^2 + \frac{p^2}{b} + \lambda\Big[\frac{y}{\lambda\sqrt{D}} + \Big(-\frac{bx}{\sqrt{D}} + \frac{1}{\lambda\sqrt{D}} - \frac{\sqrt{D}}{\lambda} \\
& - \frac{b}{\lambda\sqrt{D}}\Big)x + \Big(\frac{1}{\sqrt{D}} + \frac{1}{\lambda\sqrt{D}}\Big)p\Big]^2 \\
\le& -(\lambda+2)Dy^2 - bx^2 + \frac{p^2}{b} + \frac{W}{\lambda}y^2 + \frac{W}{\lambda}(\lambda^2b^2 + 1 + D + b^2)x^2 \\
& + \frac{W}{\lambda}(1+\lambda^2)p^2 \\
=& \Big(-(\lambda+2)D + \frac{W}{\lambda}\Big)y^2 + \Big(-b + \frac{W}{\lambda}(\lambda^2b^2 + 1 + D + b^2)\Big)x^2 \\
& + \Big(\frac{1}{b} + \lambda W + \frac{W}{\lambda}\Big)p^2,
\end{aligned}
$$

for some $W > 0$. Thus, for sufficiently large λ and sufficiently small $\xi(>0)$ and some $H > 0$,

$$
\dot{U} \le -\xi(y^2 + x^2) + H, \tag{2.5}
$$

where $-(\lambda+2)D + \frac{W}{\lambda} \le -\xi$, $-b + \frac{W}{\lambda}(1 + \lambda^2b^2 + D + b^2) \le -\xi$ and $H = \Big(\frac{1}{b} + \lambda W + \frac{W}{\lambda}\Big)p^2$. Obviously, there exists $1 \ge \eta > 0$ such that

$$
U \ge \eta(y^2 + x^2). \tag{2.6}
$$

For (2.4), $\dot{U}_{(2.4)}$ is also similar. In fact,

$$\begin{aligned}
\dot{U}_{(2.4)} =&2\lambda y(-Dy-ax+p)+2(x+y)(y-Dy-ax+p)\\
=&-(2\lambda+2)Dy^2-2ax^2+2px+(-2\lambda axy+2\lambda py+2xy-2Dxy\\
&+2y^2-2axy+2py)\\
=&-(\lambda+2)Dy^2-ax^2-a\big(x^2-2x\frac{p}{a}\big)-\lambda\Big[(\sqrt{D}y)^2+2(\sqrt{D}y)\frac{ax}{\sqrt{D}}\\
&-2(\sqrt{D}y)\frac{p}{\sqrt{D}}-2(\sqrt{D}y)\frac{x}{\lambda\sqrt{D}}+2(\sqrt{D}y)\frac{\sqrt{C}x}{\lambda}-2(\sqrt{D}y)\frac{y}{\lambda\sqrt{D}}\\
&+2(\sqrt{D}y)\frac{ax}{\lambda\sqrt{D}}-2(\sqrt{D}y)\frac{p}{\lambda\sqrt{D}}\Big]\\
=&-(\lambda+2)Dy^2-ax^2-a\big(x-\frac{p}{a}\big)^2+a\big(\frac{p}{a}\big)^2-\lambda\Big[\sqrt{D}y-(-\frac{a}{\sqrt{D}}x\\
&+\frac{p}{\sqrt{D}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}+\frac{y}{\lambda\sqrt{D}}-\frac{ax}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}})\Big]^2\\
&+\lambda\Big[-\frac{a}{\sqrt{D}}x+\frac{p}{\sqrt{D}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}+\frac{y}{\lambda\sqrt{D}}-\frac{ax}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}}\Big]^2\\
\le&-(\lambda+2)Dy^2-ax^2+\frac{p^2}{a}+\lambda\Big[-\frac{a}{\sqrt{D}}x+\frac{p}{\sqrt{C}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}\\
&+\frac{y}{\lambda\sqrt{D}}-\frac{ax}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}}\Big]^2\\
\le&-(\lambda+2)Dy^2-ax^2+\frac{p^2}{a}+\lambda\Big[\frac{y}{\lambda\sqrt{D}}+\Big(-\frac{a}{\sqrt{D}}+\frac{1}{\lambda\sqrt{D}}\\
&-\frac{\sqrt{D}}{\lambda}-\frac{a}{\lambda\sqrt{D}}\Big)x+\Big(\frac{1}{\lambda\sqrt{D}}+\frac{1}{\sqrt{D}}\Big)e\Big]^2\\
\le&-(\lambda+2)Dy^2-ax^2+\frac{p^2}{a}+\frac{W}{\lambda}y^2+\frac{W}{\lambda}(\lambda^2a^2+1+D\\
&+a^2)x^2+\frac{W}{\lambda}(1+\lambda^2)p^2\\
=&\Big(-(\lambda+2)D+\frac{W}{\lambda}\Big)y^2+\Big(-a+\frac{W}{\lambda}(\lambda^2a^2+1+D\\
&+a^2)\Big)x^2+\Big(\frac{1}{a}+\lambda W+\frac{W}{\lambda}\Big)p^2
\end{aligned}$$

for some $W>0$.

Applying Gronwall-Bellman inequality to (2.5), we get

$$U(x(t),y(t))\le U(x_0,y_0)e^{\int_0^t[-\xi(x^2(t)+y^2(t))+H]ds},$$

the any solution $\omega(t,\omega_0)=(x(t,x_0,y_0),y(t,x_0,y_0))$ of (2.2) with the initial value $\omega(0)=(x(0),y(0))=(x_0,y_0)$, which together with (2.6), we have

$$\eta\omega^2(t,\omega_0)\le U(\omega(t,\omega_0))\le U(\omega_0)e^{\int_0^t(-\xi\omega^2(s,\omega_0)+H)ds}, \tag{2.7}$$

that is to say, for any ω_0, the solution $\omega(t,\omega_0)$ exists on $\mathbb{R}^1_+$. That is,

$$\omega^2(t,\omega_0)\ge 0,$$

$$-\xi\omega^2(t,\omega_0)+H \le H,$$
$$e^{\int_0^t(-\xi\omega^2(s,\omega_0)+H)ds} \le e^{Ht}.$$

Take

$$\hat{M} = \max\left\{U(x,y)||x|^2+|y|^2 \le \frac{H+1}{\xi}\right\}, \tag{2.8}$$
$$\bar{M} = \max\left\{U(x,y)||x|^2+|y|^2 \le R^2\right\}, \quad R^2 \ge \frac{H+1}{\xi}. \tag{2.9}$$

Then

$$-\xi\omega^2(t,\omega_0)+H \ge -1,$$
$$e^{\int_0^t(-\xi\omega^2(s,\omega_0)+H)ds} \ge e^{-t}.$$

According to the above,

$$e^{-t} \le e^{\int_0^t(-\xi\omega^2(s,\omega_0)+H)ds} \le e^{Ht},$$

and according to (2.7) and (2.8), $\exists N(R)\in\mathbb{N}$, we have

$$\eta\omega^2(t,\omega_0) \le Me^{HN(R)T}, \quad \forall t \ge N(R)T,$$

so

$$\omega^2(t,\omega_0) \le \frac{M}{\eta}e^{HN(R)T},$$

i.e.

$$\omega^2(t,\omega_0) \le B, \quad \forall t \ge N(R)T, \tag{2.10}$$

where $B^2 = \frac{M}{\eta}e^{HN(R)T}$. Set

$$\Omega = \{\omega_0 \in \mathbb{R}^2 ||\omega_0| < B+1\}. \tag{2.11}$$

And define Poincaré map P, we have

$$P(\omega_0) = \omega(T,\omega_0), \tag{2.12}$$

then

$$P^i(\omega_0) = \omega(iT,\omega_0) \quad \forall i \ge 1.$$

By (2.10), we get

$$|P^i(\omega_0)| = |\omega(iT,\omega_0)| \le B \quad \forall i = N(B+1), N(B+1)+1, \quad \forall\omega_0 \in \partial\Omega. \tag{2.13}$$

By (2.11) and (2.13), we get

$$P^i(\omega_0) \in \Omega, \quad i = N, N+1. \tag{2.14}$$

We set N to be a prime number. By (2.11) – (2.14) and the Rothe theorem, we get

$$\deg(\mathrm{id} - P^N, \Omega, 0) = 1,$$

according to the modular degree theorem [16, 23], we have

$$\deg(\mathrm{id} - P, \Omega, 0) = 1 \neq 0,$$

so

$$P(\omega^*) = \omega^*, \quad \omega^* \in \Omega.$$

According to (2.12),

$$\omega^* = \omega(T, \omega^*).$$

For any t, $\omega(t + T, \omega^*) = \omega(t, \omega^*)$. Hence, $\omega(t, \omega^*)$ is a T-periodic solution of equation (2.2). □

3. An application to a suspension bridge model

Due to its remarkable flexibility, the suspension bridge is extensively employed in both practical and engineering applications. However, in the face of adverse weather conditions like thunderstorms or storms, the suspension bridge may undergo significant oscillations with large amplitudes, potentially resulting in detrimental consequences, such as the catastrophic failure witnessed in the Tacoma Narrows suspension bridge [3, 11].

So, we consider the steady state of a one-dimensional suspended bridge in reference [4, 11] as follows:

$$m\ddot{z} + \delta\dot{z} + c(\pi/L)^4 z + dz^+ = mg + h(t), \tag{3.1}$$

where m is the mass per unit of length, δ is a small viscous damping coefficient, c and L represent the flexibility and length of the bridge respectively, d is the stiffness of nonlinear springs and m, δ, c, L, $d > 0$, h is continuous T-periodic.

Let $\alpha = \dfrac{\delta}{m}$, $\mu = \dfrac{c}{m}\big(\dfrac{\pi}{L}\big)^4 + \dfrac{d}{m}$, $\nu = \dfrac{c}{m}\big(\dfrac{\pi}{L}\big)^4$, $f(t) = \dfrac{1}{m}(mg + h(t))$, then system (3.1) becomes

$$\ddot{z} + \alpha\dot{z} + \mu z^+ - \nu z^- = f(t), \tag{3.2}$$

that is

$$\ddot{z} + \alpha\dot{z} + \mu z = f(t), \quad x \geq 0, \tag{3.3}$$
$$\ddot{z} + \alpha\dot{z} + \nu z = f(t), \quad x < 0. \tag{3.4}$$

Here, we will merely state the following conclusion without proving it.

Theorem 3.1. *system* (3.2) *has a T-periodic solution if f is continuous and satisfies T-periodicity with respect to t.*

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