A New Efficient Transform Mechanism with Convergence Analysis of the Space-Fractional Telegraph Equations

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Abstract

This article's goal is to investigate the space-fractional telegraph equation using an effective method called the Adomian natural decomposition method (ANDM), which is a combination of the Adomian decomposition method (ADM) and the natural transform method (NTM). Using the Banach fixed point theorem, we explore proofs for the existence and uniqueness theorems applying it to a nonlinear differential equation. Using our method, exact solutions of the space-fractional telegraph equation and time-fractional diffusion problems have been obtained. To demonstrate the effectiveness of the suggested scheme, four examples are provided.

Keywords Fractional Liouville-Caputo Derivative. Adomian Natural Decomposition Method. Space-fractional Telegraph Equation. Diffusion Equations. Banach Fixed Point Theorem.

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1 Introduction

A general version of a partial differential equation is a fractional partial differential equation, where the fractional order derivatives are substituted for the integer order derivatives. A number of researchers have focused on solving fractional differential equations because of their widespread use in numerous scientific and engineering disciplines. A fundamental illustration of a fractional partial differential equation is the fractional telegraph equation. Many approximate and analytical methods have been developed to find their solutions, see [1, 2, 3].

The modeling of anomalous diffusion systems, the description of fractional random walks, and the unification of the diffusion and wave propagation phenomena have all benefited from the widespread application of anomalous diffusion equations in recent years, see [4] and the references therein. One of these crucial anomalous diffusion equations that may be derived from the classical diffusion-wave equation is the fractional diffusion-wave equation. It is commonly recognized that a wave equation depicts a process where a disturbance propagates at a constant pace, but a diffusion equation describes a process where a disturbance spreads infinitely quickly. The fractional diffusion-wave equation therefore, in a sense, interpolates between these two dissimilar behaviors in terms of their reaction to a localized disturbance [5].

There are works accessible in the topic of fractional diffusion equation theoretical analysis. The diffusion and wave equations were described by Schneider and Wyss [6] in terms of integro-differential equations, and the appropriate Green's functions in closed form for any number of spatial dimensions were produced using fox functions. The Cauchy problem was strictly interpreted by Fujita [7] as the intermediary phenomenon between the heat equation and the wave equation:

$$D_z^{\alpha}\ (\chi(s,z)) = D_z^{\beta}\ (\chi(s,z))\,,\ 1 \leq \alpha,\ \beta \leq 2.$$

He subsequently looked into integro-differential equations [8] that have properties related to heat diffusion and wave propagation. Additionally, he discovered certain crucial characteristics of the fundamental solution, including one that is comparable to the wave equation's characteristic: the fundamental solution's maximum points propagate at a constant pace. In addition, M. Garg and P. Manohar, gave numerical solution of fractional diffusion-wave equation with two space variables by matrix method, see [9]. Agrawal [10] offered a generic solution for a fractional distortion-wave equation developed in a bounded space domain utilizing the approach of finite sine transform and Laplace transform.

He then looked at the fractional distortion-wave system [11] when it was subject to a nonhomogeneous field, which may be stochastic or deterministic. Luchko, et al. [12] investigated the basic Cauchy problem solution for the fractional diffusion-wave equation and discovered some significant properties of the solution, including its highest point, etc. Such a fractional diffusion-wave equation, which has fractional derivatives of the same order α, $1 \leq \alpha \leq 2$ both in space and time, was also taken into consideration by Luchko [4, 12, 13].

The space-fractional telegraph equations have recently been considered by Prakash [14], Momani [15], Hashmi, M. S. etal. [16], Eltayeb, H. etal. [17], M. Garg and A. Sharma

[18], Orsingher and Zhao [19], Liu, Z., & Sun, S. [20], Shen, L. B., & Han, B. S. [21] and Al-Shara, S., see [22-23]. The NADM is based on the natural transform technique (NTM) [24, 25] and the Adomian decomposition method (ADM) [26], and it offers solutions in the form of infinite series that, if the exact solution exists, may converge to a closed form solution. Rawashdeh was the first to use the fractional natural decomposition method, see [27-29]. Moreover, Obeidat, use the tempered fractional natural transform method to solve tempered fractional diffusion equations, see [30, 31].

In general, it is difficult to solve the fractional differential equation analytically, hence it is crucial to find exact solutions to these problems. As a result, we examine the space-fractional telegraph and fractional diffusion equations below in this research work.

We shall study in our research the following space-fractional telegraph equation of the form [10]:

$$^{c}D_{s}^{\alpha}\;(\chi(s,z)) = \chi_{zz}(s,z) + a\,\chi_{z}(s,z) + b\,\chi^{n}(s,z) + h(s,z),\; 1<\alpha\leq 2, z\geq 0,$$

accompanied with its initial and boundary conditions:

$$\chi(0,z) = \phi(z), \chi_{s}(0,z) = \varphi(z),\; \chi(s,0) = g(s),\; 0<s<1.$$

where a, b, n are constant and $h(s,z)$ is a given function.

Also, we shall study the following fractional Diffusion equation of the form [10]:

$$^{c}D_{z}^{\alpha}\;(\chi(s,z)) = \chi_{ss}(s,z) + h(s,z),\; 1<\alpha<2, [0,T]\times[0,L], T>0,$$

accompanied with its initial and boundary conditions:

$$\begin{aligned} \chi(s,0) &= \phi(s), \chi_{z}(s,0) = \varphi(s),\; s\in[0,L], \\ \chi(0,z) &= \psi_{1}(z), \chi(L,0) = \psi_{1}(z),\; z\in[0,T]. \end{aligned}$$

It is the most widely known definition of the fractional derivative; it is usually called the Riemann-Liouville definition. Here $^{c}D_{z}^{\alpha}\;(\chi(s,z))$ is a Caputo fractional derivative and $h(s,z)$ denotes the source term.

The structure of this paper's content is as follows. In Section 2, we give a brief introduction to some basic definitions of fractional calculus. Section 3 provides background information on the natural transform, including definitions and important ANDM properties. Section 4 provides the uniqueness, existence theorems, and error estimate applied to the nonlinear fractional-order differential equation. Section 5 is devoted to applying the Adomian natural decomposition method (ANDM) to solve four applications, such as: the the space-fractional order Telegraph equation and time-fractional diffusion equations. The discussion and conclusion of this paper are found in Section 6.

Four points can be used to summarize the contribution of the study discussed here:

- Introducing the adequate condition (see Theorem 4.1) that ensures that Eq. (4.1) has a unique solution.
- The convergence of ANDM is explained based on the discussion below and formula (4.8) (see Theorem 4.2).
- Using point two, the maximum absolute truncated error of the Adomian series solution (4.8) is estimated (see Theorem 4.3).
- Mathematica 12 package is used to prepare an algorithm that generates the two types of Adomian polynomials, performs a comparative analysis, and solves the associated numerical examples.

2 Fractional Calculus Background Materials

The properties and definitions that are associated with fractional calculus will be presented here; for further information, see [27-29].

Definition 2.1 *Let $\chi(\upsilon) \in \mathbb{R}$, where $\upsilon > 0$. Then $\chi(\upsilon)$ is in the space C_δ, $\delta \in \mathbb{R}$ if $\exists p \in \mathbb{R}$ such that $p > \delta$ and $\chi(\upsilon) = \upsilon^p g(\upsilon)$, where $g(\upsilon)$ in $C[0,\infty)$, and $\chi(\upsilon) \in C_\delta^j$ if $\chi^{(j)} \in C_\delta$, $j = 1, 2, \ldots$.*

Definition 2.2 *For a function $\chi \in C_\delta$, the Riemann-Liouville of order $\eta \geq 0$ for the fractional integral operator is presented in the form:*

$$J^\eta\left(\chi(\upsilon)\right) = \frac{1}{\Gamma(\eta)} \int_0^\upsilon (\upsilon - z)^{\eta-1} \chi(z)\, dz, \quad \eta, \upsilon > 0, \ \text{and } J^0\chi(\upsilon) = \chi(\upsilon). \tag{2.1}$$

Definition 2.3 *The fractional derivative of f in the Liouville-Caputo sense can be defined as*

$$^cD^\eta\left(\chi(\upsilon)\right) = J^{m-\eta} D^m\left(\chi(\upsilon)\right) = \frac{1}{\Gamma(m-\eta)} \int_0^\upsilon (\upsilon - z)^{m-\eta-1} \chi^{(m)}(z) dz, \tag{2.2}$$

for $m - 1 < \eta \leq m$, $m \in \mathbb{N}$, $\upsilon > 0$, $\chi \in C_{-1}^m$.

Definition 2.4 *According to [32], the Mittag-Leffler two-parameter function is given in the form:*

$$\mathbb{E}_{\nu,\beta}(\vartheta) = \sum_{m=0}^{\infty} \frac{\vartheta^m}{\Gamma(m\,\nu + \beta)}, \ \nu > 0, \ \beta > 0, \ \vartheta \in \mathbb{C}, \tag{2.3}$$

where the gamma function is defined by:

$$\Gamma(\upsilon) = \int_0^\infty e^{-r}\, r^{\upsilon-1}\, dr, \ \upsilon > 0. \tag{2.4}$$

3 Adomian Polynomials and the Natural Transform Review

We advise readers to find out more about the general integral transform's history, the Laplace, Sumudu, and natural transform methods, as well as the features that are associated with them, for any given function $\zeta(\upsilon)$, $\upsilon \in \mathbb{R}$; see, for example [11].

Definition 3.1 *Let $\zeta(\upsilon)$ be a piece-wise continuous function on $\mathbb{R}$. If $D_1, D_2, c, d > 0$ with $c < d$, define $\mathbb{A} = \left\{\zeta(\upsilon) : |\zeta(\upsilon)| < D_1 e^{c\upsilon}\chi_{(\upsilon_2,\infty)}(\upsilon) + D_2 e^{d\upsilon}\chi_{(-\infty,\upsilon_1)}(\upsilon)\right\}$. So, $|\zeta(\upsilon)| \leq D_1 e^{c\upsilon}$ for $\upsilon \longrightarrow \infty$ i.e. $\upsilon > \upsilon_2$ and $|\zeta(\upsilon)| \leq D_2 e^{d\upsilon}$ for $\upsilon \longrightarrow -\infty$ i.e. $\upsilon < \upsilon_1$.*

Note that for any $\zeta(\upsilon)$ in the class $\mathbb{A}$ with $r, w > 0$ we have:

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{-r\upsilon}\zeta(\upsilon w)\, d\upsilon\right| &\leq D_1 \int_0^{\infty} e^{-r\upsilon} e^{c|\upsilon w|} d\upsilon + D_2 \int_{-\infty}^{0} e^{-r\upsilon} e^{d|\upsilon w|} d\upsilon \\ &= D_1 \int_{\upsilon_2}^{\infty} e^{(cw-r)\upsilon} d\upsilon + D_2 \int_{-\infty}^{\upsilon_1} e^{(dw-r)\upsilon} d\upsilon.\end{aligned}$$

Which is convergent provided that $cw - r < 0$ and $dw - r > 0$, which implies that $cw < r < dw$ i.e. $c < \frac{r}{w} < d$. Loosely speaking, $\zeta(\upsilon)$ is a function of exponential order.

Then, one can define the natural transformation (N-transformation) as:

$$\aleph\left(\zeta(\upsilon)\right) = L(r, w) = \int_{-\infty}^{\infty} e^{-r\upsilon}\, \zeta(w\upsilon) d\upsilon,\ r, w > 0, \tag{3.1}$$

where $\aleph$ is the N-transform of $\zeta(\upsilon)$ and r and w are the N-transformation variables.

Note that one can write Eq. (3.1) as,

$$\aleph\left(\zeta(\upsilon)\right) = \aleph^{+}\left(\zeta(\upsilon)\right) + \aleph^{-}\left(\zeta(\upsilon)\right) = L^{+}(r, w) + L^{-}(r, w),$$

where,

$$\begin{aligned}\aleph^{+}\left(\zeta(\upsilon)\right) &= L^{+}(r, w) = \textstyle\int_0^{\infty} e^{-r\upsilon}\, \zeta(w\upsilon)\, d\upsilon,\ r, w \in (0, \infty), \\ \aleph^{-}\left(\zeta(\upsilon)\right) &= L^{-}(r, w) = \textstyle\int_{-\infty}^{0} e^{-r\upsilon}\, \zeta(w\upsilon)\, d\upsilon,\ r, w \in (0, \infty).\end{aligned} \tag{3.2}$$

Moreover,

$$\aleph^{-1}\left[L(r, w)\right] = \zeta(\upsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{r\upsilon}{w}} L(r, w)\, dr. \tag{3.3}$$

Thus, Eq. (3.2) is the natural transformation, and Eq. (3.3) is the inverse natural transformation.

Significant Properties

In this study, we will take advantage of the following valuable N-transform features (see [20, 21]):

1. $\aleph\left[z^{\alpha}\right]=\frac{\Gamma(\eta+1)\,v^{\alpha}}{r^{\eta+1}},\;\; \eta>-1.$

2. $\aleph^{-1}\left[\frac{v^{\beta-1}r^{\eta-\beta}}{r^{\alpha}+\lambda v^{\eta}}\right]=z^{\beta-1}E_{\eta,\beta}(-\lambda z^{\eta})$, where $\alpha,\;\beta>0,\;\lambda\in\mathbb{R}$ and $|\lambda|<\frac{r^{\alpha}}{v^{\alpha}}$.

3. Given $k\in\mathbb{Z}^{+}$, where $k-1<\eta\leq k$, then the natural transform of the Liouville-Caputo fractional derivative of the function $h(z)$ of order η denoted by ${}^{c}D^{\eta}h(z)$ is given by:

$$\aleph\left[{}^{c}D^{\eta}h(z)\right]=\frac{r^{\eta}}{v^{\eta}}L(r,v)-\sum_{j=0}^{k-1}\frac{r^{\eta-(j+1)}}{u^{\eta-j}}\left(D^{j}h(z)\right)_{z=0}.$$

Adomian Polynomials Calculations

We now present the Adomian polynomials, which can be used to quickly divide a complicated nonlinear component into more manageable elements that can be integrable in the form of a Taylor series. As shown in [24], the unknown function ϕ can be expressed as:

$$\phi=\sum_{i=0}^{\infty}\phi_{i}, \tag{3.4}$$

where a recursive relation must be constructed to determine the components ϕ_i , $i\geq 0$. When dealing with nonlinear terms, $F(\phi)$, can be expressed as an infinite series known as Adomian polynomials A_i, which are denoted by the following formula:

$$F(\phi)=\sum_{i=0}^{\infty}A_{i}(\phi_{0},\,\phi_{1},....,\phi_{i}), \tag{3.5}$$

where the formula in [19] can be used to calculate the A_i of the nonlinear term $F(\phi)$:

$$A_{i}=\frac{1}{i!}\frac{d^{i}}{d\mu^{i}}\left[F\left(\sum_{k=0}^{i}\mu^{k}\phi_{k}\right)\right]_{\mu=0},i=0,1,2,... \tag{3.6}$$

Then one can express the generic formula for Eq. (3.5) as follows:

Let $F(\phi)$ represent the nonlinear function. Applying Eq. (3.6) and utilizing the Adomian polynomial definition, one can produce:

$$\begin{aligned}
&A_{0}=F(\phi_{0}),\\
&A_{1}=\phi_{1}F'(\phi_{0}),\\
&A_{2}=\phi_{2}F'(\phi_{0})+\tfrac{1}{2!}\phi_{1}^{2}F''(\phi_{0}).
\end{aligned} \tag{3.7}$$

Finally, one can produce the remaining terms in a similar manner. The polynomials mentioned previously in Eq. (3.7) provide two crucial observations. A_0 depends exclusively on ϕ_0, A_1 depends solely on ϕ_0 and ϕ_1, A_2 depends solely on ϕ_0,ϕ_1 and ϕ_2, and so forth.

Also, by inserting Eq. (3.7) into Eq. (3.5), we arrive at:

$$\begin{aligned}
F(\phi) &= A_0 + A_1 + A_2 + ... \\
&= F(\phi_0) + (\phi_1 + \phi_2 + \phi_3 + ...)F'(\phi_0) \\
&+ \frac{1}{2!}(\phi_1^2 + 2\phi_1\phi_2 + 2\phi_1\phi_3 + \phi_2^2 + ...)F''(\phi_0) \\
&+ \frac{1}{3!}(\phi_1^3 + 3\phi_1^2\phi_2 + 3\phi_1^2\phi_3 + 6\phi_1\phi_2\phi_3 + ...)F'''(\phi_0) + ... \\
&= F(\phi_0) + (\phi - \phi_0)F'(\phi_0) + \frac{1}{2!}(\phi - \phi_0)^2 F''(\phi_0) + ...
\end{aligned}$$

4 Convergence Analysis of the ANDM

In this section, we shall present proofs for the uniqueness and convergence theorems, and then we will provide an estimate error using the ANDM. Assume we have the following initial value problem for the nonlinear fractional equation:

$$ {}^cD_z^\eta \chi(s,z) + F\left(\chi(s,z)\right) + L(\chi(s,z)) = \phi(s,z), \quad 0 < \eta \leq 1. \tag{4.1}$$

And its I.C:

$$\chi(s,0) = g(s). \tag{4.2}$$

Note that the nonlinear term is $F\left(\chi(s,z)\right)$, the linear term is $L(\chi(s,z))$, and $\phi(s,z)$ is the source term.

Eq. (4.1) is then subjected to the N-transform and property 3, where $\aleph(\chi(s,z)) = \mathrm{X}\left(s,r,v\right)$ to produce:

$$\mathrm{X}\left(s,r,v\right) = \tfrac{g(s)}{r} - \left(\tfrac{v}{r}\right)^\eta \aleph\left[L\left(\chi(s,z)\right) + F(\chi(s,z)) - \phi(s,z)\right]. \tag{4.3}$$

Now we interpret Eq. (4.3) using the inverse N-transform to get:

$$\chi(s,z) = \Phi\left(s,z\right) + \aleph^{-1}\left[\left(\frac{v}{r}\right)^\eta \aleph\left[L(\chi(s,z)) + F(\chi(s,z))\right]\right]. \tag{4.4}$$

The nonhomogeneous component and the initial condition are represented by $\Phi\left(s,z\right)$. Suppose that the unknown function $\chi(s,z)$ has an infinite series solution of the form:

$$\chi(s,z) = \sum_{j=0}^{\infty} \chi_j(s,z). \tag{4.5}$$

The Adomian polynomials are the A_i's in the nonlinear term $F(\chi(s,z)) = \sum_{i=0}^{\infty} A_i$. We rewrite Eq. (4.4) as follows using Eq. (4.5):

$$\sum_{j=0}^{\infty} \chi_j(s,z) = \Phi\left((s,z)\right) + \aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta} \aleph\left[\sum_{j=0}^{\infty} A_j + \sum_{j=0}^{\infty} \chi_j(s,z)\right]\right]. \tag{4.6}$$

Eq. (4.6)'s two sides are compared, and the result is $\chi_0(s,z) = \Phi(s,z)$. Then, one can produce this general relation:

$$\chi_{j+1}(s,z) = \aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta} \aleph\left[A_j + \chi_j(s,z)\right]\right],\ j \geq 0. \tag{4.7}$$

The final expression of the anticipated exact solution is as follows:

$$\chi(s,z) = \sum_{j=0}^{\infty} \chi_j(s,z). \tag{4.8}$$

Theorem 4.1 ***(Uniqueness Theorem).*** *Eq. (4.1) has a unique solution, provided that* $0 < \sigma < 1$ *where* $\sigma = \frac{(C_1+C_2)z^{\eta}}{\Gamma(\eta+1)}$, *for all* $z \in [0,T]$.

Proof: Assume that $\mathbb{B} = (C[\Delta], \|.\|)$ is the Banach space for all continuous functions on $\Delta = [0,T]$ and the norm $\|.\|$, then define $\Pi : \mathbb{B} \to \mathbb{B}$ such that

$$\chi_{j+1}(s,z) = \Phi\left((s,z)\right) + \aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta} \aleph\left[M\left(\chi_j\left((s,z)\right)\right) + L\left(\chi_j\left((s,z)\right)\right)\right]\right].$$

Suppose that $L[\chi(s,z)] = \chi(s,z)$ and $M[\chi(s,z)] = F(\chi((s,z)))$. Further, let $|M(\chi) - M(\tilde{\chi})| < C_1 |\chi - \tilde{\chi}|$ and $|L(\theta) - L(\tilde{\chi})| < C_2 |\theta - \tilde{\chi}|$, where C_1, C_2 are the Lipschitz constants with $0 \leq C_1, C_2 < 1$ and $\chi, \tilde{\chi}$ are two different solutions of Eq. (4.1). Then,

$$\begin{aligned}
\|\Pi(\chi) - \Pi(\tilde{\chi})\| &= \max_{z\in\Delta}\left|\aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[L(\chi) + M(\tilde{\chi})\right]\right] - \aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[L(\tilde{\chi}) + M(\tilde{\chi})\right]\right]\right| \\
&= \max_{z\in\Delta}\left|\aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[L(\chi) - L(\tilde{\chi})\right]\right] + \aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[M(\chi) - M(\tilde{\chi})\right]\right]\right| \\
&\leq \max_{z\in\Delta}\left[C_1\aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[|\chi - \tilde{\chi}|\right]\right] + C_2\aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[|\chi - \tilde{\chi}|\right]\right]\right] \\
&\leq \max_{z\in\Delta}(C_1 + C_2)\left[\aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[|\chi - \tilde{\chi}|\right]\right]\right] \\
&\leq (C_1 + C_2)\left[\aleph^{-1}\left[\left(\frac{v}{r}\right)^{\eta}\aleph\left[\|\chi(z) - \tilde{\chi}(z)\|\right]\right]\right] \\
&= \|\chi - \tilde{\chi}\|\frac{(C_1 + C_2)}{\Gamma(\eta + 1)}z^{\eta}.
\end{aligned}$$

Hence, there exists a unique solution to Eq. (4.1) according to the theorem of Banach fixed-point for contraction [11] since $0 < \sigma < 1$, which implies that Π is a mapping for contraction. Theorem 4.1 has been proved. ∎

Theorem 4.2 ***(Convergence Theorem).*** *Provided both* $0 < \sigma < 1$ *and* $|\chi_1| < \infty$ *remain true, the series solution in Eq. (4.8) of Eq. (4.1) converges.*

Proof: Assume that the q_m is the m-th partial sum, i.e., $q_m = \sum_{i=0}^{m} \chi_i(s,z)$. It is our task to demonstrate that $\{q_m\}$ is a Cauchy sequence in the Banach space $\mathbb{B}$. Take into account the revised formulation of the Adomian polynomial form (see [26]). $M(q_m) = \tilde{A}_m + \sum_{i=0}^{m-1} \tilde{A}_i$. The two partial sums q_n and q_m can be any two partial sums with $m \geq n$. Then,

$$\begin{aligned}
\|q_m - q_n\| &= \max_{z\in\Delta} |q_m - q_n| \\
&= \max_{z\in\Delta} \left| \sum_{i=n+1}^{m} \tilde{\chi}_i(s,z) \right|, \; m = 1, 2, ... \\
&\leq \max_{z\in\Delta} \left| \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[K \left(\sum_{i=n+1}^{m} \chi_{i-1}(s,z) \right) \right] \right] + \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[\sum_{i=n+1}^{m} A_{i-1}(s,z) \right] \right] \right| \\
&= \max_{z\in\Delta} \left| \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[K \left(\sum_{i=n}^{m-1} \chi_{i}(s,z) \right) \right] \right] + \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[\sum_{i=n}^{m-1} A_{i}(s,z) \right] \right] \right| \\
&\leq \max_{z\in\Delta} \left| \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[K(q_{m-1}) - K(q_{n-1}) \right] \right] + \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[M(q_{m-1}) - M(q_{n-1}) \right] \right] \right| \\
&\leq C_1 \max_{z\in\Delta} \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[\|q_{m-1} - q_{n-1}\| \right] \right] + C_2 \max_{z\in\Delta} \aleph^{-1} \left[\left(\frac{v}{r}\right)^{\eta} \aleph \left[\|q_{m-1} - q_{n-1}\| \right] \right] \\
&= \frac{(C_1 + C_2)\, z^{\eta}}{\Gamma(\eta + 1)} \|q_{m-1} - q_{n-1}\|.
\end{aligned}$$

Thus, $\|q_m - q_n\| \leq \sigma \|q_{m-1} - q_{n-1}\|$. Choose $m = n + 1$, then

$$\|q_{n+1} - q_n\| \leq \sigma \|q_n - q_{n-1}\| \leq \sigma^2 \|q_{n-1} - q_{n-2}\| \leq ... \leq \sigma^n \|q_1 - q_0\|.$$

Likewise, using the triangle inequality, one can arrive at:

$$\begin{aligned}
\|q_m - q_n\| &\leq \|q_{n+1} - q_n\| + \|q_{n+2} - q_{n+1}\| + ... + \|q_m - q_{m-1}\| \\
&\leq \left[\sigma^n + \sigma^{n+1} + ... + \sigma^{m-1} \right] \|q_1 - q_0\| \\
&\leq \sigma^n \left[\frac{1 - \sigma^{m-n}}{1 - \sigma} \right] \|\chi_1\|.
\end{aligned}$$

But, $0 < \sigma < 1$, then $1 - \sigma^{m-n} < 1$. Thus,

$$\|q_m - q_n\| \leq \frac{\sigma^n}{1 - \sigma} \max_{z\in\Delta} |\chi_1|. \tag{4.9}$$

Since $\chi(s,z)$ is bounded, then $|\chi_1| < \infty$. So, as $n \to \infty$, then $\|q_m - q_n\| \to 0$. Thus, the sequence $\{q_m\}$ is a Cauchy in $\mathbb{B}$. Hence, $\chi(s,z) = \sum_{j=0}^{\infty} \chi_j(s,z)$ converges. Theorem 4.2 has been established. ■

Theorem 4.3 ***(Error Estimate).*** *For the series solution in Eq. (4.8) to Eq. (4.1), the maximum absolute cutoff error is anticipated to be:*

$$\max_{z\in\Delta}\left|\chi(s,z)-\sum_{m=0}^{n}\chi_m(s,z)\right|\le\frac{\sigma^n}{1-\sigma}\max_{z\in\Delta}|\chi_1|.$$

Proof: Eq. (4.9) in Theorem 2 leads us to the conclusion that:

$\|q_m - q_n\| \le \frac{\sigma^n}{1-\sigma}\max\limits_{z\in\Delta}|\chi_1|$. So as $m \to \infty$, we have $q_m \to \chi(s,z)$. Then, $\|\chi(s,z) - q_n\| \le \frac{\sigma^n}{1-\sigma}\max\limits_{z\in\Delta}|\chi_1(s,z)|$. Consequently, Δ's maximum absolute truncation error is:

$$\max_{z\in\,\Delta}\left|\chi(s,z)-\sum_{m=0}^{n}\chi_m(s,z)\right|\le\max_{t\in\,\Delta}\frac{\sigma^n}{1-\sigma}\left|\chi_1(s,z)\right|=\frac{\sigma^n}{1-\sigma}\left\|\chi_1(s,z)\right\|.$$

Theorem 4.3 has been established. ■

5 Numerical Simulation

This section compares the outcomes with the current solutions after applying the ANDM to two models. The procedure for the ANDM is first presented. Given the generic form of a nonlinear fractional differential equations (FODEs):

$$D_z^{\alpha}\,(\chi(s,z))+R\,(\chi(s,z))+F\,(\chi(s,z))=g(s,z),\tag{5.1}$$

where $D_t^{\alpha}\,(\chi(s,z))$ is the Caputo fractional derivative of the function $\chi(s,z)$, R is the linear differential operator, F represent the general nonlinear differential operator and $g(s,z)$ is the source term.

Employing the N-transform to Eq. (5.1) to arrive at:

$$\mathrm{X(s,r,w)}=\frac{\mathrm{v}^{\alpha}}{\mathrm{r}^{\alpha}}\sum_{\mathrm{k=0}}^{\mathrm{n-1}}\frac{\mathrm{v^{k-\alpha}}}{\mathrm{r^{(k+1)-\alpha}}}\left[\mathrm{D^k}\;\chi(\mathrm{s,z})\right]_{\mathrm{t=0}}+\frac{\mathrm{v}^{\alpha}}{\mathrm{r}^{\alpha}}\aleph\left[\mathrm{g(s,z)}\right]-\frac{\mathrm{v}^{\alpha}}{\mathrm{r}^{\alpha}}\aleph\left[\mathrm{R}\,\chi(\mathrm{s,z})+\mathrm{F}\,\chi(\mathrm{s,z})\right].\tag{5.2}$$

The inverse natural transform of Eq. (5.2) is now used to produce:

$$\chi(s,z)=G(s,z)-\aleph^{-1}\left[\frac{v^{\alpha}}{r^{\alpha}}\mathbb{N}^{+}\left[R\;\chi(s,z)+F\;\chi(s,z)\right]\right].\tag{5.3}$$

The nonhomogeneous term and the necessary conditions produced $G(s,z)$. Now let's suppose that the solutions form an infinite series:

$$\chi(s,z)=\sum_{n=0}^{\infty}\chi_n(s,z). \tag{5.4}$$

We may now rewrite Eq. (5.3) using Eq. (5.4) to produce:

$$\sum_{n=0}^{\infty}\chi_n(s,z)=G(s,z)-\aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[R\sum_{n=0}^{\infty}\chi_n(s,z)\right]+\sum_{n=0}^{\infty}A_n\right], \tag{5.5}$$

where the polynomials A_n stand for the nonlinear terms $F\,\chi(s,z)$.

Eq. (5.5)'s two sides are compared, and the following result is reached:

$$\chi_0(s,z)=G(s,z),$$

$$\chi_1(s,z)=-\aleph^{-1}\left[\tfrac{v^\alpha}{r^\alpha}\aleph\left[R\,\chi_0(s,z)\right]+A_0\right],$$

$$\chi_2(s,z)=-\aleph^{-1}\left[\tfrac{v^\alpha}{r^\alpha}\aleph\left[R\,\chi_1(s,z)\right]+A_1\right].$$

As we proceed, one can obtain the general recursive relation given by:

$$\chi_{n+1}(s,z)=-\aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[R\,\chi_n(s,z)\right]+A_n\right],\quad n\geq 0. \tag{5.6}$$

Lastly, the approximate values are provided by:

$$\chi(s,z)=\sum_{n=0}^{\infty}\chi_n(s,z).$$

Example 5.1 Consider the nonhomogeneous space-fractional telegraph equation of the form:

$${}^cD_s^\alpha\left(\chi(s,z)\right)=\chi_{zz}(s,z)+\chi_z(s,z)+\chi(s,z)-s^2-z+1,\ s,\ z>0,\ 1<\alpha\leq 2. \tag{5.7}$$

Accompanied by its conditions:

$$\chi(0,z)=z,\ \ \chi_s(0,z)=0. \tag{5.8}$$

Employing the N-transformation in Eq. (5.7), one can arrive at:

$$\aleph\left[{}^cD_s^\alpha\left(\chi(s,z)\right)\right]=\aleph\left[\chi_{zz}(s,z)+\chi_z(s,z)+\chi(s,z)-s^2-z+1\right]. \tag{5.9}$$

Substitute in Eq. (5.8) using Eq. (5.9) to produce:

$$\aleph\left[\chi(s,z)\right]\ =\ \sum_{k=0}^{n-1}\frac{v^k}{r^{k+1}}\left[D_s^k\left(\chi(s,z)\right)\right]_{s=0}+\frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{zz}(s,z)+\chi_z(s,z)+\chi(s,z)-s^2-z+1\right].$$

$$
\begin{aligned}
&= \frac{1}{r}\chi(0,z) + \frac{v}{r^2}\chi_z(0,z) + \frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{zz}(s,z) + \chi_z(s,z) + \chi(s,z) - s^2 - z + 1\right] \\
&= \frac{z}{r} - 2\frac{v^{\alpha+2}}{r^{\alpha+3}} + (1-z)\frac{v^\alpha}{r^{\alpha+1}} + \frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{zz}(s,z) + \chi_z(s,z) + \chi(s,z)\right]. \qquad (5.10)
\end{aligned}
$$

For our purposes below, we use the N-inverse transformation of Eq. (5.10):

$$
\begin{aligned}
\chi(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{zz}(s,z) + \chi_z(s,z) + \chi(s,z)\right]\right] + \aleph^{-1}\left[\frac{z}{r} - 2\frac{v^{\alpha+2}}{r^{\alpha+3}} + (1-z)\frac{v^\alpha}{r^{\alpha+1}}\right] \\
&= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{zz}(s,z) + \chi_z(s,z) + \chi(s,z)\right]\right] + z - \frac{2z^{\alpha+2}}{\Gamma\left[\alpha+3\right]} + \frac{(1-z)z^\alpha}{\Gamma\left[\alpha+1\right]}. \qquad (5.11)
\end{aligned}
$$

Suppose our intended solutions are of the form:

$$
\chi(s,z) = \sum_{n=0}^{\infty}\chi_n(s,z). \qquad (5.12)
$$

Putting Eq. (5.12) in place of Eq. (5.11) results in:

$$
\begin{aligned}
\sum_{n=0}^{\infty}\chi_n(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\sum_{n=0}^{\infty}(\chi_n)_{zz}(s,z) + \sum_{n=0}^{\infty}(\chi_n)_z(s,z) + \sum_{n=0}^{\infty}\chi(s,z)\right]\right] \\
&+ z - \frac{2z^{\alpha+2}}{\Gamma\left[\alpha+3\right]} + \frac{(1-z)z^\alpha}{\Gamma\left[\alpha+1\right]}. \qquad (5.13)
\end{aligned}
$$

We continue in a similar manner to obtain:

$$
\begin{aligned}
\chi_0(s,z) &= z - \frac{2z^{\alpha+2}}{\Gamma\left[\alpha+3\right]} + \frac{(1-z)z^\alpha}{\Gamma\left[\alpha+1\right]}, \\
\chi_1(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_0)_{zz}(s,z) + (\chi_0)_z(s,z) + \chi_0(s,z)\right]\right], \\
\chi_2(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_1)_{zz}(s,z) + (\chi_1)_z(s,z) + \chi_1(s,z)\right]\right], \\
\chi_3(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_2)_{zz}(s,z) + (\chi_2)_z(s,z) + \chi_2(s,z)\right]\right].
\end{aligned}
$$

Finally,

$$
\chi_{n+1}(s,z) = \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\sum_{n=0}^{\infty}(\chi_n)_{zz}(s,z) + \sum_{n=0}^{\infty}(\chi_n)_z(s,z) + \sum_{n=0}^{\infty}\chi(s,z)\right]\right]. \qquad (5.14)
$$

Then using Eq. (5.23) we can arrive at:

$$\begin{aligned}\chi_1(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_0)_{zz}(s,z)+(\chi_0)_z(s,z)+\chi_0(s,z)\right]\right]\\ &= \left(\frac{(1+z)s^\alpha}{\Gamma(\alpha+1)}-\frac{zs^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2s^{2\alpha+2}}{\Gamma(2\alpha+3)}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\chi_2(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_1)_{zz}(s,z)+(\chi_1)_z(s,z)+\chi_1(s,z)\right]\right]\\ &= \frac{(2+z)s^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{(1+z)s^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{2s^{3\alpha+2}}{\Gamma(3\alpha+3)}.\end{aligned}$$

$$\begin{aligned}\chi_3(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_2)_{zz}(s,z)+(\chi_2)_z(s,z)+\chi_2(s,z)\right]\right]\\ &= \frac{(3+z)s^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{(2+z)z^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{2s^{4\alpha+2}}{\Gamma(4\alpha+3)}.\end{aligned}$$

Hence, the exact solution $\chi(s,z)$ is given by:

$$\begin{aligned}\chi(s,z) &= \sum_{n=0}^{\infty}\chi_n(s,z)\\ &= \chi_0(s,z)+\chi_1(s,z)+\chi_2(s,z)+\chi_3(s,z)+...\\ &= z-\frac{2z^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{(1-z)z^\alpha}{\Gamma(\alpha+1)}+\frac{(1+z)s^\alpha}{\Gamma(\alpha+1)}-\frac{zs^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2s^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{(2+z)s^{2\alpha}}{\Gamma(2\alpha+1)}\\ &- \frac{(1+z)s^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{2s^{3\alpha+2}}{\Gamma(3\alpha+3)}+\frac{(3+z)s^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{(2+z)s^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{2s^{4\alpha+2}}{\Gamma(4\alpha+3)}\\ &= z-2+s^2+2E_{\alpha,1}(s^\alpha)-s^2E_{\alpha,3}(s^\alpha).\end{aligned}\tag{5.15}$$

Note that when $\alpha=2$, the exact solution is:

$$\chi(s,z) = s^2+z. \tag{5.16}$$

Hence, using the NADM, our exact solution is in excellent agreement with the one that exists in the literature.

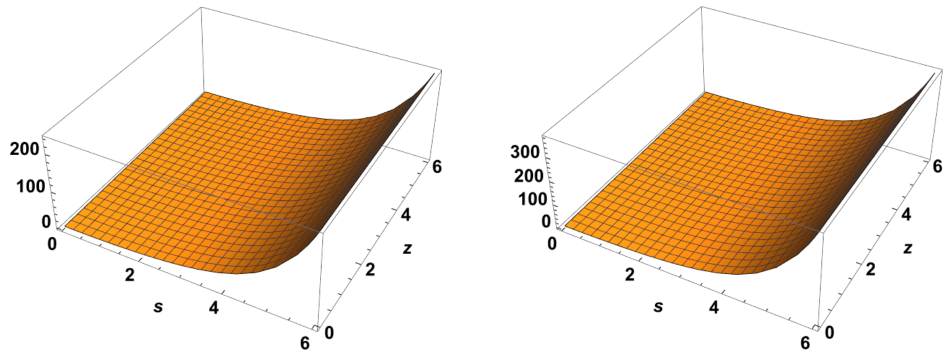

Figure 1: Exact solution to $\chi(s, z)$ for $\alpha = 2$ and $\alpha = 1.2$, respectively.

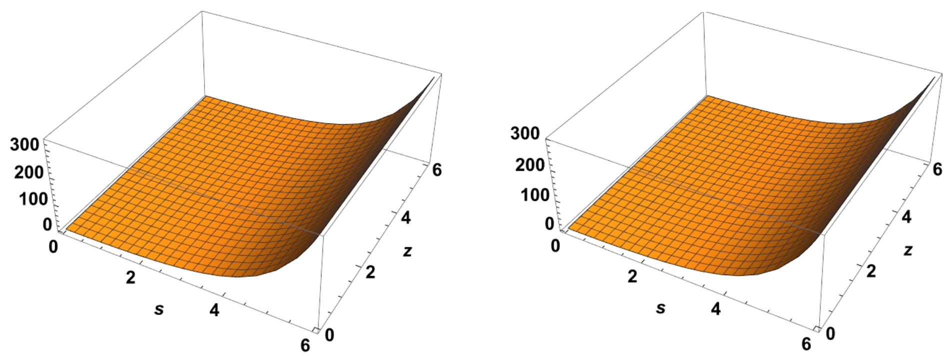

Figure 2: Exact solution to $\chi(s, z)$ for $\alpha = 1.4$ and $\alpha = 1.6$, respectively.

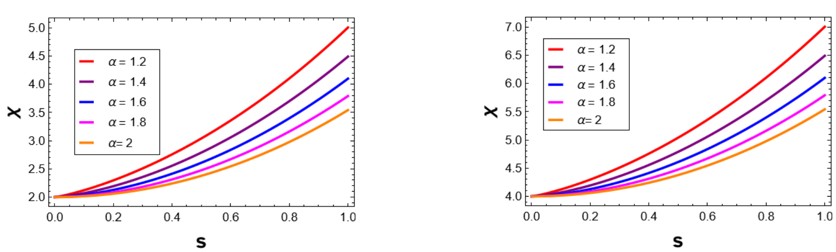

Figure 3: Exact solution to $\chi(s, z)$ for $\alpha = 1.2, 1.4, 1.6, 1.8, 2$ with $z = 2$ and $z = 4$, respectively.

Table 1. Obtained results for $\chi(s,z)$ of example (5.1) for multiple values of α, s, z.

s	z	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.8$	$\alpha = 2$
0.2	.02	0.36841	0.15664	0.08604	0.06
	.05	0.39841	0.18664	0.11604	0.09
	.08	0.42841	0.21664	0.14604	0.12
0.4	.02	0.82073	0.415443	0.25136	0.18
	.05	0.85073	0.445443	0.28136	0.21
	.08	0.88073	0.475443	0.31136	0.24
0.8	.02	1.97299	1.1975	0.8381	0.66
	.05	2.00299	1.2275	0.8681	0.69
	.08	2.03299	1.2575	0.8981	0.72

Example 5.2 Consider time-fractional diffusion equation of the form:

$$ {}^{c}D_{z}^{\alpha}\left(\chi(s,z)\right) = \chi_{ss}(s,z) + e^{s}\left(\Gamma(\alpha+1) - z^{\alpha} - z - 1\right),\ 0 < z \leq 1,\ 0 < s \leq \pi,\ 1 < \alpha \leq 2. \tag{5.17}$$

Accompanied by its conditions:

$$\chi(s,0) = e^{s}, \ \ \chi_{z}(s,0) = e^{s}. \tag{5.18}$$

Employing the N-transformation in Eq. (5.17), one can arrive at:

$$\aleph\left[{}^{c}D_{z}^{\alpha}\left(\chi(s,z)\right)\right] = \aleph\left[\chi_{ss}(s,z) + e^{s}\left(\Gamma(\alpha+1) - z^{\alpha} - z - 1\right)\right]. \tag{5.19}$$

Substitute in Eq. (5.19) using Eq. (5.18) to produce:

$$\begin{aligned}
\aleph\left[\chi(s,z)\right] &= \sum_{k=0}^{n-1}\frac{v^{k}}{r^{k+1}}\left[D_{s}^{k}\left(\chi(s,z)\right)\right]_{z=0} + \frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z) + e^{s}\left(\Gamma(\alpha+1) - z^{\alpha} - z - 1\right)\right] \\
&= \frac{1}{r}\chi(s,0) + \frac{v}{r^{2}}\chi_{z}(s,0) + \frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z) + e^{s}\left(\Gamma(\alpha+1) - z^{\alpha} - z - 1\right)\right] \\
&= e^{s}\left(\frac{1}{r} + \frac{v}{r^{2}}\right) + \frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z) + e^{s}\left(\Gamma(\alpha+1) - z^{\alpha} - z - 1\right)\right].
\end{aligned} \tag{5.20}$$

For our purposes below, we use the N-inverse transformation of Eq. (5.20):

$$\begin{aligned}
\chi(s,z) &= \aleph^{-1}\left[\frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z)\right]\right] + e^{s}(1+z) \\
&+ e^{s}\left(z^{\alpha} - \frac{\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{z^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{z^{\alpha}}{\Gamma(\alpha+1)}\right).
\end{aligned} \tag{5.21}$$

Suppose our intended solutions are of the form:

$$\chi(s,z)=\sum_{n=0}^{\infty}\chi_n(s,z). \tag{5.22}$$

Putting Eq. (5.22) in place of Eq. (5.21) results in:

$$\begin{aligned}\sum_{n=0}^{\infty}\chi_n(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{ss}(s,z)\right]\right]+e^s(1+z)\\ &+ e^s\left(z^\alpha-\frac{\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{z^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{z^\alpha}{\Gamma(\alpha+1)}\right). \end{aligned}\tag{5.23}$$

We continue in a similar manner to obtain:

$$\begin{aligned}\chi_0(s,z) &= e^s\left(1+z+z^\alpha-\frac{\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{z^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{z^\alpha}{\Gamma(\alpha+1)}\right),\\ \chi_1(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_0)_{ss}(s,z)\right]\right],\\ \chi_2(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_1)_{ss}(s,z)\right]\right],\\ \chi_3(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_2)_{ss}(s,z)\right]\right].\end{aligned}$$

Finally,

$$\chi_{n+1}(s,z)=\aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\sum_{n=0}^{\infty}(\chi_n)_{ss}(s,z)\right]\right]. \tag{5.24}$$

Then using Eq. (5.24) we can arrive at:

$$\begin{aligned}\chi_1(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_0)_{ss}(s,z)\right]\right]\\ &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[e^s\left(1+z+z^\alpha-\frac{\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{z^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{z^\alpha}{\Gamma(\alpha+1)}\right)\right]\right]\\ &= e^s\left(\frac{z^\alpha}{\Gamma(\alpha+1)}+\frac{z^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{\left(\Gamma(\alpha+1)-1\right)z^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{z^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{\Gamma(\alpha+1)z^{3\alpha}}{\Gamma(3\alpha+1)}\right).\end{aligned}$$

Similarly,

$$\chi_2(s,z) = e^s\left(\frac{z^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{z^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{\left(\Gamma(\alpha+1)-1\right)z^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{z^{3\alpha+1}}{\Gamma(3\alpha+2)}-\frac{\Gamma(\alpha+1)z^{4\alpha}}{\Gamma(4\alpha+1)}\right).$$

$$\chi_3(s,z) = e^s\left(\frac{z^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{z^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{(\Gamma(\alpha+1)-1)\,z^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{z^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{\Gamma(\alpha+1)z^{5\alpha}}{\Gamma(5\alpha+1)}\right).$$

Hence, the exact solution $\chi(s,z)$ is given by:

$$\begin{aligned}
\chi(s,z) &= \sum_{n=0}^{\infty}\chi_n(s,z) \\
&= \chi_0(s,z)+\chi_1(s,z)+\chi_2(s,z)+\chi_3(s,z)+... \\
&= e^s\left(1+z+z^{\alpha}-\frac{\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{z^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{z^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&+ e^s\left(\frac{z^{\alpha}}{\Gamma(\alpha+1)}+\frac{z^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{(\Gamma(\alpha+1)-1)\,z^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{z^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{\Gamma(\alpha+1)z^{3\alpha}}{\Gamma(3\alpha+1)}\right) \\
&+ ... \\
&= e^s\left(1+z+z^{\alpha}\right).
\end{aligned} \tag{5.25}$$

Note that when $\alpha = 2$, the exact solution is:

$$\chi(s,z) = e^s\left(1+z+z^2\right). \tag{5.26}$$

Hence, using the NADM, our exact solution is in excellent agreement with the one that exists in the literature.

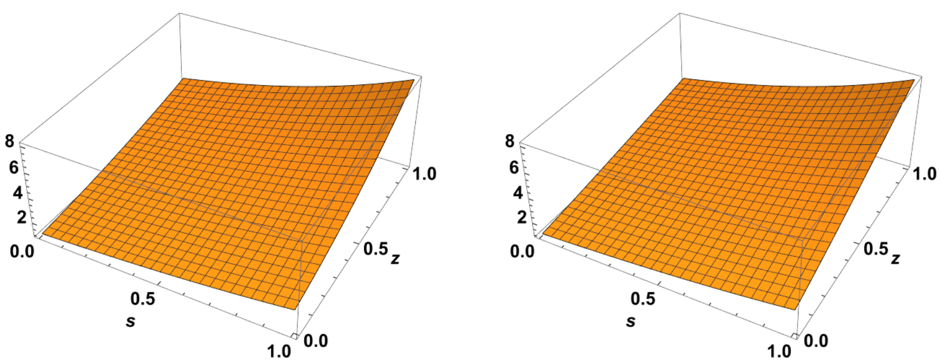

Figure 4: Exact solution to $\chi(s,z)$ for $\alpha = 2$ and $\alpha = 1.2$, respectively.

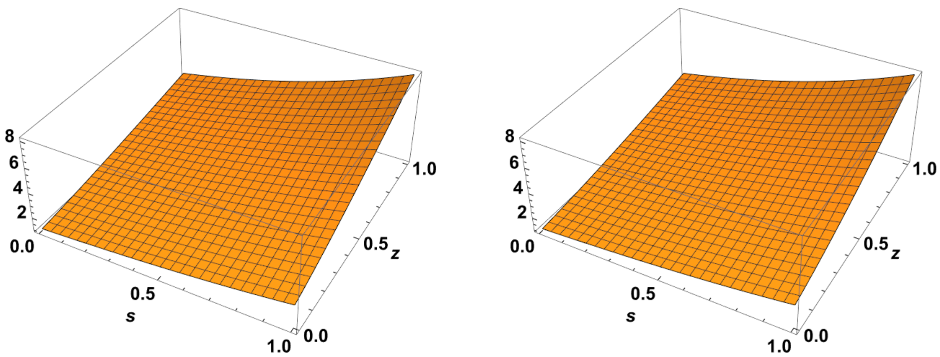

Figure 5: Exact solution to $\chi(s,z)$ for $\alpha = 1.4$ and $\alpha = 1.6$, respectively.

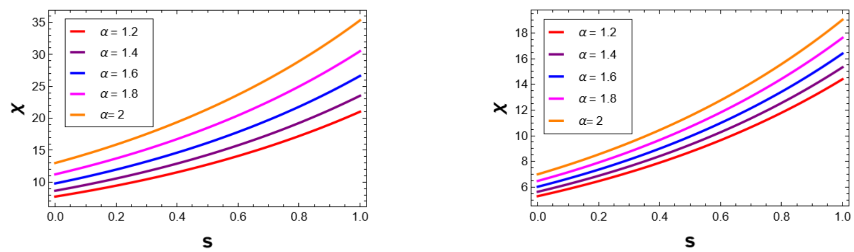

Figure 6: Exact solution to $\chi(s,z)$ for $\alpha = 1.2, 1.4, 1.6, 1.8, 2$ with $z = 2$ and $z = 3$, respectively.

Table 2. Obtained results for $\chi(s,z)$ of example (5.2) for multiple values α.

s	z	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.8$	$\alpha = 2$
0.2	.02	-0.031956	-0.17819	-0.6403	-1.1967
	.05	0.003831	-0.13976	-0.60166	-1.1588
	.08	0.037442	-0.10174	-0.56206	-1.1198
0.4	.02	-0.039031	-0.21765	-0.78206	-1.4616
	.05	0.004679	-0.1707	-0.73487	-1.4154
	.08	0.045732	-0.12427	-0.6865	-1.3678
0.8	.02	-0.05822	-0.32469	-1.1667	-2.1805
	.05	0.006981	-0.25466	-1.0963	-2.1115
	.08	0.06822	-0.18539	-1.0241	-2.0405

Example 5.3 Consider the time-fractional diffusion equation below:

$$^{c}D_z^{\alpha}\left(\chi(s,z)\right) + \chi_{ss}(s,z) + s\,\chi_s(s,z) = 2z^{\alpha} + 2(s^2+1),\ \ 0 < z,\ s \le 1,\ 0 < \alpha \le 1. \quad (5.27)$$

Accompanied by its condition:

$$\chi(s,0) = s^2. \quad (5.28)$$

Employing the N-transformation in Eq. (5.27), one can arrive at:

$$\aleph\left[{}^{c}D_z^{\alpha}\left(\chi(s,z)\right)\right] = -\aleph\left[\chi_{ss}(s,z) + s\,\chi_s(s,z) - 2z^{\alpha} - 2(s^2+1)\right]. \tag{5.29}$$

Substitute in Eq. (5.29) using Eq. (5.28) to produce:

$$\begin{aligned}
\aleph\left[\chi(s,z)\right] &= \sum_{k=0}^{n-1}\frac{v^k}{r^{k+1}}\left[D_s^k\left(\chi(s,z)\right)\right]_{z=0} - \frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z) + s\,\chi_s(s,z) - 2z^{\alpha} - 2(s^2+1)\right] \\
&= \frac{1}{r}\chi(s,0) - \frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z) + s\,\chi_s(s,z) - 2z^{\alpha} - 2(s^2+1)\right] \\
&= \frac{s^2}{r} - \frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z) + s\,\chi_s(s,z) - 2z^{\alpha} - 2(s^2+1)\right].
\end{aligned} \tag{5.30}$$

For our purposes below, we use the N-inverse transformation of Eq. (5.30):

$$\chi(s,z) = s^2 - \aleph^{-1}\left[\frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\chi_{ss}(s,z) + s\,\chi_s(s,z)\right]\right] + \frac{2\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(s^2+1)z^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.31}$$

Suppose our intended solutions are of the form:

$$\chi(s,z) = \sum_{n=0}^{\infty}\chi_n(s,z). \tag{5.32}$$

Putting Eq. (5.31) in place of Eq. (5.30) results in:

$$\begin{aligned}
\sum_{n=0}^{\infty}\chi_n(s,z) &= s^2 - \aleph^{-1}\left[\frac{v^{\alpha}}{r^{\alpha}}\aleph\left[\sum_{n=0}^{\infty}(\chi_n)_{ss}(s,z) + s\,(\chi_n)_s(s,z)\right]\right] \\
&+ \frac{2\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(s^2+1)z^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{5.33}$$

We continue in a similar manner to obtain:

$$\begin{aligned}
\chi_0(s,z) &= s^2 + \frac{2\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(s^2+1)z^{\alpha}}{\Gamma(\alpha+1)}, \\
\chi_1(s,z) &= -\aleph^{-1}\left[\frac{v^{\alpha}}{r^{\alpha}}\aleph\left[(\chi_0)_{ss}(s,z) + s\,(\chi_0)_s(s,z)\right]\right], \\
\chi_2(s,z) &= -\aleph^{-1}\left[\frac{v^{\alpha}}{r^{\alpha}}\aleph\left[(\chi_1)_{ss}(s,z) + s\,(\chi_1)_s(s,z)\right]\right], \\
\chi_3(s,z) &= -\aleph^{-1}\left[\frac{v^{\alpha}}{r^{\alpha}}\aleph\left[(\chi_2)_{ss}(s,z) + s\,(\chi_2)_s(s,z)\right]\right].
\end{aligned}$$

Finally,

$$\chi_{n+1}(s,z) = -\aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\sum_{n=0}^{\infty}(\chi_n)_{ss}(s,z) + s\,(\chi_n)_s(s,z)\right]\right]. \tag{5.34}$$

Then using Eq. (5.34) we can arrive at:

$$\begin{aligned}
\chi_1(s,z) &= -\aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_0)_{ss}(s,z) + s\,(\chi_0)_s(s,z)\right]\right] \\
&= -\aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[2(s^2+1) + \frac{2(s^2+1)z^\alpha}{\Gamma(\alpha+1)}\right]\right] \\
&= -\aleph^{-1}\left[\frac{2(s^2+1)v^\alpha}{r^{\alpha+1}} + \frac{2v^{2\alpha}(s^2+1)}{r^{2\alpha+1}}\right] \\
&= -\frac{2(s^2+1)z^\alpha}{\Gamma(\alpha+1)} - \frac{4(s^2+1)z^{2^\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\chi_2(s,z) &= \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_1)_{ss}(s,z) + s\,(\chi_1)_s(s,z)\right]\right] \\
&= \frac{4(s^2+1)z^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{8(s^2+1)z^{3\alpha}}{\Gamma(3\alpha+1)},
\end{aligned}$$

$$\chi_3(s,z) = -\frac{8(s^2+1)z^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{16(s^2+1)z^{4\alpha}}{\Gamma(4\alpha+1)},$$

$$\chi_4(s,z) = \frac{16(s^2+1)z^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{32(s^2+1)z^{5\alpha}}{\Gamma(5\alpha+1)}.$$

Hence, the exact solution $\chi(s,z)$ is given by:

$$\begin{aligned}
\chi(s,z) &= \sum_{n=0}^{\infty}\chi_n(s,z) \\
&= \chi_0(s,z) + \chi_1(s,z) + \chi_2(s,z) + \chi_3(s,z) + ... \\
&= s^2 + \frac{2\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(s^2+1)z^\alpha}{\Gamma(\alpha+1)} \\
&- \frac{2(s^2+1)z^\alpha}{\Gamma(\alpha+1)} - \frac{4(s^2+1)z^{2^\alpha}}{\Gamma(2\alpha+1)} + ...
\end{aligned}$$

$$= \quad s^2 + \frac{2\Gamma(\alpha+1)z^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{5.35}$$

Note that when $\alpha = 1$, the exact solution is:

$$\chi(s,z) \quad = \quad s^2 + z^2. \tag{5.36}$$

Hence, using the NADM, our exact solution is in excellent agreement with the one that exists in the literature.

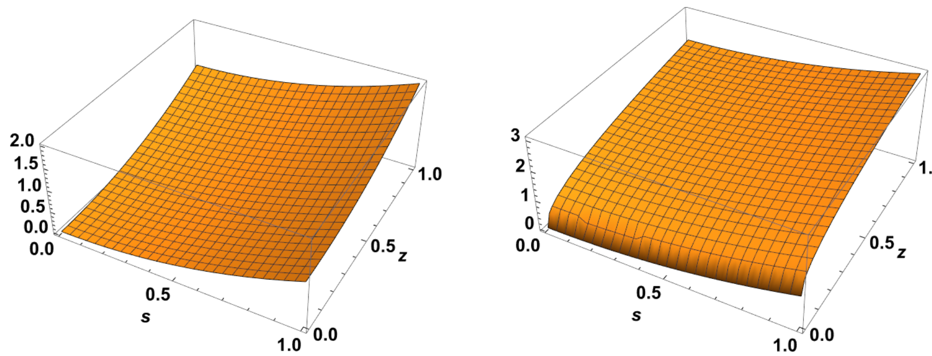

Figure 7: Exact solution to $\chi(s,z)$ for $\alpha = 2$ and $\alpha = 1.2$, respectively.

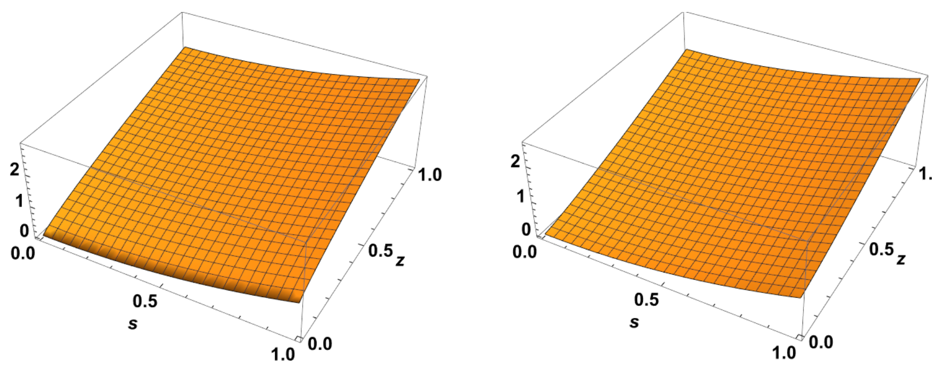

Figure 8: Exact solution to $\chi(s,z)$ for $\alpha = 1.4$ and $\alpha = 1.6$, respectively.

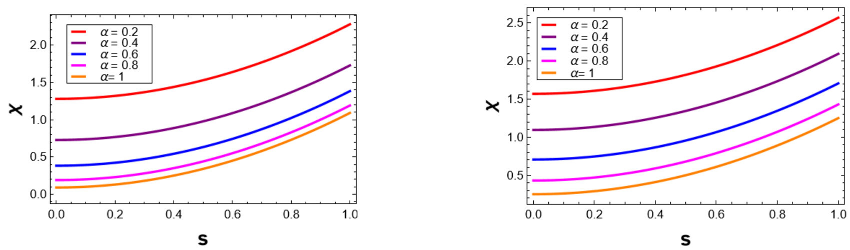

Figure 9: Exact solution to $\chi(s,z)$ for $\alpha = 0.2, 0.4, 0.6, 0.8, 1$ with $z = 0.3$ and $z = 0.5$, respectively.

Table 3. Obtained results for $\chi(s,z)$ of example (5.3) for multiple values α.

s	z	$\alpha = 1.1$	$\alpha = 1.4$	$\alpha = 1.9$	$\alpha = 2$
0.2	0.02	0.9876	0.075449	0.042492	0.0404
	0.04	1.1285	0.11089	0.04755	0.0416
	0.06	1.2205	0.14634	0.054453	0.0436
	0.08	1.2904	0.18179	0.062902	0.0464
0.4	0.02	1.1076	0.19544	0.16249	0.1604
	0.04	1.2485	0.23089	0.16755	0.1616
	0.06	1.3405	0.26634	0.17445	0.1636
	0.08	1.4104	0.30179	0.1829	0.1664
0.8	0.02	1.5876	0.67544	0.64249	0.6404
	0.04	1.7285	0.71089	0.64755	0.6416
	0.06	1.8205	0.74634	0.65445	0.6436
	0.08	1.8904	0.78179	0.6629	0.6464

Example 5.4 Consider time-fractional diffusion equation of the form:

$$^{c}D_{z}^{\alpha}\left(\chi(s,z)\right) = k\ \chi_{ss}(s,z),\ \ z > 0,\ s \in \mathbb{R},\ 1 < \alpha \leq 2. \tag{5.37}$$

Accompanied by its conditions:

$$\chi(s,0) = s^2,\ \ \chi_z(s,0) = 0. \tag{5.38}$$

Employing the N-transformation in Eq. (5.37), one can arrive at:

$$\aleph\left[^{c}D_{z}^{\alpha}\left(\chi(s,z)\right)\right] = \aleph\left[k\ \chi_{ss}(s,z)\right]. \tag{5.39}$$

Substitute in Eq. (5.39) using Eq. (5.38) to produce:

$$\aleph\left[\chi(s,z)\right] \ = \ \sum_{k=0}^{n-1} \frac{v^k}{r^{k+1}}\left[D_s^k\left(\chi(s,z)\right)\right]_{z=0} + k\ \frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{ss}(s,z)\right]$$

$$
\begin{aligned}
&= \frac{1}{r}\chi(s,0) + \frac{v}{r^2}\chi_z(s,0) + k\ \frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{ss}(s,z)\right] \\
&= \left(\frac{s^2}{r}\right) + k\ \frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{ss}(s,z)\right]. \qquad (5.40)
\end{aligned}
$$

For our purposes below, we use the N-inverse transformation of Eq. (5.40):

$$
\chi(s,z) = s^2 + k\ \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{ss}(s,z)\right]\right]. \qquad (5.41)
$$

Suppose our intended solutions are of the form:

$$
\chi(s,z) = \sum_{n=0}^{\infty}\chi_n(s,z). \qquad (5.42)
$$

Putting Eq. (5.42) in place of Eq. (5.41) results in:

$$
\sum_{n=0}^{\infty}\chi_n(s,z) = s^2 + k\ \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\chi_{ss}(s,z)\right]\right]. \qquad (5.43)
$$

We continue in a similar manner to obtain:

$$
\begin{aligned}
\chi_0(s,z) &= s^2, \\
\chi_1(s,z) &= k\ \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_0)_{ss}(s,z)\right]\right], \\
\chi_2(s,z) &= k\ \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_1)_{ss}(s,z)\right]\right], \\
\chi_3(s,z) &= k\ \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_2)_{ss}(s,z)\right]\right].
\end{aligned}
$$

Finally,

$$
\chi_{n+1}(s,z) = k\ \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[\sum_{n=0}^{\infty}(\chi_n)_{ss}(s,z)\right]\right]. \qquad (5.44)
$$

Then using Eq. (5.44) we can arrive at:

$$
\begin{aligned}
\chi_1(s,z) &= k\ \aleph^{-1}\left[\frac{v^\alpha}{r^\alpha}\aleph\left[(\chi_0)_{ss}(s,z)\right]\right] \\
&= 2k\ \aleph^{-1}\left[\frac{v^\alpha}{r^{\alpha+1}}\right]
\end{aligned}
$$

$$= \frac{2k\, z^{\alpha}}{\Gamma(\alpha+1)}.$$

Similarly,

$$\chi_2(s,z) = 0.$$

$$\chi_3(s,z) = 0.$$

Hence, the exact solution $\chi(s,z)$ is given by:

$$\begin{aligned}
\chi(s,z) &= \sum_{n=0}^{\infty} \chi_n(s,z) \\
&= \chi_0(s,z) + \chi_1(s,z) + \chi_2(s,z) + \chi_3(s,z) + \ldots \\
&= s^2 + \frac{2k\, z^{\alpha}}{\Gamma(\alpha+1)}. \qquad (5.45)
\end{aligned}$$

Note that when $\alpha = 2$, the exact solution is:

$$\chi(s,z) = s^2 + kz^2. \qquad (5.46)$$

Hence, using the NADM, our exact solution is in excellent agreement with the one that exists in the literature.

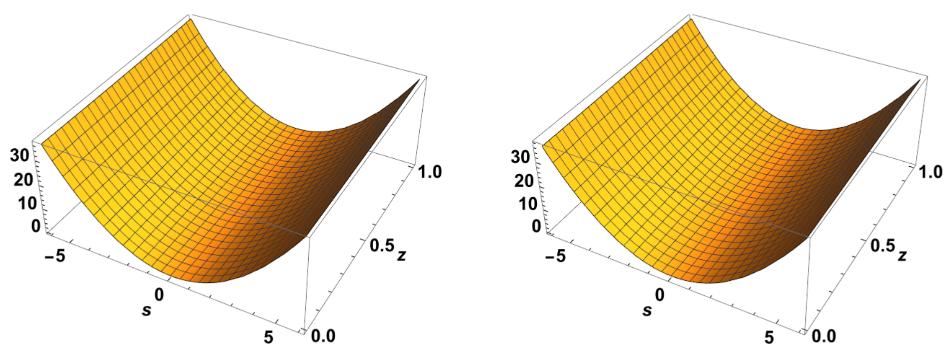

Figure 10: Exact solution to $\chi(s,z)$ for $\alpha = 2$ and $\alpha = 1.2$, $k = 1$.

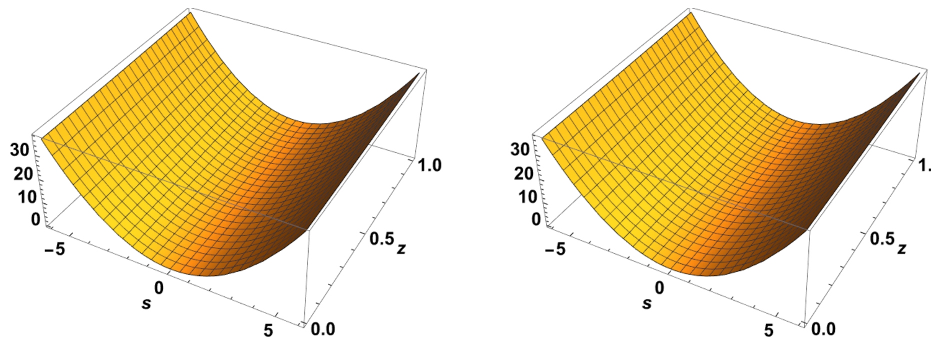

Figure 11: Exact solution to $\chi(s, z)$ for $\alpha = 1.4$ and $\alpha = 1.6$, $k = 1$.

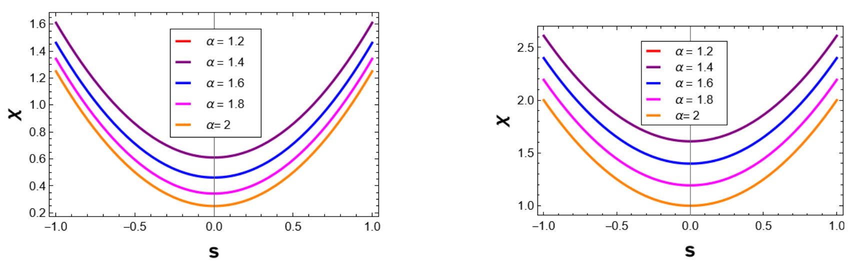

Figure 12: Exact solution to $\chi(s, z)$ for $\alpha = 1.2, 1.4, 1.6, 1.8, 2$ with $z = 0.5$ and $z = 1$, $k = 1$.

Table 4. Obtained results for $\chi(s, z)$ of example (5.4) for multiple values α and $k = 1$.

s	z	$\alpha = 1.1$	$\alpha = 1.4$	$\alpha = 1.9$	$\alpha = 2$
0.2	0.02	0.9876	0.075449	0.042492	0.0404
	0.04	1.1285	0.11089	0.04755	0.0416
	0.06	1.2205	0.14634	0.054453	0.0436
	0.08	1.2904	0.18179	0.062902	0.0464
0.4	0.02	1.1076	0.19544	0.16249	0.1604
	0.04	1.2485	0.23089	0.16755	0.1616
	0.06	1.3405	0.26634	0.17445	0.1636
	0.08	1.4104	0.30179	0.1829	0.1664
0.8	0.02	1.5876	0.67544	0.64249	0.6404
	0.04	1.7285	0.71089	0.64755	0.6416
	0.06	1.8205	0.74634	0.65445	0.6436
	0.08	1.8904	0.78179	0.6629	0.6464

6 Concluding Remarks

In the last thirty years, the subject of fractional calculus has seen the development of numerous numerical techniques. In this article, we developed a new method called the fractional natural Adomian method (ANDM) to handle applications that arise in engineering and science. We successfully found exact solutions to the space fractional-order telegraph equation and time fractional diffusion problems using an efficient scheme called the ANDM. Using the Banach fixed point theorem, we provided proofs for the existence and uniqueness theorems along with the error estimates and applied them to a nonlinear partial differential equation. Using the current technique, new exact solutions for four applications have been obtained. One can conclude that the NADM has shown a high level of improvement over the existing methods because of their accuracy and simplicity. The applicability of the ANDM proved its importance in the fields of applied science and engineering. Hence, the fractional natural Adomian decomposition method is an alternative method to the existing methods.

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8 Declarations

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Conflicts of interest. We all certify that we have no financial or other conflicts of interest with regard to the publication of this paper.

Consent to participate. Participants are informed that if they have any questions or complaints about how the research is being performed, they can get in touch with the University of Vermont Ethics Officer.

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