Finite time stability and optimal control of a stochastic reaction diffusion echinococcosis model with impulse and time-varying delay

Bin Liu[1,3], Jing Hu[1*], Libo Liu[2*]

1.College of Mathematics and Statistics, Ningxia University,

Yinchuan, 750021 ,Ningxia, China

2.College of Information Engineering, Ningxia University,

Yinchuan, 750021 ,Ningxia, China

3.College of Information Engineering, Yinchuan University of Science and Technology,

Yinchuan, 750021 ,Ningxia, China

Abstract: This paper presents a model for echinococcosis which incorporates stochastic reaction diffusion, impulse, and time-varying delay. First, the existence and uniqueness of global positive solution is proved through the construction of a Lyapunov function. Then, by applying the bounded impulse interval method, several sufficient conditions for finite time stability(FTS) are obtained. Finally, from the angle of cost-benefit, the issue of optimal control of echinococcosis is presented with the aim of minimizing infection and controlling costs. The validity of the analytical results is verified by numerical simulations.

Keywords: Finite-time stability; Lyapunov functional; Optimal control; Time-varying delay

1 Introduction

Cystic echinococcosis, also known as hydatid disease, is a zoonotic parasitic disease caused by infection with a species complex centred on echinococcus granulous. It is widely distributed on all continents except Antarctica. There can be more than 50 cases of human cystic echinococcosis per $100,000$ people annually and prevalence rates range from $5\% - 10\%$ in certain regions of Argentina, Peru, East Africa, Central Asia [1].

*Corresponding author. E-mail address: liulib@nxu.edu.cn, hujing94@nxu.enu.cn

Particularly in China, cystic echinococcosis (CE) has been reported from 22 provinces, autonomous regions and municipalities which was caused by echinococcus granulosus and echinococcus multilocularis [2,3]. The main endemic areas are in the western provinces, such as Xinjiang, Qinghai and Tibet [4], where the widely developed livestock industry has maintained a stable transmission cycle of echinococcus granulosus. The number of domestic animals being faced with the infection of echinococcosis is more than 10^8 in which the amount of dogs is at least $5*10^6$ [5]. Because of its wide prevalence and serious harm, the spread of echinococcosis has attracted extensive attention in the world. The study on the endemic areas, influencing factors and transmission mechanism of echinococcosis has been a hot topic in recent decades.

Various mathematical models have played an important role in exploring the transmission of echinococcosis. Most of the work focuses on statistical models to study the echinococcus granulosus [6-12]. For example, Gemmell et al.[6] and Roberts et al.[7] constructed a mathematical model of the life cycle of echinococcus granulosus in dogs and sheep in New Zealand and used it to discuss previously published experimental and survey data. Wang et al.[13] presented a new echinococcosis model drived by ordinary differential equations for predicting the epidemic trend of echinococcus in Xinjiang Uygur Autonomous Region of China. Liu et al.[14] presented a time-delayed echinococcosis transmission model to explore effective control and prevention strategies. Xu et al.[15] established a reaction diffusion equation with time delay to describe the transmission mechanism of echinococcosis. However, these papers mentioned considering constant delay. In the reality of biology, the delay is influenced by various factors such as season and so on. Therefore, the delay in penetration is time-dependent and the introduction of time-varying delay makes more practical sense in the echinococcosis system. On the other hand, the infectious disease system is often disturbed by human activities such as vaccinations and deworming treatments. These phenomena can be more precisely depicted using impulse differential equations. Some results were presented on modeling impulsive infectious disease systems[16-18]. Such as zhang et al.[16] established a mathematical model that incorporates periodic transmission and impulse intervention to describe the transmission dynamics of echinococcosis and explored the effectiveness of prevention and control measures.

It is well-known that various environmental factors can influence the spread of disease. Such as extreme climate, seasonal changes, and sunshine duration can affect the survival of parasitic eggs and the activity and efficacy involved in their dispersal. Hot and dry weather can shorten the lifespan of the eggs but increase their chances of dispersal. Some research has already been conducted on the impact of noise on the spread of diseases[19,20]. For example, Tornatore et al. [19] discussed the asymptotic stability of the disease-free equilibrium of a stochastic SIR model. Gray et al. [20] considered a stochastic SIS model and studied the stationary distribution in the case of disease persistence. However, there have been few studies that incorporate stochastic factors into the analysis of the dynamic behavior of cysticercosis systems.

In recent decades, scholars have conducted research on the long-term dynamic behavior of echinococcosis. However, when an infectious disease outbreak occurs, our goal is to control the spread of the disease within a limited time. The study of FTS in diseases is a very intriguing topic. Finite-time stability means that when a disease outbreak occurs, for a finite period of time, the population size is controlled within a certain range. The study of finite time stability helps the government and health departments formulate targeted interventions to block the chain of disease transmission in a timely manner and to safeguard people's health and safety. In summary, the study of finite time stability is of great significance in epidemiology. Clearly, FTS of the system does not imply asymptotic stabilization. Similarly, if the system is asymptotically stabilised, its state value may surpass a certain threshold within a finite time period. Moreover, for certain complex systems where the equilibrium point is not easily determined, as a result, the asymptotic stability of the systems cannot be ensured, despite the system can maintain a good performance in a finite time interval. This has been explored in other areas of research [21-23] but has not been specifically studied in the context of echinococcosis. Therefore, in this paper, we considered stochastic factors based on the model proposed by Xu et al. [15]. The existence and uniqueness of global positive solution is proved through the construction of a Lyapunov function and the FTS of the system is discussed using the stochastic comparison principle. From another perspective, determining the optimal control strategy for echinococcosis to balance costs and benefits due to limited resources is also an important and meaningful issue. To sum up, our primary contributions are outlined as follows:

- We introduce a time-varying delay impulsive stochastic reaction-diffusion model for echinococcosis, which extends the work in references [14, 15, 16].
- The theoretical results provide sufficient conditions for finite-time stability, which reflect the effects of diffusion, delay, impulse, and noise disturbance.
- Control strategies are applied to impulse stochastic echinococcosis systems with delay, such as vaccination, deworming therapy, and cleaning the environment. An explicit expression for the optimal control is obtained through the principle of minimum.

The remaining structure of this paper is organized as follows. In Sect. 2, a stochastic reaction diffusion echinococcosis system is constructed with impulse and time-varying delay. According to an equivalent system, Section 3 yields the well-posedness of positive solution of the system. In Sect. 4, the sufficient conditions associated with white noise, impulse, and time-varying delay for the FTS are presented. In Sect. 5, we investigate the optimal control problem for the echinococcosis system using the minimum principle. In Sect. 6, the theoretical results are illustrated through a numerical simulation. The last section gives the conclusion of this paper and discusses further work.

2 Model derivations

In this part, the hydatid transmission model can be descried by Xu et al. [15] as follows:

$$\begin{cases} \frac{\partial}{\partial t}S_1(t,x) = d_1\triangle S_1(t,x) + A_1 - \beta_1 S_1(t,x)I_2(t,x) - \mu_1 S_1(t,x) + \gamma_1 I_1(t,x), \\ \frac{\partial}{\partial t}E_1(t,x) = d_2\triangle E_1(t,x) + \beta_1 S_1(t,x)I_2(t,x) - \mu_1 E_1(t,x) \\ \qquad\qquad - \beta_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x), \\ \frac{\partial}{\partial t}I_1(t,x) = d_3\triangle I_1(t,x) + \beta_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x) - (\mu_1+\gamma_1)I_1(t,x), \\ \frac{\partial}{\partial t}S_2(t,x) = d_4\triangle S_2(t,x) + A_2 - \beta_2 S_2(t,x)I_1(t,x) - \mu_2 S_2(t,x) + \gamma_2 I_2(t,x), \\ \frac{\partial}{\partial t}E_2(t,x) = d_5\triangle E_2(t,x) + \beta_2 S_2(t,x)I_1(t,x) - \mu_2 E_2(t,x) \\ \qquad\qquad - \beta_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x), \\ \frac{\partial}{\partial t}I_2(t,x) = d_6\triangle I_2(t,x) + \beta_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x) - (\mu_2+\gamma_2)I_2(t,x). \end{cases} \tag{1}$$

All parameters are assumed nonnegative. For the definitive hosts population (mainly the dogs), A_1 describes the menstrual recruitment rate; μ_1 is the death rate of the definitive hosts; γ_1 denotes the recovery rate of transition from infected to susceptible definitive hosts, including the natural recovery rate and recovery due to anthelmintic treatment; $\beta_1 S_1(t)I_2(t)$ describes the transmission of echinococcosis between susceptible definitive hosts and infectious intermediate hosts after the ingestion of cyst containing organs of the infected intermediate hosts. For the intermediate hosts, A_2 is the menstrual recruitment rate; μ_2 is the death rate of the intermediate hosts; γ_2 denotes the recovery rate of transition from infected to susceptible intermediate hosts; $\beta_2 S_2(t)I_1(t)$ describes the transmission of echinococcosis to intermediate hosts by the ingestion of echinococcus eggs in the environment. $\tau(t)$ is the time needed for eggs to develop into adult worms. $S_1(t,x), E_1(t,x)$ and $I_1(t,x)$ represent the densities of the subpopulations of the susceptible, exposed and infected definitive hosts individuals at time t and position $x \in \Omega$, respectively. $S_2(t,x), E_2(t,x)$ and $I_2(t,x)$ represent the densities of the subpopulations of susceptible, exposed and infected intermediate hosts individuals at time t and position $x \in \Omega$, respectively. The positive constants $d_i (i = 1, 2, 3, 4, 5, 6)$ are diffusion coefficient. Δ is the Laplacian operator and $\partial\Omega$ is the boundary of $\Omega \in R^2$.

In this paper, that system (1) satisfies the following initial value conditions:

$$\begin{aligned} &S_1(s,x) = \phi_{S_1}(s,x), E_1(s,x) = \phi_{E_1}(s,x), I_1(s,x) = \phi_{I_1}(s,x), \\ &S_2(s,x) = \phi_{S_2}(s,x), E_2(s,x) = \phi_{E_2}(s,x), I_2(t,s) = \phi_{I_2}(s,x), \end{aligned} \tag{2}$$

where $\phi_i(s,x), i = S_1, E_1, I_1, S_2, E_2, I_2$, are bounded and continuous on $[-\hat{\tau}, 0] \times \Omega$, the Neumann boundary condition

$$\frac{\partial S_1(t,x)}{\partial n} = \frac{\partial E_1(t,x)}{\partial n} = \frac{\partial I_1(t,x)}{\partial n} = \frac{\partial S_2(t,x)}{\partial n} = \frac{\partial E_2(t,x)}{\partial n} = \frac{\partial I_2(t,x)}{\partial n} = 0, \tag{3}$$

for all $(s,x) \in (0, +\infty) \times \partial\Omega$, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$. The Neumann boundary conditions (3) imply that the diseases do not move across the boundary $\partial\Omega$.

In real life, the incidence of echinococcosis is often influenced by environmental factors such as seasons and climate change. In some regions, the incidence of echinococcosis may be related to specific seasons, for example, during certain seasons, the number of echinococcosis eggs increases, making it easier to spread the disease between susceptible individuals and infected individuals. Therefore, considering the disturbance of incidence rates can enhance the authenticity of models, we assume the incidence rate is stochastic perturbed. That is $\beta_i \to \beta_i + \sigma_i \dot{B}_i(t), i = 1, 2$, where $B_i(t)$ is standard Brownian motion and $\sigma_i^2 > 0$ represents the intensity of $B_i(t)$. By this way, the model (1) will be deduced to the form:

$$\begin{cases} dS_1(t,x) = [d_1 \triangle S_1(t,x) + A_1 - \beta_1 S_1(t,x) I_2(t,x) - \mu_1 S_1(t,x) + \gamma_1 I_1(t,x)]dt \\ \qquad\qquad - \sigma_1 S_1(t,x) I_2(t,x) dB_1(t), \\ dE_1(t,x) = [d_2 \triangle E_1(t,x) + \beta_1 S_1(t,x) I_2(t,x) - \mu_1 E_1(t,x) \\ \qquad\qquad - \beta_1 S_1(t-\tau(t),x) I_2(t-\tau(t),x)]dt + [\sigma_1 S_1(t,x) I_2(t,x) \\ \qquad\qquad - \sigma_1 S_1(t-\tau(t),x) I_2(t-\tau(t),x)]dB_1(t), \\ dI_1(t,x) = [d_3 \triangle I_1(t,x) + \beta_1 S_1(t-\tau(t),x) I_2(t-\tau(t),x) - (\mu_1 + \gamma_1) I_1(t,x)]dt \\ \qquad\qquad \sigma_1 S_1(t-\tau(t),x) I_2(t-\tau(t),x)]dB_1(t), \\ dS_2(t,x) = [d_4 \triangle S_2(t,x) + A_2 - \beta_2 S_2(t,x) I_1(t,x) - \mu_2 S_2(t,x) + \gamma_2 I_2(t,x)]dt \\ \qquad\qquad - \sigma_2 S_2(t,x) I_1(t,x) dB_2(t), \\ dE_2(t,x) = [d_5 \triangle E_2(t,x) + \beta_2 S_2(t,x) I_1(t,x) - \mu_2 E_2(t,x) \\ \qquad\qquad - \beta_2 S_2(t-\tau(t),x) I_1(t-\tau(t),x)]dt + [\sigma_2 S_2(t,x) I_1(t,x) \\ \qquad\qquad - \sigma_2 S_2(t-\tau(t),x) I_1(t-\tau(t),x)]dB_2(t), \\ dI_2(t,x) = [d_6 \triangle I_2(t,x) + \beta_2 S_2(t-\tau(t),x) I_1(t-\tau(t),x) - (\mu_2 + \gamma_2) I_2(t,x)]dt \\ \qquad\qquad \sigma_2 S_2(t-\tau(t),x) I_1(t-\tau(t),x) dB_2(t). \end{cases} \tag{4}$$

To account for the effect of impulsive, we regularly vaccinate the susceptible population of intermediate hosts and the susceptible population of the final host in the echinococcosis system, while administering deworming treatment to the latent population. Vaccination and deworming treatment are considered as impulsive perturbation and incorporate it into system (4) and derive the follow system.

$$\begin{cases} \left.\begin{aligned} dS_1(t,x) &= [d_1\triangle S_1(t,x) + A_1 - \beta_1 S_1(t,x)I_2(t,x) - \mu_1 S_1(t,x) + \gamma_1 I_1(t,x)]dt \\ &\quad - \sigma_1 S_1(t,x)I_2(t,x)dB_1(t), \\ dE_1(t,x) &= [d_2\triangle E_1(t,x) + \beta_1 S_1(t,x)I_2(t,x) - \mu_1 E_1(t,x) \\ &\quad - \beta_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dt + [\sigma_1 S_1(t,x)I_2(t,x) \\ &\quad - \sigma_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dB_1(t), \\ dI_1(t,x) &= [d_3\triangle I_1(t,x) + \beta_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x) - (\mu_1+\gamma_1)I_1(t,x)]dt \\ &\quad \sigma_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dB_1(t), \\ dS_2(t,x) &= [d_4\triangle S_2(t,x) + A_2 - \beta_2 S_2(t,x)I_1(t,x) - \mu_2 S_2(t,x) + \gamma_2 I_2(t,x)]dt \\ &\quad - \sigma_2 S_2(t,x)I_1(t,x)dB_2(t), \\ dE_2(t,x) &= [d_5\triangle E_2(t,x) + \beta_2 S_2(t,x)I_1(t,x) - \mu_2 E_2(t,x) \\ &\quad - \beta_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x)]dt + [\sigma_2 S_2(t,x)I_1(t,x) \\ &\quad - \sigma_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x)]dB_2(t), \\ dI_2(t,x) &= [d_6\triangle I_2(t,x) + \beta_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x) - (\mu_2+\gamma_2)I_2(t,x)]dt \\ &\quad \sigma_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x)dB_2(t), \end{aligned}\right\} \begin{array}{l} t\neq t_k, \\ t>0, \\ x\in\Omega \end{array} \\ \left.\begin{aligned} S_1(t_k^+,x) &= (1-\rho_{1k})S_1(t_k,x), \\ E_1(t_k^+,x) &= (1-\rho_{2k})E_1(t_k,x), \\ S_2(t_k^+,x) &= (1-\rho_{3k})S_2(t_k,x), \\ E_2(t_k^+,x) &= (1-\rho_{4k})E_2(t_k,x), \end{aligned}\right\} t = t_k, \end{cases} \tag{5}$$

where $\{t_k\}(k \in N)$ is impulsive sequence satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots$, as well as $\lim_{k\to+\infty} t_k = +\infty$ and $z(t_k^+,x) = \lim_{t\to t_k^+} z(t,x)$. $z(t,x) = (S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ is a solution of system (5).
Assign

$\mathbb{H} = H^1(\Omega) \equiv \{\varphi|\varphi \in L^2(\Omega), \frac{\partial\varphi}{\partial x_i} \in L^2(\Omega), i = 1,2,3,4,5,6,$
where $\frac{\partial\varphi}{\partial x_i}$ are generalized partial derivative$\}$,

where $\mathbb{H}^{-1} = H^{-1}(\Omega)$ denotes the dual space of $\mathbb{H}$. The *bracket*$\langle\cdot,\cdot\rangle$ signifies the duality product between $\mathbb{H}$ and $\mathbb{H}^{-1}$. $\mathbb{M}_+ = L^2([0,+\infty)\times\Omega, R_+^6)$ represents the set of square integrable functions defined on $[0,+\infty)\times\Omega$, which is equipped with the norm $\|\cdot\|$, where $\|z(t,x)\| = (\int_\Omega z^T(t,x)z(t,x)dx)^{\frac{1}{2}}$. $z(t,x) = (S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a complete filtered probability apace with a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ satisfying the usual conditions of completeness and right-continuity. $\mathbb{E}$ denotes the probability expectation corresponding to $\mathbb{P}$.

Additionally, there are some hypothesis that needs to be given.

(H1) $\tau(t)$ is a continuously differentiable time-varying delay with $0 \leq \tau(t) \leq \hat{\tau}$, $\tau^{'}(t) \leq \tilde{\tau} \leq 1$ for $t \in R$, $\hat{\tau}$ and $\tilde{\tau}$ are constants.

(H2) $\rho_{ik}(i = 1, 2, 3, 4)$ is impulsive strength. $0 \leq \rho_{ik} < 1$, the impulse interference indicates the proportion of susceptible populations vaccinated and the proportion of latent populations treated with anthelmintic. We suppose that $1 - \rho_{ik} > 0$.

Remark 2.1.(H1) implies that echinococcus granulosus eggs have a limited growth cycle in the host, thus delay is bounded. (H2) is based on the inability of existing treatments to completely eliminate echinococcosis.

3 Well-posedness

In this section, the well-posedness of the global positive solution $(S_1(t, x), E_1(t, x), I_1(t, x), S_2(t, x), E_2(t, x), I_2(t, x))$ to the system(5) can be proved. To get the conclusion, we first give some lemmas and take a close look at the ensuing system:

$$
\begin{cases}
dy_1(t,x) = [d_1\triangle y_1(t,x) + A_1h_1^{-1}(t) - \beta_1 y_1(t,x)h_6(t)y_6(t,x) \\
\qquad - (\mu_1 - ln(1+\rho_{1k}))y_1(t,x) + \gamma_1 h_1^{-1}(t)h_3(t)y_3(t,x)]dt \\
\qquad - \sigma_1 y_1(t,x)h_6(t)y_6(t,x)dB_1(t), \\
dy_2(t,x) = [d_2\triangle y_2(t,x) + \beta_1 h_2^{-1}(t)h_1(t)y_1(t,x)h_6(t)y_6(t,x) \\
\qquad - (\mu_1 - ln(1+\rho_{2k}))y_2(t,x) \\
\qquad - \beta_1 h_2^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t),x)h_6(t-\tau(t))y_6(t-\tau(t),x)]dt \\
\qquad + [\sigma_1 h_2^{-1}(t)h_1(t)y_1(t,x)h_6(t)y_6(t,x) \\
\qquad - \sigma_1 h_2^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t),x)h_6(t-\tau(t))y_6(t-\tau(t),x)]dB_1(t), \\
dy_3(t,x) = [d_3\triangle y_3(t,x) + \beta_1 h_3^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t),x)h_6(t-\tau(t))y_6(t-\tau(t),x) \\
\qquad - (\mu_1 + \gamma_1)y_3(t,x)]dt \\
\qquad + \sigma_1 h_3^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t),x)h_6(t-\tau(t))y_6(t-\tau(t),x)dB_1(t), \\
dy_4(t,x) = [d_4\triangle y_4(t,x) + A_2h_4^{-1}(t) - \beta_2 y_4(t,x)h_3(t)y_3(t,x) \\
\qquad - (\mu_2 - ln(1+\rho_{3k}))y_4(t,x) + \gamma_2 h_4^{-1}(t)h_6(t)y_6(t,x)]dt \\
\qquad - \sigma_2 y_4(t,x)h_3(t)y_3(t,x)dB_2(t), \\
dy_5(t,x) = [d_5\triangle y_5(t,x) + \beta_2 h_5^{-1}(t)h_4(t)y_4(t,x)h_3(t)y_3(t,x) \\
\qquad - (\mu_2 - ln(1+\rho_{4k}))y_5(t,x) \\
\qquad - \beta_2 h_5^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t),x)h_3(t-\tau(t))y_3(t-\tau(t),x)]dt \\
\qquad + [\sigma_2 h_5^{-1}(t)h_4(t)y_4(t,x)h_3(t)y_3(t,x) \\
\qquad - \sigma_2 h_5^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t),x)h_3(t-\tau(t))y_3(t-\tau(t),x)]dB_2(t), \\
dy_6(t,x) = [d_6\triangle y_6(t,x) + \beta_2 h_6^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t),x)h_3(t-\tau(t))y_3(t-\tau(t),x) \\
\qquad - (\mu_2 + \gamma_2)y_6(t,x)]dt \\
\qquad + \sigma_2 h_6^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t),x)h_3(t-\tau(t))y_3(t-\tau(t),x)dB_2(t),
\end{cases}
\tag{6}
$$

with initial value

$$
\begin{aligned}
&(y_1(0,x), y_2(0,x), y_3(0,x), y_4(0,x), y_5(0,x), y_6(0,x)) \\
&= (S_{1,0}(x), E_{1,0}(x), I_{1,0}(x), S_{2,0}(x), E_{2,0}(x), I_{2,0}(x)), \qquad (7)
\end{aligned}
$$

and boundary condition

$$
\frac{\partial y_1(t,x)}{\partial n} = \frac{\partial y_2(t,x)}{\partial n} = \frac{\partial y_3(t,x)}{\partial n} = \frac{\partial y_4(t,x)}{\partial n} = \frac{\partial y_5(t,x)}{\partial n} = \frac{\partial y_6(t,x)}{\partial n} = 0, x \in \partial\Omega, t > 0, \qquad (8)
$$

where

$$
h_i(t) = \begin{cases} 1, t \in [-\hat{\tau}, 0], \\ (1-\rho_{ik})^{[t]-t}, t \neq t_k, t > 0, \\ (1-\rho_{ik})^{-1}, t = t_k, i = 1,2,3,4. \end{cases}
$$

Obviously, $h_i(t)$ are left continuous. In order to study well-posedness of the solution of system (5), the following lemmas are proposed.

Lemma 3.1. The system (6) described by initial value (7) and boundary condition (8) can be transformed into an equivalent system (5) represented by boundary condition (3). The proof of Lemma 3.1 is shown in Appendix A.

Lemma 3.2. For any initial data $(S_{1,0}, E_{1,0}, I_{1,0}, S_{2,0}, E_{2,0}, I_{2,0})$, the solution $z(t,x) = S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ of system (5), satisfies that

$$
\limsup_{t\to\infty}(S_1(t,x) + E_1(t,x) + I_1(t,x) + S_2(t,x) + E_2(t,x) + I_2(t,x)) < B,
$$

where $B = \frac{(A_1+A_2)|\Omega|}{\Lambda}$ and $|\Omega|$ represents the volume of Ω.

The proof of Lemma 3.2 is shown in Appendix B.

Theorem 3.1. For any initial value (7) and $t \geq 0$, system (5) has a unique positive solution $z(t,x)$.

The proof is given in the Appendix C. This is the basis of the whole paper, which makes the subsequent analysis meaningful.

4 Finite-time stability

Before giving sufficient conditions to ensure FTS, the following necessary lemma is introduced.

Lemma 4.1.[21] Assume that there are positive constants $m_i, i = 1,2,\cdots,r$, such that $x \in \Omega$ and $|x_i| < m_i$. Given a function $z(x) \in R^n$ which belongs to $C^2(\Omega)$ and vanishes on $\partial\Omega$, has

$$
\int_\Omega z^T(x)\frac{\partial^2 z(x)}{\partial x^2}dx \leq -\sum_{i=1}^{r}\frac{1}{m_i^2}\int_\Omega z^T(x)z(x)dx.
$$

For ease of further research, alternative ways to express the definitions of FTS and bounded impulsive interval mentioned in Ref [22]and Ref [23] can be provided. The bounded impulse interval method restricts the duration and intensity of impulse disturbances within a certain range, allowing for precise stability analysis of the system under these limitations.

Definition 4.1 Given positive numbers T, B_1 and B_2 with $B_1 < B_2$, system (5) is guaranteed to be finite time stable concerning (T, B_1, B_2), if for any $t \in [0, T]$,

$$\sup_{-\hat{\tau}\leq s\leq 0}(\int_\Omega S_1^2(s,x)dx + \int_\Omega E_1^2(s,x)dx + \int_\Omega I_1^2(s,x)dx + \int_\Omega S_2^2(s,x)dx + \int_\Omega E_2^2(s,x)dx + \int_\Omega I_2^2(s,x)dx) \leq B_1,$$

it can be known that

$$E(\int_\Omega S_1^2(s,x)dx + \int_\Omega E_1^2(s,x)dx + \int_\Omega I_1^2(s,x)dx + \int_\Omega S_2^2(s,x)dx + \int_\Omega E_2^2(s,x)dx + \int_\Omega I_2^2(s,x)dx) \leq B_2. \tag{9}$$

With the utilization of the bounded impulsive interval method, the research aims to establish the sufficient conditions of FTS with respect to (T, B_1, B_2). These sufficient conditions show the effects of spatial diffusion, impulsive effect, delay and white noise on the FTS of system (5). Assign

$D_1 = (A_1^2 + A_2^2 + 2\sigma_1^2 B^4 + 2\sigma_2^2 B^4)$,

$D_2 = \max\{1 + \gamma_1 + 2\sigma_1^2 B^2 + \gamma_2 + 2\sigma_2^2 B^2 + \beta_2^2 B^2 - 2\sum_{i=1}^r \frac{1}{m_i^2} d_j\}, i = j = 1,2,3,4,5,6$

$D_3 = 2C(\beta_1 + \beta_2), \omega = lnB_2 - ln(B_1 + \frac{D_1}{|D_2|})(D_2 \neq 0), \mu = \max\{(1-\rho_{ik})^2\}$,

$$\theta = \begin{cases} \frac{1}{(1-\tilde{\tau})}, 0 < \tilde{\tau} < 1, \\ 1, \tilde{\tau} < 0. \end{cases}$$

Theorem 4.2. Assume that $H(1)$ and $H(2)$ are valid, system (5) is FTS with respect to (T, B_1, B_2), if any of the following conditions is met

$(C1) 0 < \mu < 1, \frac{ln\mu}{h_M} + |D_2| \leq -\frac{D_3}{\mu} e^{-\frac{ln\mu}{h_M}\hat{\tau}} < 0, -ln\mu \leq \omega$,

$(C2) 0 < \mu < 1, -\frac{D_3}{\mu} e^{-(\frac{ln\mu}{h_M} + |D_2|)\hat{\tau}} < \frac{ln\mu}{h_M} < 0$,

$(\frac{D_3\theta}{\mu} e^{-\frac{ln\mu}{h_M}\hat{\tau}} + \frac{ln\mu}{h_M} + |D_2|)T + \frac{D_3\hat{\tau}\theta}{\mu} e^{-\frac{ln\mu}{h_M}\hat{\tau}} - ln\mu \leq \omega$,

$(C3) 0 < \mu < 1, \frac{ln\mu}{h_M} + |D_2| > 0, (\frac{D_3}{\mu} + \frac{ln\mu}{h_M} + |D_2|)T - ln\mu \leq \omega$.

Proof. Define

$$\sup_{-\hat{\tau}\leq s\leq 0}(\int_\Omega \phi_{S_1}^2(s,x)dx + \int_\Omega \phi_{E_1}^2(s,x)dx + \int_\Omega \phi_{I_1}^2(s,x)dx + \int_\Omega \phi_{S_2}^2(s,x)dx + \int_\Omega \phi_{E_2}^2(s,x)dx + \int_\Omega \phi_{I_2}^2(s,x)dx) \leq B_1.$$

Choose

$$U(t) = \int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx + \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx). \tag{10}$$

For $t \neq t_k$, taking the differential of $U(t)$ along system (5)

$$\begin{aligned} dU_1(t) \leq\; & -2d_1 \sum_{i=1}^{r} \frac{1}{m_i^2} \int_\Omega S_1^2(t,x)dxdt + \int_\Omega [2A_1 S_1(t,x) - 2\beta_1 S_1^2(t,x) I_2(t,x) \\ & -2\mu_1 S_1^2(t,x) + 2\gamma_1 S_1(t,x) I_1(t,x) + \sigma_1^2 S_1^2(t,x) I_2^2(t,x)]dxdt \\ & -2\int_\Omega \sigma_1 S_1^2(t,x) I_2(t,x) dxdB_1(t), \end{aligned} \tag{11}$$

and

$$\begin{aligned} dU_2(t) \leq\; & -2d_2 \sum_{i=1}^{r} \frac{1}{m_i^2} \int_\Omega E_1^2(t,x)dxdt + \int_\Omega [2\beta_1 S_1(t,x) E_1(t,x) I_2(t,x) - 2\mu_1 E_1^2(t,x) \\ & +\beta_1 E_1(t,x)(S_1^2(t-\tau(t),x) + I_2^2(t-\tau(t),x) + \sigma_1^2 S_1^2(t,x) I_2^2(t,x) \\ & +\sigma_1^2 S_1^2(t-\tau(t),x) I_2^2(t-\tau(t),x)]dxdt + 2\int_\Omega [\sigma_1 S_1(t,x) E_1(t,x) I_2(t,x) \\ & -\sigma_1 E_1(t,x) S_1(t-\tau(t),x) I_2(t-\tau(t),x)]dxdB_1(t), \end{aligned} \tag{12}$$

further

$$\begin{aligned} dU_3(t) \leq\; & -2d_3 \sum_{i=1}^{r} \frac{1}{m_i^2} \int_\Omega I_1^2(t,x)dxdt + \int_\Omega \beta_1 I_1(t,x)(S_1^2(t-\tau(t),x) + I_2^2(t-\tau(t),x) \\ & -2(\mu_1+\gamma_1) I_1^2(t,x) + \sigma_1^2 S_1^2(t-\tau(t),x) I_2^2(t-\tau(t),x)]dxdt \\ & +2\int_\Omega \sigma_1 I_1(t,x) S_1(t-\tau(t),x) I_2(t-\tau(t),x) dxdB_1(t), \end{aligned} \tag{13}$$

also

$$\begin{aligned} dU_4(t) \leq\; & -2d_4 \sum_{i=1}^{r} \frac{1}{m_i^2} \int_\Omega S_2^2(t,x)dxdt + \int_\Omega [2A_2 S_2(t,x) - 2\beta_2 S_2^2(t,x) I_1(t,x) \\ & -2\mu_2 S_2^2(t,x) + 2\gamma_2 S_2(t,x) I_2(t,x) + \sigma_2^2 S_2^2(t,x) I_1^2(t,x)]dxdt \\ & -2\int_\Omega \sigma_2 S_2^2(t,x) I_1(t,x) dxdB_2(t), \end{aligned} \tag{14}$$

likewise

$$\begin{aligned}
dU_5(t) \;\leq\; & -2d_5\sum_{i=1}^{r}\frac{1}{m_i^2}\int_\Omega E_2^2(t,x)dxdt+\int_\Omega[2\beta_2S_2(t,x)E_2(t,x)I_1(t,x)-2\mu_2E_2^2(t,x)\\
&+\beta_2E_2(t,x)(S_2^2(t-\tau(t),x)+I_1^2(t-\tau(t),x))+\sigma_2^2S_2^2(t,x)I_1^2(t,x)\\
&+\sigma_2^2S_2^2(t-\tau(t),x)I_1^2(t-\tau(t),x)]dxdt+2\textstyle\int_\Omega[\sigma_2S_2(t,x)E_2(t,x)I_1(t,x)\\
&-\sigma_2E_2(t,x)S_2(t-\tau(t),x)I_1(t-\tau(t),x)]dxdB_2(t),
\end{aligned} \tag{15}$$

moreover

$$\begin{aligned}
dU_6(t) \;\leq\; & -2d_6\sum_{i=1}^{r}\frac{1}{m_i^2}\int_\Omega I_2^2(t,x)dxdt+\int_\Omega \beta_2I_2(t,x)(S_2^2(t-\tau(t),x)+I_1^2(t-\tau(t),x)\\
&-2(\mu_2+\gamma_2)I_2^2(t,x)+\sigma_2^2S_2^2(t-\tau(t),x)I_1^2(t-\tau(t),x)]dxdt\\
&+2\textstyle\int_\Omega \sigma_2I_2(t,x)S_2(t-\tau(t),x)I_1(t-\tau(t),x)dxdB_2(t).
\end{aligned} \tag{16}$$

Then, let's substitute equations (11) to (16) into equation (10) and arrange it

$$\begin{aligned}
dU(t) \;\leq\; & -2\textstyle\sum_{i=1}^{r}\frac{1}{m_i^2}\int_\Omega[d_1S_1^2(t,x)+d_2E_1^2(t,x)+d_3I_1^2(t,x)+d_4S_2^2(t,x)+d_5E_2^2(t,x)\\
&+d_6I_2^2(t,x)]dxdt+\textstyle\int_\Omega[2A_1S_1(t,x)+2\gamma_1S_1(t,x)I_1(t,x)+2\sigma_1^2S_1^2(t,x)I_2^2(t,x)\\
&+2\beta_1S_1(t,x)E_1(t,x)I_2(t,x)+2\sigma_2^2S_2^2(t,x)I_1^2(t,x)+2\beta_2S_2(t,x)E_2(t,x)I_1(t,x)\\
&+2\sigma_1^2S_1^2(t-\tau(t),x)I_2^2(t-\tau(t),x)+2A_2S_2(t,x)+2\gamma_2S_2(t,x)I_2(t,x)\\
&+\beta_1(E_1(t,x)+I_1(t,x))(S_1^2(t-\tau(t),x)+I_2^2(t-\tau(t),x))\\
&+\beta_2(E_2(t,x)+I_2(t,x))(S_2^2(t-\tau(t),x)+I_1^2(t-\tau(t),x))\\
&+2\sigma_2^2S_2^2(t-\tau(t),x)I_1^2(t-\tau(t),x)]dxdt-2\textstyle\int_\Omega \sigma_1S_1^2(t,x)I_2(t,x)dxdB_1(t)\\
&+2\textstyle\int_\Omega[\sigma_1S_1(t,x)E_1(t,x)I_2(t,x)\\
&-\sigma_1E_1(t,x)S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dxdB_1(t)\\
&+2\textstyle\int_\Omega \sigma_1I_1(t,x)S_1(t-\tau(t),x)I_2(t-\tau(t),x)dxdB_1(t)\\
&-2\textstyle\int_\Omega \sigma_2S_2^2(t,x)I_1(t,x)dxdB_2(t)+2\textstyle\int_\Omega[\sigma_2S_2(t,x)E_2(t,x)I_1(t,x)\\
&-\sigma_2E_2(t,x)S_2(t-\tau(t),x)I_1(t-\tau(t),x)]dxdB_2(t)\\
&+2\textstyle\int_\Omega \sigma_2I_2(t,x)S_2(t-\tau(t),x)I_1(t-\tau(t),x)dxdB_2(t).
\end{aligned} \tag{17}$$

According to Lemma 3.2, Theorem 3.1 and fundamental inequality, for $t \neq t_k$,

$$\begin{aligned}
dU(t) \leq & -2\sum_{i=1}^{r}\frac{1}{m_i^2}\int_\Omega [d_1 S_1^2(t,x) + d_2 E_1^2(t,x) + d_3 I_1^2(t,x) + d_4 S_2^2(t,x) + d_5 E_2^2(t,x) \\
& + d_6 I_2^2(t,x)]dxdt + \int_\Omega [A_1^2 + A_2^2 + 2\sigma_1^2 B^4 + 2\sigma_2^2 B^4 + (1+\gamma_1 + 2\sigma_1^2 B^2 \\
& + \beta_1^2 B^2) S_1^2(t,x) + (1+\gamma_1) I_1^2(t,x) + (1+\gamma_2 + 2\sigma_2^2 B^2 + \beta_2^2 B^2) S_2^2(t,x) \\
& + (1+\gamma_2) I_2^2(t,x) + C\beta_1 (S_1^2(t-\tau(t),x) + I_2^2(t-\tau(t),x)) \\
& + C\beta_2 (S_2^2(t-\tau(t),x) + I_1^2(t-\tau(t),x))]dxdt - 2\int_\Omega \sigma_1 S_1^2(t,x) I_2(t,x) dxdB_1(t) \\
& + 2\int_\Omega [\sigma_1 S_1(t,x) E_1(t,x) I_2(t,x) \\
& - \sigma_1 E_1(t,x) S_1(t-\tau(t),x) I_2(t-\tau(t),x)]dxdB_1(t) \\
& + 2\int_\Omega \sigma_1 I_1(t,x) S_1(t-\tau(t),x) I_2(t-\tau(t),x) dxdB_1(t) \\
& - 2\int_\Omega \sigma_2 S_2^2(t,x) I_1(t,x) dxdB_2(t) \\
& + 2\int_\Omega [\sigma_2 S_2(t,x) E_2(t,x) I_1(t,x) \\
& - \sigma_2 E_2(t,x) S_2(t-\tau(t),x) I_1(t-\tau(t),x)]dxdB_2(t) \\
& + 2\int_\Omega \sigma_2 I_2(t,x) S_2(t-\tau(t),x) I_1(t-\tau(t),x) dxdB_2(t)
\end{aligned}$$

where
$D_1 = (A_1^2 + A_2^2 + 2\sigma_1^2 B^4 + 2\sigma_2^2 B^4)$,
$D_2 = \max\{2+\gamma_1 + 2\sigma_1^2 B^2 + \gamma_2 + 2\sigma_2^2 B^2 + \beta_2^2 B^2 - 2\sum_{i=1}^{r}\frac{1}{m_i^2} d_j\}, i = j = 1,2,3,4,5,6,$
$D_3 = 2C(\beta_1 + \beta_2)$.

$$\begin{aligned}
dU(t) \leq & (\int_\Omega D_1 dxdt + D_2 \int_\Omega [S_1^2(t,x) + E_1^2(t,x) + I_1^2(t,x) + S_2^2(t,x) + E_2^2(t,x) + I_2^2(t,x)]dxdt \\
& + D_3 \int_\Omega [S_1^2(t-\tau(t),x) + E_1^2(t,x) + I_1^2(t-\tau(t),x) + S_2^2(t-\tau(t),x) \\
& + E_2^2(t,x) + I_2^2(t-\tau(t),x)]dxdt - 2\int_\Omega \sigma_1 S_1^2(t,x) I_2(t,x) dxdB_1(t) \\
& + 2\int_\Omega [\sigma_1 S_1(t,x) E_1(t,x) I_2(t,x) - \sigma_1 E_1(t,x) S_1(t-\tau(t),x) I_2(t-\tau(t),x)]dxdB_1(t) \\
& + 2\int_\Omega \sigma_1 I_1(t,x) S_1(t-\tau(t),x) I_2(t-\tau(t),x) dxdB_1(t) \\
& - 2\int_\Omega \sigma_2 S_2^2(t,x) I_1(t,x) dxdB_2(t) \\
& + 2\int_\Omega [\sigma_2 S_2(t,x) E_2(t,x) I_1(t,x) - \sigma_2 E_2(t,x) S_2(t-\tau(t),x) I_1(t-\tau(t),x)]dxdB_2(t) \\
& + 2\int_\Omega \sigma_2 I_2(t,x) S_2(t-\tau(t),x) I_1(t-\tau(t),x) dxdB_2(t),
\end{aligned} \tag{18}$$

For $t = t_k$, obviously

$$\begin{aligned} U(t_k^+) &= \int_\Omega S_1^2(t_k^+,x)dx + \int_\Omega E_1^2(t_k^+,x)dx + \int_\Omega I_1^2(t_k,x)dx + \int_\Omega S_2^2(t_k^+,x)dx \\ &\quad + \int_\Omega E_2^2(t_k^+,x)dx + \int_\Omega I_2^2(t_k,x)dx \\ &= \int_\Omega (1-\rho_{1k})^2 S_1^2(t_k,x)dx + \int_\Omega (1-\rho_{2k})^2 E_1^2(t_k,x)dx + \int_\Omega I_1^2(t_k,x)dx \\ &\quad + \int_\Omega (1-\rho_{3k})^2 S_2^2(t_k,x)dx + \int_\Omega (1-\rho_{4k})^2 E_2^2(t_k,x)dx + \int_\Omega I_2^2(t_k,x)dx \\ &\le \max\{(1-\rho_{ik})^2\}(\int_\Omega S_1^2(t_k,x)dx + \int_\Omega E_1^2(t_k,x)dx + \int_\Omega I_1^2(t_k,x)dx \\ &\quad + \int_\Omega S_2^2(t_k,x)dx + \int_\Omega E_2^2(t_k,x) + \int_\Omega I_2^2(t_k,x)dx) \\ &= \mu U(t_k). \end{aligned} \tag{19}$$

The solution $X(t)$ satisfies the following system

$$\begin{cases} dX(t) = [D_1 + D_2X(t) + D_3X(t-\tau(t))]dt - 2\int_\Omega \sigma_1 S_1^2(t,x)I_2(t,x)dxdB_1(t) \\ \quad +2\int_\Omega [\sigma_1 S_1(t,x)E_1(t,x)I_2(t,x) \\ \quad -\sigma_1 E_1(t,x)S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dxdB_1(t) \\ \quad +2\int_\Omega \sigma_1 I_1(t,x)S_1(t-\tau(t),x)I_2(t-\tau(t),x)dxdB_1(t) \\ \quad -2\int_\Omega \sigma_2 S_2^2(t,x)I_1(t,x)dxdB_2(t) \\ \quad +2\int_\Omega [\sigma_2 S_2(t,x)E_2(t,x)I_1(t,x) \\ \quad -\sigma_2 E_2(t,x)S_2(t-\tau(t),x)I_1(t-\tau(t),x)]dxdB_2(t) \\ \quad +2\int_\Omega \sigma_2 I_2(t,x)S_2(t-\tau(t),x)I_1(t-\tau(t),x)dxdB_2(t), \\ X(t_k^+) = \mu X(t_k), \\ X(s) = \int_\Omega \phi_{S_1}^2(s,x)dx + \int_\Omega \phi_{E_1}^2(s,x)dx + \int_\Omega \phi_{I_1}^2(s,x)dx + \int_\Omega \phi_{S_2}^2(s,x)dx \\ \quad \int_\Omega \phi_{E_2}^2(s,x)dx + \int_\Omega \phi_{I_2}^2(s,x)dx, -\hat{\tau} \le s \le 0. \end{cases} \tag{20}$$

Due to

$$\begin{aligned} U(s) = \int_\Omega \phi_{S_1}^2(s,x)dx + \int_\Omega \phi_{E_1}^2(s,x)dx + \int_\Omega \phi_{I_1}^2(s,x)dx + \int_\Omega \phi_{S_2}^2(s,x)dx \\ + \int_\Omega \phi_{E_2}^2(s,x)dx + \int_\Omega \phi_{I_2}^2(s,x)dx = X(s), -\hat{\tau} \le s \le 0, \end{aligned}$$

This can be derived from equations (18) and (19), as well as the comparison theorem, resulting in

$$U(t) \le X(t).$$

From another aspect, the method of variation of constants suggests that the solution of the system (20) takes the form

$X(t) = -\frac{D_1}{D_2}\mu^{Y(t,0)} + (X(0) + \frac{D_1}{D_2})\beta(t,0) + \int_0^t \beta(t,s)D_3X(s-\tau(s))ds + Z(s), a.s$
where $\beta(t,s) = \mu^{Y(t,s)}e^{D_2(t-s)}, t > s \geq 0$,

$$\begin{aligned} Z(s) &= -2\int_0^t \beta(t,s)\int_\Omega \sigma_1 S_1^2(s,x)I_2(s,x)dxdB_1(s) \\ &\quad +2\int_0^t \beta(t,s)\int_\Omega [\sigma_1 S_1(s,x)E_1(s,x)I_2(s,x) \\ &\quad -\sigma_1 E_1(s,x)S_1(s-\tau(s),x)I_2(s-\tau(s),x)]dxdB_1(s) \\ &\quad +2\int_0^t \beta(t,s)\int_\Omega \sigma_1 I_1(s,x)S_1(s-\tau(s),x)I_2(s-\tau(s),x)dxdB_1(s) \\ &\quad -2\int_0^t \beta(t,s)\int_\Omega \sigma_2 S_2^2(s,x)I_1(s,x)dxdB_2(s) \\ &\quad +2\int_0^t \beta(t,s)\int_\Omega [\sigma_2 S_2(s,x)E_2(s,x)I_1(s,x) \\ &\quad -\sigma_2 E_2(s,x)S_2(s-\tau(s),x)I_1(s-\tau(s),x)]dxdB_2(s) \\ &\quad +2\int_0^t \beta(t,s)\int_\Omega \sigma_2 I_2(s,x)S_2(s-\tau(s),x)I_1(s-\tau(s),x)dxdB_2(s). \end{aligned}$$

For all $t \geq s$ within the interval $[0,T]$, it can be easily inferred that

$$\beta(t,s) = \mu^{Y(t,s)}e^{D_2(t-s)} = e^{Y(t,s)ln\mu + D_2(t-s)}.$$

Let $Y(T,t)$ denote the number of impulsive moments in the time sequence t_k during the interval $(t,T]$. Then, we have

$$\frac{t-s-h_M}{h_M} \leq Y(t,s) \leq \frac{t-s}{h_m},$$

where $h_M = \max_{k\in K}(t_k - t_{k-1})$ and $h_m = \min_{k\in K}(t_k - t_{k-1}), k \in K = 1,2,...,Y(T,0)$. When $0 < \mu < 1$, in the following, two important cases will be considered for our analysis.

Due to

$\beta(t,s) = e^{Y(t,s)ln\mu + D_2(t-s)} \leq e^{\frac{t-s-h_M}{h_M}ln\mu + D_2(t-s)} = e^{(\frac{ln\mu}{h_M}+D_2)(t-s)-ln\mu}$,
$\mu^{Y(t,0)} = e^{Y(t,0)ln\mu} \leq e^{\frac{t-h_M}{h_M}ln\mu} = e^{\frac{ln\mu}{h_M}t - ln\mu}$.

Thus, we have

$$\begin{aligned} E(&\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx + \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx) \\ &\leq EX(t) \leq -\frac{D_1}{D_2}\mu^{Y(t,0)} + (X(0) + \frac{D_1}{D_2})\beta(t,0) + E\int_0^t \beta(t,s)D_3X(s-\tau(s))ds \\ &\leq -\frac{D_1}{\mu D_2}e^{\frac{ln\mu}{h_M}t} + \frac{1}{\mu}(X(0) + \frac{D_1}{D_2})e^{(\frac{ln\mu}{h_M}+D_2)t} + \frac{D_3}{\mu}E\int_0^t e^{(\frac{ln\mu}{h_M}+D_2)(t-s)}X(s-\tau(s))ds. \end{aligned} \tag{21}$$

Case 1: $D_2 > 0$. Then

$$E(\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx + \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx)$$
$$\leq \frac{1}{\mu}(X(0) + \frac{D_1}{D_2})e^{(\frac{ln\mu}{h_M}+D_2)t} + \frac{D_3}{\mu}E\int_0^t e^{(\frac{ln\mu}{h_M}+D_2)(t-s)}X(s-\tau(s))ds. \tag{22}$$

Case 1.1: $\frac{ln\mu}{h_M} + D_2 \leq 0$. In this scenario, for $-\hat{\tau} \leq t \leq 0$, it is evident that

$$E(\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx + \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx)$$
$$\leq \frac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{\lambda t}, \tag{23}$$

Subsequently, we aim to validate that, for $t \geq 0$,, equation (23) holds. If the statement is incorrect, then there exists a t^* such that

$$\begin{aligned} E(&\int_\Omega S_1^2(t^*,x)dx + \int_\Omega E_1^2(t^*,x)dx + \int_\Omega I_1^2(t^*,x)dx + \int_\Omega S_2^2(t^*,x)dx \\ &+ \int_\Omega E_2^2(t^*,x)dx + \int_\Omega I_2^2(t^*,x)dx) \geq \frac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{\lambda t^*}, \end{aligned} \tag{24}$$

moreover, for $t < t^*$

$$\begin{aligned} E(&\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx \\ &+ \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx) \leq \frac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{\lambda t}, \end{aligned} \tag{25}$$

therefore, we can deduce from (25) and (22) that

$$\begin{aligned} &E(\int_\Omega S_1^2(t^*,x)dx + \int_\Omega E_1^2(t^*,x)dx + \int_\Omega I_1^2(t^*,x)dx + \int_\Omega S_2^2(t^*,x)dx \\ &\quad + \int_\Omega E_2^2(t^*,x)dx + \int_\Omega I_2^2(t^*,x)dx) \\ &\leq \frac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{(\frac{ln\mu}{h_M}+D_2)t^*} + \frac{D_3}{\mu}E\int_0^{t^*} e^{(\frac{ln\mu}{h_M}+D_2)(t^*-s)}X(s-\tau(s))ds \\ &\leq e^{(\frac{ln\mu}{h_M}+D_2)t^*}[\frac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2}) + \frac{D_3}{\mu}\int_0^{t^*} e^{-(\frac{ln\mu}{h_M}+D_2)s}(\frac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{\lambda(s-\tau(s))})ds] \\ &\leq e^{(\frac{ln\mu}{h_M}+D_2)t^*}\frac{1}{\mu}[(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2}) + \frac{D_3}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{-\lambda\hat{\tau}}\int_0^{t^*} e^{[\lambda-(\frac{ln\mu}{h_M}+D_2)]s})ds] \\ &\leq e^{(\frac{ln\mu}{h_M}+D_2)t^*}\frac{1}{\mu}[(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2}) + \frac{\frac{D_3}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{-\lambda\hat{\tau}}}{\lambda - (\frac{ln\mu}{h_M} + D_2)}(e^{[\lambda-(\frac{ln\mu}{h_M}+D_2)]t^*} - 1)]. \end{aligned} \tag{26}$$

Case 1.1.1: From $(C1)$, $\frac{ln\mu}{h_M} + D_2 \leq -\frac{D_3}{\mu} e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}} < 0, -(\frac{ln\mu}{h_M} + D_2) > \frac{D_3}{\mu}$, assign

$$h(\theta) = -\theta + (\frac{ln\mu}{h_M} + D_2) + \frac{D_3}{\mu} e^{-\theta\hat{\tau}},$$

we can directly derive that $h(0) = (\frac{ln\mu}{h_M}+D_2)+\frac{D_3}{\mu} < 0$ and $h(-\infty) = +\infty$. From $h^{'}(\theta) = -1-\frac{D_3\hat{\tau}}{\mu}e^{-\theta\hat{\tau}} < 0$, there exists a unique $\lambda < 0$ that $h(\lambda) = 0 \Leftrightarrow \lambda-(\frac{ln\mu}{h_M}+D_2) = \frac{D_3}{\mu}e^{-\lambda\hat{\tau}}$. It can be deduced follows from (26),

$$\begin{aligned} E(\int_\Omega S_1^2(t^*, x)dx + \int_\Omega E_1^2(t^*, x)dx + \int_\Omega I_1^2(t^*, x)dx + \int_\Omega S_2^2(t^*, x)dx \\ + \int_\Omega E_2^2(t^*, x)dx + \int_\Omega I_2^2(t^*, x)dx) \leq \frac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2})e^{\lambda t^*}, \end{aligned} \tag{27}$$

which contradicts (24), then

$$\begin{aligned} & E(\textstyle\int_\Omega S_1^2(t, x)dx + \int_\Omega E_1^2(t, x)dx + \int_\Omega I_1^2(t, x)dx + \int_\Omega S_2^2(t, x)dx + \int_\Omega E_2^2(t, x)dx + \int_\Omega I_2^2(t, x)dx) \\ \leq & \tfrac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \tfrac{D_1}{D_2})e^{\lambda t} \leq \tfrac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \tfrac{D_1}{D_2}) \\ \leq & \tfrac{1}{\mu}(B_1 + \tfrac{D_1}{D_2}). \end{aligned} \tag{28}$$

For $\omega = lnB_2 - ln(B_1 + \frac{D_1}{|D_2|})$, and $-ln\mu \leq \omega$,

$$\begin{aligned} E(\int_\Omega S_1^2(t, x)dx + \int_\Omega E_1^2(t, x)dx + \int_\Omega I_1^2(t, x)dx + \int_\Omega S_2^2(t, x)dx \\ + \int_\Omega E_2^2(t, x)dx + \int_\Omega I_2^2(t, x)dx) \leq B_2. \end{aligned} \tag{29}$$

Case 1.1.2: If condition $(C2)$ is valid, it results in $-\frac{D_3}{\mu}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}} \leq \frac{ln\mu}{h_M} + D_2 \leq 0$, from (22),

$$\begin{aligned} & E[e^{-(\frac{ln\mu}{h_M}+D_2)t}X(t)] \\ \leq & \tfrac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2}) + \frac{D_3}{\mu}E\int_0^t e^{-(\frac{ln\mu}{h_M}+D_2)s}X(s-\tau(s))ds \\ \leq & \tfrac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2}) + \frac{D_3}{\mu}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}}E\int_0^t e^{-(\frac{ln\mu}{h_M}+D_2)(s-\tau(s))}X(s-\tau(s))ds \\ \leq & \tfrac{1}{\mu}(\sup_{-\hat{\tau}\leq s\leq 0} X(s) + \frac{D_1}{D_2}) + \frac{D_3}{\mu}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}}\frac{1}{1-\tilde{\tau}}E\int_{-\hat{\tau}}^t e^{-(\frac{ln\mu}{h_M}+D_2)s}X(s)ds. \end{aligned} \tag{30}$$

By applying the Gronwall inequality, we obtain

$$E[e^{-(\frac{ln\mu}{h_M}+D_2)t}X(t)] \le \frac{1}{\mu}(\sup_{-\hat{\tau}\le s\le 0} X(s) + \frac{D_1}{D_2})exp[\frac{D_3}{\mu(1-\tilde{\tau})}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}}(t+\hat{\tau})]. \quad (31)$$

Then

$$E(\int_\Omega S_1^2(t,x)dx+\int_\Omega E_1^2(t,x)dx+\int_\Omega I_1^2(t,x)dx+\int_\Omega S_2^2(t,x)dx+\int_\Omega E_2^2(t,x)dx+\int_\Omega I_2^2(t,x)dx) \le EX(t)$$
$$\le \begin{cases} \frac{1}{\mu}(\sup_{-\hat{\tau}\le s\le 0} X(s) + \frac{D_1}{D_2})exp[(\frac{D_3}{\mu(1-\tilde{\tau})}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}} + \frac{ln\mu}{h_M} + D_2)t + \frac{D_3\hat{\tau}}{\mu(1-\tilde{\tau})}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}}], 0<\tilde{\tau}<1, \\ \frac{1}{\mu}(\sup_{-\hat{\tau}\le s\le 0} X(s) + \frac{D_1}{D_2})exp[(\frac{D_3}{\mu}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}} + \frac{ln\mu}{h_M} + D_2)t + \frac{D_3\hat{\tau}}{\mu}e^{-(\frac{ln\mu}{h_M}+D_2)\hat{\tau}}], \tilde{\tau}<0. \end{cases} \quad (32)$$

It can be obtained by $(C2)$ that

$$E(\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx + \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx) \le B_2.$$

Case 1.2: $\frac{ln\mu}{h_M} + D_2 > 0$. Let $X(t)$ be the solution to the following system

$$\begin{cases} P(t) = \frac{1}{\mu}(\sup_{-\hat{\tau}\le s\le 0} P(s) + \frac{D_1}{D_2})e^{(\frac{ln\mu}{h_M}+D_2)t} + \frac{D_3}{\mu}\int_0^t e^{(\frac{ln\mu}{h_M}+D_2)(t-s)}P(s-\tau(s))ds, t>0, \\ P(s) = \int_\Omega \phi_{S_1}^2(s,x)dx + \int_\Omega \phi_{E_1}^2(s,x)dx + \int_\Omega \phi_{I_1}^2(s,x)dx + \int_\Omega \phi_{S_2}^2(s,x)dx \\ \quad + \int_\Omega \phi_{E_2}^2(s,x)dx + \int_\Omega \phi_{I_2}^2(s,x)dx, -\hat{\tau}\le s\le 0. \end{cases} \quad (33)$$

From (22),

$$0 < E(\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx + \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx) \le EX(t), fort > -\hat{\tau}.$$

For $0<t<\tau(t), \tau(t)\in[0,\hat{\tau}]$,

$$\begin{aligned} P(t) - P(t-\tau(t)) &\ge P(t) - \tfrac{1}{\mu}(\sup_{-\hat{\tau}\le s\le 0} P(s) + \frac{D_1}{D_2}) \\ &= \tfrac{1}{\mu}\sup_{-\hat{\tau}\le s\le 0} P(s)(e^{(\frac{ln\mu}{h_M}+D_2)t} - 1) + \frac{D_3}{\mu}\int_0^t e^{(\frac{ln\mu}{h_M}+D_2)(t-s)}P(s-\tau(s))ds \ge 0. \end{aligned} \quad (34)$$

For $t > \tau(t), \tau(t) \in [0, \hat{\tau}]$, and $t \in (\hat{\tau}, T]$,

$$
\begin{aligned}
& P(t) - P(t-\tau(t)) \\
= \ & \tfrac{1}{\mu}(\sup_{-\hat{\tau} \le s \le 0} P(s)(e^{(\frac{ln\mu}{h_M}+D_2)t} - e^{(\frac{ln\mu}{h_M}+D_2)(t-\tau(t))}) + \frac{D_3}{\mu}\int_0^t e^{(\frac{ln\mu}{h_M}+D_2)(t-s)} P(s-\tau(s))ds \\
& -\tfrac{D_3}{\mu}\int_0^{t-\tau(t)} e^{(\frac{ln\mu}{h_M}+D_2)(t-\tau(t)-s)} P(s-\tau(s))ds \\
= \ & \tfrac{1}{\mu}(\sup_{-\hat{\tau} \le s \le 0} P(s)(e^{(\frac{ln\mu}{h_M}+D_2)t}(1-e^{-\tau(t)}) \\
& +\tfrac{D_3}{\mu} e^{(\frac{ln\mu}{h_M}+D_2)(t-\tau(t))} \int_{t-\tau(t)}^t e^{-(\frac{ln\mu}{h_M}+D_2)s} P(s-\tau(s))ds \ge 0.
\end{aligned} \tag{35}
$$

Then $P(t) \ge P(t-\tau(t))$, when $t > 0$. This is deduced from (33) that

$$
P(t) \le \frac{1}{\mu}(\sup_{-\hat{\tau} \le s \le 0} P(s) + \frac{D_1}{D_2})e^{(\frac{ln\mu}{h_M}+D_2)t} + \frac{D_3}{\mu}\int_0^t e^{(\frac{ln\mu}{h_M}+D_2)(t-s)} P(s)ds, fort > 0. \tag{36}
$$

According to the Gronwall inequality, it can be get

$$
P(t)e^{-(\frac{ln\mu}{h_M}+D_2)t} \le \tfrac{1}{\mu}(\sup_{-\hat{\tau} \le s \le 0} P(s) + \frac{D_1}{D_2})exp(\frac{D_3}{\mu}t), fort > 0.
$$

This implies that

$$
\begin{aligned}
& E(\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx + \int_\Omega E_2^2(t,x)dx \\
& \quad + \int_\Omega I_2^2(t,x)dx) \le \frac{1}{\mu}(\sup_{-\hat{\tau} \le s \le 0} P(s) + \frac{D_1}{D_2})exp[(\frac{D_3}{\mu} + \frac{ln\mu}{h_M} + D_2)t].
\end{aligned} \tag{37}
$$

For $(C3)$ holds, it can be known that

$$
\begin{aligned}
& E(\int_\Omega S_1^2(t,x)dx + \int_\Omega E_1^2(t,x)dx + \int_\Omega I_1^2(t,x)dx + \int_\Omega S_2^2(t,x)dx. \\
& \quad + \int_\Omega E_2^2(t,x)dx + \int_\Omega I_2^2(t,x)dx) \le B_2.
\end{aligned}
$$

Case 2:$D_2 < 0$. It can be deduced

$$
\begin{aligned}
EX(t) & \le -\tfrac{D_1}{D_2}\mu^{Y(t,0)} + (X(0) + \tfrac{D_1}{D_2})\mu^{Y(t,0)}e^{D_2 t} + E\int_0^t \beta(t,s)D_3X(s-\tau(s))ds \\
& \le \tfrac{1}{\mu}(\sup_{0 \le s \le \hat{\tau}} X(s) - \frac{D_1}{D_2})e^{\frac{ln\mu}{h_M}t} + \frac{D_3}{\mu}E\int_0^t e^{\frac{ln\mu}{h_M}(t-s)} X(s-\tau(s))ds.
\end{aligned} \tag{38}
$$

Case 2.1.1: For$(C1)$, if $\frac{ln\mu}{h_M} \le -\frac{D_3}{\mu}e^{\frac{ln\mu}{h_M}\hat{\tau}} < 0, i.e. -\frac{ln\mu}{h_M} > \frac{D_3}{\mu}$. Similar to the discussions in case 1.1.1, when $-ln\mu \le \omega$, we obtain

$$E(\int_{\Omega} S_1^2(t,x)dx + \int_{\Omega} E_1^2(t,x)dx + \int_{\Omega} I_1^2(t,x)dx + \int_{\Omega} S_2^2(t,x)dx + \int_{\Omega} E_2^2(t,x)dx + \int_{\Omega} I_2^2(t,x)dx) \leq B_2. \tag{39}$$

Case 2.1.2: For$(C1)$, if $-\frac{D_3}{\mu} e^{-\frac{ln\mu}{h_M}\hat{\tau}} < \frac{ln\mu}{h_M} < 0, 0 < \tilde{\tau} < 1$, and

$$(\frac{D_3}{\mu(1-\tilde{\tau})} e^{-\frac{ln\mu}{h_M}\hat{\tau}} + \frac{ln\mu}{h_M})T + \frac{D_3\tilde{\tau}}{\mu(1-\tilde{\tau})} e^{-\frac{ln\mu}{h_M}\hat{\tau}} - ln\mu \leq \omega,$$

or $\hat{\tau} < 0$, and

$$(\frac{D_3 T}{\mu} e^{-\frac{ln\mu}{h_M}\hat{\tau}} + \frac{ln\mu}{h_M})T + \frac{D_3\hat{\tau}}{\mu} e^{-\frac{ln\mu}{h_M}\hat{\tau}} - ln\mu \leq \omega.$$

Similar discussions as case 1.1.2, yield

$$E(\int_{\Omega} S_1^2(t,x)dx + \int_{\Omega} E_1^2(t,x)dx + \int_{\Omega} I_1^2(t,x)dx + \int_{\Omega} S_2^2(t,x)dx + \int_{\Omega} E_2^2(t,x)dx + \int_{\Omega} I_2^2(t,x)dx) \leq B_2.$$

Remark 4.2 This theorem provides criteria for the FTS of system (5). Based on the sufficient conditions given in the theorem, it can be seen that impulsive disturbance, spatial diffusion, environmental noise, and time delay all have an impact on the FTS of echinococcosis system. Later, we will provide detailed explanations through numerical simulations.

5 Optimal control strategies

To achieve the optimal control strategy, we consider the control of echinococcosis from a cost effectiveness point of view. The goal of the optimal control problem is to determine a control set that minimizes the infected dogs and the infected livestock when minimizing the control costs. Therefore, the following strategy is proposed.

(i)$v_1(t,x)$ denotes the reduction in the number of infections in the final host (dogs, etc.) by preventing herders from feeding the organs of intermediate hosts (cows, sheep, etc.) to domestic dogs through hygiene education for herders, as well as trapping and killing stray dogs.

(ii)$v_2(t,x)$ denotes the reduction of the number of eggs of the echinococcosis pathogen in the environment through proper sanitation, which prevents the ingestion of eggs by intermediate hosts and thus reduces the number of infections in intermediate hosts.

(iii)$v_1(t_k,x), v_3(t_k,x)$ denote the enhancement of immunity of susceptible populations against infection through human intervention by vaccinating the final host susceptible population and the intermediate host susceptible population, respectively. $v_2(t_k,x), v_4(t_k,x)$ denote the reduction in the number of population infections by deworming the final host-exposed population and the intermediate host-exposed population, respectively.

Considering the above control strategy, the following control model is developed

$$
\begin{cases}
\left.\begin{aligned}
dS_1(t,x) &= [d_1\triangle S_1(t,x) + A_1 - \beta_1 S_1(t,x)I_2(t,x) - \mu_1 S_1(t,x) + \gamma_1 I_1(t,x)]dt \\
&\quad - \sigma_1 S_1(t,x)I_2(t,x)dB_1(t), \\
dE_1(t,x) &= [d_2\triangle E_1(t,x) + \beta_1 S_1(t,x)I_2(t,x) - \mu_1 E_1(t,x) \\
&\quad - \beta_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dt + [\sigma_1 S_1(t,x)I_2(t,x) \\
&\quad - \sigma_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dB_1(t), \\
dI_1(t,x) &= [d_3\triangle I_1(t,x) + \beta_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x) - (\mu_1+\gamma_1)I_1(t,x) \\
&\quad - v_1(t,x)I_1(t,x)]dt + \sigma_1 S_1(t-\tau(t),x)I_2(t-\tau(t),x)]dB_1(t), \\
dS_2(t,x) &= [d_4\triangle S_2(t,x) + A_2 - \beta_2 S_2(t,x)I_1(t,x) - \mu_2 S_2(t,x) + \gamma_2 I_2(t,x)]dt \\
&\quad - \sigma_2 S_2(t,x)I_1(t,x)dB_2(t), \\
dE_2(t,x) &= [d_5\triangle E_2(t,x) + \beta_2 S_2(t,x)I_1(t,x) - \mu_2 E_2(t,x) \\
&\quad - \beta_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x)]dt + [\sigma_2 S_2(t,x)I_1(t,x) \\
&\quad - \sigma_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x)]dB_2(t), \\
dI_2(t,x) &= [d_6\triangle I_2(t,x) + \beta_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x) - (\mu_2+\gamma_2)I_2(t,x) \\
&\quad - v_2(t,x)I_2(t,x)]dt + \sigma_2 S_2(t-\tau(t),x)I_1(t-\tau(t),x)dB_2(t),
\end{aligned}\right\}
\begin{aligned} & t \neq t_k, \\ & t > 0, \\ & x \in \Omega \end{aligned} \\
\left.\begin{aligned}
S_1(t_k^+,x) &= (1-\rho_{1k}(v_1(t_k,x)))S_1(t_k,x), \\
E_1(t_k^+,x) &= (1-\rho_{2k}(v_2(t_k,x)))E_1(t_k,x), \\
S_2(t_k^+,x) &= (1-\rho_{3k}(v_3(t_k,x)))S_2(t_k,x), \\
E_2(t_k^+,x) &= (1-\rho_{4k}(v_4(t_k,x)))E_2(t_k,x),
\end{aligned}\right\} t = t_k, k \in 1,2,...,N.
\end{cases}
\tag{40}
$$

Where $y(t,x) = (S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))^T$ is the solution of model (40) with positive initial values for the control. $v(t,x) = (v_1(t,x), v_2(t,x))^T$ is an ordinary control variable and $v(t_k,x) = (v_1(t_k,x), v_2(t_k,x), v_3(t_k,x), v_4(t_k,x))^T$ is an impulse control variable. The admissible control sets κ_v and κ_{v_k} are bounded and convex, defined as follows

$\kappa_v = \{v(t,x) \in L^\infty(\Omega \times [0,T]; R^2) | 0 \leq v_i(t,x) \leq v_{i\max}, \forall t \in [0,T], x \in \Omega\}$

$\kappa_{v_k} = \{v(t_k,x) \in L^\infty(\Omega \times [t_1,...,t_N]; R^4) | 0 \leq v_i(t_k,x) \leq v_{ik\max}, \forall t \in [t_1,...,t_N], x \in \Omega\}$

$\rho_{ik}(v(t_k,x)) = v_i(t_k,x)\rho_{ik}, i = 1,2,3,4$. The aim is to reduce the number of infected populations of echinococcus granulosus through minimal control efforts.

Therefore, we define an objective functional:

$$\begin{aligned} J(u(t,x)) &= E[\int_0^T \int_\Theta L(t,y(t,x),v(t,x))dxdt + \sum_{k=1}^N \int_\Theta G(t_k,y(t_k,x),(v(t_k,x))dx \\ &\quad + \int_\Theta h(y(t,x))dx], \end{aligned}$$

with

$$L(t,y(t,x),v(t,x)) = O_1I_1(t,x) + O_2I_2(t,x) + \sum_{i=1}^{2} \frac{R_i}{2} v_i^2(t,x),$$
$$G(t_k,y(t_k,x),(v(t_k,x)) = \sum_{i=1}^4 \frac{\tilde{R}_i}{2} v_i^2(t_k,x).$$

Where $L(t,y(t,x),v(t,x))$ denotes the cost at moment$t(t \neq t_k)$, $G(t_k,y(t_k,x),(v(t_k,x))$ is the cost associated with the $k-th$ impulse jump, and $h(y(T,x)$ is a function of the $y(t,x)$ at time T. The positive parameters O_1, O_2, R_1 and R_2 are weight constants of the final host infected population, intermediate host infected population and control strategies, respectively; $\tilde{R}_i(i = 1,2,3,4)$ are positive weight constants of impulse control of the $k-th$ jump, respectively.

Theorem 5.1. Assume that $J(u(t,x))$ is subject to model (40) with positive initial conditions. Then there exist an optimal control u and the corresponding state solution $\bar{y}(t,x)$ such that $J(u)$ is minimized.

See Ref.[26] for the similar proof. So, the proof for Theorem 5.1 is omitted for simplicity.

Further, Pontryagin's Maximum Principle [27] transforms the optimal control problem into minimizing the Hamiltonian function from the objective functional subject to the state system (40), define the following Hamiltonian function:

$$\begin{aligned} &H(y(t,x),v(t,x),p(t,x),q(t,x)) \\ &= p_1(t,x)[d_1\triangle S_1(t,x) + A_1 - \beta_1S_1(t,x)I_2(t,x) - \mu_1S_1(t,x) + \gamma_1I_1(t,x)] \\ &\quad + p_2(t,x)[d_2\triangle E_1(t,x) + \beta_1S_1(t,x)I_2(t,x) - \mu_1E_1(t,x) - \beta_1S_1(t-\tau(t),x)I_2(t-\tau(t),x)] \\ &\quad + p_3(t,x)[d_3\triangle I_1(t,x) + \beta_1S_1(t-\tau(t),x)I_2(t-\tau(t),x) - (\mu_1+\gamma_1)I_1(t,x) - v_1(t,x)I_1(t,x)] \\ &\quad + p_4(t,x)[d_4\triangle S_2(t,x) + A_2 - \beta_2S_2(t,x)I_1(t,x) - \mu_2S_2(t,x) + \gamma_2I_2(t,x)] \\ &\quad + p_5(t,x)[d_5\triangle E_2(t,x) + \beta_2S_2(t,x)I_1(t,x) - \mu_2E_2(t,x) - \beta_2S_2(t-\tau(t),x)I_1(t-\tau(t),x)] \\ &\quad + p_6(t,x)[d_6\triangle I_2(t,x) + \beta_2S_2(t-\tau(t),x)I_1(t-\tau(t),x) - (\mu_2+\gamma_2)I_2(t,x) - v_2(t,x)I_2(t,x)] \\ &\quad - q_1(t,x)\sigma_1S_1(t,x)I_2(t,x) + q_2(t,x)[\sigma_1S_1(t,x)I_2(t,x) - \sigma_1S_1(t-\tau(t),x)I_2(t-\tau(t),x)] \\ &\quad + q_3(t,x)\sigma_1S_1(t-\tau(t),x)I_2(t-\tau(t),x) - q_4(t,x)\sigma_2S_2(t,x)I_1(t,x) \\ &\quad + q_5(t,x)[\sigma_2S_2(t,x)I_1(t,x) - \sigma_2S_2(t-\tau(t),x)I_1(t-\tau(t),x)] \\ &\quad + q_6(t,x)\sigma_2S_2(t-\tau(t),x)I_1(t-\tau(t),x) + L(t,y(t,x),v(t,x)), \end{aligned}$$

and an impulse Hamiltonian :

$$
\begin{aligned}
&HI(y(t_k,x),v(t_k,x),p(t_k,x),q(t_k,x)) \\
&= p_1(t_k,x)\rho_{1k}v_1(t_k,x)S_1(t_k,x) + p_2(t_k,x)\rho_{2k}v_2(t_k,x)E_1(t_k,x) \\
&\quad + p_4(t_k,x)\rho_{3k}v_3(t_k,x)S_2(t_k,x) + p_5(t_k,x)\rho_{4k}v_4(t_k,x)E_2(t_k,x) \\
&\quad + G(t_k,y(t_k,x),v_k).
\end{aligned}
$$

Moreover there are adjoint functions $P_i(\cdot,\cdot)(i=1,2,3,4,5,6)$ such as

$$
\begin{aligned}
dp_1(t,x) &= [-d_1\triangle p_1(t,x) + (\beta_1 I_2(t,x)+\mu_1)p_1(t,x) - \beta_1 I_2(t,x)p_2(t,x) \\
&\quad + q_1(t,x)\sigma_1 I_2(t,x) - q_2(t,x)\sigma_1 I_2(t,x) \\
&\quad + 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\beta_1 I_2(t+\tau(\acute{t}),x)p_2(t+\tau(\tilde{t}),x) \\
&\quad - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\beta_1 I_2(t+\tau(\acute{t}),x)p_3(t+\tau(\tilde{t}),x) \\
&\quad + 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\sigma_1 I_2(t+\tau(\acute{t}),x)q_2(t+\tau(\tilde{t}),x) \\
&\quad - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\sigma_1 I_2(t+\tau(\acute{t}),x)q_3(t+\tau(\tilde{t}),x)]dt \\
&\quad + q_1(t,x)dB_1(t) \\
dp_2(t,x) &= [-d_2\triangle p_2(t,x) + \mu_1 p_2(t,x)]dt + q_2(t,x)dB_1(t) \\
dp_3(t,x) &= [-d_3\triangle p_3(t,x) + (\mu_1+\gamma_1) + v_1(t,x))p_3(t,x) \\
&\quad + \beta_2 S_2(t,x)p_4(t,x) - \beta_2 S_2(t,x)p_5(t,x) \\
&\quad + q_4(t,x)\sigma_2 S_2(t,x) - q_5(t,x)\sigma_2 S_2(t,x) - \gamma_1 p_1(t,x) - O_1 \\
&\quad + 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\beta_2 S_2(t+\tau(\acute{t}),x)p_5(t+\tau(\tilde{t}),x) \\
&\quad - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\beta_2 S_2(t+\tau(\acute{t}),x)p_6(t+\tau(\tilde{t}),x) \\
&\quad - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\sigma_2 S_2(t+\tau(\acute{t}),x)q_6(t+\tau(\tilde{t}),x)]dt \\
&\quad + q_3(t,x)dB_1(t) \\
dp_4(t,x) &= [-d_4\triangle p_4(t,x) + (\beta_2 I_1(t,x)+\mu_2)p_4(t,x) - \beta_2 I_1(t,x)p_5(t,x) \\
&\quad + q_4(t,x)\sigma_2 I_1(t,x) - q_5(t,x)\sigma_2 I_1(t,x) \\
&\quad + 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\beta_2 I_1(t+\tau(\acute{t}),x)p_5(t+\tau(\tilde{t}),x) \\
&\quad - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\beta_2 I_1(t+\tau(\acute{t}),x)p_6(t+\tau(\tilde{t}),x) \\
&\quad + 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\sigma_2 I_1(t+\tau(\acute{t}),x)q_5(t+\tau(\tilde{t}),x) \\
&\quad - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\tau(\acute{t}))}\sigma_2 I_1(t+\tau(\acute{t}),x)q_6(t+\tau(\tilde{t}),x)]dt \\
&\quad + q_4(t,x)dB_2(t) \\
dp_5(t,x) &= [-d_5\triangle p_5(t,x) + \mu_2 p_5(t,x)]dt + q_5(t,x)dB_2(t)
\end{aligned}
$$

$$
\begin{aligned}
dp_6(t,x) = & [-d_6\triangle p_6(t,x) + (\mu_2+\gamma_2) + v_2(t,x)]p_6(t,x) \\
& + \beta_1 S_1(t,x)p_1(t,x) - \beta_1 S_1(t,x)p_2(t,x) \\
& + q_1(t,x)\sigma_1 S_1(t,x) - q_2(t,x)\sigma_1 S_1(t,x) - \gamma_2 P_4(t,x) - O_2 \\
& + 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\acute{\tau}(t))}\beta_1 S_1(t+\acute{\tau}(t),x)p_2(t+\tilde{\tau}(t),x) \\
& - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\acute{\tau}(t))}\beta_1 S_1(t+\acute{\tau}(t),x)p_3(t+\tilde{\tau}(t),x) \\
& - 1_{[0,T-\tau(T)]}\frac{1}{1-\dot{\tau}(t-\tau(t)+\acute{\tau}(t))}\sigma_1 S_1(t+\acute{\tau}(t),x)q_3(t+\tilde{\tau}(t),x)]dt \\
& + q_6(t,x)dB_2(t)
\end{aligned}
$$

where the time lead function $\acute{\tau}(\cdot)$ is introduced to take into account the functional dependence of the delay $\tau(\cdot)$ on time. If $s = t - \tau(t)$ for $0 \le t \le T$, by solving for $t, \acute{\tau}(s)$ is given by $t = s + \acute{\tau}(s)$[28]. At the impulse or jump point, we have

$$
\begin{aligned}
p_1(t_k^+,x) &= (1+\rho_{1k}(v_1(t_k,x)))p_1(t_k,x), \\
p_2(t_k^+,x) &= (1+\rho_{2k}(v_2(t_k,x)))p_2(t_k,x), \\
p_4(t_k^+,x) &= (1+\rho_{3k}(v_3(t_k,x)))p_4(t_k,x), \\
p_5(t_k^+,x) &= (1+\rho_{4k}(v_4(t_k,x)))p_5(t_k,x),
\end{aligned}
$$

with transverse conditions

$$
P_i(T) = \int_\Omega h_{y_i}(y(T,x))dx, P_i(T^+) = \int_\Omega h_{y_i}(y(T^+,x))dx, i = 1,2,3,4.
$$

Moreover,
$\bar{v}_i(t,x) = \max\{0, \min\{\tilde{v}_i(t,x), v_{i\max}\}\}, \bar{v}_i(t_k,x) = \max\{0, \min\{\tilde{v}_i(t_k,x), v_{ik\max}\}\}$
within
$\tilde{v}_1(t,x) = \frac{p_3(t,x)\bar{I}_1(t,x)}{R_1}$, $\tilde{v}_2(t,x) = \frac{p_6(t,x)\bar{I}_2(t,x)}{R_2}$, $\tilde{v}_1(t_k,x) = -\frac{p_1(t_k,x)\rho_{1k}\bar{S}_1(t_k,x)}{\tilde{R}_1}$,
$\tilde{v}_2(t_k,x) = -\frac{p_2(t_k,x)\rho_{2k}\bar{E}_1(t_k,x)}{\tilde{R}_2}$, $\tilde{v}_3(t_k,x) = -\frac{p_4(t_k,x)\rho_{3k}\bar{S}_2(t_k,x)}{\tilde{R}_3}$, $\tilde{v}_4(t_k,x) = -\frac{p_5(t_k,x)\rho_{4k}\bar{E}_2(t_k,x)}{\tilde{R}_4}$.

To date, there is no method to completely resolve echinococcosis. Therefore, the development of a control strategy only provides some recommendations for prevention and control, which will form the basis for the development of a new control programme.

6 Numerical simulations

In this section, we analyze the system (5) by utilizing its numerical solution. To accomplish this, we opt for the Milstein method [27] to simulate the behavior of the system (5). This approach allows us to effectively model and study the system's dynamics and behavior in a numerical framework. The values of the parameters of the system are taken from [28,29] and are listed in Table 1. Thus, we set the spatial variable dimension as $r = 1$, with x belonging to the domain $\Omega = [-0.4, 0.4]$, and choose

Table 1: Parameters value.

Parameters	Biological definition	Value	Source
A_1	Recruitment rate of susceptible dogs	$1.67\times10^4 month^{-1}$	Estimated
β_1	Transmission rate form livestock to dogs	$4.6\times10^{-7} month^{-1}$	[9]
μ_1	Natural death rate of dogs	$0.02 month^{-1}$	Estimated
γ_1	Recovery rate of infected dogs	$0.34 month^{-1}$	Estimated
A_2	Recruitment rate of susceptible livestock	$4.52\times10^4 month^{-1}$	Estimated
β_2	Transmission rate form dogs to livestock	7.9×10^{-7}	[9]
μ_2	Natural death rate of livestock	$0.07 month^{-1}$	[10]
γ_2	Recovery rate of infected livestock	$0.51 month^{-1}$	Estimated

a function as

$$\tau(t) = \frac{T}{2\pi}\sin(\frac{\pi t}{T}), t \in [-\hat{\tau}, T],$$

which gives $\tau(t) \leq \hat{\tau} = \frac{T}{2\pi}$ and $\acute{\tau(t)} \leq \tilde{\tau} = \frac{1}{2} < 1$. Setting the impulse sequence

$$\{t_k\} = \{0.4, 0, 8, 1.2, 1.6, 2.0, 2.4, 2.8, 3.2, 3.6, 4.0, 4.4\}, i.e. t_M = t_m = 0.4 month.$$

6.1 Analysis of FTS

Set $T = 4.5 month, B_1 = 1.944\times10^5, B_2 = 2.5\times10^5, \rho_{1k} = 0.2, \rho_{2k} = 0.3, \rho_{3k} = 0.2, \rho_{4k} = 0.2$. $\mu = \max\{(1-\rho_{ik})^2\} \in (0,1)$ and choosing $\sigma_1 = 4\times10^{-6}, \sigma_2 = 3\times10^{-6}$, we have $\frac{ln\mu}{h_M} + |D_2| + \frac{D_3}{\mu}e^{-\frac{ln\mu}{h_M}\hat{\tau}} = -0.263$, and $-ln\mu - \omega = -2.79\times10^8$, as a result, the fulfillment of condition (C1) in Theorem 4.2 suggests that the system exhibits FTS with respect to the parameters $(4.5, 1.944\times10^5, 2.5\times10^5)$ as depicted in Fig.1.

In this scenario, we study the impact of impulse, noise and time delay on the system(5). By selecting parameter values that meet the requirements of condition(C1) in Theorem 4.2, we observe that the system displays FTS with respect to the parameters $(4.5, 1.944\times10^5, 2.5\times10^5)$.

- **The role of impulsive**

When examining the effect of impulse interference on system (5), we maintain the parameter values from Figure 1 and set $\rho_{ik} = 0$. Computational analysis reveals that none of the conditions in Theorem 4.2 are met. In other words, the system, in relation to $(4.5, 1.944\times10^5, 2.5\times10^5)$, does not fulfill the requirements for FTS, as depicted in Figure 2.

In this section, when we solely modify the condition of the impulse compared to Figure 1, we observe that the system is unable to achieve FTS without the influence of the impulse. These findings indicate that the impulse perturbation significantly impacts the system's stability within a finite time frame.

- **The role of noise**

To demonstrate the effect of white noise on System (5), we have selected the same parameters as those used for generating Figure 1 and assigned values of $\sigma_1 = \sigma_2 =$

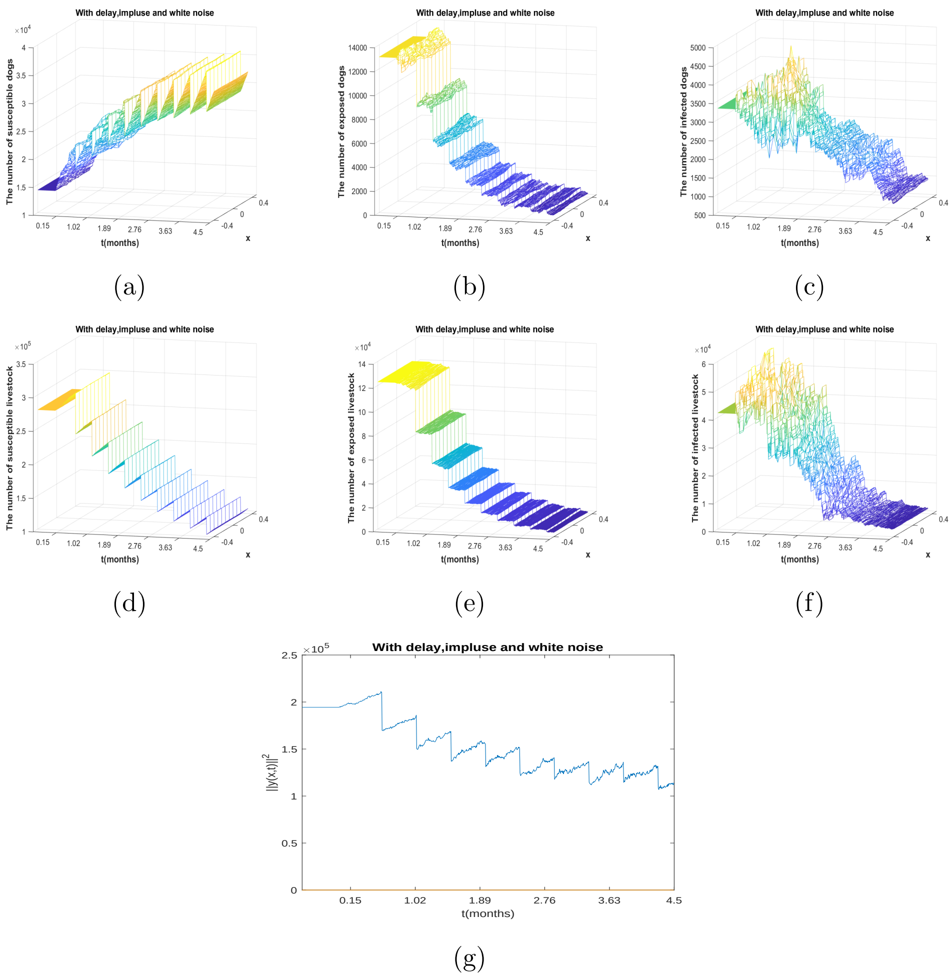


Figure 1: The state trajectories of $y(x,t) = (S_1(x,t), E_1(x,t), I_1(x,t), S_2(x,t), E_2(x,t), I_2(x,t))$ for system (5) with initial value $y(x,0) = (1.405 \times 10^4, 1.3 \times 10^4, 3.3 \times 10^3, 2.78 \times 10^5, 1.23 \times 10^5, 4.16 \times 10^4)$

0. The calculations indicate that the system is not finite time stability relative to $(4.5, 1.944 \times 10^5, 2.5 \times 10^5)$, as shown in Figure 3.

In regards to Figure 3, we are not considering the effect of white noise on System (5). Upon comparing Figure 3 to Figure 1, It is clear that the system fails to sustain its finite-time stability under identical conditions. Thus, it is apparent that white noise exerts a notable influence on the system's finite-time stability, which is only assured when the conditions outlined in Theorem 4.2 are met.

- **The role of delay**

As per Theorem 4.2, it is important to acknowledge that delay also contributes to the FTS of System (5). To demonstrate this influence, we have selected a delay that is different shown in Figure 1 for comparison, while keeping the other conditions the

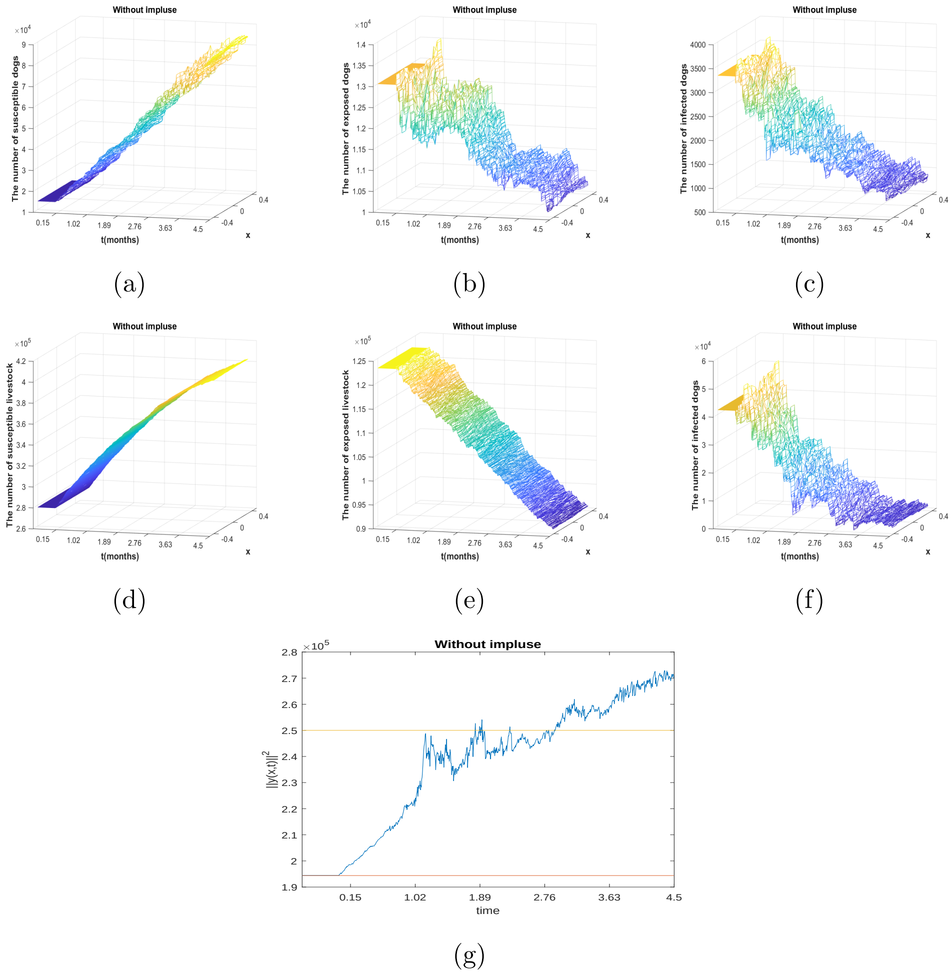


Figure 2: The system (5) displays state trajectories for $y(t,x) = (S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ with an initial value of $y(0,x) = (1.405 \times 10^4, 1.3 \times 10^4, 3.3 \times 10^3, 2.78 \times 10^5, 1.23 \times 10^5, 4.16 \times 10^4)$ without impulse perturbation.

same. Our calculations indicate that under these conditions, the system does not meet the criteria for finite time stability, as illustrated in Figure 4.

The figure shows that, under the same conditions, an unnecessarily large time delay is not beneficial for the finite-time stability of the system.

6.2 Determination of optimal control

In this section, our goal is to observe the impact of control strategies on System (5), particularly their impact on the number of infections in the final and intermediate hosts. We have decided to select a time delay of $\tau(t) = \tau = 2.5$ and weight constants become $O_1 = 1.5 \times 10^4, O_2 = 0.8 \times 10^4, R_1 = 1.9 \times 10^7, R_2 = 0.2 \times 10^4, R_3 = 2.1 \times 10^2, R_4 = 0/4 \times 10^2, R_5 = 2.3 \times 10^3, R_6 = 0.1 \times 10^3$. Addi-

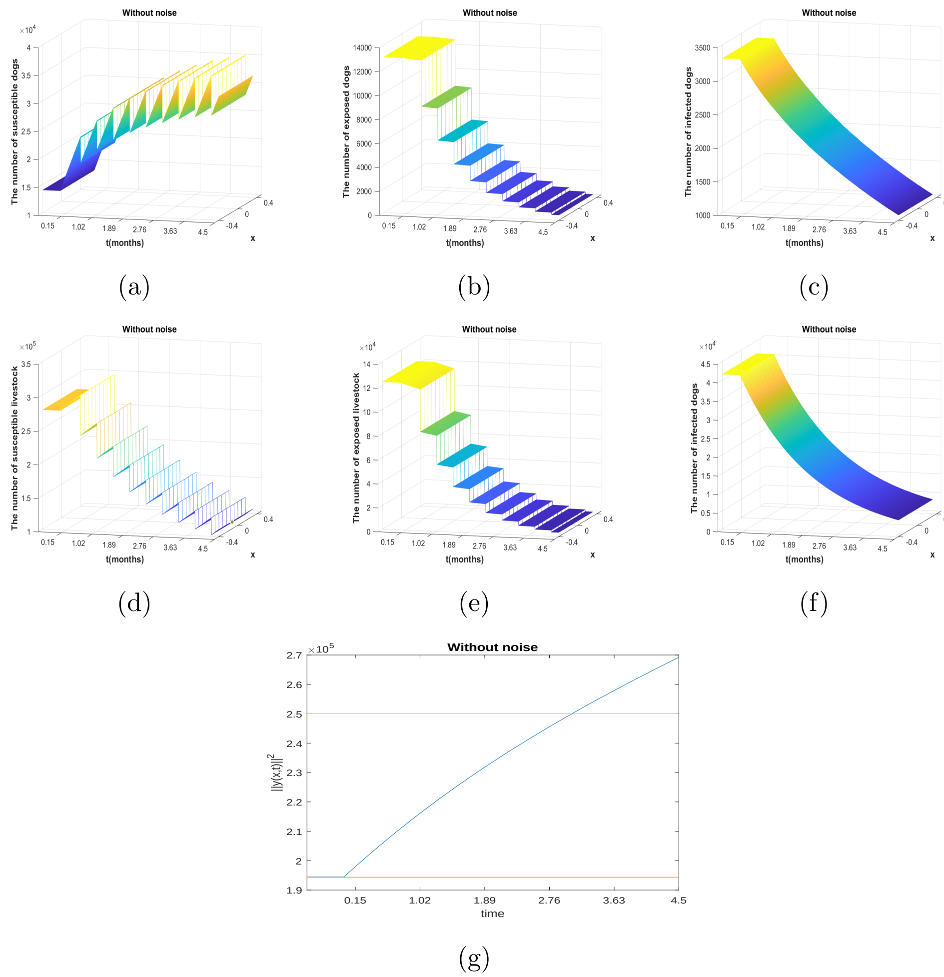


Figure 3: The system (5) displays state trajectories for $y(t,x) = (S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ with an initial value of $y(0,x) = (1.405 \times 10^4, 1.3 \times 10^4, 3.3 \times 10^3, 2.78 \times 10^5, 1.23 \times 10^5, 4.16 \times 10^4)$ without white noise.

tionally, we select $\Delta_t = 0.1$, delay $\tau = 2.5 = \ell\Delta_t$ and $T = 40 = \jmath\Delta_t$, $\Delta_x = 1$ and $X = 10 = \alpha\Delta_x$, where $\ell > 0, \jmath > 0$, and $\alpha > 0$ are integers. The values of $S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x), p_k(t,x), k = 1,2,3,4,5,6,$ at nodal points are provided as $S1_{i,j}, E1_{i,j}, I1_{i,j}, S2_{i,j}, E2_{i,j}, I2_{i,j}, p^k_{i,j}$, where $-\ell \leq j \leq \jmath, 0 \leq i \leq \alpha$. We employ forward and backward difference methods to approximate the state equation and adjoint equation. Therefore, the parameter values outlined in Table 1, $\rho_{1k} = 0.2, \rho_{2k} = 0.3, \rho_{3k} = 0.2, \rho_{4k} = 0.3$. We set $v_{i\,\max} = 1$ and $v_{ik\,\max} = 1$, and the control measures, for $v_i \in [0,1]$, are designed to reduce the number of infected hosts. For $v_{ik} \in (0,1]$, they represent an increase in the frequency of vaccinations and deworming. Furthermore, setting the spatial variable $x = 6$, we can generate the time trajectory plot for the optimal control, as depicted in Fig.5. Addi-

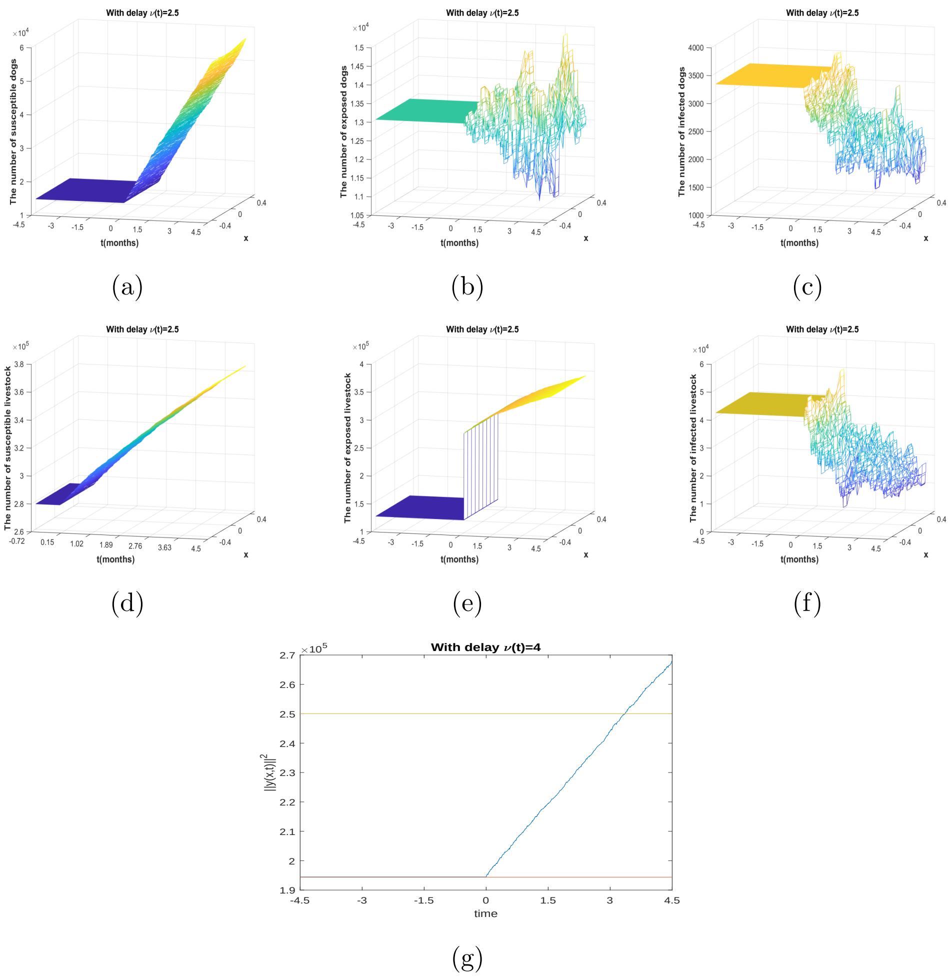
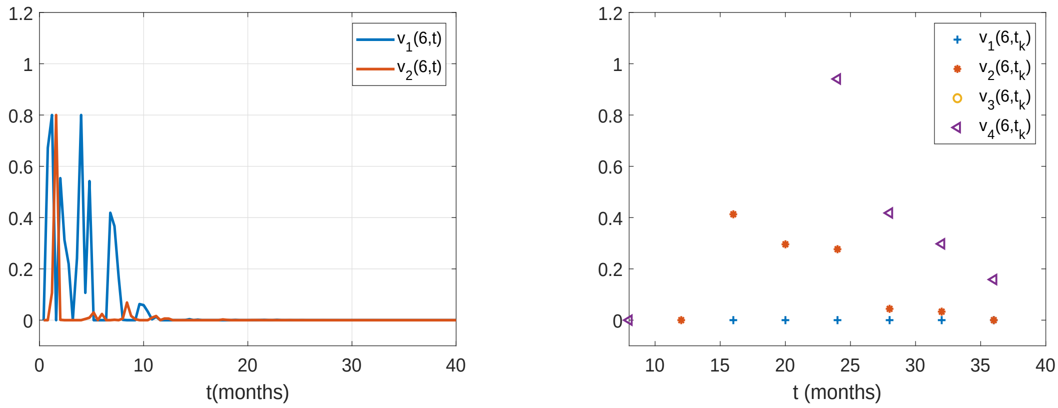


Figure 4: The system (5) displays state trajectories for $y(t,x) = (S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ with an initial value of $y(0,x) = (1.405 \times 10^4, 1.3 \times 10^4, 3.3 \times 10^3, 2.78 \times 10^5, 1.23 \times 10^5, 4.16 \times 10^4)$ with time delay$\tau(t) = 2.5$.

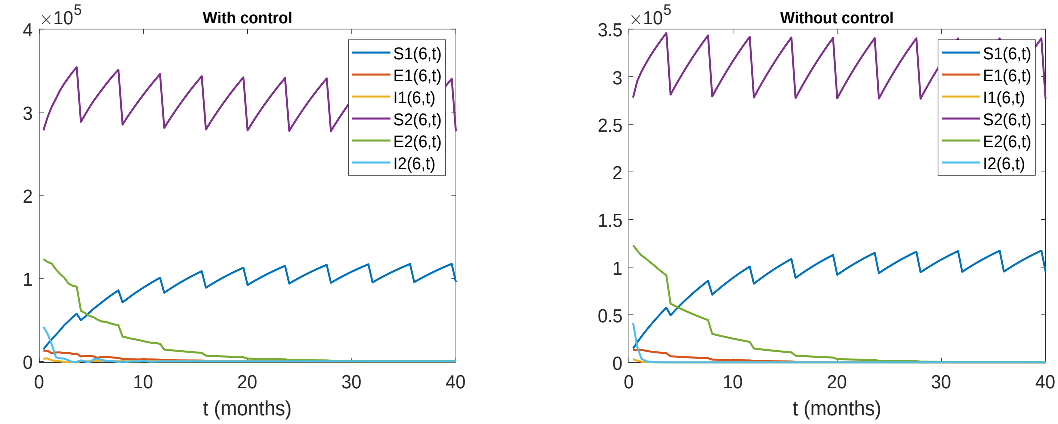
tionally, to demonstrate the role of control variables, we have also plotted the paths $(S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ with and without control in Fig.6.

Figure 6 demonstrates that targeted control measures can significantly decrease the number of infections in both final and intermediate hosts approaching zero (as shown Fig.6(a). However, if only a pulse therapeutic method is adopted, such an effect cannot be achieved (as shown Fig.6(b). Targeted control strategies are crucial for controlling costs and improving treatment outcomes.



(a) (b)

Figure 5: The trajectories of $v_1(t,x), v_2(t,x)$ for system (40), with a constant spatial variable $x = 6$ and initial value $y(0,x) = (1.405 \times 10^4, 1.3 \times 10^4, 3.3 \times 10^3, 2.78 \times 10^5, 1.23 \times 10^5, 4.16 \times 10^4)$.



(a) (b)

Figure 6: The paths of $(S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ for system (40) , with a constant spatial variable $x = 6$ and initial value $y(0,x) = (1.405 \times 10^4, 1.3 \times 10^4, 3.3 \times 10^3, 2.78 \times 10^5, 1.23 \times 10^5, 4.16 \times 10^4)$, with and without control, respectively.

7 Conclusions

In this paper, we expand upon a new ISPDDE model that considers noise, time-varying delay, and impulse interference related to echinococcosis. Sufficient conditions for the FTS of the system by using the Lyapunov function and the bounded impulse interval method. Numerical simulations illustrate the validity of the theoretical results (Fig.1) and explain the effects of impulse (Fig.2), noise (Fig.3), and delay (Fig.4) on the

FTS of the system. In addition, the control variables introduced lead to a reduction in the number of infected hosts in the intermediate and final hosts and minimize the corresponding costs from the point of view of timely control. It is also crucial to consider finite-time control as an important aspect. Although our article only considered FTS, future research will look into finite-time control. Moreover, different from the Lyapunov asymptotical stability, finite-time stability can ensure systems state trajectories converge to the ideal state in a finite time, and the finite time is said to be settling time or convergence time. Estimating the settling time is one of the most important problems of finite-time stability and a point of concern for our further research. Our research mainly focuses on the theoretical impact of various environmental factors (including randomness, impulse, and time delay) on the development of echinococcosis. It can be seen that by combining sheep immunization, dog deworming treatment, and environmental cleaning, it is possible to reduce the level of echinococcus granulosis infection in two hosts in the transmission chain to a very low level. In particular, the environmental disinfection control strategy could be often ignored. Thus, when developing echinococcosis control and prevention, environmental disinfection needs to put special emphasis on planning.

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Conflict of interest

The authors declare there is no conflict of interest.

Appendix A. Proof of Lemma 3.1

Proof. Owing to

$$(S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x)) = (h_1(t)y_1(t,x), h_2(t)y_2(t,x),$$

$$h_3(t)y_3(t,x), h_4(t)y_4(t,x), h_5(t)y_5(t,x), h_6(t)y_6(t,x)).$$

It is clearly that $(S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$ is continuous on

$(t_k, t_{k+1}) \subset [0, +\infty)$. For every $t \neq t_k$,

$$\begin{aligned}
dS_1(t,x) &= h_1'(t)y_1(t,x) + h_1(t)dy_1(t,x) \\
&= h_1(t)\{[d_1\triangle y_1(t,x) + A_1h_1^{-1}(t) - \beta_1 y_1(t,x)h_6(t)y_6(t,x) \\
&\quad -\mu_1 y_1(t,x) + \gamma_1 h_1^{-1}(t)h_3(t)y_3(t,x)]dt \\
&\quad -\sigma_1 y_1(t,x)h_6(t)y_6(t,x)dB_1(t)\} \\
&= [d_1\triangle S_1(t,x) + A_1 - \beta_1 S_1(t,x)I_2(t,x) \\
&\quad -\mu_1 S_1(t,x) + \gamma_1 I_1(t,x)]dt - \sigma_1 S_1(t,x)I_2(t,x)dB_1(t).
\end{aligned}$$

Furthermore, for $t_k \in [0, +\infty)$,

$$S_1(t_k^-, x) = \lim_{t\to t_k^-} h_1(t)y_1(t,x) = (1-\rho_{1k})^{(t_k-1)-t_k}y_1(t_k,x) = (1-\rho_{1k})^{-1}y_1(t_k,x) = S_1(t_k,x),$$

and

$$S_1(t_k^+, x) = \lim_{t\to t_k^+} h_1(t)y_1(t,x) = (1-\rho_{1k})^{t_k-t_k}y_1(t_k,x) = y_1(t_k,x).$$

Therefore, we know that

$$S_1(t_k^+, x) = (1-\rho_{1k})S_1(t_k,x), fort = t_k.$$

With regard to $E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x)$, we can infer the same result using the same method as for $S_1(t,x)$. It implies that $(h_1(t)y_1(t,x), h_2(t)y_2(t,x), h_3(t)y_3(t,x), h_4(t)y_4(t,x), h_5(t)y_5(t,x), h_6(t)y_6(t,x))$ satisfies the equivalent system (5) for almost every $t \in (0,+\infty)\backslash\{t_k\}$. It also satisfies the impulsive conditions at $t = t_k$. Proving that system (5) has a unique global positive solution $(S_1(t,x), E_1(t,x), I_1(t,x), S_2(t,x), E_2(t,x), I_2(t,x))$, requires demonstrating the existence and uniqueness of the global positive solution $(y_1(t,x), y_2(t,x), y_3(t,x), y_4(t,x), y_5(t,x), y_6(t,x))$ for system (6).

Appendix B. Proof of Lemma 3.2

Proof. Let

$$W(t) = \int_\Omega (S_1(t,x) + E_1(t,x) + I_1(t,x) + S_2(t,x) + E_2(t,x) + I_2(t,x))dx,$$

by (5), we have

$$\begin{aligned}
\frac{dW(t)}{dt} &= \int_\Omega (\frac{\partial S_1(t,x)}{\partial t} + \frac{\partial E_1(t,x)}{\partial t} + \frac{\partial I_1(t,x)}{\partial t} + \frac{\partial S_2(t,x)}{\partial t} + \frac{\partial E_2(t,x)}{\partial t} + \frac{\partial I_2(t,x)}{\partial t})dx \\
&= \int_\Omega (d_1 \triangle S_1(t,x) + A_1 - \mu_1 S_1(t,x) + d_2 \triangle E_1(t,x) - \mu_1 E_1(t,x) \\
&\quad + d_3 \triangle I_1(t,x) - \mu_1 I_1(t,x) + d_4 \triangle S_2(t,x) + A_2 - \mu_2 S_2(t,x) \\
&\quad + d_5 \triangle E_2(t,x) - \mu_2 E_2(t,x) + d_6 \triangle I_2(t,x) - \mu_2 I_2(t,x))dt \\
&\leq \int_\Omega (d_1 \triangle S_1(t,x) + d_2 \triangle E_1(t,x) + d_3 \triangle I_1(t,x) + d_4 \triangle S_2(t,x) \\
&\quad + d_5 \triangle E_2(t,x) + d_6 \triangle I_2(t,x) + A_1 + A_2 \\
&\quad - \Lambda((S_1(t,x) + E_1(t,x) + I_1(t,x) + S_2(t,x) + E_2(t,x) + I_2(t,x)))dx \\
&\leq d_1 \int_{\partial\Omega} \frac{\partial S_1(t,x)}{\partial n} dx + d_2 \int_{\partial\Omega} \frac{\partial E_1(t,x)}{\partial n} dx + d_3 \int_{\partial\Omega} \frac{\partial I_1(t,x)}{\partial n} dx + d_4 \int_{\partial\Omega} \frac{\partial S_2(t,x)}{\partial n} dx \\
&\quad + d_5 \int_{\partial\Omega} \frac{\partial E_2(t,x)}{\partial n} dx + d_6 \int_{\partial\Omega} \frac{\partial I_2(t,x)}{\partial n} dx + \int_\Omega (A_1 + A_2) dx - \Lambda W(t) \\
&= (A_1 + A_2)|\Omega| - \Lambda W(t),
\end{aligned}$$

where $\Lambda = min(\mu_1, \mu_2)$.

$$\lim_{t\to\infty} W(t) \leq \frac{(A_1 + A_2)|\Omega|}{\Lambda} = B.$$

Appendix C. Proof of Theorem 3.1

Proof. Since the coefficients of system (6) satisfy the local Lipschitz condition, for any given initial value $(y_1(s,x), y_2(s,x), y_3(s,x), y_4(s,x), y_5(s,x), y_6(s,x)) \in L^2([-\hat{\tau}, 0] \times \Omega; R_+^6)$, the system (6) has a unique solution $y(t,x)$ on $[-\hat{\tau}, \tau_e)$, where τ_e is the explosion time.

Let $q_0 > 0$ be sufficiently large number. For each $q \geq q_0$, define a stopping time

$$\tau_q = \inf \left\{ t \in [0, \tau_e) : min\{y_i(t,x)\} \leq \frac{1}{q} or \max\{y_i(t,x)\} \geq q \right\}, i = 1,2,3,4,5,6$$

set $\inf \emptyset = \infty$. It can be seen that τ_q is increasing as $q \to +\infty$. Let $\tau_\infty = \lim_{t\to+\infty}$, then $\tau_\infty \leq \tau_e a.s$ and $y(t,x) > 0$. To verify that $\tau_e = \infty a.s.$ We only need to demonstrate that $\tau_\infty = +\infty a.s.$ Before confirming its validity, we need to determine the boundedness of the solution. For simplicity, we denote $y_i = y_i(t,x), i = 1,2,3,4,5,6$. For arbitrary $T > 0, t \in [0, t_q \wedge T)$, let

$$V_1(t) = \int_\Omega y_1^2 dx, V_2(t) = \int_\Omega y_2^2 dx, V_3(t) = \int_\Omega y_3^2 dx,$$

$$V_4(t) = \int_\Omega y_4^2 dx, V_5(t) = \int_\Omega y_5^2 dx, V_6(t) = \int_\Omega y_6^2 dx.$$

Applying the Itô formula and basic inequality we deduce that

$$\begin{aligned} dV_1(t) &\leq 2\int_\Omega y_1[d_1\triangle y_1 + A_1h_1^{-1}(t) - \beta_1 y_1 h_6(t)y_6 - (\mu_1 - ln(1-\rho_{1k}))y_1(t,x) \\ &+\gamma_1 h_1^{-1}(t)h_3(t)y_3]dxdt + \int_\Omega \sigma_1^2 y_1^2 h_6^2(t)y_6^2 dxdt \\ &-2\int_\Omega \sigma_1 y_1^2 h_6(t)y_6 dxdB_1(t) \\ &\leq 2\int_\Omega[-d_1(\nabla y_1)^2 + A_1h_1^{-1}(t)y_1 - \beta_1 y_1^2 h_6(t)y_6 - \mu_1 y_1^2 + |ln(1-\rho_{1k})|y_1^2 \\ &+\gamma_1 h_1^{-1}(t)h_3(t)y_1y_3 + \tfrac{1}{2}\sigma_1^2 y_1^2 h_6^2(t)y_6^2]dxdt \\ &-2\int_\Omega \sigma_1 y_1^2 h_6(t)y_6 dxdB_1(t), \end{aligned} \tag{41}$$

and

$$\begin{aligned} dV_2(t) &\leq 2\int_\Omega y_2[d_2\triangle y_2 + \beta_1 h_2^{-1}(t)h_1(t)h_6(t)y_1y_6 - (\mu_1 - ln(1-\rho_{2k}))y_2 \\ &-\beta_1 h_2^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t))h_6(t-\tau(t))y_6(t-\tau(t))]dxdt \\ &+\int_\Omega[\sigma_1^2 h_2^{-2}(t)h_1^2(t)y_1^2h_6^2(t)y_6^2 \\ &+\sigma_1^2 h_2^{-2}(t)h_1^2(t-\tau(t))y_1^2(t-\tau(t))h_6^2(t-\tau(t))y_6^2(t-\tau(t))]dxdt \\ &+2[\int_\Omega \sigma_1 h_2^{-1}(t)h_1(t)y_1y_2h_6(t)y_6 \\ &-\sigma_1 h_2^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t))y_2h_6(t-\tau(t))y_6(t-\tau(t))]dxdB_1(t) \\ &\leq 2\int_\Omega[-d_2(\nabla y_2)^2 + \beta_1 h_2^{-1}(t)h_1(t)h_6(t)y_1y_2y_6 - \mu_1 y_2^2 + |ln(1-\rho_{2k})|y_2^2 \\ &+\tfrac{\beta_1 h_2^{-1}(t)\hat{h_1}\hat{h_6}y_2}{2}(y_1^2(t-\tau(t)) + y_6^2(t-\tau(t)) + \tfrac{1}{2}\sigma_1^2h_2^{-2}h_1^2y_1^2h_6^2y_6^2 \\ &+\tfrac{1}{2}\sigma_1^2h_2^{-2}\hat{h}_1^2y_1^2(t-\tau(t))\hat{h}_6^2y_6^2(t-\tau(t))]dtdx \\ &+2\int_\Omega \sigma_1 h_2^{-1}h_1h_6y_1y_2y_6 - \sigma_1 h_2^{-1}\hat{h_1}\hat{h_6}y_1(t-\tau(t))y_2y_6(t-\tau(t))]dxdB_1(t), \end{aligned} \tag{42}$$

moreover,

$$\begin{aligned}
dV_3(t) &\leq 2\int_\Omega y_3[d_3\triangle y_3+\beta_1 h_3^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t))h_6(t-\tau(t))y_6(t-\tau(t))\\
&-(\mu_1+\gamma_1)y_3]dxdt+\int_\Omega \sigma_1^2 h_3^{-2}(t)\hat{h_1^2}y_1^2(t-\tau(t))\hat{h_6}^2y_6^2(t-\tau(t))dxdt\\
&+2\int_\Omega \sigma_1 h_3^{-1}(t)h_1(t-\tau(t))y_1(t-\tau(t))y_3h_6(t-\tau(t))y_6(t-\tau(t))dxdB_1(t)\\
&\leq 2\int_\Omega[-d_3(\nabla y_3)^2+\frac{\beta_1 h_3^{-1}\hat{h_1}\hat{h_6}y_3}{2}(y_1^2(t-\tau(t))+y_6^2(t-\tau(t)))\\
&-(\mu_1+\gamma_1)y_3^2+\frac{1}{2}\sigma_1^2h_3^{-2}\hat{h_1^2}y_1^2(t-\tau(t))\hat{h_6^2}y_6^2(t-\tau(t))]dtdx\\
&+2\int_\Omega \sigma_1 h_3^{-1}(t)\hat{h_1}y_1(t-\tau(t))y_3\hat{h_6}y_6(t-\tau(t))dxdB_1(t),
\end{aligned} \tag{43}$$

$$\begin{aligned}
dV_4(t) &\leq 2\int_\Omega y_4[d_4\triangle y_4+A_2h_4^{-1}(t)-\beta_2y_4h_3(t)y_3-(\mu_2-ln(1-\rho_{3k}))y_4(t,x)\\
&+\gamma_2h_4^{-1}(t)h_6(t)y_6]dxdt+\int_\Omega\sigma_2^2y_4^2h_3^2(t)y_3^2dxdt\\
&-2\int_\Omega\sigma_2y_4^2h_3(t)y_3dxdB_2(t)\\
&\leq 2\int_\Omega[-d_4(\nabla y_4)^2+A_2h_4^{-1}(t)y_4-\beta_2y_4^2h_3(t)y_3-\mu_2y_4^2+|ln(1-\rho_{3k})|y_4^2\\
&+\gamma_2h_4^{-1}(t)h_6(t)y_4y_6+\frac{1}{2}\sigma_2^2y_4^2h_3^2(t)y_3^2]dxdt\\
&-2\int_\Omega\sigma_2y_4^2h_3(t)y_3dxdB_2(t),
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
dV_5(t) &\leq 2\int_\Omega y_5[d_5\triangle y_5+\beta_2h_5^{-1}(t)h_4(t)y_4h_3(t)y_3-(\mu_2-ln(1-\rho_{4k}))y_5\\
&-\beta_2h_5^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t))h_3(t-\tau(t))y_3(t-\tau(t))]dxdt\\
&+\int_\Omega[\sigma_2^2h_5^{-2}(t)h_4^2(t)y_4^2h_3^2(t)y_3^2\\
&+\sigma_2^2h_5^{-2}(t)h_4^2(t-\tau(t))y_4^2(t-\tau(t))h_3^2(t-\tau(t))y_3^2(t-\tau(t))]dxdt\\
&+2[\int_\Omega\sigma_2h_5^{-1}(t)h_4(t)y_4y_5h_3(t)y_3\\
&-\sigma_2h_5^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t))y_5h_3(t-\tau(t))y_3(t-\tau(t))]dxdB_2(t)\\
&\leq 2\int_\Omega[-d_5(\nabla y_5)^2+\beta_2h_5^{-1}h_3(t)h_4(t)y_3y_4y_5-\mu_2y_5^2+|ln(1-\rho_{4k})|y_5^2\\
&+\frac{\beta_2h_5^{-1}(t)\hat{h_4}\hat{h_3}y_5}{2}(y_3^2(t-\tau(t))+y_4^2(t-\tau(t)))+\frac{1}{2}\sigma_2^2h_5^{-2}h_4^2y_4^2h_3^2y_3^2\\
&+\frac{1}{2}\sigma_2^2h_5^{-2}\hat{h_4^2}y_4^2(t-\tau(t))\hat{h_3^2}y_3^2(t-\tau(t))]dxdt\\
&+2\int_\Omega\sigma_2h_5^{-1}h_3h_4y_3y_4y_5-\sigma_2h_5^{-1}\hat{h_4}\hat{h_3}y_4(t-\tau(t))y_5y_3(t-\tau(t))]dxdB_2(t),
\end{aligned} \tag{45}$$

and

$$\begin{aligned} dV_6(t) &\leq 2\int_\Omega y_6[d_6\triangle y_6+\beta_2 h_6^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t))h_3(t-\tau(t))y_3(t-\tau(t)) \\ &\quad -(\mu_2+\gamma_2)y_6]dxdt+\int_\Omega \sigma_2^2 h_6^{-2}(t)\hat{h_4^2}y_4^2(t-\tau(t))\hat{h_3}^2 y_3^2(t-\tau(t))dxdt \\ &\quad +2\int_\Omega \sigma_2 h_6^{-1}(t)h_4(t-\tau(t))y_4(t-\tau(t))y_6 h_3(t-\tau(t))y_3(t-\tau(t))dxdB_2(t) \\ &\leq 2\int_\Omega[-d_6(\nabla y_6)^2+\tfrac{\beta_2 h_6^{-1}\hat{h_3}\hat{h_4}y_6}{2}(y_3^2(t-\tau(t))+y_4^2(t-\tau(t))) \\ &\quad -(\mu_2+\gamma_2)y_6^2+\tfrac{1}{2}\sigma_2^2 h_6^{-2}\hat{h_3^2}y_3^2(t-\tau(t))\hat{h_4^2}y_4^2(t-\tau(t))]dtdx \\ &\quad +2\int_\Omega \sigma_2 h_6^{-1}\hat{h_3}\hat{h_4}y_3(t-\tau(t))y_6 y_4(t-\tau(t))dxdB_2(t), \end{aligned} \tag{46}$$

then, for $V(t)=\sum_{i=1}^{6}V_i(t)$, we have

$$\begin{aligned} dV(t) &\leq 2\int_\Omega[A_1 h_1^{-1}(t)y_1+|ln(1-\rho_{1k})|y_1^2+\gamma_1 h_1^{-1}h_3 y_1 y_3+\frac{1}{2}\sigma_1^2 y_1^2 h_6^2 y_6^2 \\ &\quad +\beta_1 h_1 h_2^{-1}h_6 y_1 y_2 y_6+|ln(1-\rho_{2k})|y_2^2+\tfrac{\beta_1 h_2^{-1}\hat{h_1}\hat{h_6}y_2}{2}(y_1^2(t-\tau(t))+y_6^2(t-\tau(t))) \\ &\quad +\tfrac{1}{2}\sigma_1^2 h_2^{-2}h_1^2 h_6^2 y_1^2 y_6^2++\tfrac{1}{2}\sigma_1^2 h_2^{-2}\hat{h_1^2}\hat{h_6^2}y_1^2(t-\tau(t))y_6^2(t-\tau(t)) \\ &\quad +\tfrac{\beta_1 h_3^{-1}\hat{h_1}\hat{h_6}y_3}{2}(y_1^2(t-\tau(t))+y_6^2(t-\tau(t)))+\tfrac{1}{2}\sigma_1^2 h_3^{-2}\hat{h_1^2}y_1^2(t-\tau(t))\hat{h_6^2}y_6^2(t-\tau(t)) \\ &\quad +A_2 h_4^{-1}y_4+|ln(1-\rho_{3k})|y_4^2+\gamma_2 h_4^{-1}h_6 y_4 y_6+\tfrac{1}{2}\sigma_2^2 h_3^2 y_3^2 y_4^2 \\ &\quad +\beta_2 h_5^{-1}h_3 h_4 y_3 y_4 y_5+|ln(1-\rho_{4k})|y_5^2+\tfrac{\beta_2 h_5^{-1}\hat{h_4}\hat{h_3}y_5}{2}(y_3^2(t-\tau(t))+y_4^2(t-\tau(t))) \\ &\quad +\tfrac{1}{2}\sigma_2^2 h_5^{-2}h_3^2 h_4^2 y_3^2 y_4^2+\tfrac{1}{2}\sigma_2^2 h_5^{-2}\hat{h_4^2}y_4^2(t-\tau(t))\hat{h_3^2}y_3^2(t-\tau(t)) \\ &\quad +\tfrac{\beta_2 h_6^{-1}\hat{h_3}\hat{h_4}y_6}{2}(y_3^2(t-\tau(t))+y_4^2(t-\tau(t)))+\tfrac{1}{2}\sigma_2^2 h_6^{-2}\hat{h_3^2}y_3^2(t-\tau(t))\hat{h_4^2}y_4^2(t-\tau(t))]dxdt \\ &\quad -2\int_\Omega \sigma_1 y_1^2 h_6 y_6 dxdB_1(t)+2\int_\Omega[\sigma_1 h_2^{-1}h_1 h_6 y_1 y_2 y_6 \\ &\quad -\sigma_1 h_2^{-1}\hat{h_1}\hat{h_6}y_1(t-\tau(t))y_2 y_6(t-\tau(t))]dxdB_1(t) \\ &\quad +2\int_\Omega \sigma_1 h_3^{-1}\hat{h_1}\hat{h_6}y_1(t-\tau(t))y_3 y_6(t-\tau(t))dxdB_1(t) \\ &\quad -2\int_\Omega \sigma_2 y_4^2 h_3 y_3 dxdB_2(t)+2[\int_\Omega \sigma_2 h_5^{-1}h_3 h_4 y_3 y_4 y_5 \\ &\quad -\sigma_2 h_5^{-1}\hat{h_3}\hat{h_4}y_3(t-\tau(t))y_5 y_4(t-\tau(t))]dxdB_2(t) \\ &\quad +2\int_\Omega \sigma_2 h_6^{-1}\hat{h_3}\hat{h_4}y_3(t-\tau(t))y_6 y_4(t-\tau(t))dxdB_2(t). \end{aligned} \tag{47}$$

Taking the integral of both sides of equation (47) from 0 to $\tau_q \wedge T$, and then finding the expected value, yields the following. Integrating both sides of (47) from and taking expectations gives that

$$
\begin{aligned}
&E[V(\tau_q\Lambda T)]-V(0)\\
\leq\ &E\int_0^{\tau_q\Lambda T}\int_\Omega 2[A_1h_1^{-1}(t)y_1+|ln(1-\rho_{1k})|y_1^2+\gamma_1h_1^{-1}h_3y_1y_3+\tfrac{1}{2}\sigma_1^2y_1^2h_6^2y_6^2\\
&+\beta_1h_1h_2^{-1}h_6y_1y_2y_6+|ln(1-\rho_{2k})|y_2^2+\tfrac{\beta_1h_2^{-1}\hat{h_1}\hat{h_6}y_2}{2}(y_1^2(t-\tau(t))+y_6^2(t-\tau(t)))\\
&+\tfrac{1}{2}\sigma_1^2h_2^{-2}h_1^2h_6^2y_1^2y_6^2+\tfrac{1}{2}\sigma_1^2h_2^{-2}\hat{h_1^2}\hat{h_6^2}y_1^2(t-\tau(t))y_6^2(t-\tau(t))\\
&+\tfrac{\beta_1h_3^{-1}\hat{h_1}\hat{h_6}y_3}{2}(y_1^2(t-\tau(t))+y_6^2(t-\tau(t)))+\tfrac{1}{2}\sigma_1^2h_3^{-2}\hat{h_1^2}y_1^2(t-\tau(t))\hat{h_6^2}y_6^2(t-\tau(t))\\
&+A_2h_4^{-1}y_4+|ln(1-\rho_{3k})|y_4^2+\gamma_2h_4^{-1}h_6y_4y_6+\tfrac{1}{2}\sigma_2^2h_3^2y_3^2y_4^2\\
&+\beta_2h_5^{-1}h_3h_4y_3y_4y_5+|ln(1-\rho_{4k})|y_5^2+\tfrac{\beta_2h_5^{-1}\hat{h_4}\hat{h_3}y_5}{2}(y_3^2(t-\tau(t))+y_4^2(t-\tau(t)))\\
&+\tfrac{1}{2}\sigma_2^2h_5^{-2}h_3^2h_4^2y_3^2y_4^2+\tfrac{1}{2}\sigma_2^2h_5^{-2}\hat{h_4^2}y_4^2(t-\tau(t))\hat{h_3^2}y_3^2(t-\tau(t))\\
&+\tfrac{\beta_2h_6^{-1}\hat{h_3}\hat{h_4}y_6}{2}(y_3^2(t-\tau(t))+y_4^2(t-\tau(t)))+\tfrac{1}{2}\sigma_2^2h_6^{-2}\hat{h_3^2}y_3^2(t-\tau(t))\hat{h_4^2}y_4^2(t-\tau(t))]dxdt,
\end{aligned}
$$

there exists positive constants L_i such that $L_i=sup\{2|ln(1-\rho_{ik})|\}, t>0, i=1,2,3,4.$ Then, we can calculate

$$
\begin{aligned}
&E[V(\tau_q\Lambda T)]\\
\leq\ &V(0)+E\int_0^{\tau_q\Lambda T}\int_\Omega\{[A_1^2h_1^{-2}(t)+A_2^2h_4^{-2}(t)+[1+2|ln(1-\rho_{1k})|+\gamma_1h_1^{-1}h_3\\
&+\sigma_1^2h_6^2B^2+\sigma_1^2h_1^2h_2^{-2}h_6^2B^2+\beta_1^2h_1^2h_2^{-2}h_6^2B^2]y_1^2+2|ln(1-\rho_{2k})|y_2^2\\
&+\gamma_1h_1^{-1}h_3y_3^2+[1+2|ln(1-\rho_{3k})|+\gamma_2h_4^{-1}h_6+\sigma_2^2h_3^2B^2\\
&+\sigma_2^2h_5^{-2}h_3^2h_4^2B^2+\beta_2^2h_3^2h_4^2h_5^{-2}B^2]y_4^2+[1+2|ln(1-\rho_{4k})|]y_5^2+[1+\gamma_2h_4^{-1}h_6]y_6^2\\
&+\beta_1\hat{h_1}\hat{h_6}(h_2^{-1}y_2+h_3^{-1}y_3)[y_1^2(t-\tau(t))+y_6^2(t-\tau(t))]\\
&+\sigma_1^2\hat{h_1^2}\hat{h_6^2}(h_2^{-2}+h_3^{-2})y_1^2(t-\tau(t))y_6^2(t-\tau(t))\\
&+\beta_2\hat{h_3}\hat{h_4}(h_5^{-1}y_5+h_6^{-1}y_6)[y_3^2(t-\tau(t))+y_4^2(t-\tau(t))]\\
&+\sigma_2^2\hat{h_3^2}\hat{h_4^2}(h_5^{-2}+h_6^{-2})y_3^2(t-\tau(t))y_4^2(t-\tau(t))\}dxdt\\
\leq\ &V(0)+E\int_0^{\tau_q\Lambda T}\int_\Omega\ C_1[1+y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2+y_1^2(t-\tau(t))+y_6^2(t-\tau(t))\\
&+y_3^2(t-\tau(t))+y_4^2(t-\tau(t))+y_1^2(t-\tau(t))y_6^2(t-\tau(t))+y_3^2(t-\tau(t))y_4^2(t-\tau(t))]dxdt\\
\leq\ &V(0)+E\int_0^{\tau_q\Lambda T}[C_1|\Omega|+\int_\Omega\ C_1[y_1^2(t-\tau(t))+y_6^2(t-\tau(t))+y_3^2(t-\tau(t))+y_4^2(t-\tau(t))\\
&+y_1^2(t-\tau(t))y_6^2(t-\tau(t))+y_3^2(t-\tau(t))y_4^2(t-\tau(t))dx]dt\\
&+E\int_0^{\tau_q\Lambda T}\int_\Omega\ [C_1(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2)dx]dt,
\end{aligned}
\tag{48}
$$

where

$C_1=\max\{A_1^2h_1^{-2}+A_2^2h_4^{-2},1+|L_i|+\gamma_1h_1^{-1}h_3+\sigma_1^2h_6^2B^2+\sigma_1^2h_1^2h_2^{-2}h_6^2B^2+\beta_1^2h_1^2h_2^{-2}h_6^2B^2,1+2|ln(1+\rho_{3k})|+\gamma_2h_4^{-1}h_6+\sigma_2^2h_3^2B^2+\sigma_2^2h_5^{-2}h_3^2h_4^2B^2+\beta_2^2h_3^2h_4^2h_5^{-2}B^2,$
$\beta_1\hat{h_1}\hat{h_6}(h_2^{-1}y_2+h_3^{-1}y_3),\sigma_1^2\hat{h_1^2}\hat{h_6^2}(h_2^{-2}+h_3^{-2}),\beta_2\hat{h_3}\hat{h_4}(h_5^{-1}y_5+h_6^{-1}y_6),\sigma_2^2\hat{h_3^2}\hat{h_4^2}(h_5^{-2}+h_6^{-2})\},i=1,2,3,4.$
Since

$$\begin{aligned}
&E\int_0^{\tau_q\Lambda T}\int_\Omega\ C_1y_1^2(t-\tau(t))dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\ C_1y_3^2(t-\tau(t))dxdt\\
&\quad+E\int_0^{\tau_q\Lambda T}\int_\Omega\ C_1y_4^2(t-\tau(t))+E\int_0^{\tau_q\Lambda T}\int_\Omega\ C_1y_6^2(t-\tau(t))dxdt\\
\le\ & E\int_{-\hat{\tau_1}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_1^2(t)dxdt+E\int_{-\hat{\tau_2}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_3^2(t)dxdt\\
&+E\int_{-\hat{\tau_3}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_4^2(t)dxdt+E\int_{-\hat{\tau_4}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_6^2(t)dxdt\\
&+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_1^2(t)dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_3^2(t)dxdt\\
&+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_4^2(t)dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_6^2(t)dxdt\\
\le\ & C_2+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_1^2(t)dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_3^2(t)dxdt\\
&+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_4^2(t)dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_6^2(t)dxdt,
\end{aligned}$$

where

$$\begin{aligned}
C_2=&\ E\int_{-\hat{\tau_1}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_1^2(t)dxdt+E\int_{-\hat{\tau_2}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_3^2(t)dxdt\\
&+E\int_{-\hat{\tau_3}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_4^2(t)dxdt+E\int_{-\hat{\tau_4}}^0\int_\Omega\frac{C_1}{1-\tilde{\tau}}y_6^2(t)dxdt,
\end{aligned}$$

since

$$\begin{aligned}
&E\int_0^{\tau_q\Lambda T}\int_\Omega\ C_1y_1^2(t-\tau(t))y_6^2(t-\tau(t))dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\ C_1y_3^2(t-\tau(t))y_4^2(t-\tau(t))dxdt\\
\le\ & E\int_{-\hat{\tau_5}}^0\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}y_1^2(t)y_6^2(t)dxdt+E\int_{-\hat{\tau_6}}^0\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}y_3^2(t)y_4^2(t)dxdt\\
&+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}y_1^2(t)y_6^2(t)dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}y_3^2(t)y_4^2(t)dxdt\\
\le\ & C_3+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}B^2y_1^2(t)dxdt+E\int_0^{\tau_q\Lambda T}\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}B^2y_3^2(t)dxdt,
\end{aligned}$$

where
$C_3=E\int_{-\hat{\tau_5}}^0\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}B^2y_1^2(t)dxdt+E\int_{-\hat{\tau_6}}^0\int_\Omega\frac{C_1}{(1-\tilde{\tau})^2}B^2y_3^2(t)dxdt.$
Thus, we have

$$\begin{aligned}
&E[V(x,\tau_q\Lambda T)]\\
\le\ & C_4+C_5E\int_0^{\tau_q\Lambda T}\int_\Omega(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2)dxdt\\
=\ & C_4+C_5\int_0^T E[V(x,\tau_q\Lambda T)]dt
\end{aligned}\tag{49}$$

where $C_4=V(x,0)+(C_1|\Omega|+C_2+C_3)\tau_q\Lambda T, C_5=C_1+\max\{\frac{C_1}{1-\tilde{\tau_i}},\frac{C_1B^2}{(1-\tilde{\tau_i})^2})\},i=1,2,3,4,5,6.$ It follows from Gronwall inequalities that

$E[\int_\Omega(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2)]dx \le C_4e^{C_5T}$.

Define

$\mu_q = inf\{V(t,x), \|y(t,x)\| \ge q\}, forq \ge q_0$.

Obviously,

$$\lim_{q\to+\infty}\mu_q = +\infty.$$

$\mu_q P(\tau_q \le T) \le C_4e^{C_5T}$. Therefore, choosing $q \to +\infty$ yields that $P(\tau_q \le T) = 0$, that is $P(\tau_\infty > T) = 1$.

References

[1] Echinococcosis Fact Sheet No. 377, World Health Organization, March 2014 (retrieved 19 March 2014).

[2] W. Zhu, Progress and present status of hydatid disease control in China in the recent eight decades. In:Hami Prefectural Institute of Scientific and Technological Information, Hami, China, p.5, 1985.

[3] K. Chu, Multiple pulmonary hydatid disease in left lung:report of a case. In:A Collection of Abstracts of Papers Presented at the Second National Symposium on Hydatidosis, Urumqi, China, p.73, 1992.

[4] J. Chai, Epidemiological studies on cystic echinococcosis in China-a review. Biomed.Environ.Sci.8, 122-136, 1995.

[5] Z. Wang, X. Wang, X. Liu, echinococcos is in China, a review of the epidemiology of echinococcus spp.EcoHealth 5, 115-126, 2008.

[6] M.A. Gemmell, J.R. Lawson, M.G. Roberts. Population dynamics in echinococcosis and cysticercosis:evaluation of the biological parameters of taenia hydatigena and T.ovis and comparison with those of echinococcus grataulosus. Parasitology 94, 161-180, 1987.

[7] M.G. Roberts, J.R. Lawson, M.A. Gemmell. Population dynamics in echinococcosis and cysticercosis:mathematical model of the life-cycles of taenia hydatigena and T.ovis.Parasitology 94, 181-197, 1987.

[8] P.A. Cabrera, G. Haran, U. Benavidez, S. Valledor, G.- Perera, S. Lloyd, M.A. Gemmell, M. Baraibar, A. Morana, J. Maissonave, M. Carballo. Transmission dynamics of echinococcus grataulosus, taenia hydatigena and taenia ovis in sheep in Uruguay.Int.J.Parasitol. 25, 807-813, 1995.

[9] P.R. Torgerson, I. Ziadinov, D. Aknazarov, R. Taurgaziev, P. Deplazes. Modelling the age variation of larval protoscoleces of echinococcus grataulosus in sheep. Int. J. Parasitol. 39, 1031-1035, 2009.

[10] S. Lahmar, H. Debbek, L.H. Zhang, D.P. McMataus, A. Souissi, S. Chelly, P.R. Torgerson. Transmission dynamics of the echinococcus grataulosus sheepCdog strain(G1 genotype) in camels in Tunisia. Vet. Parasitol. 121, 151-156, 2004.

[11] R. Azlaf, A. Dakkak, A. Chentou and M. Berrahmani, Modelling the transmission of echinococcus grataulosus in dogs in the northwest and in the southwest of Morocco, Vet. Parasitol. 145, 297-303, 2007.

[12] L. Huang, Y. Huang, Q. Wang, N. Xiao, D. Yi, W. Yu, D. Qiu:An agent-based model for control strategies of echinococcus grataulosus. Vet. Parasitol. 179(1-3), 84-91.

[13] K. Wang, K. Zhang, Z. Jin et al., Modeling and analysis of the transmission of echinococcosis with application to Xinjiang Uygur autonomous region of China, J. Theor. Biol. 333, 78-90, 2013.

[14] J. Liu, L. Liu, X. Fang and J. Fang, Global dynamics of a time-delayed echinococcosis transmission model, Adv. Differ. Equations(2015)1-16, 2015.

[15] Z. Xu, C. Ai, A spatial echinococcosis transmission model with time delays:Stability and traveling waves, Int. J. Biomath. 10, 1-32, 2017.

[16] Y. Zhang, Y. Xiao, Modeling and analyzing the effects of fixed-time intervention on transmission dynamics of echinococcosis in Qinghai province, Math Meth Appl Sci. (2020)1-21, 2020.

[17] T. Pan, D. Jiang, H.Tasawar, A. Ahmed, Extinction and periodic solutions for an impulsive SIR model with incidence rate stochastically perturbed, Physica A 505, 385-397, 2018.

[18] J. Hu, Q. Zhang, M.B.Anke, M.Ye, Finite-time stability and optimal control of a stochastic reaction-diffusion model for Alzheimers disease with impulse and time-varying delay, Applied Mathematical Modelling 102, 511-539, 2022.

[19] E. Tornatore, S. M. Buccellato, P. Vetro, Stability of a stochastic SIR system, Physica. A. 354, 111-126, 2005.

[20] A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A Stochastic Differential Equation SIS Epidemic Model, Siam. J. Appl. Math. 71(3), 876-902, 2011.

[21] K. Wu, B. Chen, Synchronization of partial differential systems via diffusion coupling, IEEE Trans. Circuits Syst. I Regul. Pap. 59(11), 2655-2668, 2012.

[22] K. Wu, M. Na, L. Wang, X. Ding, B. Wu, Finite-time stability of implusive reaction-diffusion systems with and without time delay, Applied Mathematucs and Computatuin 363, 124591, 2019.

[23] J. Lu, D.W.C. Ho, J. Cao, A unified synchronization criterion for impulsive dynamical networks, Automatica 46(7), 1215-1221, 2010.

[24] I.A. Baba, R.A. Abdulkadir, P. Esmaili, Analysis of tuberculosis model with saturated incidence rate and optimal control, Physica A 540, 123237, 2020.

[25] S. Lenhart, J.T. Workman, Optimal Control Applied to Biological Models, John Chapman and Hall, 2007.

[26] J.F. Banas, A.G. Vacroux, Optimal piecewise constant control of contitauous time systems with time-varying delay, Automatica 6(6), 809-811, 1970.

[27] D.J. Higham, An algorithmic introduction to taumerical simulation of stochastic differential equations, SIAM Rev. 43(3), 525-546, 2001.

[28] M.L. Greer, L. Pujomenjouet, G.F. Webb, A mathematical analysis of the dynamics of prion proliferation, J.Theor.Biol. 242(3), 598-606, 2006.

[29] S. Portet,J. Arino, An in vivo intermediate filament assembly model, Mathematical Biosciences Engineering 6(1), 117-134, 2008.