Finite-time stability of discrete descriptor systems with time-varying delay and nonlinear uncertainties

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Abstract

The finite-time stability for discrete descriptor systems with time-varying delay and nonlinear uncertainties is studied. A new discrete inequality is obtained. On this basis, by combining exponential weighted Lyapunov-like functional (LLF) and convex combination techniques, the sufficient conditions for the system to be finite-time stable are obtained. Finally, we demonstrate the effectiveness of our method through three specific examples.

Keywords: Finite-time stability, descriptor systems, delay, discrete weighted inequality.

1. Introduction

Descriptor systems(also referring to singular systems), have special properties different from normal system, which used in many fields widely. For example, economic system, power system and robot system. In the past decades, correlated study on descriptor systems has become more and more in-depth and extensive, and many excellent results have emerged [1–12]. As everyone knows, in most actual production systems, delay may cause unstable, oscillatory or other poor system performance. Descriptor systems with time-delay have achieved rich research results with respect to Lyapunov asymptotic stability (LAS), the filtering problems, the controller design and the stability analysis [4–9]. We noticed that most of the relevant results are

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about LAS. However, in some practical systems, such as systems related to chemical fields or missile launching process [13]. In these practical systems, what we need is for the system to remain stable within a certain time interval. This inspires us to study the finite-time stability of the system.

Finite-time stability (FTS) refers to that the state within a certain range for a given initial state within the specified time interval. By definition, FTS is different from LAS fundamentally. Indeed, in some cases, a system may be FTS but not LAS, and vice versa [14]. Recently, there have been rich achievements focusing on FTS [14–19]. Among them, relatively, there are few studies for discrete descriptor systems.

There are two important issues to be considered in the research of FTS for discrete descriptor systems.

Q-1: What kind of Lyapunov functional can effectively reduce conservatism?

Q-2: How to handle finite-sum term to get less conservative results when using Lyapunov functional method?

Regarding the first question, the LLF considering the influence of exponential weighting is established in [19]. On the second, unfortunately, at present, there is no suitable method for the FTS of discrete systems. Considering the specific research objectives of FTS, we believe the improvement is possible. This is also the motivation for us to conduct relevant research.

The contributions of the paper mainly include the following two points: firstly, we obtain a new inequality suitable for handle finite-sum term. Secondly, we introduce a new LLF, which allows us to achieve less conservative results. The new LLF makes the derivation process easier to implement.

Notations: R^n means the n dimensional Euclidean space, $R^{m\times n}$ denotes the set of matrices with $m \times n$ dimensions. $Q > 0$ means Q is positive definite. $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ are the maximum and minimum eigenvalues of Q respectively. N stands for the set of natural numbers. $*$ denotes the symmetric block in symmetric matrix.

2. Problem Formulation

The discrete descriptor systems under consideration is as follows:

$$Ex(k+1) = Ax(k) + A_d x(k-d(k)) + g_1(k, x(k)) + g_2(k, x(k-d(k))) \quad (1)$$

$x(k) = \psi(k), k \in [-d_M, \ -d_M+1, \ \cdots, \ 0],$

where $E \in R^{n\times n}$ is a singular matrix with $rank(E) = r < n$, $A \in R^{n\times n}$ and $A_d \in R^{n\times n}$ are two known constant matrices, $x(k) \in R^n$ is state vector, $g_1(k, x(k))$ and $g_2(k, x(k-d(k)))$ are nonlinear uncertainties that satisfy the following assumptions,

$$\|g_1(k, x(k))\| \le l_1 \|x(k)\|, \tag{2}$$

$$\|g_2(k, x(k-d(k)))\| \le l_2 \|x(k-d(k))\|, \tag{3}$$

where $l_i \, (i = 1, 2)$ are known contants. The initial condition $\psi(k)$ satisfies

$$(\psi(k+1) - \psi(k))^T (\psi(k+1) - \psi(k)) \leqslant c_0, k \in [-d_M, \ -d_M+1, \ \cdots, \ 0],$$

in which c_0 is a known positive integer. The time-varying delay $d(k)$ satisfies $0 < d_m \leqslant d(k) \leqslant d_M$, where d_m and d_M are two known positive integers. For simplicity, we denote $d = d_M - d_m$. Without loss of generality, we consider the case where $E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Based on the previously mentioned questions $Q-1$ and $Q-2$, our aim is to study the FTS of system (1), and we assume that the system (1) is regular and causal.

Firstly, let's introduce the following definition and lemmas.

Definition 1. (FTS) The descriptor system (1) is said to be finite-time stable with respect to (c_1, c_2, N), in which $0 < c_1 < c_2$, $N \in \mathrm{N}$, if it is regular, causal such that:

$$\sup_{k\in[-d_M,-d_M+1,\cdots,0]} \psi^T(k)\psi(k) \leqslant c_1 \Rightarrow x^T(k) E^T E x(k) < c_2, k = 0, 1, \cdots N.$$

Lemma 1. [20] For a given matrix $R > 0$ and integers $h_j \, (j = 1, 2)$ satisfying $0 < h_1 \leqslant h_2 \leqslant i$, we denote $y(s) = x(s+1) - x(s)$,

$$\chi(i, h_1, h_2) = \begin{cases} \dfrac{1}{h_2 - h_1} \left[2 \displaystyle\sum_{s=i-h_2}^{i-h_1-1} x(s) + x(i-h_1) - x(i-h_2) \right], & h_1 < h_2, \\ 2x(t - h_1), & h_1 = h_2, \end{cases}$$

then we have

$$- (h_2 - h_1) \sum_{s=i-h_2}^{i-h_1-1} y^T(s) R y(s) \leqslant -\Omega_0^T R \Omega_0 - 3\Omega_1^T R \Omega_1,$$

where

$$\Omega_0 = x(i-h_1) - x(i-h_2),$$
$$\Omega_1 = x(i-h_1) + x(i-h_2) - \chi(i, h_1, h_2).$$

Lemma 2. [21] For given matrices U and $P > 0$, real scalars ν_i, vectors ζ_i, satisfying $\begin{pmatrix} P & U \\ U^T & P \end{pmatrix} \geqslant 0$, $\sum_{i=1}^{2} \nu_i = 1$, we have

$$-\sum_{i=1}^{2} \frac{1}{\nu_i} \zeta_i^T P \zeta_i \leqslant -\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}^T \begin{pmatrix} P & U \\ U^T & P \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

Lemma 3. (Cauchy matrix inequality) For any vectors x,y and scalar $\rho > 0$, one has
$2x^T X y \leq \rho^{-1} x^T X^T X x + \rho y^T y$.

3. Main results

3.1. Discrete weighted inequality

First, we introduce the relevant knowledge about discrete orthogonal polynomials to be used in the subsequent proof.

For integers $h_1 < h_2$ and vector function $f_1(i)$, $f_2(i) \in \{Z[h_1, h_2-1] \to R^n\}$, we define $(f_1, f_2)_\omega = \sum_{i=h_1}^{h_2-1} \omega(i) f_1(i) f_2(i)$, where $\omega(i)$ is the weighted function. $f_1(i)$ and $f_2(i)$ are orthogonal with respect to $\omega(i)$ if $(f_1, f_2)_\omega = 0$.

Lemma 4. For any matrix $R > 0$ and integers $h_1 < h_2$, vector-valued function $\chi(i) \in \{Z[h_1, h_2-1] \to R^n\}$, one has

$$\sum_{i=h_1}^{h_2-1} \chi^T(i) R \chi(i) w(i) \geqslant \sum_{k=0}^{\infty} \mu_k \Omega_k^T(\chi) R \Omega_k(\chi), \tag{4}$$

in which $\Omega_k(\chi) = \sum\limits_{i=h_1}^{h_2-1} l_k(i)\,\chi(i)$, $\mu_k = \dfrac{1}{\lambda_k}$, where $\lambda_k = (l_k, l_k)_{\omega^{-1}}$, $l_k(i)$ is a discrete orthogonal monic polynomial sequence with respect to $\omega^{-1}(i)$, $k = 0, 1, \cdots$.

Proof. Upon introducing

$$z(i) = \sum_{k=0}^{\infty} \mu_k \Omega_k(\chi)\, S_k(i),$$

where $S_k(i) = \omega^{-\frac{1}{2}}(i)\, l_k(i)$, $k = 0, 1, \cdots$, one has

$$\begin{aligned}
0 &\leqslant \sum_{i=h_1}^{h_2-1} \left(\sqrt{\omega(i)}\chi(i) - z(i)\right)^T R \left(\sqrt{\omega(i)}\chi(i) - z(i)\right) \\
&= \sum_{i=h_1}^{h_2-1} \chi^T(i) R \chi(i)\, \omega(i) - 2 \sum_{i=h_1}^{h_2-1} z(i)^T R \chi(i) \sqrt{\omega(i)} + \sum_{i=h_1}^{h_2-1} z^T(i) R z(i) \\
&= \sum_{i=h_1}^{h_2-1} \chi^T(i) R \chi(i)\, \omega(i) - 2 \sum_{i=h_1}^{h_2-1} \left(\sum_{k=0}^{\infty} \mu_k \Omega_k(\chi)\, S_k(i)\right)^T R \chi(i) \sqrt{\omega(i)} \\
&\quad + \sum_{i=h_1}^{h_2-1} \left(\sum_{k=0}^{\infty} \mu_k \Omega_k(\chi)\, S_k(i)\right)^T R \left(\sum_{k=0}^{\infty} \mu_k \Omega_k(\chi)\, S_k(i)\right) \\
&= \sum_{i=h_1}^{h_2-1} \chi^T(i) R \chi(i)\, \omega(i) - 2 \sum_{k=0}^{\infty} \mu_k {\Omega_k}^T(\chi) R \sum_{i=h_1}^{h_2-1} S_k(i) \chi(i) \sqrt{\omega(i)} \\
&\quad + \sum_{i=h_1}^{h_2-1} \left(\sum_{k=0}^{\infty} \mu_k \Omega_k(\chi)\, S_k(i)\right)^T R \left(\sum_{k=0}^{\infty} \mu_k \Omega_k(\chi)\, S_k(i)\right) \\
&= \sum_{i=h_1}^{h_2-1} \chi^T(i) R \chi(i)\, \omega(i) - 2 \sum_{k=0}^{\infty} \mu_k \Omega_k^T(\chi) R \Omega_k(\chi)
\end{aligned}$$

$$+\sum_{i=h_1}^{h_2-1}\left(\sum_{k=0}^{\infty}\mu_k^2 S_k^2\left(i\right)\Omega_k\left(\chi\right)\right)^T R\Omega_k\left(\chi\right)$$

$$\sum_{i=h_1}^{h_2-1}\chi^T\left(i\right)R\chi\left(i\right)w\left(i\right)-\sum_{k=0}^{\infty}\mu_k\Omega_k^T\left(\chi\right)R\Omega_k\left(\chi\right)$$

which means (4). The proof is completed.

Remark 1. For the weighting function $\omega\left(i\right)$, according to the properties and operating rules of orthogonal polynomials, we obtain the calculation formula of $l_k\left(i\right)$ in Lemma 4.

$$l_k\left(i\right)=i^k+\sum_{j=0}^{k-1}\kappa_{kj}l_j\left(i\right),k=1,2,\cdots,n,\cdots, \tag{5}$$

where $\kappa_{kj}=-\frac{\left(i^k,l_j\right)_{\omega^{-1}}}{\left(l_j,l_j\right)_{\omega^{-1}}}$, $l_0\left(i\right)=1$,

As mentioned in Q-1 above, for introducing an appropriate LLF, the weighting function should be appropriately selected. Then, one has

Corollary 1. For integers $h_1<h_2-1$ and matrix $R>0$, vector function $\chi\left(i\right)\in\left\{Z\left[h_1,h_2-1\right]\rightarrow R^n\right\}$, by choosing the weighted function $\omega\left(i\right)=\sigma^{h_2-i}$ with scalar σ, one has

$$\sum_{i=h_1}^{h_2-1}\chi^T\left(i\right)R\chi\left(i\right)\sigma^{h_2-i}\geqslant\sum_{k=0}^{\infty}\mu_k\Omega_k^T\left(\chi\right)R\Omega_k\left(\chi\right), \tag{6}$$

where μ_k is the same as that defined in Lemma 4.

Remark 2. According to actual needs, we can increase the number of terms in (6). It should be noted that μ_k could be obtained easily according to (5). Here for $k=0,1,2$, the following results can be obtained by careful calculation,

$$\lambda_0=\frac{\sigma^h-1}{\sigma^h\left(\sigma-1\right)},$$

$$\lambda_1 = \frac{\sigma + \sigma^{2h+1} - \sigma^h h^2 - 2\sigma^{h+1} - \sigma^{h+2} h^2 + 2\sigma^{h+1} h^2}{\sigma^h \left(\sigma^h - 1\right) \left(\sigma - 1\right)^3},$$

$$\begin{aligned}\lambda_2 = & \frac{6\sigma^3 h^4 - 4\sigma^4 h^3 + \sigma^5 h^2 - 4\sigma^4 h^4 + 2\sigma^5 h^3}{\left(\sigma - 1\right)^5 \left(\sigma + \sigma^{1+2h} - \sigma^h h^2 - 2\sigma^{1+h} - \sigma^{2+h} h^2 + 2\sigma^{1+h} h^2\right)} \\ & + \frac{18\sigma^{3+h} h^2 - 8\sigma^{4+2h} h^2 - 6\sigma^{3+h} h^4 - 4\sigma^{4+h} h^3}{\left(\sigma - 1\right)^5 \left(\sigma + \sigma^{1+2h} - \sigma^h h^2 - 2\sigma^{1+h} - \sigma^{2+h} h^2 + 2\sigma^{1+h} h^2\right)} \\ & + \frac{4\sigma^{4+h} h^4 + 2\sigma^{5+h} h^3 - \sigma^{5+h} h^4 + 8\sigma^2 h^2 + 4\sigma^2 h^3}{\left(\sigma - 1\right)^5 \left(\sigma + \sigma^{1+2h} - \sigma^h h^2 - 2\sigma^{1+h} - \sigma^{2+h} h^2 + 2\sigma^{1+h} h^2\right)} \\ & - \frac{\left(2\sigma h^3 - \sigma h^2 - 12\sigma^3 - \sigma h^4 + 12\sigma^{h+1} - 4\sigma^{2h+3}\right)}{\left(\sigma - 1\right)^5 \left(\sigma + \sigma^{1+2h} - \sigma^h h^2 - 2\sigma^{1+h} - \sigma^{h+2} h^2 + 2\sigma^{1+h} h^2\right)} \\ & - \frac{\sigma^{1-h} \left(\sigma^{2h} h^4 + 4\sigma^2 - 8\sigma^5 h^2 + 8\sigma^{1+2h} h^2 - 4\sigma^{1+2h} h^3 - 4\sigma^{1+2h} h^4\right)}{\left(\sigma - 1\right)^5 \left(\sigma + \sigma^{1+2h} - \sigma^h h^2 - 2\sigma^{1+h} - \sigma^{h+2} h^2 + 2\sigma^{1+h} h^2\right)} \\ & + \frac{\left(\sigma^5 h^4 - \sigma^{5+h} h^2 - 4\sigma^2 h^4 - \sigma^{1+h} h^2 - 2\sigma^{1+h} h^3 - 18\sigma^3 h^2\right)}{\left(\sigma - 1\right)^5 \left(\sigma + \sigma^{1+2h} - \sigma^h h^2 - 2\sigma^{1+h} - \sigma^{h+2} h^2 + 2\sigma^{1+h} h^2\right)},\end{aligned}$$

where $h = h_2 - h_1$. According to Lemma 4, $\mu_k = \dfrac{1}{\lambda_k}$, naturally, we have

$$\sum_{i=h_1}^{h_2-1} \chi^T(i) R \chi(i) \gamma^{h_2-1} \geqslant \sum_{k=0}^{2} \mu_k \Omega_k^T(\chi) R \Omega_k(\chi) \tag{7}$$

Remark 3. By setting μ_k and Ω_k as defined in Lemma 4, we can get $\lim\limits_{\sigma \to 1} \mu_0 = \dfrac{1}{h}$. Then, for $k = 0$, (7) reduces to the discrete Jensen-type inequality in [22].

Especially for $y(i) = x(i+1) - x(i)$ and $k = 0, 1$, we have
Corollary 2. As μ_k and Ω_k are defined in Lemma 4. Then, the following

inequality holds

$$\sum_{i=h_1}^{h_2-1} \sigma^{h_2-i} y^T(i)\, R y(i) \geqslant \sum_{i=0}^{1} \mu_k \Omega_i^T(\chi)\, R \Omega_i(y),$$

where

$$\Omega_0(y) = x(h_2) - x(h_1),$$
$$\Omega_1(y) = (h_2 + c_{10} - 1)\, x(h_2) - (h_1 + c_{10} - 1)\, x(h_1) - \sum_{i=h_1}^{h_2-1} x(i),$$

in which κ_{10} is defined in Remark 1.

3.2. FTS analysis

By constructing a new LLK, applying convex combination technique and the weighted inequality obtained previously, sufficient conditions guaranteeing FTS of the system are obtained. For the sake of conciseness in the following Theorem 1, there are the following notations.

$$e_i = \left[0_{n\times(i-1)n}\, I_n\, 0_{n\times(7-i)n}\right], i = 1, 2, ..., 7,$$
$$\Psi_1 = (Ae_1 + A_d e_3)^T P_1 (Ae_1 + A_d e_3) + e_1^T \left(-\sigma E^T P_1 E + MSA + A^T S^T M^T\right) e_1$$
$$+ e_3^T \left(MSA + A^T S^T M^T\right) e_1,$$
$$S = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix},$$
$$\Psi_2 = e_1^T (Q_1 + Q_2)\, e_1 - \sigma^{d_m} e_2^T Q_1 e_2 - \sigma^{d_M} e_4^T Q_2 e_4,$$
$$\Psi_3 = (Ae_1 + A_d e_3 - Ee_1)^T (d_m - 1) P_2 (Ae_1 + A_d e_3 - Ee_1)$$
$$-(e_1 - e_2)^T \mu_1 E^T P_2 E (e_1 - e_2)$$
$$-[E * ((\mu + d_m)\, e_1 - \mu e_2) - e_5]^T \mu_2 P_2 \,[E * ((\mu + d_m)\, e_1 - \mu e_2) - e_5],$$

in which

$$\mu = \frac{\sigma^{d_m}(1 + d_m + \sigma d_m) - 1}{(\sigma^{d_m} - 1)(\sigma - 1)},$$
$$\mu_1 = \frac{\sigma^d\,(\sigma - 1)}{\sigma^d - 1},$$

$$\mu_2 = \frac{\sigma^d \left(\sigma^d - 1\right) (\sigma - 1)^3}{\sigma - d^2 \sigma^d - 2\sigma^{d+1} + 2 d^2 \sigma^{d+1} - d^2 \sigma^{d+2} + \sigma^{2d+1}},$$

$$\Psi_4 = (Ae_1 + A_d e_3 - Ee_1)^T (d^2 - 1) P_3 (Ae_1 + A_d e_3 - Ee_1)$$

$$-\sigma^{d_m+1} \Omega^T \begin{bmatrix} \overline{P_3} & U \\ * & \overline{P_3} \end{bmatrix} \Omega,$$

$$\Omega = [E * (e_2 - e_3); E * (e_2 + e_3) - e_6; E * (e_3 - e_4); E * (e_3 + e_4) - e_7)],$$

$$\overline{P}_3 = \begin{pmatrix} P_3 & 0 \\ 0 & 3P_3 \end{pmatrix},$$

$$t_1 = \frac{\sigma^{d_m} - 1}{\sigma - 1},$$

$$t_2 = \frac{\sigma^{d_M} - 1}{\sigma - 1},$$

$$t_3 = \frac{d_m - \sigma + \sigma\,\sigma^{d_m} - d_m\,\sigma}{(\sigma - 1)^2},$$

$$t_4 = \frac{\sigma^{d_M+1}}{(\sigma - 1)^2} - \frac{\sigma^{d_m} \left(d_m - d_M + \sigma + d_M\,\sigma - d_m\,\sigma\right)}{(\sigma - 1)^2},$$

$$\Psi_5 = e_1{}^T \left[l_1^2 (\rho_1 + \rho_3 + \rho_5) I + l_1^2 P_1\right] e_1 + e_3{}^T \left[l_2^2 (\rho_2 + \rho_4) I + P_1\right] e_3. \tag{8}$$

Theorem 1. For given c_1, c_2 and N, the system (1) is finite-time stable, if there exist scalar $\sigma > 1$ and $\upsilon_i > 0$ $(i = 1, 2, ..., 6)$, $\rho_i > 0\,(i = 1, 2, 3, 4, 5)$, matrices $Q_i \in R^{n\times n}(i = 1, 2)$, $U \in R^{2n\times 2n}$, $M = \begin{pmatrix} * & * \\ * & M_4 \end{pmatrix} \left(M_4 \in R^{(n-r)\times(n-r)}\right)$, and $P_i = \begin{pmatrix} P_{i1} & * \\ * & * \end{pmatrix}$ $(P_{i1} \in R_{r\times r}, i = 1, 2, 3)$ satisfying the following inequalities

$$\begin{pmatrix} \overline{P_3} & U \\ * & \overline{P_3} \end{pmatrix} > 0, \tag{9}$$

$$\Psi = \sum_{i=1}^{5} \Psi_i < 0, \tag{10}$$

$$\begin{pmatrix} \Psi & e_1^T PA & e_1^T PA & e_3^T PA_d & e_3^T PA_d & l_2 e_3^T P \\ * & -\rho_1 I & 0 & 0 & 0 & 0 \\ * & * & -\rho_2 I & 0 & 0 & 0 \\ * & * & * & -\rho_3 I & 0 & 0 \\ * & * & * & * & -\rho_4 I & 0 \\ * & * & * & * & * & -\rho_5 I \end{pmatrix} < 0, \tag{11}$$

$$\upsilon_1 I_r < P_{11} < \upsilon_2 I_r, \tag{12}$$
$$0 < Q_1 < \upsilon_3 I_n, \tag{13}$$
$$0 < Q_2 < \upsilon_4 I_n, \tag{14}$$
$$0 < P_{21} < \upsilon_5 I_r, \tag{15}$$
$$0 < P_{31} < \upsilon_6 I_r, \tag{16}$$
$$\sigma^N \left(\upsilon_2 + \upsilon_3 t_1 + \upsilon_4 t_2\right) c_1 + c_0 \left(\upsilon_5 t_3 + d\upsilon_6 t_4\right) < \upsilon_1 c_2, \tag{17}$$

where Φ_i and t_i are defined in (8).

Proof. Consider the following LLK

$$V(k) = \sum_{i=1}^{3} V_i(k), \tag{18}$$

in which

$$V_1(k) = x^T(k) E^T P_1 E x(k),$$

$$V_2(k) = \sum_{i=k-d_m}^{k-1} \sigma^{k-1-i} x^T(i) Q_1 x(i) + \sum_{i=k-d_M}^{k-1} \sigma^{k-1-i} x^T(i) Q_2 x(i),$$

$$V_3(k) = \sum_{i=k-1-d_m}^{k-1} \sigma^{k-1-i}(d_m - k + 1 + i)y^T(i)E^T P_2 E y(i)$$
$$+ d \sum_{i=k-1-d_M}^{k-d_m-1} \sigma^{k-1-i}(d_M - k + 1 + i)y^T(i)\, E^T P_3 E y(i).$$

Taking the difference of (18), one has

$$\begin{aligned} \Delta V_1 &= V_1(k+1) - V_1(k) \\ &= x^T(k+1)E^T P_1 E x(k+1) - x^T(k)E^T P_1 E x(k) \end{aligned} \tag{19}$$

$$\begin{aligned} &= x^T(k)(A^T P_1 A - \sigma E^T P_1 E)x(k) + 2x^T(k)A^T P_1 A_d x(k - d(k)) \\ &+ x^T(k - d(k))A_d^T P_1 A_d x(k - d(k)) + (\sigma - 1)\, V_1(k) \\ &+ 2x^T(k)MSAx(k) + 2x^T(k)MSA_d x(k - d(k)) \\ &+ 2x^T(k)A^T P_1 g_1(k, x(k)) + 2x^T(k)A^T P_1 g_2(k, x(k - d(k))) \\ &+ 2x^T(k - d(k))\, {A_d}^T P_1 g_1(k, x(k)) \\ &+ 2x^T(k - d(k))\, {A_d}^T P_1 g_2(k, x(k - d(k))) \\ &+ {g_1}^T(k, x(k))\, P_1 g_1(k, x(k)) + 2{g_1}^T(k, x(k))\, P_1 g_2(k, x(k - d(k))) \\ &+ {g_2}^T(k, x(k - d(k)))\, P_1 g_2(k, x(k - d(k))) \end{aligned}$$

According to Lemma 3, combined with (2) and (3), there exists $\rho_i\,(i = 1, 2, 3, 4, 5)$, such that

$$\begin{aligned} &2x^T(k)A^T P_1 g_1(k, x(k)) + 2x^T(k)A^T P_1 g_2(k, x(k - d(k))) \\ &+ 2x^T(k - d(k))\, {A_d}^T P_1 g_1(k, x(k)) \\ &+ 2x^T(k - d(k))\, {A_d}^T P_1 g_2(k, x(k - d(k))) \\ &+ {g_1}^T(k, x(k))\, P_1 g_1(k, x(k)) + 2{g_1}^T(k, x(k))\, P_1 g_2(k, x(k - d(k))) \\ &+ {g_2}^T(k, x(k - d(k)))\, P_1 g_2(k, x(k - d(k))) \end{aligned}$$

$$
\begin{aligned}
&\leq l_1^2 x^T(k)\left[(\rho_1+\rho_3+\rho_5)I+P_1\right]x(k)\\
&\quad + l_2^2 x^T(k-d(k))\left[(\rho_2+\rho_4)I+P_1\right]x(k-d(k))\\
&\quad + \left(\rho_1^{-1}+\rho_3^{-1}\right)x^T(k)P_1AA^TP_1^Tx(k)\\
&\quad + \left(\rho_2^{-1}+\rho_4^{-1}\right)x^T(k-d(k))P_1A_dA_d^TP_1^Tx(k-d(k))\\
&\quad + \rho_5^{-1}l_2^2x^T(k-d(k))P_1P_1^Tx(k-d(k))
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\Delta V_2(k) &= V_2(k+1)-V_2(k)\\
&= \sum_{i=k-d_m+1}^{k}\sigma^{k-i}x^T(i)Q_1x^T(i)-\sum_{i=k-d_m}^{k-1}\sigma^{k-i-1}x^T(i)Q_1x(i)\\
&\quad + \sum_{i=k-d_M+1}^{k}\sigma^{k-i}x^T(i)Q_2x(i)-\sum_{i=k-d_M}^{k-1}\sigma^{k-i-1}x^T(i)Q_2x(i)\\
&= x^T(k)Q_1x(k)-\sigma^{d_m}x^T(k-d_m)Q_1x(k-d_m)+x^T(k)Q_2x(k)\\
&\quad -\sigma^{d_M}x^T(k-d_M)Q_2x(k-d_M)+(\sigma-1)V_2(k),
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\Delta V_3(k) = V_3(k+1)-V_3(k)\\
&=(d_m-1)y^T(k)E^TP_2Ey(k)+(d^2-1)y^T(k)E^TP_3Ey(k)\\
&\quad -\sum_{i=k-d_m}^{k-1}\sigma^{k-i}y^T(i)E^TP_2Ey(i)-d\sum_{i=k-d_M}^{k-d_m-1}\sigma^{k-i}y^T(i)E^TP_3Ey(i)\\
&\quad +(\sigma-1)V_3(k)\\
&\leqslant (d_m-1)y^T(k)E^TP_2Ey(k)+(d^2-1)y^T(k)E^TP_3Ey(k)+(\sigma-1)V_3(k)\\
&\quad -\sum_{i=k-d_m}^{k-1}\sigma^{k-i}y^T(i)E^TP_2Ey(k)-d\sigma^{d_m+1}\sum_{i=k-d_M}^{k-d_m-1}y^T(i)E^TP_3Ey(i),
\end{aligned}
\tag{22}
$$

then, by Corollary 2 we obtain:

$$
-\sum_{i=k-d_m}^{k-1}\sigma^{k-i}y^T(i)E^TP_2Ey(i)\leqslant-\sum_{i=1}^{2}\mu_iU_i^T(k)E^TP_2EU_i(k),
\tag{23}
$$

where

$$U_1(k) = x(k) - x(k-d_m),$$
$$U_2(k) = (\mu + d_m)x(k) - \mu x(k-d_m) - G_1(k),$$

$$G_1(k) = \begin{cases} \sum_{i=k-d_m}^{k-1} Ex(i), & d_m > 0, \\ Ex(k), & d_m = 0. \end{cases}$$

Based on Lemma 1, when $d_m < d(k) < d_M$ we have

$$-d\left[\sum_{i=k-d(k)}^{k-d_m-1} y^T(i) E^T P_3 E y(i) + \sum_{i=k-d_m}^{k-d(k)-1} y^T(i) E^T P_3 E y(i)\right]$$

$$\leqslant -\frac{d}{d(k)-d_m}\left(W_1^T(k) E^T P_3 E W_1(k) + 3W_2^T(k) E^T P_3 E W_2(k)\right) \quad (24)$$
$$-\frac{d}{d_M - d(k)}\left(W_3^T(k) E^T P_3 E W_3(k) + 3W_4^T(k) E^T P_3 E W_4(k)\right),$$

where

$$W_1(k) = x(k-d_m) - x(k-d(k)),$$

$$W_2(k) = x(k-d_m) + x(k-d(k)) - G_2(k),$$

$$G_2(k)$$
$$= \begin{cases} \frac{1}{\beta_1}\left(2\sum_{i=k-d(k)}^{k-d_m-1} Ex(i) + Ex(k-d_m) - Ex(k-d(k))\right), & d_m < d(k), \\ 2Ex(d_m), & d_m = d(k), \end{cases}$$

$W_3(k) = x(k-d(k)) - x(k-d_M),$

$W_4(k) = x\left(k-d(k)\right) + x\left(k-d_M\right) - G_3(k),$

$G_3(k)$

$$= \begin{cases} \dfrac{1}{\beta_2}\left(2\displaystyle\sum_{i=k-d_M}^{k-d(k)-1} Ex\left(i\right) + Ex\left(k-d\left(k\right)\right) - Ex(k-d_M)\right), & d\left(k\right) < d_M, \\ 2Ex\left(d_M\right), & d_M = d\left(k\right), \end{cases}$$

where $\beta_1 = d\left(k\right) - d_m$, $\beta_2 = d_M - d\left(k\right)$.

Obviously, $\dfrac{\beta_1}{d} + \dfrac{\beta_2}{d} = 1$, then, by Lemma 2, one has

$$-d\sum_{i=k-d_M}^{k-d_m-1} y^T(i)E^T P_3 Ey\left(i\right) \leqslant W^T\left(k\right)\begin{bmatrix} \overline{P_3} & U \\ * & \overline{P_3} \end{bmatrix} W\left(k\right), \tag{25}$$

where

$W^T\left(k\right) = \begin{bmatrix} W_1^T\left(k\right), & W_2^T\left(k\right), & W_3^T\left(k\right), & W_4^T\left(k\right) \end{bmatrix},$

$\overline{P_3} = \begin{bmatrix} P_3 & 0 \\ 0 & 3P_3 \end{bmatrix}.$

Especially when $d\left(k\right) = d_m$ or $d\left(k\right) = d_M$, one has $W_i\left(k\right) = 0\left(i = 1, 2\right)$ or $W_i\left(k\right) = 0\left(i = 3, 4\right)$ respectively. Therefore, we can obtain that (25) still holds.

Then, from (18) to (25), we can obtain

$$\begin{aligned} &\Delta V(k) - (\sigma - 1)V(k) \leqslant \xi^T\left(k\right)\Psi\xi\left(k\right) \\ &+ l_1^2 x^T\left(k\right)\left[\left(\rho_1 + \rho_3 + \rho_5\right)I + P_1\right]x\left(k\right) \\ &+ l_2^2 x^T\left(k-d\left(k\right)\right)\left[\left(\rho_2 + \rho_4\right)I + P_1\right]x\left(k-d\left(k\right)\right) \\ &+ \left(\rho_1^{-1} + \rho_3^{-1}\right)x^T\left(k\right)P_1 AA^T P_1^T x\left(k\right) \\ &+ \left(\rho_2^{-1} + \rho_4^{-1}\right)x^T\left(k-d\left(k\right)\right)P_1 A_d A_d^T P_1^T x\left(k-d\left(k\right)\right) \\ &+ \rho_5^{-1} l_2^2 x^T\left(k-d\left(k\right)\right)P_1 P_1^T x\left(k-d\left(k\right)\right) \end{aligned} \tag{26}$$

where

$\xi^T(k) =$

$\left[x^T(k), x^T(k-d_m), x^T(k-d(k)), x^T(k-d_M), G_1^T(k), G_2^T(k), G_3^T(k)\right].$

Combining (10) and (26) one has

$$\Delta V(k) - (\sigma - 1)V(k) < 0, \tag{27}$$

then we get

$$V(k) < \sigma V(k-1), \tag{28}$$

further obtaining

$$V(k) < \sigma V(k-1) < \sigma^2 V(k-2) < \cdots < \sigma^k V(0). \tag{29}$$

According to (18), we have

$$\lambda_{\min}(P_{11}) x^T(k) E^T E x(k) \leqslant V(k), \tag{30}$$

and

$$\begin{aligned} V(0) &= x^T(0)E^T P_1 E x(0) + \sum_{i=-d_m}^{-1} \sigma^{-1-i} x^T(i) Q_1 x(i) \\ &+ \sum_{i=-d_m}^{-1} \sigma^{-1-i} x^T(i) Q_2 x(i) + \sum_{i=-d_m-1}^{-1} \sigma^{-1-i} (d_m + i + 1) y_i^T E^T P_2 E y(i) \\ &+ d \sum_{i=-d_M-1}^{-d_m-1} \sigma^{-1-i} (d_M + i + 1) y_i^T(k) E^T P_3 E y(i), \end{aligned} \tag{31}$$

after careful calculation, we have

$$\begin{aligned} V(0) \leqslant\ & \lambda_{\max}(P_{11}) c_1 + \lambda_{\max}(Q_1) c_1 t_1 + \lambda_{\max}(Q_2) c_1 t_2 \\ &+ \delta\left[\lambda_{\max}(P_{21}) t_3 + d\lambda_{\max}(P_{31}) t_3\right]. \end{aligned} \tag{32}$$

From (30) to (32) it can be deduced that

$$x^T(k)E^TEx(k) \leqslant \frac{\sigma^N(\gamma_1+\gamma_2)}{\lambda_{\min}(P_{11})},$$

in which

$$\begin{aligned}\gamma_1 &= [\lambda_{\max}(P_{11}) + \lambda_{\max}(Q_1)t_1 + \lambda_{\max}(Q_2)t_2]c_1,\\ \gamma_2 &= [\lambda_{\max}(P_{21})t_3 + d\lambda_{\max}(P_{31})t_4]c_0 \qquad .\end{aligned}$$

By (11) to (17), it can be obtained that

$$x^T(k)E^TEx(k) < c_2,$$

then, the system (1) is finite-time stable from Definition 1, which completes the proof.

Remark 4. In Theorem 1, different from [18], a new LLK (18) for FTS of discrete system is established. Less conservative results can be obtained. Because in this way $V(k) < \sigma V(k-1)$ could be obtained without enlarging the inequality, unlike simple scaling $V(k) - \sigma V(k-1) < V(k) - V(k-1)$ used in [18].

Remark 5. Compared with the LLF used in [19], we added the term

$$\begin{aligned}V_3(k) = &\sum_{i=k-d_m-1}^{k-1} \sigma^{k-1-i}(d_m - k + 1 + i)y^T(i)E^TP_2Ey(i)\\ &+ d\sum_{i=k-d_m-1}^{k-d_m-1} \sigma^{k-1-i}(d_M - k + 1 + i)y^T(i)E^TP_3E^TEy(i),\end{aligned}$$

in this way, more information from the system (1) is used. Moreover, about the handling of the term $\sum\limits_{i=k-d_m}^{k-1} \sigma^{k-i}y^T(i)E^TP_2Ey(i)$, we have fully considered the influence of weighting functions instead of directly reducing it to $\sigma\sum\limits_{i=k-d_m}^{k-1} y^T(i)E^TP_2Ey(i)$. The inequality therefore has theoretical improvements in analysis.

Remark 6. In Theorem 1, the parameters c_1, c_2, σ, N are involved. It is necessary for us to explain the relationship between the relevant parameters. Firstly, (10) and (17) are not in the form of LMIs respect to σ. But we note that (10) and (17) are LMIs for fixed σ, then, the LMIs (9) to (17) can be solved for the given σ. Moreover, based on the actual background of FTS, minimizing c_2 for given c_1 and N is a meaningful optimization problem.

Especially when $E = I_n$, system (1) reduces to a normal time-delay system. Based on Theorem 1, one has

Corollary 3. When $E = I_n$, the system (1) is finite-time stable respect to (c_1, c_2, N), $0 < c_1 < c_2$, if there exist positive scalar$\sigma > 1$ and $\upsilon_i > 0\,(i = 1, 2, ..., 6)$, $\rho_i > 0\,(i = 1, 2, 3, 4, 5)$ matrices P_1, P_2, P_3, Q_1, Q_2, M and U, such that

$$\begin{pmatrix} \overline{P_3} & U \\ * & \overline{P_3} \end{pmatrix} > 0, \tag{33}$$

$$\Psi = \sum_{i=1}^{5} \Psi_i < 0, \tag{34}$$

$$\begin{pmatrix} \Psi & e_1^T PA & e_1^T PA & e_3^T PA_d & e_3^T PA_d & l_2 e_3^T P \\ * & -\rho_1 I & 0 & 0 & 0 & 0 \\ * & * & -\rho_2 I & 0 & 0 & 0 \\ * & * & * & -\rho_3 I & 0 & 0 \\ * & * & * & * & -\rho_4 I & 0 \\ * & * & * & * & * & -\rho_5 I \end{pmatrix} < 0, \tag{35}$$

$$\upsilon_1 I_r < P_{11} < \upsilon_2 I_r, \tag{36}$$

$$0 < Q_1 < \upsilon_3 I_n, \tag{37}$$

$$0 < Q_2 < \upsilon_4 I_n, \tag{38}$$

$$0 < P_2 < \upsilon_5 I_r, \tag{39}$$

$$0 < P_3 < \upsilon_6 I_r, \tag{40}$$

$$\sigma^N (\upsilon_2 + \upsilon_3 t_1 + \upsilon_4 t_2)\, c_1 + c_0 (\upsilon_5 t_3 + d\upsilon_6 t_4) < \upsilon_1 c_2, \tag{41}$$

in which the definition of Ψ is the same as in Theorem 1.

Remark 7. When studying the FTS of linear normal time-delay systems, less conservative results can be obtained by Corollary 3. Because unlike [23], a new LLK (18) for FTS of discrete system is established. In this way $\Delta V(k) < (\sigma - 1)V(k)$ could be obtained without enlarging the inequality, unlike simple scaling $\Delta V(k) < (\sigma - 1)V_1(k) < (\sigma - 1)V(k)$ used in [23].

4. Numerical examples

Example 1. Considering the system (1) with

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A=\begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 \\ 0.2 & 1 & 0.2 \end{bmatrix},\ A_d=\begin{bmatrix} 0.1 & 0.03 & 0.1 \\ 0.1 & 0 & 0.02 \\ 1 & 0 & 0 \end{bmatrix},$$

$d_m = 2$, $d_M = 5$, $c_1 = 3$, $c_0 = 2$, (l_1, l_2)=$(0.01, 0.01)$.

According to Theorem 1, the system is finite-time stable. In addition, in order to observe the impact of nonlinear uncertainties on the finite-time stability of the system, we validated different parameters. In Table 1 we obtained the minimum allowable value of c_2 for different (l_1, l_2).

Table 1: Minimum allowable c_2 different l_1 and l_2

(l_1, l_2)	$(0.01, 0.01)$	$(0.03, 0.03)$	$(0.05, 0.05)$
c_2	79.67	86.34	92.53

We can see that the minimum allowable value of c_2 increases as the degree of nonlinear uncertainties increases.

Example 2. Considering the system (1) with

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A=\begin{bmatrix} 1 & 1 & 0.5 \\ -1.5 & 0.5 & 3.75 \\ 1 & 2 & 3 \end{bmatrix},\ A_d=\begin{bmatrix} 0.5 & 0.2 & -0.2 \\ -0.15 & -0.6 & 0.05 \\ 1 & 0 & 0 \end{bmatrix},$$

$d_m = 1$, $d_M = 3$, $c_1 = 1$, $c_0 = 2$.

Based on the actual background of FTS, we hope that c_2 is as small as possible for a given initial state and time interval. About the problem, [18] achieved good results. In Table 2, we compare the result with [18].

We can see that c_2 obtained from Theorem 1 is smaller than that in [18],

Table 2: Minimum allowable c_2 compared with [18]

[18]	$c_1 = 1$	$N = 4$	$c_2 = 372.49$
Theorem 1	$c_1 = 1$	$N = 4$	$c_2 = 109.08$

verifying that our result is less conservative than [18].

Example 3. Considering the system (1) with

$$A = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.4 \\ 0.2 & 1 & 0.2 \end{bmatrix}, \ A_d = \begin{bmatrix} 0.1 & 0.01 & 0.1 \\ 0.1 & 0.01 & 0.1 \\ 0.1 & 0.1 & 0.01 \end{bmatrix},$$

$d_m = 2$, $d_M = 5$, $c_1 = 3$, $c_0 = 1.1$.

In Table 3, we compare the result with [23].

Table 3: Minimum allowable c_2 compared with [23]

[23]	$c_1 = 3$	$N = 5$	$c_2 = 56$
Corollary 3	$c_1 = 3$	$N = 5$	$c_2 = 22$

We can see that c_2 obtained from Corollary 3 is smaller than that in [23], which illustrates that our result is less conservative than [23].

5. Conclusion

In this paper, we have studied the finite-time stability for singular time-delay systems. Upon constructing a new Lyapunov-like functional (LLF) and a new weighted integral inequality, we are able to establish some sufficient conditions such that the underlying system is finite-time stable. Two numerical examples have been presented to illustrate the efficiency of the proposed method. Based on the results of this paper, we can study the synthesis problem of finite-time stability for singular time-delay systems, which is also our future work

6. Statements and Declarations

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6.2. Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

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