Journal of Applied Analysis and Computation
Volume *, Number *, * *, 1–22
Website:http://www.jaac-online.com
DOI:10.11948/***

HOPF BIFURCATION AND CONTROL FOR THE DELAYED PREDATOR-PREY MODEL WITH NONLINEAR PREY HARVESTING *

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Abstract In our study, we focused on investigating a delayed differential-algebraic system. The system incorporates a square root functional response and non-linear prey harvesting. Employing the normal form of differential algebraic systems and the central manifold theory, we conducted a detailed analysis of the system's stability and bifurcation phenomena, with time delay identified as a critical bifurcation parameter. When the time delay reached a critical value, the system's equilibrium points underwent the Hopf bifurcation, resulting in system instability. To achieve stability, we introduced a feedback controller, successfully transitioning the system from an unstable to a stable state. Through subsequent numerical simulations, we validated the accuracy and correctness of our research conclusions.

Keywords Stability, Predator–prey system, Time delay, Hopf bifurcation, Periodic solution.

MSC(2010) 34D20, 92D25.

1. Introduction

In the fields of ecology and dynamic system theory, the study of delayed predator-prey systems has a profound background. In the past few decades, numerous researchers [1–4, 7, 9, 14, 15, 17, 20–26] have delved into the dynamic behavior of interactions between different species in ecosystems, especially in predator-prey relationships. Researchers are gradually realizing that time delay plays a crucial role in these interactions. Time delay often leads to new dynamic behaviors in the system, such as periodic oscillations, stable coexistence, or system collapse. Research in this area is crucial for understanding the behavior of complex natural ecosystems, the formation and maintenance of ecological balance, and the impact of environmental changes on biodiversity. Further research on delayed predatory systems can help better predict and manage ecosystem responses, especially in the context of global climate change and increasing human interference. The in-depth exploration in

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[*]The authors were supported by the National Science Foundation of China under Grant no. 61976228 and no. 62373383.
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this field provides an important theoretical basis for formulating effective ecological protection strategies and sustainable management plans in the future.

In recent years, the academic community has shown strong interest in the following research questions. Jiao et al. [13] extended the Leslie Gower model in non smooth Filippov control systems, introduced time delay to investigate the influence of predator maturation time, and conducted in-depth research on the stability of system equilibrium points and the existence of Hopf bifurcation. Chakraborty et al. [5] focused on studying the bioeconomic model of predator-prey systems with Holling III functional response, which includes continuous pregnancy time delay and delves into the system instability caused by time delay. Zhang et al. [27] are dedicated to studying a class of differential algebraic predator-prey systems with time delays. They use time delay as bifurcation parameter and use normal form theory and central manifold theory to study the stability direction of Hopf bifurcation. Liu et al. [16] proposed a Gause predator-prey model that includes pregnancy delay and Michaelis Menten type harvest.

In this study, we introduce a delayed bioeconomic system characterized by differential algebraic equations, following the methods of Jiao et al. [13], Chakraborty et al. [5], Zhang et al. [27] and Liu et al. [16]. The system contains a square root functional response and non-linear prey harvesting, with time delay as the bifurcation parameter. Through the application of central manifold theory and normal form theory, we conducted an in-depth analysis of the stability of the system and determined the direction of Hopf bifurcation. We delve deeper into the complex dynamic behavior of systems under the influence of time delay.

Mortuja et al. [18] delved into the dynamic properties of predator-prey interactions, specifically focusing on systems characterized by nonlinear prey harvesting. The system which they studied is given by the following equation:

$$\begin{cases} \frac{dx(t)}{dt} = rx\left(1-\frac{x}{k}\right) - \frac{\varrho\sqrt{x}y}{1+t_h\varrho\sqrt{x}} - \frac{qEx}{m_1E+m_2x}, \\ \frac{dy(t)}{dt} = -\beta y + \frac{e\varrho\sqrt{x}y}{1+t_h\varrho\sqrt{x}}, \end{cases} \tag{1.1}$$

where, the population density of prey is denoted by x, the population density of predator is denoted by the variable y, the prey population growth rate is represented by r, the environmental carrying capacity is represented by k. The growth rate of the prey population is denoted by r, and the environmental carrying capacity is represented by k. The average handling time of captured prey is expressed by t_h, the depletion rate is expressed by e, the efficiency in searching for prey is expressed by ϱ, and the natural mortality rate of the predator in the absence of prey is expressed by β. Additionally, the model incorporates nonlinear prey harvesting, where the coefficient of harvesting capacity is denoted by q, harvesting effort is represented by E, and m_1 and m_2 are intrinsic constants.

Simultaneously, taking practical significance into account, our model incorporates algebraic equations to account for the economic dimension of harvesting activities. This new model comprehensively considers various factors related to the profitability of harvesting activities, providing a more holistic understanding of the dynamics of predator-prey systems by simultaneously integrating ecological and economic factors. According to the economic theory of Gordon [10]: Net Economic Revenue (NER) is calculated as the value obtained by subtracting Total Cost (TC) from Total Revenue (TR).

In the framework of the system (1.1), the expressions for Total Revenue (TR) and Total Cost (TC) are as follows:

$$TR = \frac{qEx}{m_1E + m_2x}p,$$
$$TC = \frac{qE}{m_1E + m_2x}c,$$

where p represents the unit price and c represents the unit harvesting cost, economic profit (m) is equivalent to Net Economic Revenue (NER). Mathematically, this relationship can be expressed by the following equation:

$$NER = TR - TC = \frac{qE}{m_1E + m_2x}(px - c) = m.$$

By amalgamating the aforementioned algebraic equation concerning the biological-economic aspect with system (1.1), the system can be expressed through differential-algebraic equations as follows:

$$\begin{cases} \frac{dx(t)}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\varrho\sqrt{x}y}{1+t_h\varrho\sqrt{x}} - \frac{qEx}{m_1E+m_2x}, \\ \frac{dy(t)}{dt} = -\beta y + \frac{e\varrho\sqrt{x}y}{1+t_h\varrho\sqrt{x}}, \\ 0 = \frac{qE}{m_1E+m_2x}(px-c) - m. \end{cases} \tag{1.2}$$

It is a special case for system(1.2). In the real world, time delay exists in various phenomena, such as the transmission of electricity, the transmission and reception of signals, the gestation cycle and reaction time of biological individuals, and so on. Therefore, studying systems with time delay is more in line with practical needs and has greater significance. Now we add time delay to system (1.2):

$$\begin{cases} \frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t-\tau)}{k}\right) - \frac{\varrho\sqrt{x(t)}y(t)}{1+t_h\varrho\sqrt{x(t)}} - \frac{qE(t)x(t)}{m_1E(t)+m_2x(t)}, \\ \frac{dy(t)}{dt} = -\beta y(t) + \frac{e\varrho\sqrt{x(t)}y(t)}{1+t_h\varrho\sqrt{x(t)}}, \\ 0 = \frac{qE(t)}{m_1E(t)+m_2x(t)}(px(t)-c) - m. \end{cases} \tag{1.3}$$

For simplicity, let

$$f(X,E) = \begin{bmatrix} f_1(X,E) \\ f_2(X,E) \end{bmatrix} = \begin{bmatrix} rx\left(1 - \frac{x(t-\tau)}{k}\right) - \frac{\varrho\sqrt{x}y}{1+t_h\varrho\sqrt{x}} - \frac{qEx}{m_1E+m_2x} \\ -\beta y + \frac{e\varrho\sqrt{x}y}{1+t_h\varrho\sqrt{x}} \end{bmatrix},$$

$$g(X,E) = \frac{qE}{m_1E + m_2x}(px - c) - m,$$

where $X = [x,y]^T$, time delay $\tau > 0$ serves as a bifurcation parameter, and its specific definition will be elucidated subsequently.

This paper predominantly focuses on analyzing the model system (1.3) within the domain $R_+^3 = \left\{[x,y,E]^T \mid x > 0, y > 0, E > 0\right\}$. The region R_+^3 refers to the presence of prey density (x), predator density (y), and harvesting effort (E), reflecting the ecological relevance and feasibility of the system in practical biological significance.

The paper is structured as follows: Treating τ as the bifurcation parameter, we explore the stability and Hopf bifurcation at the equilibrium point of system (1.3).

In Section 2, we investigate the stability and Hopf bifurcation at the equilibrium point under variations in time delay. In Section 3, we draw inspiration from the normal form theory and central manifold theory which are introduced by Hassard et al. [12], and derive formulas characterizing the Hopf bifurcation in system (1.3). In Section 4, we introduce a feedback controller that successfully transitions the system from an unstable to a stable state. In Section 5, we present numerical simulations to validate and complement our analytical findings. In Section 6, we conclude and outline future prospects.

Remark 1.1. In contrast to the work by Jiao et al. [13], our study incorporates a time delay into the system. Distinguishing itself from the investigations of Zhang et al. [27] and Chakraborty et al. [5], our model introduces nonlinear harvesting dynamics. Furthermore, in deviation from Liu et al. [16], we employ a distinct response function and incorporate a feedback controller into the system. This unique combination of elements adds a novel dimension to our analysis, allowing us to explore a more comprehensive and nuanced set of dynamics in the considered bioeconomic system.

2. Local stability analysis

Highlighting our exclusive attention to the internal balance represented by $Y_0 = (x_0, y_0, E_0)$ in the model system (1.3), it is noteworthy that this equilibrium point holds biological significance. The presence of prey, predator, and harvesting in this interior equilibrium aligns with the core aspects of our study. A thorough analysis of the model system (1.3) indicates the presence of an equilibrium within the positive region R^3_+ only when the following equations are met:

$$\begin{aligned} 0 =& rx(t)\left(1 - \frac{x(t-\tau)}{k}\right) - \frac{\varrho\sqrt{x(t)}y(t)}{1 + t_h\varrho\sqrt{x(t)}} - \frac{qE(t)x(t)}{m_1E(t) + m_2x(t)}, \\ 0 =& -\beta y(t) + \frac{e\varrho\sqrt{x(t)}y(t)}{1 + t_h\varrho\sqrt{x(t)}}, \\ 0 =& \frac{qE(t)}{m_1E(t) + m_2x(t)}(px(t) - c) - m. \end{aligned} \tag{2.1}$$

Considering the biological significance of the above internal equilibrium, prey, predators, and harvesting can coexist in the system. To ensure the existence of internal equilibrium, certain inequalities must be satisfied, specifically: $r - \frac{r}{k}x_0 - \frac{qE_0}{m_1E_0+m_2x_0} > 0$ and $q - px_0 - qc - mm_1 > 0$. Therefore, it can be affirmed that the equations (2.1) have a unique internal equilibrium point $Y_0 = (x_0, y_0, E_0)$.

Where:

$$\begin{aligned} x_0 =& \left(\frac{\beta}{\varrho(e - t_h\beta)}\right)^2, \\ y_0 =& \frac{\sqrt{x_0}(1 + t_h\varrho\sqrt{x_0})}{\varrho}\left(r - \frac{r}{k}x_0 - \frac{qE_0}{m_1E_0 + m_2x_0}\right), \\ E_0 =& \frac{mm_2x_0}{qpx_0 - qc - mm_1} \end{aligned}$$

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To investigate the characteristics of the equilibrium points in the model system (1.3), we employed an approach similar to that proposed in the literature [6]. Initially, we focus on the local parameter Φ associated with the final equation of the system (1.3), defined as follows:

$$[x(t), y(t), E(t)]^T = \Phi(\aleph(t)) = Y_0^T + U_0\aleph(t) + V_0\hbar(\aleph(t)), \quad g(\Phi(\aleph(t))) = 0,$$

where $U_0 = \begin{pmatrix} 1\ 0 \\ 0\ 1 \\ 0\ 0 \end{pmatrix}$,and $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\aleph(t) = (\eta_1(t), \eta_2(t))^T$, $Y_0 = (x_0, y_0, E_0)$,
$\hbar(\aleph(t)) = \hbar_3(\eta_1(t), \eta_2(t)) : R^2 \to R$ is a smooth mapping, that is

$$x(t) = x_0 + \eta_1(t), \quad y(t) = y_0 + \eta_2(t), \quad E(t) = E_0 + \hbar_3(\eta_1(t), \eta_2(t)).$$

Consequently, we obtain the subsequent parametric system within the framework of the model system (1.3):

$$\begin{aligned} \frac{dx(t)}{dt} &= r(x_0+\eta_1(t))\left(1 - \frac{(x_0+\eta_1(t))}{k}\right) - \frac{\varrho\sqrt{(x_0+\eta_1(t))}(y_0+\eta_2(t))}{1+t_h\varrho\sqrt{(x_0+\eta_1(t))}} \\ &\quad - \frac{q(E_0+\hbar_3(\eta_1(t),\eta_2(t)))(x_0+\eta_1(t))}{m_1(E_0+\hbar_3(\eta_1(t),\eta_2(t))) + m_2(x_0+\eta_1(t))}, \\ \frac{dy(t)}{dt} &= -\beta(y_0+\eta_2(t)) + \frac{e\varrho\sqrt{(x_0+\eta_1(t))}(y_0+\eta_2(t))}{1+t_h\varrho\sqrt{(x_0+\eta_1(t))}}. \end{aligned} \tag{2.2}$$

Due to the condition $g(\Phi(\aleph(t))) = 0$, we are now able to derive the linearized system associated with the parametric system (2.2) at $(0, 0)$:

$$\begin{aligned} \frac{\mathrm{d}\eta_1(t)}{\mathrm{d}t} &= \left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)\eta_1(t) \\ &\quad - \frac{rx_0}{k}\eta_1(t-\tau) - \frac{\varrho\sqrt{x_0}}{1+t_h\varrho\sqrt{x_0}}\eta_2(t), \\ \frac{\mathrm{d}\eta_2(t)}{\mathrm{d}t} &= \frac{e\varrho y_0}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\eta_1(t). \end{aligned}$$

Lemma 2.1. *For the positive equilibrium point Y_0 of the system (1.3),*

(i) If $0 < m < \frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0 - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - 2\sqrt{\frac{e\varrho^2 y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$, the non-negative equilibrium point Y_0 of system (1.3) demonstrates asymptotic stability.

(ii)If $m > \frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0 - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + 2\sqrt{\frac{e\varrho^2 y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$, the positive equilibrium point Y_0 is unstable .

Proof. To begin with, we easily derive the characteristic equation for the linearized system associated with the parametric system (1.3) when $\tau = 0$ at the point (0, 0). This equation is expressed as follows:

$$\lambda^2+(\frac{r}{k}x_0-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} -\frac{qpx_0E_0}{(px_0-c)\,(m_1E_0+m_2x_0)})\lambda+\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}=0. \tag{2.3}$$

We denote Δ by

$$\Delta=(\frac{r}{k}x_0-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}-\frac{qpx_0E_0}{(px_0-c)\,(m_1E_0+m_2x_0)})^2-\frac{2e\varrho^2y_0}{(1+t_h\varrho\sqrt{x_0})^3}.$$

Clearly, if $0<m<\frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}-2\sqrt{\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$, all roots of the equation (2.3) have negative real parts. Conversely, when $m>\frac{(px_0-c)^2}{px_0}$ $\left(\frac{r}{k}x_0-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}+2\sqrt{\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$, all roots of the equation (2.3) have positive real parts. Consequently, both part (i) and part (ii) hold true. □

Remark 2.1. To ensure the existence of an internal balance point, we have $0<m<\frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\right.$ $\left.-2\sqrt{\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$. When $\tau=0$ and $0<m=\frac{r(3x_0+4t_h\varrho x_0^{\frac{3}{2}}-2kt_h\varrho\sqrt{x_0}-k)(px_0-c)^2}{k(c+2ct_h\varrho\sqrt{x_0}+px_0)}<\frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\right.$ $\left.-2\sqrt{\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$, all the roots of equation (2.3) have zero real part. Consequently, the positive equilibrium point of system (1.3) becomes a center.

Moreover, considering m as the bifurcation parameter, the Hopf bifurcation occurs in the model system (1.3) when m reaches the bifurcation value $m_0=\frac{r(3x_0+4t_h\varrho x_0^{\frac{3}{2}}-2kt_h\varrho\sqrt{x_0}-k)(px_0-c)^2}{k(c+2ct_h\varrho\sqrt{x_0}+px_0)}$. This bifurcation scenario can be analyzed similarly as discussed in the academic paper [11].

Now, we delve into the local stability in the vicinity of Y_0 and examine the possible emergence of the Hopf bifurcation at Y_0 for $\tau>0$. To initiate our exploration, we introduce the following Lemma .

Lemma 2.2. *For the model system (1.3), if* $0<m<\frac{(px_0-c)^2}{px_0}(\frac{r}{k}x_0-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}-2\sqrt{\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}})$ *, then,*

(i)if $\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}+\frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2>\frac{e\varrho^2y_0}{(1+t_h\varrho\sqrt{x_0})^3}+(\frac{rx_0}{k})^2$ *, for all* $\tau\geq0$*, the real parts of every root in Eq. (2.5) consistently have negative values.*

(ii)if $\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}+\frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2<\frac{e\varrho^2y_0}{(1+t_h\varrho\sqrt{x_0})^3}+(\frac{rx_0}{k})^2$ *and* $\left[\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}+\frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2-\frac{e\varrho^2y_0}{(1+t_h\varrho\sqrt{x_0})^3}-(\frac{rx_0}{k})^2\right]^2>(\frac{e\varrho^2y_0}{(1+t_h\varrho\sqrt{x_0})^3})^2$ *, Eq.(2.8) possesses two positive roots denoted as* ϖ^+ *and* ϖ^-*. Upon substituting these roots into (2.7), we obtain:*

$$\tau_n^{\pm}=\frac{1}{\varpi^{\pm}}\arccos\left[\frac{(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}+\frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)})k}{rx_0}\right]+\frac{2n\pi}{\varpi^{\pm}},\quad n=0,1,2,\ldots \tag{2.4}$$

Proof. Firstly, we readily obtain the characteristic equation for the linearized system associated with the parametric system (1.3) at the point (0, 0). This equation is expressed as follows:

$$\begin{aligned} &\lambda^2 + (\frac{r}{k}x_0\mathrm{e}^{-\lambda\tau} - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \\ &- \frac{qpx_0E_0}{(px_0-c)\,(m_1E_0+m_2x_0)})\lambda + \frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3} = 0, \end{aligned} \tag{2.5}$$

we consider $\pm\mathrm{i}\varpi$ as the pair of purely imaginary roots for equation (2.5). Substituting $\mathrm{i}\varpi$ (where ϖ is a positive real value) into equation (2.5), we get:

$$\begin{aligned} &-\varpi^2 + \mathrm{i}\varpi(\frac{r}{k}x_0(\cos\varpi\tau - \mathrm{i}\sin\varpi\tau) - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \\ &- \frac{qpx_0E_0}{(px_0-c)\,(m_1E_0+m_2x_0)}) + \frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3} = 0. \end{aligned}$$

Upon separating the real and imaginary parts, we obtain two transcendental equations as follows:

$$\frac{rx_0\varpi}{k}\sin\varpi\tau = \varpi^2 - \frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}, \tag{2.6}$$

$$\frac{rx_0\varpi}{k}\cos\varpi\tau = \varpi\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right). \tag{2.7}$$

Squaring and adding (2.6) and (2.7), the calculation yields:

$$\begin{aligned} &\varpi^4 + \left[\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2\right. \\ &\left.-\frac{e\varrho^2y_0}{(1+t_h\varrho\sqrt{x_0})^3} - (\frac{rx_0}{k})^2\right]\varpi^2 + (\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3})^2 = 0. \end{aligned} \tag{2.8}$$

When $\tau = 0$, the condition $0 < m < \frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0 - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} -2\sqrt{\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$ ensures that all roots of equation (2.5) have negative real parts.

Furthermore, when $\tau = 0$, in the case where $0 < m < \frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0 - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - 2\sqrt{\frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$ holds, it implies that all the roots of equation (2.5) have negative real parts. According to Rouche's theorem [19], the sum of the order of the zeros of $P\left(\lambda, \mathrm{e}^{-\lambda\tau}\right) = \lambda^2 + \left(\frac{rx_0}{k}\mathrm{e}^{-\lambda\tau} - \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)} - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\right)\lambda + \frac{e\varrho^2y_0}{2(1+t_h\varrho\sqrt{x_0})^3}$ on the open right half plane can only change if a zero appears on or crosses the imaginary axis.

Therefore, based on the above discussions, it can be concluded that equation (2.5) with $\tau > 0$ maintains the same number of roots with a negative real part as equation (2.5) with $\tau = 0$. In conclusion, when $\tau > 0$ and if $0 < m < \frac{(px_0-c)^2}{px_0}\left(\frac{r}{k}x_0 - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - 2\sqrt{\frac{e\varrho^2 y_0}{2(1+t_h\varrho\sqrt{x_0})^3}}\right)$ and $\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2 > \frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3} + \left(\frac{rx_0}{k}\right)^2$ hold, all the roots of equation (2.5) also have negative real parts.

When $\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2 < \frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3} + (\frac{rx_0}{k})^2$ and $\left[\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2 - \frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3} - (\frac{rx_0}{k})^2\right]^2 > (\frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3})^2$, it can be easily deduced that equation (2.8) has two positive roots ϖ^+ and ϖ^-. Substituting $\varpi^\pm$ into (2.7), we obtain $\tau_n^\pm$. With this, the demonstration of Lemma 2.2 concludes. □

Now from (2.5) we obtain

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{2\lambda - \frac{rx_0}{k}\lambda\tau \mathrm{e}^{-\lambda\tau} + \left(\frac{rx_0}{k}\mathrm{e}^{-\lambda\tau} - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{(x_0)}(1+t_h\varrho\sqrt{x_0})^2} - \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)}{\frac{rx_0}{k}\lambda^2\mathrm{e}^{-\lambda\tau}}.$$

Thus,

$$\operatorname{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right\}_{\lambda=\mathrm{i}\varpi} = \operatorname{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right\}_{\lambda=\mathrm{i}\varpi}$$
$$= \operatorname{sign}\left\{\frac{\varpi^4 - \left(\frac{e\varrho^2 y_0}{2(1+t_h\varrho\sqrt{x_0})^3}\right)^2}{\varpi^2\left[\varpi^2\left(-\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{(x_0)}(1+t_h\varrho\sqrt{x_0})^2} - \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2 + \left(\frac{e\varrho^2 y_0}{2(1+t_h\varrho\sqrt{x_0})^3} - \varpi^2\right)^2\right]}\right\}.$$

The following transversality conditions can be easily verified: $\operatorname{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right\}_{\tau=\tau_n^+,\varpi=\varpi^+} > 0$ and $\operatorname{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right\}_{\tau=\tau_n^-,\varpi=\varpi^-} < 0$.

In summary of the aforementioned results, the following theorem is presented regarding the stability and Hopf bifurcation of system (1.3).

Theorem 2.1. *For system (2.2), if* $0 < m < \frac{(px_0-c)^2}{px_0}(\frac{r}{k}x_0 - \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - 2\sqrt{\frac{e\varrho^2 y_0}{2(1+t_h\varrho\sqrt{x_0})^3}})$ *, then,*

(i) When $\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2 > \frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3} + (\frac{rx_0}{k})^2$*, then, for all* $\tau \geq 0$*, the real parts of all roots of Equation (2.5) are negative, thereby confirming the asymptotic stability of the equilibrium point* Y_0 *in the system (1.3).*

(ii) When $\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2 < \frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3} + (\frac{rx_0}{k})^2$ *and* $\left[\left(\frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)}\right)^2 - \frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3} - (\frac{rx_0}{k})^2\right]^2 > (\frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3})^2$*, then, for any positive integer* M*, there exist* M *intervals where the stability of the equilibrium point* Y_0 *in system (1.3) alternates between stable and unstable. More precisely, when* $\tau \in [0,\tau_0^+), (\tau_0^-,\tau_1^+), \ldots, (\tau_{M-1}^-,\tau_M^+)$*, the equilibrium point* Y_0 *is stable. Conversely, when* $\tau \in [\tau_0^+,\tau_0^-), (\tau_1^+,\tau_1^-), \ldots, (\tau_M^+,\tau_M^-)$*, the equilibrium*

point Y_0 is unstable. Hence, bifurcations occur at the equilibrium point Y_0 of system (1.3) when $\tau = \tau_n^{\pm}, n = 0, 1, 2, \ldots, M$.

3. Direction and the stability of Hopf bifurcation

In this section, we extensively investigate the direction of Hopf bifurcation and the stability of bifurcating periodic solutions though employing the normal form theory and the central manifold theory [12].

In the subsequent analysis, we assume that system (1.3) undergoes the Hopf bifurcation at the positive equilibrium point Y_0 for $\tau = \tau_n$, where $i\varpi$ represents the corresponding purely imaginary root of the characteristic equation at the positive equilibrium Y_0. We employ the parameterized form (2.2) of system (1.3) to investigate the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions in system (1.3). Initially, by employing the transformations $\eta_1 = x - x_0$, $\eta_2 = y - y_0$, $t = \frac{t}{\tau}$, $\tau = \tau_n + \wp$, the parameterized form (2.2) of system (1.3) can be equivalently expressed as the following Functional Differential Equation (FDE) system in $D = D\left([-1, 0], R^2\right)$,

$$\dot{\aleph}(t) = L_{\wp}\left(\aleph_t\right) + f\left(\wp, \aleph_t\right), \tag{3.1}$$

where $\aleph(t) = (\eta_1(t), \eta_2(t))^T$ and $L_{\wp} : D \to \mathbb{R}, f : \mathbb{R} \times D \to \mathbb{R}$ are given:

$$L_{\wp}(\varphi) = (\tau_n + \wp)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix}\varphi^T(0) + (\tau_n + \wp)\begin{bmatrix} b_{11} & 0 \\ 0 & 0 \end{bmatrix}\varphi^T(-1),$$

where $a_{11} = \frac{qpx_0y_0}{(px_0-c)(m_1E_0+m_2x_0)} + \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}$, $a_{12} = -\frac{\varrho\sqrt{x_0}}{1+t_h\varrho\sqrt{x_0}}$,
$a_{21} = \frac{e\varrho y_0}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}$, $b_{11} = -\frac{rx_0}{k}$, and $f(\wp, \varphi) = (\tau_n + \wp)\begin{bmatrix} f_{11} \\ f_{22} \end{bmatrix}$, where

$$\begin{aligned} f_{11} = (&\frac{\varrho y_0(1 + 3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1 + t_h\varrho\sqrt{x_0})^3} + \frac{m_1m_2qE_0^2}{(m_1E_0 + m_2x_0)^3} \\ &- \frac{E_0q(m_1pE_0 + m_2c)(cm_1E_0 + m_2p^2x_0)}{(px_0 - c)^2(m_1E_0 + m_2x_0)^3})\varphi_1^2(0) \\ &- \frac{r}{k}\varphi_1(0)\varphi_1(-1) - \frac{\varrho}{2\sqrt{x_0}(1 + t_h\varrho\sqrt{x_0})^2}\varphi_1(0)\varphi_2(0) + \cdots, \\ f_{22} = -&\frac{e\varrho y_0(1 + 3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1 + t_h\varrho\sqrt{x_0})^3}\varphi_1^2(0) + \frac{e\varrho}{2\sqrt{x_0}(1 + t_h\varrho\sqrt{x_0})^2}\varphi_1(0)\varphi_2(0) + \cdots, \end{aligned}$$

and $\varphi = (\varphi_1, \varphi_2) \in D$. On the basis of the Riesz representation theorem, there exists a matrix function whose components are functions $\phi(\zeta, \wp)$ of bounded variation in $\zeta \in [-1, 0]$, such that:

$$L_{\wp}\varphi = \int_{-1}^{0} \mathrm{d}\phi(\zeta, \wp)\varphi(\zeta), \quad \varphi \in D.$$

To be precise, we can select

$$\phi(\zeta,\wp)=(\tau_n+\wp)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix}\delta(\zeta)+(\tau_n+\wp)\begin{bmatrix} -b_{11} & 0 \\ 0 & 0 \end{bmatrix}\delta(\zeta+1),$$

where $\delta(\zeta)=\begin{cases} 0, \zeta\neq 0, \\ 1, \zeta=0 \end{cases}$ For $\varphi\in D^1\left([-1,0],\mathbb{R}^2\right)$, define

$$\Im(\wp)\varphi(\zeta)=\begin{cases} \frac{\mathrm{d}\varphi(\zeta)}{\mathrm{d}\zeta}, & -1\leq\zeta<0, \\ \int_{-1}^{0}\ \mathrm{d}\phi(\zeta,\wp)\varphi(\zeta), \zeta=0. \end{cases} \tag{3.2}$$

Then, the equivalent formulation of system (3.1) is:

$$\dot{\aleph}(t)=\Im(\wp)\aleph_t+R(\wp)\aleph_t. \tag{3.3}$$

For $\Phi\in D^1\left([0,1],\left(\mathbb{R}^2\right)^*\right)$, the adjoint operator $\Im^*$ of $\Im$ is defined as

$$\Im^*\Phi(\daleth)=\begin{cases} -\frac{\mathrm{d}\Phi(\daleth)}{\mathrm{d}\daleth}, & 0<\daleth\leq 1, \\ \int_{-1}^{0}\ \mathrm{d}\phi^T(\daleth,0)\Phi(-\daleth), \daleth=0. \end{cases} \tag{3.4}$$

and an alternative representation is provided by a bilinear inner product, expressed as:

$$\langle\Phi(\daleth),\varphi(\zeta)\rangle=\bar{\Phi}(0)\varphi(0)-\int_{\zeta=-1}^{0}\int_{\xi=0}^{\zeta}\bar{\Phi}(\xi-\zeta)\mathrm{d}\phi(\zeta)\varphi(\xi)\mathrm{d}\xi, \tag{3.5}$$

where $\phi(\zeta)=\phi(\zeta,0)$. It can be easily shown that $\Im(0)$ and $\Im^*$ constitute a pair of adjoint operators.

Building upon the exploration in Section 2, recognizing that $\pm\mathrm{i}\varpi$ are eigenvalues of $\Im(0)$, it follows that they also function as eigenvalues for $\Im^*$. Going ahead, we engage in determining the eigenvector $\vartheta(\zeta)$ of $\Im$ corresponding to $i\varpi$ and the eigenvector $\vartheta(\daleth)$ of $\Im^*$ corresponding to the eigenvalue $-i\varpi$. Subsequently, it is easy to demonstrate:

$$\vartheta(\zeta)=(1,\gamma)^T\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta},\quad \vartheta^*(\daleth)=G\left(\gamma^*,1\right)\mathrm{e}^{\mathrm{i}\varpi\tau_n\daleth},$$

where $\gamma=\frac{\mathrm{i}\varpi-\mathrm{a}_{11}-b_{11}\mathrm{e}^{-\mathrm{i}\varpi\tau_n}}{a_{12}}$, $\gamma^*=-\frac{\mathrm{i}\varpi}{a_{12}}$, $\bar{G}=\left(\gamma+\bar{\gamma}^*-\tau_n\bar{\gamma}^*b_{11}\mathrm{e}^{-\mathrm{i}\varpi\tau_n}\right)^{-1}$. Moreover, $\langle\vartheta^*(\daleth),\vartheta(\zeta)\rangle=1$ and $\left\langle\vartheta^*(\daleth),\bar{\vartheta}(\zeta)\right\rangle=0$.

Following this, we explore the stability analysis of bifurcated periodic solutions. Using notations distinct from those in [12], we first calculate the coordinates employed to characterize the center manifold D_0 at $\wp=0$. Define:

$$\dot{\varsigma}(t)=\langle\vartheta^*,\aleph_t\rangle,\quad P(t,\zeta)=\aleph_t-2\,\mathrm{Re}\{\varsigma(t)\vartheta(\zeta)\}. \tag{3.6}$$

On the center manifold D_0 , we have

$$P(t,\zeta)=P(\varsigma(t),\bar{\varsigma}(t),\zeta)=P_{20}(\zeta)\frac{\varsigma^2}{2}+P_{11}(\zeta)\varsigma\bar{\varsigma}+P_{02}(\zeta)\frac{\bar{\varsigma}^2}{2}+\cdots. \tag{3.7}$$

Indeed, ς and $\bar{\varsigma}$ serve as local coordinates for the center manifold D_0 in the directions of ϑ and $\bar{\vartheta}^*$. It is crucial to note that P is real when $\aleph_t$ is real. In this context, we exclusively focus on real solutions. For the solution $\aleph_t \in D_0$, given that $\wp = 0$ and considering (3.1), we obtain:

$$\begin{aligned} \dot{\varsigma} &= \mathrm{i}\varpi\tau_n\varsigma + \langle \vartheta^*(\zeta), f(0, P(\varsigma,\bar{\varsigma},\zeta) + 2\,\mathrm{Re}[\varsigma(t)\vartheta(\zeta)]) \rangle\,, \\ &= \mathrm{i}\varpi\tau_n\varsigma + \bar{\vartheta}^*(0) f(0, P(\varsigma,\bar{\varsigma},0) + 2\,\mathrm{Re}[\varsigma(t)\vartheta(\zeta)]), \end{aligned} \tag{3.8}$$

Rewrite this equation as

$$\dot{\varsigma} = \mathrm{i}\varpi\tau_\mathrm{n}\varsigma + g(\varsigma,\bar{\varsigma}), \tag{3.9}$$

where

$$g(\varsigma,\bar{\varsigma}) = g_{20}(\zeta)\frac{\varsigma^2}{2} + g_{11}(\zeta)\varsigma\bar{\varsigma} + g_{02}(\zeta)\frac{\bar{\varsigma}^2}{2} + \cdots\,. \tag{3.10}$$

From (3.3) and (3.8), we have

$$\begin{aligned} \dot{P} &= \dot{\aleph}_\mathrm{t} - \dot{\varsigma}\vartheta - \dot{\bar{\varsigma}}\bar{\vartheta}, \\ &= \begin{cases} \Im P - 2\,\mathrm{Re}\left\{\bar{\vartheta}^*(0) f(\varsigma,\bar{\varsigma})\vartheta(\zeta)\right\}, & -1 \le \zeta < 0, \\ \Im P - 2\,\mathrm{Re}\left\{\bar{\vartheta}^*(0) f(\varsigma,\bar{\varsigma})\vartheta(\zeta)\right\} + f,\ \zeta = 0. \end{cases} \end{aligned} \tag{3.11}$$

Rewrite (3.11) as

$$\dot{P} = \Im P + H(\varsigma,\bar{\varsigma},\zeta), \tag{3.12}$$

where

$$H(\varsigma,\bar{\varsigma},\zeta) = H_{20}(\zeta)\frac{\varsigma^2}{2} + H_{11}(\zeta)\varsigma\bar{\varsigma} + H_{02}(\zeta)\frac{\bar{\varsigma}^2}{2} + \cdots\,. \tag{3.13}$$

By substituting the corresponding series into (3.12) and comparing coefficients, we obtain expressions:

$$\begin{aligned} \left(\Im - 2\mathrm{i}\varpi\tau_n\right) P_{20}(\zeta) &= -\,H_{20}(\zeta), \\ \Im P_{11}(\zeta) &= -\,H_{11}(\zeta). \end{aligned} \tag{3.14}$$

Notice that $\vartheta(\zeta) = (1,\gamma)^\mathrm{T} \mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta}$, $\vartheta^*(0) = G\left(\gamma^*, 1\right)$, and from (3.6) we obtain

$$\begin{aligned} \eta_{1t}(0) =& \varsigma + \bar{\varsigma} + P^{(1)}(t,0), \\ \eta_{2t}(0) =& \gamma\varsigma + \bar{\gamma}\bar{\varsigma} + P^{(2)}(t,0), \\ \eta_{1t}(-1) =& \varsigma \mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} + \bar{\varsigma}\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta} + P^{(1)}(t,0), \\ \eta_{2t}(-1) =& \gamma\varsigma \mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} + \bar{\gamma}\bar{\varsigma}\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta} + P^{(2)}(t,0). \end{aligned}$$

According to (3.8) and (3.9), we know that

$$g(\varsigma,\bar{\varsigma}) = \bar{\vartheta}^*(0) f_0(\varsigma,\bar{\varsigma}) = \bar{G}\tau_n\left(\bar{\gamma}^*, 1\right) \begin{bmatrix} f_{11}^0 \\ f_{22}^0 \end{bmatrix}, \tag{3.15}$$

where

$$
\begin{aligned}
f_{11} = &\Big(\frac{\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} + \frac{m_1m_2qE_0^2}{(m_1E_0+m_2x_0)^3} \\
&- \frac{E_0q(m_1pE_0+m_2c)(cm_1E_0+m_2p^2x_0)}{(px_0-c)^2(m_1E_0+m_2x_0)^3}\Big)\eta_{1t}^2(0) \\
&- \frac{r}{k}\eta_{1t}(0)\eta_{1t}(-1) - \frac{\varrho}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\eta_{1t}(0)\eta_{2t}(0) + \cdots, \\
f_{22} = &-\frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3}\eta_{1t}^2(0) + \frac{e\varrho}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\eta_{1t}(0)\eta_{2t}(0) + \cdots.
\end{aligned}
$$

By (3.7) it follows that

$$
\begin{aligned}
g(\varsigma,\bar{\varsigma}) = \bar{G}\tau_n \Bigg\{ &\bar{\gamma}^*\left(\frac{\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} + \frac{m_1m_2qE_0^2}{(m_1E_0+m_2x_0)^3}\right) \\
&\times\left[\varsigma+\bar{\varsigma}+P_{20}^{(1)}(0)\frac{\varsigma^2}{2}+P_{11}^{(1)}(0)\varsigma\bar{\varsigma}+P_{02}^{(1)}(0)\frac{\bar{\varsigma}^2}{2}\right]^2 \\
&-\bar{\gamma}^*\left(\frac{E_0q(m_1pE_0+m_2c)(cm_1E_0+m_2p^2x_0)}{(px_0-c)^2(m_1E_0+m_2x_0)^3}\right) \\
&\times\left[\varsigma+\bar{\varsigma}+P_{20}^{(1)}(0)\frac{\varsigma^2}{2}+P_{11}^{(1)}(0)\varsigma\bar{\varsigma}+P_{02}^{(1)}(0)\frac{\bar{\varsigma}^2}{2}\right]^2 \\
&-\frac{\bar{\gamma}^*r}{k}\left[\varsigma+\bar{\varsigma}+P_{20}^{(1)}(0)\frac{\varsigma^2}{2}+P_{11}^{(1)}(0)\varsigma\bar{\varsigma}+P_{02}^{(1)}(0)\frac{\bar{\varsigma}^2}{2}\right] \\
&\times\left[\varsigma e^{-i\varpi\tau_n\zeta}+\bar{\varsigma}e^{i\varpi\nu\tau_n\zeta}+P_{20}^{(1)}(-1)\frac{\varsigma^2}{2}+P_{11}^{(1)}(-1)\varsigma\bar{\varsigma}+P_{02}^{(1)}(-1)\frac{\bar{\varsigma}^2}{2}\right] \\
&-\frac{\varrho\bar{\gamma}^*}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\left[\varsigma+P_{20}^{(1)}(0)\frac{\varsigma^2}{2}+\bar{\varsigma}+P_{11}^{(1)}(0)\varsigma\bar{\varsigma}+P_{02}^{(1)}(0)\frac{\bar{\varsigma}^2}{2}\right] \\
&\times\left[\gamma\varsigma+P_{11}^{(2)}(0)\varsigma\bar{\varsigma}+P_{20}^{(2)}(0)\frac{\varsigma^2}{2}+\bar{\gamma}\bar{\varsigma}+P_{02}^{(2)}(0)\frac{\bar{\varsigma}^2}{2}\right] \\
&-\frac{\varrho e y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3}\left[\varsigma+\bar{\varsigma}+P_{20}^{(1)}(0)\frac{\varsigma^2}{2}+P_{11}^{(1)}(0)\varsigma\bar{\varsigma}+P_{02}^{(1)}(0)\frac{\bar{\varsigma}^2}{2}\right]^2 \\
&+\frac{\varrho e}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\left[\varsigma+\bar{\varsigma}+P_{20}^{(1)}(0)\frac{\varsigma^2}{2}+P_{11}^{(1)}(0)\varsigma\bar{\varsigma}+P_{02}^{(1)}(0)\frac{\bar{\varsigma}^2}{2}\right] \\
&\times\left[\gamma\varsigma+\bar{\gamma}\bar{\varsigma}+P_{20}^{(2)}(0)\frac{\varsigma^2}{2}+P_{11}^{(2)}(0)\varsigma\bar{\varsigma}+P_{02}^{(2)}(0)\frac{\bar{\varsigma}^2}{2}\right]+\cdots\Bigg\}.
\end{aligned}
$$

For simplicity, let $H = \frac{\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} + \frac{m_1m_2qE_0^2}{(m_1E_0+m_2x_0)^3} - \frac{E_0q(m_1pE_0+m_2c)(cm_1E_0+m_2p^2x_0)}{(px_0-c)^2(m_1E_0+m_2x_0)^3}$, then, we have :

$$
\begin{aligned}
g(\varsigma,\bar{\varsigma}) = \bar{G}\tau_n \Bigg\{ & \varsigma^2 \left[H\bar{\gamma}^* - \frac{\varrho\gamma\bar{\gamma}^* - e\varrho\gamma}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r\bar{\gamma}^*}{k}\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} \right] \\
& + \varsigma\bar{\varsigma} \left[2H\bar{\gamma}^* + \frac{2e\varrho - 2\varrho\bar{\gamma}^*}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\,\mathrm{Re}(\gamma) \right. \\
& \left. - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{4x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} - \frac{2r\bar{\gamma}^*}{k}\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\varpi\pi_n\zeta}\right) \right] \\
& + \bar{\varsigma}^2 \left[H\bar{\gamma}^* - \frac{\varrho\bar{\gamma}^*\bar{\gamma} - e\varrho\bar{\gamma}}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r\bar{\gamma}^*}{k}\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} \right] \\
& + \varsigma^2\bar{\varsigma} \left[\left(2H\bar{\gamma}^* - \frac{r\bar{\gamma}^*}{k}\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{4x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} - \frac{\varrho\gamma\bar{\gamma}^* - e\varrho\gamma}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \right) \right. \\
& \times P_{11}^{(1)}(0) + \frac{(e\varrho - \varrho\bar{\gamma}^*)\,P_{11}^{(2)}(0)}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \\
& + \left(H\bar{\gamma}^* - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} - \frac{r\bar{\gamma}^*\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta}}{2k} + \frac{e\varrho\bar{\gamma} - \varrho\bar{\gamma}^*\bar{\gamma}}{4\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \right) P_{20}^{(1)}(0) \\
& \left. - \frac{r\bar{\gamma}^* P_{11}^{(1)}(-1)}{k} + \frac{(e\varrho - \varrho\bar{\gamma}^*)\,P_{20}^{(2)}(0)}{4\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r\bar{\gamma}^* P_{20}^{(1)}(-1)}{2k} \right] + \cdots \Bigg\}.
\end{aligned}
$$

By comparing the coefficients with (3.10), we can deduce:

$$
\begin{aligned}
g_{20} &= 2\bar{G}\tau_n \left[H\bar{\gamma}^* - \frac{\varrho\gamma\bar{\gamma}^* - e\varrho\gamma}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r\bar{\gamma}^*}{k}\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} \right], \\
g_{11} &= \bar{G}\tau_n \left[2H\bar{\gamma}^* + \frac{2e\varrho - 2\varrho\bar{\gamma}^*}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}\,\mathrm{Re}(\gamma) - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{4x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} \right. \\
& \left. - \frac{2r\bar{\gamma}^*}{k}\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\varpi\pi_n\zeta}\right) \right] \\
g_{02} &= 2\bar{G}\tau_n \left[H\bar{\gamma}^* - \frac{\varrho\bar{\gamma}^*\bar{\gamma} - e\varrho\bar{\gamma}}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r\bar{\gamma}^*}{k}\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} \right] \\
g_{21} &= 2\bar{G}\tau_n \left[\left(2H\bar{\gamma}^* - \frac{r\bar{\gamma}^*}{k}\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{4x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} - \frac{\varrho\gamma\bar{\gamma}^* - e\varrho\gamma}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \right) \right. \\
& \times P_{11}^{(1)}(0) + \frac{(e\varrho - \varrho\bar{\gamma}^*)\,P_{11}^{(2)}(0)}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \\
& + \left(H\bar{\gamma}^* - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} - \frac{r\bar{\gamma}^*\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta}}{2k} + \frac{e\varrho\bar{\gamma} - \varrho\bar{\gamma}^*\bar{\gamma}}{4\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} \right) P_{20}^{(1)}(0) \\
& \left. - \frac{r\bar{\gamma}^* P_{11}^{(1)}(-1)}{k} + \frac{(e\varrho - \varrho\bar{\gamma}^*)\,P_{20}^{(2)}(0)}{4\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r\bar{\gamma}^* P_{20}^{(1)}(-1)}{2k} \right]
\end{aligned}
$$

Given that $P_{20}(\zeta)$ and $P_{11}(\zeta)$ are present in g_{21}, it is necessary to calculate them.

Referring to (3.11) and (3.12), we observe that for $\zeta \in [-1,0)$, an expression can be stated as:

$$H(\varsigma,\bar{\varsigma},\zeta) = -2\operatorname{Re}\left\{\bar{\vartheta}^*(0)f(\varsigma,\bar{\varsigma})\vartheta(\zeta)\right\} = -g(\varsigma,\bar{\varsigma})\vartheta(\zeta) - \bar{g}(\varsigma,\bar{\varsigma})\bar{\vartheta}(\zeta). \tag{3.16}$$

Comparing with (3.13) yields:

$$H_{20}(\zeta) = -g_{20}\vartheta(\zeta) - \bar{g}_{02}\bar{\vartheta}(\zeta), \quad H_{11}(\zeta) = -g_{11}\vartheta(\zeta) - \bar{g}_{11}\bar{\vartheta}(\zeta). \tag{3.17}$$

It follows from (3.14) that

$$\begin{cases} \dot{P}_{20}(\zeta) = 2\mathrm{i}\varpi P_{20}(\zeta) + g_{20}\vartheta(\zeta) + \bar{g}_{02}\bar{\vartheta}(\zeta), \\ \dot{P}_{11}(\zeta) = g_{11}\vartheta(\zeta) + \bar{g}_{11}\bar{\vartheta}(\zeta). \end{cases} \tag{3.18}$$

Then, we obtain

$$\begin{cases} P_{20}(\zeta) = \frac{\mathrm{i}g_{20}}{\tau_n\varpi}\vartheta(0)\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta} + \frac{\mathrm{i}\bar{g}_{02}}{3\varpi\tau_n}\bar{\vartheta}(0)\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} + L_1\mathrm{e}^{2\mathrm{i}\varpi\tau_n\zeta}, \\ P_{11}(\zeta) = -\frac{\mathrm{i}g_{11}}{\tau_n\varpi}\vartheta(0)\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta} + \frac{\mathrm{i}\bar{g}_{11}}{\varpi\tau_n}\bar{\vartheta}(0)\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} + L_2. \end{cases} \tag{3.19}$$

In the subsequent discussion, we will look for suitable values for L_1 and L_2 in (3.19).

Referring to (3.11) and (3.15), we can express them as:

$$H_{20}(0) = -g_{20}\vartheta(0) - \bar{g}_{02}\bar{\vartheta}(0) + 2\tau_n\Im_1, \tag{3.20}$$

$$H_{11}(0) = -g_{11}\vartheta(0) - \bar{g}_{11}\bar{\vartheta}(0) + 2\tau_n\Im_2, \tag{3.21}$$

where

$$\Im_1 = \begin{bmatrix} \Im_1^{(1)} \\ \Im_1^{(2)} \end{bmatrix} = \begin{bmatrix} H - \frac{\varrho\gamma}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r}{k}\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta} \\ \frac{e\varrho\gamma}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} \end{bmatrix},$$

$$\Im_2 = \begin{bmatrix} \Im_2^{(1)} \\ \Im_2^{(2)} \end{bmatrix} = \begin{bmatrix} H - \frac{\varrho\operatorname{Re}(\gamma)}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{r}{k}\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta}\right) \\ \frac{e\varrho\operatorname{Re}(\gamma)}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2} - \frac{e\varrho y_0(1+3t_h\varrho\sqrt{x_0})}{8x_0^{\frac{3}{2}}(1+t_h\varrho\sqrt{x_0})^3} \end{bmatrix}.$$

Substituting (3.19)-(3.21) into (3.14) and noting that

$$\left(\mathrm{i}\varpi\tau_n I - \int_{-1}^{0}\mathrm{e}^{\mathrm{i}\varpi\tau_n\zeta}\mathrm{d}\eta(\zeta)\right)\vartheta(0) = 0,$$
$$\left(-\mathrm{i}\varpi\tau_n I - \int_{-1}^{0}\mathrm{e}^{-\mathrm{i}\varpi\tau_n\zeta}\mathrm{d}\eta(\zeta)\right)\vartheta(0) = 0,$$

we obtain

$$\begin{bmatrix} 2\mathrm{i}\varpi - a_{11} - b_{11}\mathrm{e}^{-2\mathrm{i}\varpi\tau_{\mathrm{n}}} & -a_{12} \\ -a_{21} & 2\mathrm{i}\varpi \end{bmatrix} L_1 = 2\Im_1, \tag{3.22}$$

$$\begin{bmatrix} -a_{11} - b_{11} & -a_{12} \\ -a_{21} & 0 \end{bmatrix} L_2 = 2\Im_2. \tag{3.23}$$

Obtaining L_1 and L_2 from (3.22) and (3.23) is a straightforward process, namely:

$$L_1^{(1)} = -\frac{4\Im_1^{(1)}\mathrm{i}\varpi + 2a_{12}\Im_1^{(2)}}{a_{12}a_{21} + 2b_{11}\mathrm{i}\varpi\mathrm{e}^{-2\mathrm{i}\varpi\tau_n} + 4\varpi^2 + 2a_{11}\mathrm{i}\varpi},$$

$$L_1^{(2)} = -\frac{2\Im_1^{(1)}a_{21} + \left(4\mathrm{i}\varpi - 2a_{11} - 2b_{11}\mathrm{e}^{-2\mathrm{i}\varpi\tau_n}\right)\Im_1^{(2)}}{a_{12}a_{21} + 2b_{11}\mathrm{i}\varpi\mathrm{e}^{-2\mathrm{i}\varpi\tau_n} + 4\varpi^2 + 2a_{11}\mathrm{i}\varpi},$$

$$L_2^{(1)} = -\frac{2\Im_2^{(2)}}{a_{21}}, \quad L_2^{(2)} = \frac{-2a_{21}\Im_2^{(1)} + 2\left(a_{11} + b_{11}\right)\Im_2^{(2)}}{a_{12}a_{21}}.$$

Therefore, we can compute the following values

$$\kappa_1(0) = \frac{\mathrm{i}}{2\varpi\tau_n}\left(g_{11}g_{20} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\chi_2 = -\frac{\mathrm{Re}\left\{\kappa_1(0)\right\}}{\mathrm{Re}\left\{\lambda'\left(\tau_n\right)\right\}}, \quad \alpha_2 = 2\,\mathrm{Re}\left\{\kappa_1(0)\right\},$$

$$T_2 = -\frac{\mathrm{Im}\left\{\kappa_1(0)\right\} + \wp_2\,\mathrm{Im}\left\{\lambda'\left(\tau_n\right)\right\}}{\varpi\tau_n},$$

The results obtained in the previous calculation determine the Hopf bifurcation orientation and the stability of bifurcated periodic solutions in the system (1.3) at the critical value τ_n.

Theorem 3.1. *(i) The orientation of the Hopf bifurcation relies on the sign of χ_2: if $\chi_2 > 0$, the bifurcation is identified as supercritical, whereas it is classified as subcritical when $\chi_2 < 0$.*

(ii) The stability of the bifurcated periodic solutions is contingent on the value of α_2: these solutions are stable when $\alpha_2 < 0$ and unstable when $\alpha_2 > 0$.

(iii) The period of the bifurcated periodic solutions is influenced by T_2: it increases with $T_2 > 0$ and decreases with $T_2 < 0$.

Remark 3.1. Drawing on the normal form introduced by [8], there is ample opportunity to explore the stability of periodic solutions and the orientation of Hopf bifurcation. We plan to delve into this matter in our upcoming research and regard it as a significant direction for future publications.

4. Control of bifurcation for uncontrolled system

A feedback controller has the ability to dynamically adjust control strategies in real-time based on the current state of the system, and to enhance adaptability to the dynamic variations in a predator-prey system. By continuously monitoring the system state and making adjustments, the feedback controller contributes to maintaining system stability, preventing the occurrence of unstable behaviors or system collapse in the predator-prey system. Additionally, the feedback controller can optimize system performance, ensuring the system achieves improved dynamic equilibrium in predator-prey interactions under different conditions, thereby enhancing overall system efficiency. In the ensuing discussion, we will introduce a feedback controller to transition the system from an unstable state to a stable one.

$$u_1(t) = k_1(x(t) - x_0), \tag{4.1}$$

where k_1 represents the feedback gains.

Alternatively, we can describe this by incorporating the controller $u_1(t)$ into the first equation of system (1.3), yielding:

$$\begin{cases} \frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t-\tau)}{k}\right) - \frac{\varrho\sqrt{x(t)}y(t)}{1+t_h\varrho\sqrt{x(t)}} - \frac{qE(t)x(t)}{m_1E(t)+m_2x(t)} - k_1(x(t) - x_0), \\ \frac{dy(t)}{dt} = -\beta y(t) + \frac{e\varrho\sqrt{x(t)}y(t)}{1+t_h\varrho\sqrt{x(t)}}, \\ 0 = \frac{qE(t)}{m_1E(t)+m_2x(t)}(px(t) - c) - m. \end{cases} \tag{4.2}$$

Theorem 4.1. *For system (4.2), When* $k_1 > 2\sqrt{\frac{e\varrho^2 y_0}{(1+t_h\varrho\sqrt{x_0})^3}} - \frac{rx_0}{k} + \frac{qpx_0E_0}{(px_0-c)(m_1E_0+m_2x_0)} + \frac{\varrho y_0(1+2t_h\varrho\sqrt{x_0})}{2\sqrt{x_0}(1+t_h\varrho\sqrt{x_0})^2}$*, the equilibrium point* Y_0 *of system (4.2) demonstrates asymptotic stability.*

The demonstration closely parallels the argumentation in Lemma 2.2 and Theorem 2.1, and is therefore omitted.

5. Numerical Simulation

In this section, we confirm the findings through simulation, employing the parameters listed below:

$$\begin{aligned} &r = 2, k = 8, t_h = 1, \beta = 2, \varrho = 1, q = 1, \\ &e = 3, c = 1, m_1 = 4, m_2 = 1, p = 1, k_1 = 1.3, m = \frac{1}{4}, \end{aligned} \tag{5.1}$$

then the system (1.3) becomes

$$\begin{cases} \frac{dx(t)}{dt} = 2x(t)\left(1 - \frac{x(t-\tau)}{8}\right) - \frac{\sqrt{x(t)}y(t)}{1+\sqrt{x(t)}} - \frac{E(t)x(t)}{4E(t)+x(t)}, \\ \frac{dy(t)}{dt} = -2y(t) + \frac{3\sqrt{x(t)}y(t)}{1+\sqrt{x(t)}}, \\ 0 = \frac{E(t)}{4E(t)+x(t)}(x(t) - 1) - \frac{1}{4}. \end{cases} \tag{5.2}$$

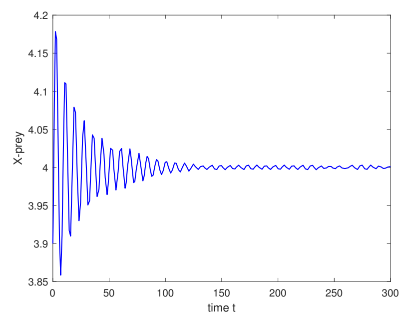
According to Sections 2 and 3, we have determined the stability of the positive equilibrium point and identified the occurrence of Hopf bifurcation. The exclusive positive equilibrium point for the model system (5.2) is denoted as $Y_0 = (4, 5.5, 0.5)$. Following calculations, we derived values of $\varpi^+ = 0.8455$ and $\varpi^- = 0.3613$. As outlined in Section 2, the critical values are $\tau^+ = 0.5978$ and $\tau^- = 1.3988$. In accordance with Theorem 2.1, the equilibrium point Y_0 demonstrates local asymptotic stability for $\tau \in [0, \tau_0^+) = [0, 0.5978)$ and instability when $\tau \in (\tau_0^+, \tau_0^-)$. Additionally, Hopf bifurcation is happened at $\tau = \tau_n^{\pm}, n = 0, 1, 2, \ldots, M$.

At $\tau = 0$, it is straightforward to show that the positive equilibrium point $Y_0 = (4, 5.5, 0.5)$ exhibits asymptotic stability.

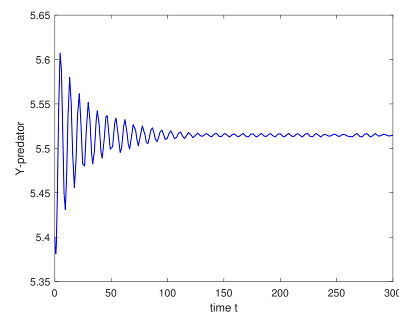
Next, we determined the direction of a Hopf bifurcation at $\tau_0 = \tau_0^+ = 0.5978$ and explored additional characteristics of periodic solutions based on the theory established by Hassard et al. [12]. Utilizing mathematical tools for computation, the resulting numerical values are as follows:

$$\kappa_1(0) = 0.0023 + 0.0012i, \quad \lambda^{'}(\tau_n) = 0.3941 - 1.8822i. \tag{5.3}$$

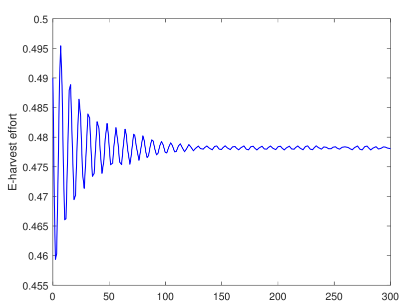
Therefore, we obtain $\chi_2 = -0.0058 < 0$, $\alpha_2 = 0.0046 > 0$, and $T_2 = 0.0194 > 0$. Utilizing these numerical results in conjunction with Theorem 3.1, we infer that the Hopf bifurcation in system (5.2) at $\tau_0 = 0.5978$ is subcritical. The bifurcated periodic solution emerges as τ transitions to the left of τ_0, and the resulting periodic solution is unstable.



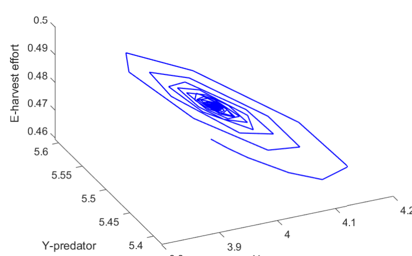
(a)



(b)



(c)



(d)

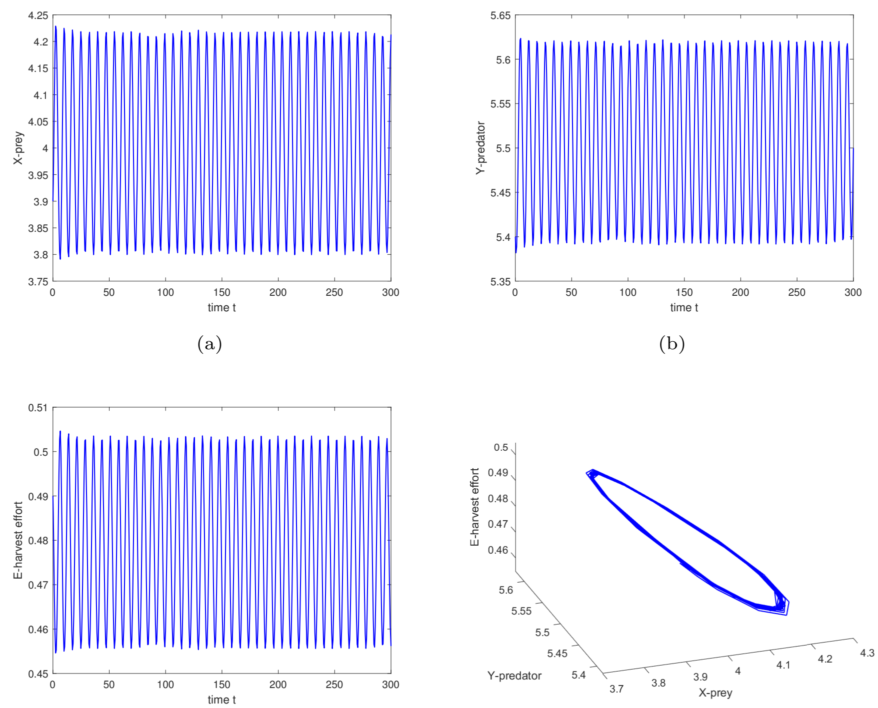
Figure 1. Under the condition $\tau = 0.45 < \tau_0$, the positive equilibrium point Y_0 demonstrates local asymptotic stability, considering the special initial conditions $x_0 = 3.9$, $y_0 = 5.4$, $E_0 = 0.49$.

The simulation outcomes can be succinctly summarized as follows:

(i) When $\tau = 0.45 < \tau_0$, the positive equilibrium point Y_0 demonstrates local asymptotic stability (refer to Fig.1).

(ii) At $\tau = 0.595 < \tau_0$, periodic solutions emerge at the positive equilibrium points Y_0 (refer to Fig.2).

(iii) When $k_1 = 1.3 > 0.98$, the controller effectively changes the hopf bifurcation behavior of the system, causing the system to transition from an unstable state to a stable state. The positive equilibrium point Y_0 exhibits local asymptotic stability (refer to Fig.3).



(a) (b) (c) (d)

Figure 2. At $\tau = 0.595 < \tau_0$, periodic solutions manifest at the positive equilibrium points Y_0 under the provided initial conditions $x_0 = 3.9$, $y_0 = 5.4$, $E_0 = 0.49$.

The simulation results reveal that the stability of the system (1.3) undergoes a switch with the variation of the parameter τ. Therefore, it is imperative for the government to adjust tax rates, formulate preferential policies, encourage fisheries production, and mitigate environmental pollution. These measures aim to maintain the ecological and economic differential-algebraic system (1.3) in a stable state, fostering the continued stable development of the ecosystem.

This study emphasizes the stability and Hopf bifurcation in a delayed predator-prey system with nonlinear predation and square root functional response, holding significant implications for ecology and biology. The results provide valuable insights into the dynamics of interacting populations, contributing to a deeper understanding of ecological and biological systems.

The impact of time delay on population dynamics is well-established. According to the theorems and simulation results, time delay in the range of $0 < \tau < \tau_0$ leads to system stability. This stability is reflected in balanced population densities of predators and prey, as well as consistent predation. A stable predator-prey system promotes ecological equilibrium, protecting the overall structure and function of the ecosystem.

The study thoroughly investigates the influence of time delay on system stability. According to Theorem 2.1 and numerical simulation results, time delay at the critical delay value ($\tau_0 = 0.5978$) induces oscillatory behavior in the system. In ecological systems, both Hopf bifurcation and instability are unsatisfactory status.

To address instability, a feedback controller is introduced, effectively transforming the system from an unstable to a stable state. By dynamically adjusting the prey population, the feedback controller contributes to maintaining a relatively stable ecosystem.

In conclusion, the investigation into the stability and Hopf bifurcation within delayed predator-prey systems offers valuable ecological insights, shedding light on the adaptive responses of biological systems to environmental changes. The significance of these findings spans various disciplines, encompassing ecology, conservation biology, and the sustainable management of resources.

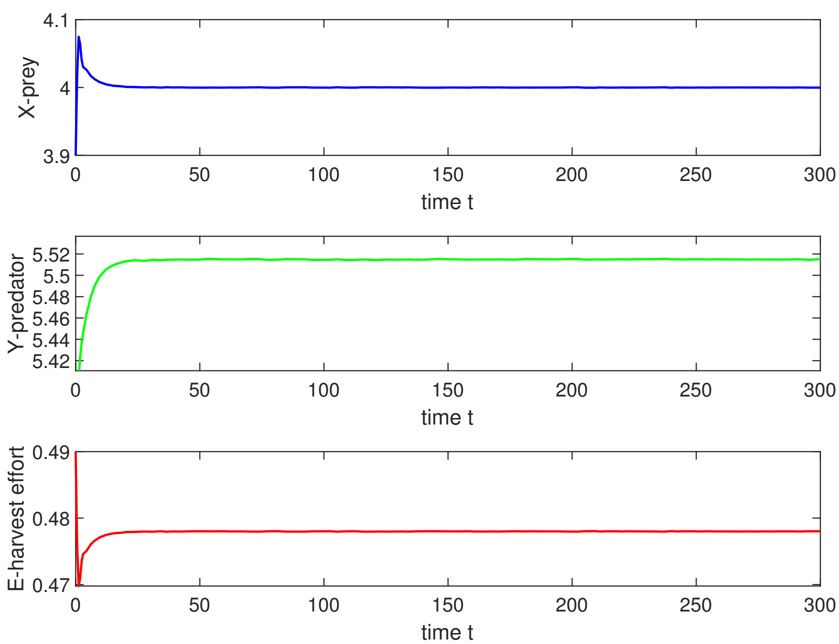


Figure 3. When $k_1 = 1.3 > 0.98$, the controller effectively changes the hopf bifurcation behavior of the system, causing the system to transition from an unstable state to a stable state. The positive equilibrium point Y_0 exhibits local asymptotic stability with the specified initial conditions $x_0 = 3.9$, $y_0 = 5.4$, $E_0 = 0.49$.

6. Discussion

This research delves into the dynamics of differential-algebraic predator-prey systems featuring nonlinear prey harvesting, with a particular emphasis on capturing the realism of nonlinear interactions. The study systematically explores the role of time delay as a bifurcation parameter, shedding light on its impact on the stability of the system. The noteworthy findings center around the emergence of Hopf bifurcation, leading to a transition from system stability to instability at the internal equilibrium point Y_0.

The principal contributions of this study encompass the incorporation of time delay, an in-depth investigation of the dynamics surrounding Hopf bifurcation, and the introduction of a feedback controller. In the future research, we could consider extending the model to incorporate nonlinear predator harvesting, refining its practical applicability, and exploring more advanced control strategies.

This research significantly contributes to the understanding of complex ecological systems and opens up avenues for further exploration at the intersection of mathematical modeling, ecology, and control theory.

In the future, we can combine time delay factors with other ecological factors (such as spatial heterogeneity and environmental changes) to study their comprehensive effects in predator-prey systems.

We can apply our understanding of time delay to practices such as resource management, biodiversity conservation, and ecosystem services to promote sustainable development and ecosystem protection.

We can consider time delay in complex networks and study the dynamic behavior of predator-prey systems in multi-level and multi-scale network structures.

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