Global smooth solution for phase transition equations in ferromagnetism

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Abstract. This paper presents a phase transition model that characterizes the thermodynamic and electromagnetic properties of ferromagnetic materials. We establish the existence of both a global weak solution and a global smooth solution for the phase transition equations in two and three dimensions.

Keywords: Phase transition equations, Galerkin method, Weak solution, Global smooth solution.

1 Introduction

It is widely recognized that magnetism can be divided into diamagnetism, paramagnetism, ferromagnetism, antiferromagnetism, and ferrimagnetism. Notably, ferromagnetism is predominantly observed in metals such as iron, cobalt, nickel, and various alloys containing these elements. This type of magnetism occurs due to the spontaneous alignment of magnetic moments, resulting in a strong and persistent magnetic field. This property makes these metals and alloys invaluable in various electromagnetic applications, such as motors and generators [1]. The most fundamental characteristic of ferromagnetic materials is the existence of spontaneous magnetization. The theory of spontaneous magnetization reveals the nature of numerous ferromagnetic properties, including the influence of temperature on ferromagnetism. As temperature increases, the distance between atoms grows, thereby decreasing the atomic exchange interaction. The distance between atoms increases when the temperature increases, which reduces the exchange action of atoms. Meanwhile, the thermodynamic motion destroys the regular orientation of the spin magnetic moments continuously, thus causing a decrease in spontaneous magnetization. Eventually, when the temperature surpasses Curie temperature θ_c, the spontaneous magnetic moment vanishes, and the material transitions from being ferromagnetic to paramagnetic [2].

In this paper, we are concerned with the theory of the paramagnetic-ferromagnetic transition [3–8]. Our investigation originates from the paper [6], in which the authors proposed a phase transition model that describes the paramagnetic-ferromagnetic transition in ferromagnetic materials and established the existence and uniqueness of weak solutions in dimensions three. The phase transition equations governing the

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evolution of the ferromagnetic material reads

$$\gamma\mathbf{M}_t = \nu\Delta\mathbf{M} - \theta_c(|\mathbf{M}|^2 - 1)\mathbf{M} - \theta\mathbf{M} + \mathbf{H}, \quad \text{in } Q_T, \tag{1.1}$$

$$c_1(\ln\theta)_t + c_2\theta_t = k_0\Delta(\ln\theta) + \mathbf{M}\cdot\mathbf{M}_t + k_1\Delta\theta + \hat{r}, \quad \text{in } Q_T, \tag{1.2}$$

where $\mathbf{M}$ denotes the magnetization vector, θ represents the absolute temperature, and $Q_T = \Omega\times[0,T]$ with $T > 0$. The constants $\gamma, \nu, c_1, c_2, k_0, k_1$ are strictly positive, while θ_c is the Curie temperature. Additionally, $\hat{r}$ is a known function of x, t. Our focus extends to combining this model with Maxwell's equations

$$\mu\mathbf{H}_t + \mathbf{M}_t = -\nabla\times\mathbf{E}, \tag{1.3}$$

$$\mathbf{E}_t + \sigma\mathbf{E} = \nabla\times\mathbf{H}, \tag{1.4}$$

$$\nabla\cdot(\mu\mathbf{H} + \mathbf{M}) = 0, \ \nabla\cdot\mathbf{E} = 0, \tag{1.5}$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{E}$ is the electric field, μ and σ are respectively the magnetic permeability and the conductivity. The existence and uniqueness of the global weak solution for (1.1)-(1.5) were proved in [6] without the displacement current $\partial_t E$. Some limiting problems for this model were explored in [9], and the fractional version of the model was studied in [10].

In this paper, we will consider the global smooth solution of (1.1)-(1.5) with the inclusion of the current $\partial_t E$ in two and three dimensions. We assume that $c_1 = k_0 = 0$. This assumption means that the heat conductivity and specific heat are dependent on the absolute temperature according to the laws: $k(\theta) = k_1\theta$ and $c(\theta) = \frac{c_2}{2}\theta^2$. We consider the phase transition equations

$$\gamma\mathbf{M}_t = \nu\Delta\mathbf{M} - \theta_c(|\mathbf{M}|^2 - 1)\mathbf{M} - \theta\mathbf{M} + \mathbf{H}, \tag{1.6}$$

$$c\theta_t = \mathbf{M}\cdot\mathbf{M}_t + k\Delta\theta + \hat{r}, \tag{1.7}$$

$$\mu\mathbf{H}_t + \mathbf{M}_t = -\nabla\times\mathbf{E}, \tag{1.8}$$

$$\mathbf{E}_t + \sigma\mathbf{E} = \nabla\times\mathbf{H}, \tag{1.9}$$

$$\nabla\cdot(\mu\mathbf{H} + \mathbf{M}) = 0, \ \nabla\cdot\mathbf{E} = 0, \tag{1.10}$$

with the periodic conditions

$$\begin{aligned} &\mathbf{M}(x + 2De_i, t) = \mathbf{M}(x,t), \ \theta(x + 2De_i, t) = \theta(x,t), \\ &\mathbf{H}(x + 2De_i, t) = \mathbf{H}(x,t), \ \mathbf{E}(x + 2De_i, t) = \mathbf{E}(x,t) \end{aligned} \tag{1.11}$$

and the initial conditions

$$\mathbf{M}(x,0) = \mathbf{M}_0(x), \ \theta(x,0) = \theta_0(x), \ \mathbf{H}(x,0) = \mathbf{H}_0(x), \ \mathbf{E}(x,0) = \mathbf{E}_0(x), \ \ x\in\Omega\in\mathbb{R}^d, \tag{1.12}$$

where $\Omega = \prod_{j=1}^{d}(-D,D), d = 2,3$.

In view of the equation (1.10), we should impose the following constraints on the initial functions $\mathbf{M}_0, \mathbf{H}_0$ and $\mathbf{E}_0$:

$$\nabla\cdot(\mu\mathbf{H}_0 + \mathbf{M}_0) = 0, \ \ \nabla\cdot\mathbf{E}_0 = 0, \tag{1.13}$$

Inspired by the ideas presented in [9–16], we aim to study the existence of the global weak solution and the global smooth solution for the phase transition equations (1.6)–(1.12). Initially, we construct the solutions of the equations (1.6)–(1.9) with (1.11) (1.12). Subsequently, we demonstrate that these constructed solutions fulfill equation (1.10) when subject to the condition specified in (1.13). As a result, we establish the existence of solutions for the problem (1.6)–(1.12). To the best of our knowledge, there are currently no available results on global smooth solutions related to this problem (1.6)–(1.12).

The main results are as follows:

Theorem 1.1. *Assume that* $(\mathbf{M}_0, \theta_0, \mathbf{H}_0, \mathbf{E}_0) \in (H^1(\Omega), L^2(\Omega), L^2(\Omega), L^2(\Omega)), \hat{r}(x,t) \in L^2(0,T;\, H^1(\Omega)), \Omega \subset \mathbb{R}^d, d = 2,3$ *and* (1.13) *is satisfied. The constants* μ, σ *are positive. Then the problem* (1.6)-(1.12) *has at least one global weak solution* $(\mathbf{M}(x,t), \theta(x,t), \mathbf{H}(x,t), \mathbf{E}(x,t))$ *such that*

$$\begin{aligned}
&\mathbf{M}(x,t) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\\
&\theta(x,t) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
&\mathbf{H}(x,t) \in L^\infty(0,T;L^2(\Omega)),\\
&\mathbf{E}(x,t) \in L^\infty(0,T;L^2(\Omega)).
\end{aligned}$$

Theorem 1.2. *Assume* $(\mathbf{M}_0, \theta_0, \mathbf{H}_0, \mathbf{E}_0) \in (H^{m+1}(\Omega), H^m(\Omega), H^m(\Omega), H^m(\Omega)), \hat{r}(x,t) \in L^\infty(0,T;H^m(\Omega)), m \geq 1, \Omega \subset \mathbb{R}^2$, *and* (1.13) *is satisfied. The constants* μ, σ *are positive. Then there exists a unique global solution* $(\mathbf{M}(x), \theta(x), \mathbf{H}(x), \mathbf{E}(x))$ *of the periodic problem* (1.6)-(1.12) *and for any* $T > 0$, *satisfying*

$$\begin{aligned}
&\mathbf{M}(x,t) \in L^\infty(0,T;H^{m+1}(\Omega)) \cap L^2(0,T;H^{m+2}(\Omega)),\\
&\theta(x,t) \in L^\infty(0,T;H^m(\Omega)) \cap L^2(0,T;H^{m+1}(\Omega)),\\
&\mathbf{H}(x,t) \in L^\infty(0,T;H^m(\Omega)),\\
&\mathbf{E}(x,t) \in L^\infty(0,T;H^m(\Omega)).
\end{aligned}$$

Additionally, the above results still hold for $d = 3$ *when* $\|\mathbf{M}_0\|_{H^1}^2 \leq \delta_0, \delta_0 \ll 1$.

This paper is organized as follows. In the next section, we will provide the definition of a weak solution for the phase transition equations, and establish the existence of the global weak solution by the Galerkin method. In Section 3, by employing a priori estimates, we obtain the existence of the smooth solution for the phase transition equations. In the last section, we show that the global solution of problem (1.6)-(1.12) is unique.

2 The existence of global weak solution

In this section, we will establish the existence of a global weak solution to the phase transition equations. Firstly, we construct the Galerkin approximate solutions of the problem (1.6)-(1.12), and establish a priori uniform estimates of these solutions. Then we provide the proof of the existence of the generalized solutions to the problem (1.6)-(1.12). Thus, Theorem 1.1 holds.

First, we introduce the definition of the weak solution to the problem (1.6)-(1.12).

Definition 2.1. *A three-dimensional vector function* $(\mathbf{M}(x,t), \theta(x,t), \mathbf{H}(x,t), \mathbf{E}(x,t)) \in (L^\infty(0,T;H^1(\Omega)), L^\infty(0,T;L^2(\Omega)), L^\infty(0,T;L^2(\Omega)), L^\infty(0,T;L^2(\Omega)))$ *is called a weak solution to* (1.6)-(1.12)*, if for any vector-valued test function* $\phi(x,t) \in C^1([0,T];H^2(\Omega))$ *with* $\phi(x,t)|_{t=T} = 0$, *and for any scalar test function* $\xi(x,t) \in C^1([0,T];C^1(\Omega))$*, the following equations hold:*

$$\begin{aligned}
&\gamma \iint_{Q_T} \mathbf{M} \cdot \phi_t dxdt - \nu \iint_{Q_T} \nabla\mathbf{M} \cdot \nabla\phi dxdt - \iint_{Q_T} \theta_c(|\mathbf{M}|^2 - 1)\mathbf{M} \cdot \phi dxdt \\
&\quad - \iint_{Q_T} \theta\mathbf{M} \cdot \phi dxdt + \iint_{Q_T} \mathbf{H} \cdot \phi dxdt + \gamma \int_\Omega \mathbf{M}_0 \cdot \phi(x,0)dx = 0,
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&c \iint_{Q_T} \theta \cdot \phi_t dxdt + \iint_{Q_T} \mathbf{M} \cdot \mathbf{M}_t \cdot \phi dxdt - k \iint_{Q_T} \nabla\theta \cdot \nabla\phi dxdt \\
&\quad + \iint_{Q_T} \hat{r} \cdot \phi dxdt + c \int_\Omega \theta_0 \cdot \phi(x,0)dx = 0,
\end{aligned} \tag{2.2}$$

$$\iint_{Q_T} (\mu\mathbf{H} + \mathbf{M}) \cdot \phi_t dxdt - \iint_{Q_T} (\nabla \times \phi) \cdot \mathbf{E} dxdt + \int_\Omega (\mu\mathbf{H}_0 + \mathbf{M}_0) \cdot \phi(x,0) dx = 0, \tag{2.3}$$

$$\iint_{Q_T} \mathbf{E} \cdot \phi_t(x,t) e^{\sigma t} dxdt + \iint_{Q_T} e^{\sigma t} (\nabla \times \phi) \cdot \mathbf{H} dxdt + \int_\Omega \mathbf{E}_0(x)\phi(x,0) dx = 0, \tag{2.4}$$

$$\iint_{Q_T} \nabla\xi \cdot (\mu\mathbf{H} + \mathbf{M})\, dxdt = 0, \quad \iint_{Q_T} \nabla\xi \cdot \mathbf{E}\, dxdt = 0, \tag{2.5}$$

$$\mathbf{M}(x,0) = \mathbf{M}_0(x), \quad \theta(x,0) = \theta_0(x), \quad \mathbf{H}(x,0) = \mathbf{H}_0(x), \quad \mathbf{E}(x,0) = \mathbf{E}(x), \qquad x \in \Omega. \tag{2.6}$$

Next, to solve the equations (1.6)-(1.12), we use the Galerkin method. First, we establish a priori estimates for the approximate solutions of (1.6)-(1.12).

Let $\omega_n(x), n = 1, 2, \cdots$ be the unit eigenfunctions satisfying the equations

$$\Delta\omega_n + \lambda_n\omega_n = 0, \ \ \omega_n(x - De_i) = \omega_n(x + De_i), \ \ i = 1, 2, ..., d,$$

where λ_n, $n = 1, 2, \cdots$ are the corresponding eigenvalues that are different from each other. The set $\{\omega_n(x)\}$ consists of an orthogonal normal basis of $L^2(\Omega)$.

Denote the approximate solution of the problem (1.6)-(1.9) by $\mathbf{M}_N(x,t), \theta_N(x,t), \mathbf{H}_N(x,t), \mathbf{E}_N(x,t)$ in the following form

$$\mathbf{M}_N(x,t) = \sum_{s=1}^{N} \alpha_{sN}(t)\omega_s(x), \quad \theta_N(x,t) = \sum_{s=1}^{N} \beta_{sN}(t)\omega_s(x),$$

$$\mathbf{H}_N(x,t) = \sum_{s=1}^{N} \gamma_{sN}(t)\omega_s(x), \quad \mathbf{E}_N(x,t) = \sum_{s=1}^{N} \zeta_{sN}(t)\omega_s(x),$$

where $\alpha_{sN}(t)$, $\beta_{sN}(t), \gamma_{sN}(t), \zeta_{sN}(t)$, $s = 1, 2, \cdots, N, N = 1, 2, \cdots$ satisfy the following system of ordinary differential equations

$$\int_\Omega \Big[\gamma\mathbf{M}_{Nt}\omega_s(x) + \nu\nabla\mathbf{M}_N\nabla\omega_s(x) + \theta_c(|\mathbf{M}_N|^2 - 1)\mathbf{M}_N\omega_s(x) + \theta_N\mathbf{M}_N\omega_s(x) - \mathbf{H}_N\omega_s(x)\Big]dx = 0, \tag{2.7}$$

$$\int_\Omega \Big[c\theta_{Nt}\omega_s(x) - \mathbf{M}_N \cdot \mathbf{M}_{Nt}\omega_s(x) + k\nabla\theta_N\nabla\omega_s(x) - \hat{r}w_s(x)\Big]dx = 0, \tag{2.8}$$

$$\int_\Omega \Big[\mu\mathbf{H}_{Nt}\omega_s(x) + \mathbf{M}_{Nt}\omega_s(x) + (\nabla \times \mathbf{E}_N)\omega_s(x)\Big]dx = 0, \tag{2.9}$$

$$\int_\Omega \Big[\mathbf{E}_{Nt}\omega_s(x) + \sigma\mathbf{E}_N\omega_s(x) - (\nabla \times \mathbf{H}_N)\omega_s(x)\Big]dx = 0 \tag{2.10}$$

and the initial conditions

$$\begin{aligned}
\alpha_{sN}(0) &= \int_\Omega \mathbf{M}_N(x,0)\omega_s(x)dx = \int_\Omega \mathbf{M}_0(x)\omega_s(x)dx = \alpha_{0s}, \\
\beta_{sN}(0) &= \int_\Omega \theta_N(x,0)\omega_s(x)dx = \int_\Omega \theta_0(x)\omega_s(x)dx = \beta_{0s}, \\
\gamma_{sN}(0) &= \int_\Omega \mathbf{H}_N(x,0)\omega_s(x)dx = \int_\Omega \mathbf{H}_0(x)\omega_s(x)dx = \gamma_{0s}, \\
\zeta_{sN}(0) &= \int_\Omega \mathbf{E}_N(x,0)\omega_s(x)dx = \int_\Omega \mathbf{E}_0(x)\omega_s(x)dx = \zeta_{0s}.
\end{aligned} \tag{2.11}$$

Obviously, there holds

$$\begin{aligned}
&\int_\Omega \mathbf{M}_{Nt}\omega_s(x)dx = \alpha'_{sN}(t), \ \ \int_\Omega \theta_{Nt}\omega_s(x)dx = \beta'_{sN}(t), \\
&\int_\Omega \mathbf{H}_{Nt}\omega_s(x)dx = \gamma'_{sN}(t), \ \ \int_\Omega \mathbf{E}_{Nt}\omega_s(x)dx = \zeta'_{sN}(t).
\end{aligned} \tag{2.12}$$

For simplicity, we introduce notation as follows

$$\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p, \quad p \geq 2. \tag{2.13}$$

It follows from the standard theory on nonlinear ordinary differential equations that the problem (2.7)-(2.11) admits a unique local solution. In order to obtain the existence and uniqueness of the solution of (2.7)-(2.11), we require additional estimates as follows.

Lemma 2.1. *Assume that* $(\mathbf{M}_0(x), \theta_0(x), \mathbf{H}_0(x), \mathbf{E}_0(x)) \in (H^1(\Omega), L^2(\Omega), L^2(\Omega), L^2(\Omega))$*,* $\hat{r}(x,t) \in L^2(0,T; H^1(\Omega))$*. For the solutions to the initial value problem* (2.7)-(2.11)*, we have the following estimates*

$$\sup_{0\leq t\leq T}\left\{\|\mathbf{M}_N\|_{H^1} + \|\theta_N\|_2 + \|\mathbf{H}_N\|_2 + \|\mathbf{E}_N\|_2\right\} \leq C, \tag{2.14}$$

$$\int_0^T [\|\mathbf{M}_{Nt}\|_2^2 + \|\nabla\theta_N\|_2^2 + \|\mathbf{E}_N\|_2^2]dt \leq C, \tag{2.15}$$

where C is a constant which is independent of N and D.

Proof. Multiplying equation (2.7) by $\alpha'_{sN}(t)$ for each s from 1 to N, and then summing up all the results, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big[\nu\|\nabla\mathbf{M}_N\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}_N\|_4^4\Big] + \gamma\|\mathbf{M}_{Nt}\|_2^2 + \int_\Omega \theta_N\mathbf{M}_N\mathbf{M}_{Nt}dx - \theta_c\int_\Omega \mathbf{M}_N\mathbf{M}_{Nt}dx - \int_\Omega \mathbf{H}_N\mathbf{M}_{Nt}dx = 0. \tag{2.16}$$

By taking the scalar product of $\beta_{sN}(t)$ with (2.8), and then summing up the outcomes for all $s = 1, 2, \cdots, N$, we derive

$$\frac{c}{2}\frac{d}{dt}\|\theta_N\|_2^2 + k\|\nabla\theta_N\|_2^2 - \int_\Omega \theta_N\mathbf{M}_N\mathbf{M}_{Nt}dx - \int_\Omega \hat{r}\theta_N dx = 0. \tag{2.17}$$

By taking the scalar product of $\gamma_{sN}(t)$ with (2.9) and the scalar product of $\zeta_{sN}(t)$ with (2.10) respectively, adding the two resulting equalities together, and subsequently summing up the outcomes for all $s = 1, 2, ..., N$, we have

$$\frac{1}{2}\frac{d}{dt}\Big[\mu\|\mathbf{H}_N\|_2^2 + \|\mathbf{E}_N\|_2^2\Big] + \sigma\|\mathbf{E}_N\|_2^2 + \int_\Omega \mathbf{H}_N\mathbf{M}_{Nt}dx = 0. \tag{2.18}$$

Adding (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\nu\|\nabla\mathbf{M}_N\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}_N\|_4^4 + c\|\theta_N\|_2^2 + \mu\|\mathbf{H}_N\|_2^2 + \|\mathbf{E}_N\|_2^2\Big] + \gamma\|\mathbf{M}_{Nt}\|_2^2 + k\|\nabla\theta_N\|_2^2 + \sigma\|\mathbf{E}_N\|_2^2\\
&= \theta_c\int_\Omega \mathbf{M}_N\mathbf{M}_{Nt}dx + \int_\Omega \hat{r}\theta_N dx\\
&\leq \frac{\gamma}{2}\|\mathbf{M}_{Nt}\|_2^2 + \frac{\theta_c^2}{2\gamma}\|\mathbf{M}_N\|_2^2 + \frac{k}{2}\|\theta_N\|_2^2 + \frac{1}{2k}\|\hat{r}\|_2^2,
\end{aligned} \tag{2.19}$$

and then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\nu\|\nabla\mathbf{M}_N\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}_N\|_4^4 + c\|\theta_N\|_2^2 + \mu\|\mathbf{H}_N\|_2^2 + \|\mathbf{E}_N\|_2^2\Big] + \frac{\gamma}{2}\|\mathbf{M}_{Nt}\|_2^2 + k\|\nabla\theta_N\|_2^2 + \sigma\|\mathbf{E}_N\|_2^2\\
&\leq C(1 + \|\mathbf{M}_N\|_4^4 + \|\theta_N\|_2^2 + \|\hat{r}\|_2^2).
\end{aligned} \tag{2.20}$$

By Gronwall inequality, we can obtain the estimates (2.14) and (2.15). This completes the proof of lemma 2.1. □

Remark 2.1. *In fact, by the equation* (2.7) *and estimates* (2.14) *and* (2.15)*, we can derive* $\int_0^T \|\Delta \mathbf{M}_N\|_2^2 dx \leq$ *C easily.*

Lemma 2.2. *Under the conditions of Lemma 2.1, for the solution* $(\mathbf{M}_N, \theta_N, \mathbf{H}_N, \mathbf{E}_N)$ *of the initial value problem* (2.7)-(2.11)*, we have the following estimates*

$$\|\mathbf{M}_{Nt}\|_{H^{-1}(\Omega)} + \|\mathbf{H}_{Nt}\|_{H^{-1}(\Omega)} + \|\mathbf{E}_{Nt}\|_{H^{-1}(\Omega)} \leq C, \tag{2.21}$$

$$\int_0^T \|\theta_{Nt}\|_{H^{-1}(\Omega)}^2 dx \leq C, \tag{2.22}$$

where C is independent of N and D, and $H^{-m}(\Omega)$ *denotes the dual space of the space* $H^m(\Omega)$*.*

Proof. For any function $\varphi \in H^2$, φ can be represented as

$$\varphi = \varphi_N + \overline{\varphi}_N, \quad \varphi_N = \sum_{s=1}^{N} \beta_s \omega_s(x), \quad \overline{\varphi}_N = \sum_{s=N+1}^{\infty} \beta_s \omega_s(x).$$

For $s \geq N+1$, we have $\int_\Omega \mathbf{M}_{Nt} \omega_s(x) dx = 0$. Then by Lemma 2.1, we get

$$\begin{aligned}
&\int_\Omega \mathbf{M}_{Nt} \varphi dx = \int_\Omega \mathbf{M}_{Nt} \varphi_N(x) dx \\
&= -\frac{\nu}{\gamma} \int_\Omega \nabla \mathbf{M}_N \nabla \varphi_N dx - \frac{\theta_c}{\gamma} \int_\Omega (|\mathbf{M}_N|^2 - 1) \mathbf{M}_N \varphi_N dx - \frac{1}{\gamma} \int_\Omega \theta_N \mathbf{M}_N \varphi_N dx + \frac{1}{\gamma} \int_\Omega \mathbf{H}_N \varphi_N dx \\
&\leq C\Big[\|\nabla \mathbf{M}_N\|_2 \|\nabla \varphi_N\|_2 + (\|\mathbf{M}_N\|_6^6 + \|\mathbf{M}_N\|_2^2) \|\varphi_N\|_2 + \|\mathbf{M}_N\|_4 \|\theta_N\|_2 \|\varphi_N\|_4 + \|\mathbf{H}_N\|_2 \|\varphi_N\|_2\Big] \\
&\leq C \|\varphi_N\|_{H^1(\Omega)} \leq C \|\varphi\|_{H^1(\Omega)}.
\end{aligned}$$

Similarly, for $s \geq N+1$, we have $\int_\Omega \mathbf{H}_{Nt} \cdot \omega_s(x) dx = 0$, $\int_\Omega \mathbf{E}_{Nt} \cdot \omega_s(x) dx = 0$. Then by Lemma 2.1, we deduce that

$$\begin{aligned}
\mu \int_\Omega \mathbf{H}_{Nt} \phi dx = \mu \int_\Omega \mathbf{H}_{Nt} \phi_N dx &= \mu \int_\Omega (\nabla \times \mathbf{E}_N \cdot \varphi_N - \mathbf{M}_{Nt} \varphi_N) dx \\
&\leq C(\|\mathbf{E}_N\|_2 \|\nabla \varphi_N\|_2 + \|\varphi\|_{H^1(\Omega)}) \\
&\leq C(\|\nabla \phi_N\|_2 + \|\varphi\|_{H^1(\Omega)}) \leq C_1 \|\varphi\|_{H^1(\Omega)}. \\
\int_\Omega \mathbf{E}_{Nt} \phi dx = \int_\Omega \mathbf{E}_{Nt} \phi_N dx &\leq C(\|\mathbf{H}_N\|_2 + \|\mathbf{E}_N\|_2)(\|\nabla \varphi_N\|_2 + \|\varphi_N\|_2) \\
&\leq C(\|\nabla \phi_N\|_2 + \|\varphi_N\|_2) \leq C_2 \|\varphi\|_{H^1(\Omega)}.
\end{aligned}$$

The above inequalities indicate that (2.21) holds true.

Let $\Phi \in L^2(0, T; H^1(\Omega))$, by (2.8) and Lemma 2.1, we have

$$\begin{aligned}
\int_0^T \int_\Omega \theta_{Nt} \Phi dx dt &\leq \frac{1}{c} \int_0^T \|\mathbf{M}_N\|_4 \|\mathbf{M}_{Nt}\|_2 \|\Phi\|_4 + k \|\nabla \theta_N\|_2 \|\nabla \Phi\|_2 + \|\hat{r}\|_2 \|\Phi\|_2 dt \\
&\leq C \int_0^T \|\Phi\|_{H^1(\Omega)}^2 dt,
\end{aligned}$$

where C is a constant independent of N. The lemma is proved. □

Lemma 2.3. *Assume that the conditions presented in Lemma 2.1 are satisfied. For the solution* $(\mathbf{M}_N(x,t), \theta_N(x,t),$ $\mathbf{H}_N(x,t), \mathbf{E}_N(x,t))$ *of the initial value problem* (2.7)-(2.11)*, there are the following estimates*

$$\|\mathbf{M}_N(\cdot,t_1) - \mathbf{M}_N(\cdot,t_2)\|_2 \le C|t_1 - t_2|^{\frac{1}{2}}, \ \forall t_1, t_2 \ge 0,$$

$$\|\theta_N(\cdot,t_1) - \theta_N(\cdot,t_2)\|_{H^{-1}} + \|\mathbf{H}_N(\cdot,t_1) - \mathbf{H}_N(\cdot,t_2)\|_{H^{-1}} + \|\mathbf{E}_N(\cdot,t_1) - \mathbf{E}_N(\cdot,t_2)\|_{H^{-1}} \le C|t_1 - t_2|^{\frac{1}{2}},$$
$$\forall t_1, t_2 \ge 0,$$

where the constant C is independent of N and D.

Proof. By Lemma 2.2 and Hölder inequality, we have

$$\begin{aligned}\|\mathbf{M}_N(\cdot,t_1) - \mathbf{M}_N(\cdot,t_2)\|_2 =& \Big\| \int_{t_1}^{t_2} \mathbf{M}_{Nt} dt \Big\|_2 \le \int_{t_1}^{t_2} \|\mathbf{M}_{Nt}\|_2 dt \\ \le& |t_2 - t_1|^{\frac{1}{2}} \Big(\int_0^T \int_\Omega |\mathbf{M}_{Nt}|^2 dx dt \Big)^{\frac{1}{2}} \\ \le& C|t_2 - t_1|^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\|\theta_N(\cdot,t_1) - \theta_N(\cdot,t_2)\|_{H^{-1}} =& \Big\| \int_{t_1}^{t_2} \dot{\theta}_N dt \Big\|_{H^{-1}} \le \int_{t_1}^{t_2} \|\dot{\theta}_N\|_{H^{-1}} dt \\ \le& |t_2 - t_1|^{\frac{1}{2}} \Big(\int_0^T \|\dot{\theta}_N\|_{H^{-1}}^2 dt \Big)^{\frac{1}{2}} \\ \le& C|t_2 - t_1|^{\frac{1}{2}}.\end{aligned}$$

Similarly, we have

$$\|H_N(\cdot,t_1) - H_N(\cdot,t_2)\|_{H^{-1}} \le C|t_2 - t_1|^{\frac{1}{2}}, \ \|\mathbf{E}_N(\cdot,t_1) - \mathbf{E}_N(\cdot,t_2)\|_{H^{-1}} \le C|t_2 - t_1|^{\frac{1}{2}}.$$

This lemma is proved. □

From ODE theory, Lemma 2.1- Lemma 2.3, we have the following lemma:

Lemma 2.4. *Under the conditions of Lemma 2.1, there exists a unique global solution* $(\alpha_{sN}(t), \beta_{sN}(t), \gamma_{sN}(t),$ $\zeta_{sN}(t))$ $(s = 1, 2, ..., N, \ t \in [0, T], \ \forall T > 0)$ *of the initial value problem for the ordinary differential equations* (2.7)-(2.11)*. Moreover, this solution is continuously differentiable.*

In the following, we will prove the existence of a global weak solution for (1.6)-(1.12).

The Proof of Theorem 1.1

From the uniform estimates of the approximate solution $\{\mathbf{M}_N(x,t), \theta_N(x,t), \mathbf{H}_N(x,t), \mathbf{E}_N(x,t)\}$ in Lemma 2.1 and Lemma 2.2, then by the Sobolev imbedding theorem and Lions-Aubin lemma, there is a subsequence which is still denoted by $\{\mathbf{M}_N(x,t), \theta_N(x,t), \mathbf{H}_N(x,t), \mathbf{E}_N(x,t)\}$ such that

$$\mathbf{M}_N(x,t) \rightharpoonup \mathbf{M}(x,t) \ \textit{weak} \ * \ \textit{in} \ L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{2.23}$$

$$\mathbf{M}_{Nt}(x,t) \rightharpoonup \mathbf{M}_t(x,t) \ \textit{weak} \ * \ \textit{in} \ L^\infty(0,T;H^{-1}(\Omega)), \tag{2.24}$$

$$\mathbf{M}_N(x,t) \to \mathbf{M}(x,t) \ \textit{strongly in} \ L^q(0,T;L^p(\Omega)), 2 \le q < \infty, \ \ 2 \le p \le \infty, \tag{2.25}$$

$$\theta_N(x,t) \rightharpoonup \theta(x,t) \ \textit{weak} \ * \ \textit{in} \ L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{2.26}$$

$$\theta_{Nt}(x,t) \rightharpoonup \theta_t(x,t) \ \textit{weak} \ * \ \textit{in} \ L^2(0,T;H^{-1}(\Omega)), \tag{2.27}$$

$$\theta_N(x,t) \to \theta(x,t) \ \textit{strongly in} \ L^2(0,T;L^2(\Omega)), \tag{2.28}$$

$$\mathbf{H}_N(x,t) \rightharpoonup \mathbf{H}(x,t) \ \textit{weak}^* \ \textit{in} \ L^\infty(0,T;L^2(\Omega)), \tag{2.29}$$

$$\mathbf{E}_N(x,t) \rightharpoonup \mathbf{E}(x,t) \ \textit{weak}^* \ \textit{in} \ L^\infty(0,T;L^2(\Omega)). \tag{2.30}$$

For any vector-valued test function $\phi(x,t) \in C^1([0,T]; H^1(\Omega))$ with $\phi(x,t)|_{t=T} = 0$, we define an approximate sequence as follows

$$\phi_N(x,t) = \sum_{n=1}^{N} a_n(t)\omega_n(x), \;\; a_n(t) = \int_\Omega \phi(x,t)\omega_n(x)dx.$$

We know that ϕ_N is uniformly convergent to $\phi(x,t)$ in $C^1([0,T]; H^1(\Omega))$, namely

$$\|\phi_N - \phi\|_{C^1([0,T];H^1(\Omega))} \to 0, \;\; as \;\; N \to \infty. \tag{2.31}$$

Taking the scaler product of $a_s(t)$ with (2.7),(2.8),(2.9), respectively, and the scaler product of $e^{\sigma t}a_s(t)$ with (2.10), summing up the products for all $s = 1, 2, \cdots, N$ and then integrating by parts, we get

$$\begin{aligned} &\gamma \iint_{Q_T} \mathbf{M}_N \cdot \phi_{Nt} dxdt - \nu \iint_{Q_T} \nabla \mathbf{M}_N \cdot \nabla \phi_N dxdt - \iint_{Q_T} \theta_c(|\mathbf{M}_N|^2 - 1)\mathbf{M}_N \cdot \phi_N dxdt \\ &\qquad - \iint_{Q_T} \theta_N \mathbf{M}_N \cdot \phi_N dxdt + \iint_{Q_T} \mathbf{H}_N \cdot \phi_N dxdt + \gamma \int_\Omega \mathbf{M}_N(x,0) \cdot \phi_N(x,0)dx = 0, \end{aligned} \tag{2.32}$$

$$\begin{aligned} &c \iint_{Q_T} \theta_N \cdot \phi_{Nt} dxdt + \iint_{Q_T} \mathbf{M}_N \mathbf{M}_{Nt} \cdot \phi_N dxdt - k \iint_{Q_T} \nabla \theta_N \cdot \nabla \phi_N dxdt \\ &\qquad + \iint_{Q_T} \hat{r}(x,t) \cdot \phi_N dxdt + c \int_\Omega \theta_N(x,0) \cdot \phi_N(x,0)dx = 0, \end{aligned} \tag{2.33}$$

$$\begin{aligned} &\mu \iint_{Q_T} \mathbf{H}_N \cdot \phi_{Nt}(x,t)dxdt + \iint_{Q_T} \mathbf{M}_N \cdot \phi_{Nt}(x,t)dxdt - \iint_{Q_T} (\nabla \times \phi_N) \cdot E_N(x)dxdt \\ &\qquad + \int_\Omega (\mu \mathbf{H}_N(x,0) + \mathbf{M}_N(x,0)) \cdot \phi_N(x,0)dx = 0, \end{aligned} \tag{2.34}$$

$$\iint_{Q_T} \mathbf{E}_N \cdot (\phi_{Nt} e^{\sigma t})dxdt + \iint_{Q_T} e^{\sigma t}(\nabla \times \phi_N) \cdot \mathbf{H}_N(x,t)dxdt + \int_\Omega \mathbf{E}_N(\cdot,0) \cdot \phi_N(\cdot,0)dx = 0. \tag{2.35}$$

From (2.23)-(2.31), it suffices to deal with the nonlinear terms in (2.32)-(2.35). From (2.23) and (2.31) we have

$$\begin{aligned} &\iint_{Q_T} (\nabla \mathbf{M}_N \cdot \nabla \phi_N - \nabla \mathbf{M} \cdot \nabla \phi)dxdt \\ &= \iint_{Q_T} (\nabla \mathbf{M}_N - \nabla \mathbf{M}) \cdot \nabla \phi_N + \nabla \mathbf{M}(\nabla \phi_N - \nabla \phi)dxdt \\ &\le \int_0^T \|\nabla \mathbf{M}_N - \nabla \mathbf{M}\|_2 \|\nabla \phi_N\|_2 dt + \int_0^T \|\nabla \mathbf{M}\|_2 \|\nabla \phi_N - \nabla \phi\|_2 dt \to 0, \;\; N \to \infty. \end{aligned}$$

From (2.23)-(2.25) and (2.31), we derive

$$\iint_{Q_T} (|\mathbf{M}_N|^2 - 1)\mathbf{M}_N \cdot \phi_N dxdt \to \iint_{Q_T} (|\mathbf{M}|^2 - 1)\mathbf{M} \cdot \phi dxdt, \;\; as \;\; N \to \infty.$$

By (2.25), (2.28) and (2.31) we obtain

$$\iint_{Q_T} \theta_N \mathbf{M}_N \cdot \phi_N dxdt \to \iint_{Q_T} \theta \mathbf{M} \cdot \phi dxdt, \;\; as \;\; N \to \infty.$$

It follows from (2.23)-(2.25) and (2.31) that

$$\iint_{Q_T} \mathbf{M}_N \mathbf{M}_{Nt} \cdot \phi_N dxdt \to \iint_{Q_T} \mathbf{M}\mathbf{M}_t \cdot \phi dxdt, \;\; as \;\; N \to \infty.$$

So by (2.30) and (2.31), we derive

$$\begin{aligned}
&\iint_{Q_T} (\nabla \times \phi_N) \cdot \mathbf{E}_N dxdt - \iint_{Q_T} (\nabla \times \phi) \cdot \mathbf{E} dxdt \\
=&\iint_{Q_T} \nabla \times (\phi_N - \phi) \cdot \mathbf{E}_N dxdt + \iint_{Q_T} \nabla \times \phi \cdot \mathbf{E}_N dxdt - \iint_{Q_T} (\nabla \times \phi) \cdot \mathbf{E} dxdt \\
=&\iint_{Q_T} \nabla \times (\phi_N - \phi) \cdot \mathbf{E}_N dxdt + \iint_{Q_T} (\nabla \times \phi) \cdot (\mathbf{E}_N - \mathbf{E}) dxdt \\
\leq&\Big(\iint_{Q_T} |\nabla(\phi_N - \phi)|^2 dxdt \Big)^{\frac{1}{2}} \|\mathbf{E}_N\|_{L^2(Q_T)} + \Big| \iint_{Q_T} (\nabla \times \phi) \cdot (\mathbf{E}_N - \mathbf{E}) dxdt \Big| \\
&\to 0, \ \ as \ \ N \to \infty.
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
&\iint_{Q_T} \mathbf{H}_N \cdot \phi_{Nt} dxdt \to \iint_{Q_T} \mathbf{H} \cdot \phi_t dxdt, \ \ as \ \ N \to \infty, \\
&\iint_{Q_T} \mathbf{E}_N \cdot (\phi_{Nt} e^{\sigma t}) dxdt \to \iint_{Q_T} \mathbf{E} \cdot (\phi_t e^{\sigma t}) dxdt, \ \ as \ \ N \to \infty, \\
&\iint_{Q_T} e^{\sigma t} (\nabla \times \phi_N) \cdot \mathbf{H}_N dxdt \to \iint_{Q_T} e^{\sigma t} (\nabla \times \phi) \cdot \mathbf{H} dxdt, \ \ as \ \ N \to \infty, \\
&\iint_{Q_T} \mathbf{M}_N \cdot \phi_{Nt} dxdt \to \iint_{Q_T} \mathbf{M} \cdot \phi_t dxdt, \ \ as \ \ N \to \infty, \\
&\iint_{Q_T} \theta_N \cdot \phi_{Nt} dxdt \to \iint_{Q_T} \theta \cdot \phi_t dxdt, \ \ as \ \ N \to \infty, \\
&\iint_{Q_T} \nabla\theta_N \cdot \nabla\phi_N dxdt \to \iint_{Q_T} \nabla\theta \cdot \nabla\phi dxdt, \ \ as \ \ N \to \infty.
\end{aligned}$$

Thus, taking $N \to \infty$ in (2.32), (2.33), (2.34) and (2.35), we obtain that the limit functions $\mathbf{M}(x,t)$, $\theta(x,t)$, $\mathbf{H}(x,t)$ and $\mathbf{E}(x,t)$ satisfy the integral equalities (2.1), (2.2), (2.3) and (2.4). Furthermore, if the initial vector functions $\mathbf{M}_0, \mathbf{H}_0, \mathbf{E}_0$ satisfy the conditions $\int_\Omega \nabla\xi \cdot (\mu\mathbf{H}_0 + \mathbf{M}_0)\, dx = 0$, $\int_\Omega \nabla\xi \cdot \mathbf{E}_0 \, dx = 0$ for all $\xi(x) \in C^1(\Omega)$, we can easily deduce that for any $\xi(x,t) \in C^1([0,T]; C^1(\Omega))$ with $\xi(x,T) = 0$ and $\xi_0 = \xi(x,0)$, we have from (2.3) and (2.4) that

$$\iint_{Q_T} \nabla\xi \cdot (\mu\mathbf{H} + \mathbf{M})\, dxdt = 0, \ \iint_{Q_T} \nabla\xi \cdot \mathbf{E}\, dxdt = 0.$$

Therefore, through the above analysis and calculations, the global weak solution of the problem (1.6)-(1.12) is obtained.

3 The existence of global smooth solution

To demonstrate the existence of a global smooth solution $(\mathbf{M}, \theta, \mathbf{H}, \mathbf{E})$ for problem (1.6)-(1.12), it is necessary to establish a priori estimates. In this section, we first consider the special case $d = 2$ and $x \in \Omega \subset \mathbb{R}^2$. Then we also find the estimates hold for $d = 3$ when the initial data is small.

Lemma 3.1. *Assume that* $(\mathbf{M}_0(x), \theta_0(x), \mathbf{H}_0(x), \mathbf{E}_0(x)) \in (H^2(\Omega), H^1(\Omega), H^1(\Omega), H^1(\Omega))$*,* $\hat{r}(x,t) \in L^2(0,T; H^1(\Omega))$*, then there exists a smooth solution* $(\mathbf{M}, \theta, \mathbf{H}, \mathbf{E})$ *for problem* (1.6)-(1.12) *satisfying the following* *estimates*

$$\sup_{0\le t\le T}\Big\{\|\mathbf{M}\|_{H^2}+\|\theta\|_{H^1}+\|\mathbf{H}\|_{H^1}+\|\mathbf{E}\|_{H^1}\Big\}$$
$$+\int_0^T[\|\nabla\Delta\mathbf{M}\|_2^2+\|\Delta\theta\|_2^2+\|\nabla\mathbf{E}\|_2^2]dt\le C. \tag{3.1}$$

Proof. Multiplying (1.6) by $\Delta^2\mathbf{M}$, and integrating the resulting equality with respect to $x\in\Omega$, we obtain

$$\begin{aligned}
&\frac{\gamma}{2}\frac{d}{dt}\|\Delta\mathbf{M}\|_2^2+\nu\|\nabla\Delta\mathbf{M}\|_2^2\\
&=\theta_c\int_\Omega\nabla[(|\mathbf{M}|^2-1)\mathbf{M}]\cdot\nabla\Delta\mathbf{M}dx+\int_\Omega\nabla(\theta_N\mathbf{M})\cdot\nabla\Delta\mathbf{M}dx-\int_\Omega\nabla\mathbf{H}\cdot\nabla\Delta\mathbf{M}dx\\
&\le\theta_c\Big(3\|\mathbf{M}\|_\infty^2+1\Big)\|\nabla\mathbf{M}\|_2\|\nabla\Delta\mathbf{M}\|_2+\Big(\|\mathbf{M}\|_\infty\|\nabla\theta\|_2+\|\theta\|_4\|\nabla\mathbf{M}\|_4\Big)\|\nabla\Delta\mathbf{M}\|_2\\
&\quad+\|\nabla\Delta\mathbf{M}\|_2\|\nabla\mathbf{H}\|_2\\
&\le C\Big(\|\mathbf{M}\|_2\|\Delta\mathbf{M}\|_2+1\Big)\|\nabla\mathbf{M}\|_2\|\nabla\Delta\mathbf{M}\|_2+C\Big(\|\mathbf{M}\|_2^{\frac12}\|\Delta\mathbf{M}\|_2^{\frac12}\|\nabla\theta\|_2\\
&\quad+\|\nabla\theta\|_2^{\frac12}\|\theta\|_2^{\frac12}\|\nabla\mathbf{M}\|_2^{\frac12}\|\Delta\mathbf{M}\|_2^{\frac12}\Big)\|\nabla\Delta\mathbf{M}\|_2+\|\nabla\Delta\mathbf{M}\|_2\|\nabla\mathbf{H}\|_2\\
&\le\frac{\nu}{16}\|\nabla\Delta\mathbf{M}\|_2^2+C(1+\|\Delta\mathbf{M}\|_2^4)+\frac{\nu}{16}\|\nabla\Delta\mathbf{M}\|_2^2+C(1+\|\Delta\mathbf{M}\|_2^2)\|\nabla\theta\|_2^2\\
&\quad+\frac{\nu}{16}\|\nabla\Delta\mathbf{M}\|_2^2+C(\|\nabla\theta\|_2^2+\|\Delta\mathbf{M}\|_2^2)+\frac{\nu}{16}\|\nabla\Delta\mathbf{M}\|_2^2+C\|\nabla\mathbf{H}\|_2^2\\
&\le\frac{\nu}{4}\|\nabla\Delta\mathbf{M}\|_2^2+C(\|\Delta\mathbf{M}\|_2^4+\|\Delta\mathbf{M}\|_2^2+\|\nabla\theta\|_2^4+\|\nabla\theta\|_2^2+\|\nabla\mathbf{H}\|_2^2+1),
\end{aligned}\tag{3.2}$$

where we have used Hölder inequality, Gagliardo-Nirenberg inequality and Lemma 2.1.

Multiplying (1.7) by $\Delta\theta$, and integrating the resulting equality with respect to $x\in\Omega$, we have

$$\begin{aligned}
&\frac{c}{2}\frac{d}{dt}\|\nabla\theta\|_2^2+k\|\Delta\theta\|_2^2\\
&=-\int_\Omega\mathbf{M}\cdot\mathbf{M}_t\cdot\Delta\theta dx-\int_\Omega\hat{r}\cdot\Delta\theta dx\\
&=-\frac{\nu}{\gamma}\int_\Omega\mathbf{M}\cdot\Delta\mathbf{M}\cdot\Delta\theta dx+\frac{\theta_c}{\gamma}\int_\Omega\mathbf{M}\cdot[(|\mathbf{M}|^2-1)\mathbf{M}]\cdot\Delta\theta dx\\
&\quad+\frac{1}{\gamma}\int_\Omega\mathbf{M}\cdot(\theta\mathbf{M})\cdot\Delta\theta dx-\frac{1}{\gamma}\int_\Omega\mathbf{M}\cdot\mathbf{H}\cdot\Delta\theta dx-\int_\Omega\hat{r}\cdot\Delta\theta dx.
\end{aligned}\tag{3.3}$$

By Lemma 2.1, Hölder inequality and Gagliardo-Nirenberg inequality, we derive

$$\begin{aligned}
\left|-\frac{\nu}{\gamma}\int_\Omega\mathbf{M}\cdot\Delta\mathbf{M}\cdot\Delta\theta dx\right|&\le C\|\mathbf{M}\|_\infty\|\Delta\mathbf{M}\|_2\|\Delta\theta\|_2\le C\|\mathbf{M}\|_2^{\frac12}\|\Delta\mathbf{M}\|_2^{\frac32}\|\Delta\theta\|_2\\
&\le\frac{k}{8}\|\Delta\theta\|_2^2+C(\|\Delta\mathbf{M}\|_2^4+1).
\end{aligned}\tag{3.4}$$

Similarly, we obtain

$$\begin{aligned}
\left|\frac{\theta_c}{\gamma}\int_\Omega\mathbf{M}\cdot[(|\mathbf{M}|^2-1)\mathbf{M}]\cdot\Delta\theta dx\right|&\le\frac{\theta_c}{\gamma}\Big(\|\mathbf{M}\|_\infty^4+\|\mathbf{M}\|_\infty^2\Big)\|\Delta\theta\|_2\\
&\le C\Big(\|\mathbf{M}\|_2^2\|\Delta\mathbf{M}\|_2^2+\|\mathbf{M}\|_2\|\Delta\mathbf{M}\|_2\Big)\|\Delta\theta\|_2\\
&\le\frac{k}{8}\|\Delta\theta\|_2^2+C\Big(\|\Delta\mathbf{M}\|_2^4+\|\Delta\mathbf{M}\|_2^2+1\Big),
\end{aligned}\tag{3.5}$$

$$\left|\frac{1}{\gamma}\int_{\Omega}\mathbf{M}\cdot(\theta\mathbf{M})\cdot\Delta\theta dx\right| \le C\|\mathbf{M}\|_\infty^2\|\theta\|_2\|\Delta\theta\|_2$$
$$\le C\|\mathbf{M}\|_2\|\Delta\mathbf{M}\|_2\|\theta\|_2\|\Delta\theta\|_2$$
$$\le\frac{k}{8}\|\Delta\theta\|_2^2 + C\|\Delta\mathbf{M}\|_2^2 \tag{3.6}$$

and

$$\left|-\frac{1}{\gamma}\int_{\Omega}\mathbf{M}\cdot\mathbf{H}\cdot\Delta\theta dx\right| \le \frac{1}{\gamma}\|\mathbf{M}\|_\infty\|\mathbf{H}\|_2\|\Delta\theta\|_2 \le \frac{k}{8}\|\Delta\theta\|_2^2 + C\Big(\|\Delta\mathbf{M}\|_2^2 + 1\Big), \tag{3.7}$$

$$\left|\int_{\Omega}\hat{r}\cdot\Delta\theta dx\right| \le C\|\nabla\hat{r}\|_2\|\nabla\theta\|_2 \le C(\|\nabla\theta\|_2^2 + 1). \tag{3.8}$$

Thus, inserting estimates (3.4)-(3.8) into (3.3), we have

$$\frac{c}{2}\frac{d}{dt}\|\nabla\theta\|_2^2 + \frac{k}{2}\|\Delta\theta\|_2^2 \le C(\|\nabla\theta\|_2^2 + \|\Delta\mathbf{M}\|_2^2 + \|\Delta\mathbf{M}\|_2^4 + 1). \tag{3.9}$$

Now taking the product of $\Delta\mathbf{H}$ with (1.8) and the product of $\Delta\mathbf{E}$ with (1.9) respectively, and summing the two resulting equalities, and then integrating the final equality with respect to $x \in \Omega$, we obtain the following inequality

$$\frac{1}{2}\frac{d}{dt}(\mu\|\nabla\mathbf{H}\|_2^2 + \|\nabla\mathbf{E}\|_2^2) + \sigma\|\nabla\mathbf{E}\|_2^2 = \int_{\Omega}\nabla\mathbf{M}_t\cdot\nabla\mathbf{H}dx \le \|\nabla\mathbf{M}_t\|_2\|\nabla\mathbf{H}\|_2$$
$$\le\frac{1}{\gamma}(\nu\|\nabla\Delta\mathbf{M}\|_2 + \theta_c\|\nabla[(|\mathbf{M}|^2-1)\mathbf{M}]\|_2 + \|\nabla\mathbf{M}\|_\infty\|\theta\|_2 + \|\mathbf{M}\|_\infty\|\nabla\theta\|_2 + \|\nabla\mathbf{H}\|_2)\|\nabla\mathbf{H}\|_2$$
$$\le\frac{\nu}{4}\|\nabla\Delta\mathbf{M}\|_2^2 + C(\|\nabla\theta\|_2^4 + \|\Delta\mathbf{M}\|_2^4 + \|\Delta\mathbf{M}\|_2^2 + \|\nabla\mathbf{H}\|_2^2 + 1), \tag{3.10}$$

where we have used Hölder inequality and Sobolev imbedding theorem.

Thus, combining (3.2), (3.9) and (3.10), we derive

$$\frac{1}{2}\frac{d}{dt}\left(\gamma\|\Delta\mathbf{M}\|_2^2 + c\|\nabla\theta\|_2^2 + \mu\|\nabla\mathbf{H}\|_2^2 + \|\nabla\mathbf{E}\|_2^2\right) + \frac{\nu}{2}\|\nabla\Delta\mathbf{M}\|_2^2 + \frac{k}{2}\|\Delta\theta\|_2^2 + \sigma\|\nabla\mathbf{E}\|_2^2$$
$$\le C(\|\Delta\mathbf{M}\|_2^4 + \|\Delta\mathbf{M}\|_2^2 + \|\nabla\theta\|_2^4 + \|\nabla\theta\|_2^2 + \|\nabla\mathbf{H}\|_2^2 + 1). \tag{3.11}$$

By employing the estimates (2.15) and (3.11) and the Gronwall inequality, we establish the estimate (3.1). □

Lemma 3.2. *Assume that* $(\mathbf{M}_0(x), \theta_0(x), \mathbf{H}_0(x), \mathbf{E}_0(x)) \in (H^{m+1}(\Omega), H^m(\Omega), H^m(\Omega), H^m(\Omega)), \hat{r} \in L^2(0,T;$ $H^m(\Omega))$, $m \ge 0$, *then there exists a smooth solution* $(\mathbf{M}, \theta, \mathbf{H}, \mathbf{E})$ *for problem* (1.6)-(1.12) *satisfying the following estimates*

$$\sup_{0\le t\le T}\Big[\|\mathbf{M}(\cdot,t)\|_{H^{m+1}}^2 + \|\theta(\cdot,t)\|_{H^m}^2 + \|\mathbf{H}(\cdot,t)\|_{H^m}^2 + \|\mathbf{E}(\cdot,t)\|_{H^m}^2\Big]$$
$$+\int_0^T(\|\mathbf{M}\|_{H^{m+2}}^2 + \|\theta\|_{H^{m+1}}^2 + \|\mathbf{E}\|_{H^m}^2)dt \le C. \tag{3.12}$$

Proof. The lemma will be proved by the induction for m. According to Lemma 2.1 and Lemma 3.1, the estimate (3.12) holds when $m = 0, 1$.

Now assume that the estimate (3.12) holds for $m = K \geq 2$, that is

$$\sup_{0\le t\le T}\Big[\|\mathbf{M}(\cdot,t)\|_{H^{K+1}}^2+\|\theta(\cdot,t)\|_{H^K}^2+\|\mathbf{E}(\cdot,t)\|_{H^K}^2+\|\mathbf{H}(\cdot,t)\|_{H^K}^2\Big]$$
$$+\int_0^T(\|\mathbf{M}\|_{H^{K+2}}^2+\|\theta\|_{H^{K+1}}^2+\|\mathbf{E}\|_{H^K}^2)dt\le C. \tag{3.13}$$

We aim to prove that (3.12) holds for $m = K + 1$.

Taking the scalar product of $\Delta^{K+2}\mathbf{M}$ with (1.6), and integrating the resulting equality with respect to $x \in \Omega$, we derive

$$\begin{aligned}&\frac{\gamma}{2}\frac{d}{dt}\|\nabla^{K+2}\mathbf{M}\|_2^2dx+\nu\|\nabla^{K+3}\mathbf{M}\|_2^2\\&=\theta_c\int_\Omega\nabla^{K+1}[(|\mathbf{M}|^2-1)\mathbf{M}]\cdot\nabla^{K+3}\mathbf{M}dx+\int_\Omega\nabla^{K+1}(\theta\mathbf{M})\cdot\nabla^{K+3}\mathbf{M}dx\\&\quad-\int_\Omega\nabla^{K+1}\mathbf{H}\cdot\nabla^{K+3}\mathbf{M}dx,\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}&\int_\Omega\nabla^{K+1}[(|\mathbf{M}|^2-1)\mathbf{M}]\cdot\nabla^{K+3}\mathbf{M}dx\\&\le\|\nabla^{K+1}(|\mathbf{M}|^2\mathbf{M})\|_2\|\nabla^{K+3}\mathbf{M}\|_2+\|\nabla^{K+1}\mathbf{M}\|_2\|\nabla^{K+3}\mathbf{M}\|_2\\&\le\|\sum_{i=0}^{K+1}\sum_{j=0}^{i}C_{K+1}^iC_i^j\nabla^j\mathbf{M}\nabla^{K+1-i}\mathbf{M}\nabla^{i-j}\mathbf{M}\|_2\|\nabla^{K+3}\mathbf{M}\|_2+\|\nabla^{K+1}\mathbf{M}\|_2\|\nabla^{K+3}\mathbf{M}\|_2\\&\le C\sum_{i_1+i_2+j_3=K+1}\|\nabla^{i_1}\mathbf{M}\|_6\|\nabla^{i_2}\mathbf{M}\|_6\|\nabla^{i_3}\mathbf{M}\|_6\|\nabla^{K+3}\mathbf{M}\|_2\\&\le\frac{\nu}{18}\|\nabla^{K+3}\mathbf{M}\|_2^2+C(1+\|\nabla^{K+2}\mathbf{M}\|_2^2).\end{aligned}\tag{3.15}$$

Similarly, we have

$$\begin{aligned}\int_\Omega\nabla^{K+1}(\theta\mathbf{M})\cdot\nabla^{K+3}\mathbf{M}dx=&C\sum_{i=0}^{K+1}\|\nabla^i\theta\|_3\|\|\nabla^{K+1-i}\mathbf{M}\|_6\|\nabla^{K+3}\mathbf{M}\|_2\\&\le\frac{\nu}{18}\|\nabla^{M+3}\mathbf{M}\|_2^2+C(1+\|\nabla^{K+1}\theta\|_2^2).\end{aligned}\tag{3.16}$$

Hence, combining (3.14)–(3.16), we have

$$\begin{aligned}&\frac{\gamma}{2}\frac{d}{dt}\|\nabla^{K+2}\mathbf{M}\|_2^2+\nu\|\nabla^{K+2}\|_2^2\\&\le\frac{\nu}{6}\|\nabla^{K+3}\mathbf{M}\|_2^2+C(1+\|\nabla^{K+1}\theta\|_2^2+\|\nabla^{K+2}\mathbf{M}\|_2^2+\|\nabla^{K+1}\mathbf{H}\|_2^2).\end{aligned}\tag{3.17}$$

Taking the scalar product of $\Delta^{K+1}\theta$ with (1.7), and integrating the resulting equality with respect to

$x \in \Omega$, we obtain

$$\begin{aligned}
&\frac{c}{2}\frac{d}{dt}\|\nabla^{K+1}\theta\|_2^2 + k\|\nabla^{K+2}\theta\|_2^2 \\
\leq &\int_\Omega \mathbf{M}\mathbf{M}_t \cdot \nabla^{K+2}\theta dx + \int_\Omega \hat{r} \cdot \nabla^{K+2}\theta dx \\
\leq &\|\mathbf{M}\|_\infty \|\mathbf{M}_t\|_2 \|\nabla^{K+2}\theta\|_2 + \|\hat{r}\|_2 \|\nabla^{K+2}\theta\|_2 \\
\leq &C\|\Delta\mathbf{M}\|_2^{\frac{1}{2}} \|\mathbf{M}\|_2^{\frac{1}{2}} (\nu\|\Delta\mathbf{M}\|_2 + \theta_c\|(|\mathbf{M}|^2-1)\mathbf{M}\|_2 \\
&+ \|\theta\|_4\|\mathbf{M}\|_4 + \|\mathbf{H}\|_2)\|\nabla^{K+2}\theta\|_2 + \|\nabla^{K+1}\hat{r}\|_2\|\nabla^{K+2}\theta\|_2 \\
\leq &\frac{\nu}{6}\|\nabla^{K+3}\mathbf{M}\|_2^2 + \frac{k_1}{2}\|\nabla^{K+2}\theta\|_2^2 + C(1 + \|\nabla^{K+2}\mathbf{M}\|_2^2 + \|\nabla^{K+1}\theta\|_2^2 + \|\nabla^{K+1}\hat{r}\|_2^2). \qquad (3.18)
\end{aligned}$$

Taking the scalar product of $\Delta^{K+1}E$ with (1.8) and the scalar product of $\Delta^{K+1}H$ with (1.9), summing the two equalities, and then integrating the resulting equality with respect to $x \in \Omega$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla^{K+1}\mathbf{E}\|_2^2 + \mu\|\nabla^{K+1}\mathbf{H}\|_2^2) + \sigma\|\nabla^{K+1}\mathbf{E}\|_2^2 = \int_\Omega \nabla^{K+1}\dot{\mathbf{M}} \cdot \nabla^{K+1}\mathbf{H} dx \\
\leq &\|\nabla^{K+1}\dot{\mathbf{M}}\|_2\|\nabla^{K+1}\mathbf{H}\|_2 \\
\leq &(\nu\|\nabla^{K+3}\mathbf{M}\|_2 + \theta_c\|\nabla^{K+1}[(|\mathbf{M}|^2-1)\mathbf{M}]\|_2 \\
&+ 2\|\nabla^{K+1}(\mathbf{M}\theta)\|_2 + \|\nabla^{K+1}\mathbf{H}\|_2)\|\nabla^{K+1}\mathbf{H}\|_2 \\
\leq &\frac{\nu}{6}\|\nabla^{K+3}\mathbf{M}\|_2^2 + C(1 + \|\nabla^{K+1}\mathbf{H}\|_2^2 + \|\nabla^{K+2}\mathbf{M}\|_2^4 + \|\nabla^{K+1}\theta\|_2^2). \qquad (3.19)
\end{aligned}$$

It follows from (3.17)-(3.19), that

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla^{K+2}\mathbf{M}\|_2^2 + \|\nabla^{K+1}\theta\|_2^2 + \|\nabla^{K+1}\mathbf{E}\|_2^2 + \|\nabla^{K+1}\mathbf{H}\|_2^2\Big) + \|\nabla^{K+2}\|_2^2 + \|\nabla^{K+2}\theta\|_2^2 + \|\nabla^{K+1}\mathbf{E}\|_2^2 \\
\leq &C(1 + \|\nabla^{K+1}\theta\|_2^2 + \|\nabla^{K+2}\mathbf{M}\|_2^2 + \|\nabla^{K+2}\mathbf{M}\|_2^4 + \|\nabla^{K+1}\mathbf{H}\|_2^2). \qquad (3.20)
\end{aligned}$$

Therefore, by employing (3.13) and applying Gronwall inequality, we can establish the estimate (3.12). □

Following a similar approach to the proof of Theorem 1.1, we can establish the local existence of the smooth solution to (1.6)-(1.12). Subsequently, by employing a priori estimates for the smooth solution, we can deduce the global existence of a smooth solution to the problem (1.6)-(1.12).

Remark 3.2. *When the dimension is set to $d = 3$, Lemma 3.1 and Lemma 3.2 can also be proven, provided that $\|\mathbf{M}_0\|_{H^1}^2$ is sufficiently small. Subsequently, by replicating the methodologies employed in the proof for the case $d = 2$, we can establish the existence of the solution $(\mathbf{M}, \theta, \mathbf{H}, \mathbf{E})$ for the $d = 3$ dimension.*

4 The uniqueness of global smooth solution

In this section, we are devoted to proving uniqueness. Let $(\mathbf{M}_j, \theta_j, \mathbf{H}_j, \mathbf{E}_j)$ $(j = 1, 2)$ be the smooth solutions for the problem (1.6)-(1.12). Denote $(\mathbf{M}, \theta, \mathbf{H}, \mathbf{E}) = (\mathbf{M}_1 - \mathbf{M}_2, \theta_1 - \theta_2, \mathbf{H}_1 - \mathbf{H}_2, \mathbf{E}_1 - \mathbf{E}_2)$. As a result, $(\mathbf{M}, \theta, \mathbf{H}, \mathbf{E})$ satisfies the following system:

$$\gamma \mathbf{M}_t = \nu\Delta\mathbf{M} - \theta_c((|\mathbf{M}_1|^2 - 1)\mathbf{M} + (\mathbf{M}_1 + \mathbf{M}_2)\mathbf{M}\mathbf{M}_2) - (\theta\mathbf{M}_1 + \theta_2\mathbf{M}) + \mathbf{H}, \tag{4.1}$$

$$c\theta_t = \mathbf{M}\mathbf{M}_{1t} + \mathbf{M}_2\mathbf{M}_t + k\Delta\theta, \tag{4.2}$$

$$\mu\mathbf{H}_t + \mathbf{M}_t = -\nabla \times \mathbf{E}, \tag{4.3}$$

$$\mathbf{E}_t + \sigma\mathbf{E} = \nabla \times \mathbf{H}, \tag{4.4}$$

$$\nabla \cdot (\mu\mathbf{H} + \mathbf{M}) = 0, \nabla \cdot \mathbf{E} = 0, \tag{4.5}$$

with periodic conditions

$$\begin{aligned} &\mathbf{M}(x + 2De_i, t) = \mathbf{M}(x, t), \quad \theta(x + 2De_i, t) = \theta(x, t), \\ &\mathbf{H}(x + 2De_i, t) = \mathbf{H}(x, t), \quad \mathbf{E}(x + 2De_i, t) = \mathbf{E}(x, t), \end{aligned} \tag{4.6}$$

and initial conditions

$$\mathbf{M}(x, 0) = 0, \quad \theta(x, 0) = 0, \quad \mathbf{H}(x, 0) = 0, \quad \mathbf{E}(x, 0) = 0. \tag{4.7}$$

Taking the scalar product of equation (4.1) with $\mathbf{M} - \Delta\mathbf{M}$, then integrating the equality obtained over Ω, we derive

$$\begin{aligned} &\frac{\gamma}{2}\frac{d}{dt}\{\|\mathbf{M}\|_2^2 + \|\nabla\mathbf{M}\|_2^2\} + \nu\{\|\nabla\mathbf{M}\|_2^2 + \|\Delta\mathbf{M}\|_2^2\} \\ =& -\theta_c \int_\Omega ((|\mathbf{M}_1|^2 - 1)\mathbf{M} + (\mathbf{M}_1 + \mathbf{M}_2)\mathbf{M}\mathbf{M}_2)(\mathbf{M} - \Delta\mathbf{M})dx \\ &- \int_\Omega (\theta\mathbf{M}_1 + \theta_2\mathbf{M})(\mathbf{M} - \Delta\mathbf{M})dx + \int_\Omega \mathbf{H}(\mathbf{M} - \Delta\mathbf{M})dx \\ \le& C(\|\mathbf{M}_1\|_\infty^2 + \|\mathbf{M}_2\|_\infty^2 + 1)\|\mathbf{M}\|_2(\|\mathbf{M}\|_2 + \|\Delta\mathbf{M}\|_2) \\ &+ (\|\mathbf{M}_1\|_\infty\|\theta\|_2 + \|\theta_2\|_2\|\mathbf{M}\|_\infty)(\|\mathbf{M}\|_2 + \|\Delta\mathbf{M}\|_2) + \|\mathbf{H}\|_2(\|\mathbf{M}\|_2 + \|\Delta\mathbf{M}\|_2)\}. \end{aligned}$$

By Gagliardo-Nirenberg inequality, we get

$$\|\mathbf{M}\|_\infty \le C\|\mathbf{M}\|_2^{1-\frac{d}{4}}\|\Delta\mathbf{M}\|_2^{\frac{d}{4}}. \tag{4.8}$$

Then it follows from the estimates (3.12) and (4.8) that

$$\begin{aligned} &\frac{\gamma}{2}\frac{d}{dt}\{\|\mathbf{M}\|_2^2 + \|\nabla\mathbf{M}\|_2^2\} + \nu\{\|\nabla\mathbf{M}\|_2^2 + \|\Delta\mathbf{M}\|_2^2\} \\ \le& \frac{\nu}{6}\|\Delta\mathbf{M}\|_2^2 + C(\|\mathbf{M}\|_2^2 + \|\theta\|_2^2 + \|\mathbf{H}\|_2^2). \end{aligned} \tag{4.9}$$

By taking the scalar product of equation (4.2) with θ, and integrating the equality with respect to $x \in \Omega$, we have

$$\begin{aligned} &\frac{c}{2}\frac{d}{dt}\|\theta\|_2^2 + k\|\nabla\theta\|_2^2 = \int_\Omega (\mathbf{M}\mathbf{M}_{1t} + \mathbf{M}_2\mathbf{M}_t)\theta dx \\ \le& C\|\mathbf{M}\|_\infty\Big(\|\Delta\mathbf{M}_1\|_2 + \theta_c\|(|\mathbf{M}_1|^2 - 1)\|_\infty\|\mathbf{M}_1\|_2 + \|\mathbf{M}_1\|_\infty\|\theta_1\|_2 + \|\mathbf{H}\|_2\Big)\|\theta\|_2 \\ &+ C\|\mathbf{M}_2\|_\infty\Big(\|\Delta\mathbf{M}\|_2 + \theta_c(\|\mathbf{M}_1\|_\infty^2 + 1)\|\mathbf{M}\|_2 + \|\mathbf{M}_1\|_\infty\|\theta\|_2 + \|\theta_2\|_2\|\mathbf{M}\|_\infty + \|\mathbf{H}\|_2\Big)\|\theta\|_2 \\ \le& \frac{\nu}{6}\|\Delta\mathbf{M}\|_2^2 + C(\|\mathbf{M}\|_2^2 + \|\theta\|_2^2 + \|\mathbf{H}\|_2^2), \end{aligned} \tag{4.10}$$

where we have used the estimates (3.12) and (4.8).

Next, by taking the scalar product of equation (4.3) with $\mathbf{H}$, and the scalar product of equation (4.4) with $\mathbf{E}$, summing the two equalities, then integrating the resulting equality with respect to $x \in \Omega$, and applying the estimates (3.12) and (4.8), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\mu\|\mathbf{H}\|_2^2 + \|\mathbf{E}\|_2^2\} + \sigma\|\mathbf{E}\|_2^2 = -\int_\Omega \mathbf{M}_t\mathbf{H}dx \\
\leq &C\Big(\|\Delta\mathbf{M}\|_2 + (\|\mathbf{M}_1\|_\infty^2 + 1)\|\mathbf{M}\|_2 + \|\mathbf{M}_1\|_\infty\|\theta\|_2 + \|\theta_2\|_2\|\mathbf{M}\|_\infty + \|\mathbf{H}\|_2\Big)\|\mathbf{H}\|_2 \\
\leq &\frac{\nu}{6}\|\Delta\mathbf{M}\|_2^2 + C(\|\mathbf{M}\|_2^2 + \|\theta\|_2^2 + \|\mathbf{H}\|_2^2).
\end{aligned} \tag{4.11}$$

By adding (4.9), (4.10) and (4.11), then applying the Gronwall inequality, we derive

$$\|\mathbf{M}\|_2^2 + \|\nabla\mathbf{M}\|_2^2 + \|\theta\|_2^2 + \|\mathbf{H}\|_2^2 + \|\mathbf{E}\|_2^2 = 0. \tag{4.12}$$

Thus the global solution $(\mathbf{M}, \theta, \mathbf{E}, \mathbf{H})$ is unique for $m \geq 1$.

Therefore by the above uniqueness result and existence result established in Section 3, we complete the proof of Theorem 1.2.

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